International Macroeconomics: Lecture 2

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Recap from last time

First with perfect foresight and two periods; then with stochastic rational expectations and infinite horizons:

- Temporary positive (negative) shocks lead to temporary current account surpluses (deficits)
- Permanent positive (negative) shocks do not affect the current account
- Cant explain persistence of current account without investment or other types of mechanisms

Today

- Puzzles:
  - Persistence
  - Deaton’s Paradox
  - Feldstein-Horioka
  - Small Response to Shocks

- Theory:
  - Large Country Case
  - Stochastic Calculus
  - Kraay and Ventura ”New Rule”

Deaton's Paradox (I)

- From last time, we have:

$$C_s = \overline{Y} + \frac{r}{1+r-\rho}\left(Y_s - \overline{Y}\right) = \overline{Y} + \frac{r}{1+r-\rho}\left(\rho\left(Y_{s-1} - \overline{Y}\right) + \varepsilon_s\right)$$

- This suggests that consumption (and the current account) should react less to shocks (and thus than current income.

- However, Deaton (1992) – in his book "Understanding Consumption" – notes that we can not reject mean reversion in growth rates not levels of GDP!

Deaton’s Paradox (II)

- In the case of mean reversion in growth rates, our stochastic income process is:

$$Y_{t+1} - Y_t = \rho\left(Y_t - Y_{t-1}\right) + \varepsilon_{t+1}$$

$$= \rho\left[\rho\left(Y_{t-1} - Y_{t-2}\right) + \varepsilon_t\right] + \varepsilon_{t+1} = \rho^2\left(Y_{t-1} - Y_{t-2}\right) + \rho\varepsilon_t + \varepsilon_{t+1}$$

- This stochastic income process is integrated of order one: I(1). In other words, it is non-stationary. Shocks to levels are more than permanent. We can rewrite current income as:

$$Y_{t+1} = Y_{t+1} + \sum_{s=-\infty}^{t+1} \rho^{t+1-s} \varepsilon_s$$

Deaton's Paradox (III)

- Calculating the new consumption function :

$$C_s = \frac{r}{1+r}\sum_{t=s}^{\infty} E_t\left(\frac{1}{1+r}\right)^{t-s} Y_t = \frac{r}{1+r}\sum_{t=s}^{\infty}\left(\frac{1}{1+r}\right)^{t-s}\left(Y_{s-1} + \sum_{k=s}^{t}\rho^{t-k}\varepsilon_s\right)$$

- Re-expressing, we get:

$$\frac{r}{1+r}\sum_{t=s}^{\infty}\left(\frac{1}{1+r}\right)^{t-s}\left(Y_{s-1} + \frac{1}{1-\rho^{t-s}}\varepsilon_s\right)$$

- However, we can bound this:

$$Y_{s-1} + \sum_{t=s}^{\infty}\left(\frac{1}{1+r}\right)^{t}\frac{1}{1-\rho^{t-s}}\varepsilon_s > Y_{s-1} + \sum_{t=s}^{\infty}\left(\frac{1}{1+r}\right)^{t}\varepsilon_s = Y_{s-1} + \frac{1+r}{r}\varepsilon_s$$

Deaton’s Paradox (IV)

- So, we have an anti-Keynesian consumption function (What about in a closed economy?):

$$C_s > Y_{s-1} + \frac{1+r}{r}\varepsilon_s$$

- Consumption (and thus the current account) should be much more volatile than income in this non-stationary case but empirically, they are not.

- Question: Why are consumption functions Keynesian?

Large Open Economies: 2 Period Model (I)

- Suppose we have investment and savings in an equilibrium model; then:

$$S_1(r) + S_1^*(r) = I_1(r) + I_1^*(r)$$

- Walrasian stability of the equilibrium implies that a small rise in the interest rate around the equilibrium should lead to excess supply of savings:

$$S_1^{'}(r) + S_1^{*'}(r) - I_1^{'}(r) - I_1^{*'}(r) > 0$$

Large Open Economies: 2 Period Model (II)

- We now define exports and imports of intertemporal goods:

$$M_1 = C_1 + I_1 - Y_1 = I_1 - S_1$$

$$X_2 = Y_2 - C_2 - I_2 = I_2 - S_2$$

- Balanced intertemporal trade then implies:

$$(1+r)M_1 = X_2$$

Large Open Economies: 2 Period Model (III)

- Going back to the stability condition, we can write it as:

$$\left(\frac{1}{1+r}\right)M_2^*(r) - X_2(r) = 0$$

$$\frac{d}{dr}\left[\left(\frac{1}{1+r}\right)M_2^*(r) - M_1(r)\right] > 0$$

- Define import elasticities:

$$\zeta = -\frac{(1+r)M_1^{'}(r)}{M_1(r)}, \zeta^* = \frac{(1+r)M_2^{*'}(r)}{M_2^*(r)}$$

Large Open Economies: 2 Period Model (IV)

- Taking the derivative with respect to r:

$$-\frac{1}{(1+r)^2}M_2^*(r)+\frac{M_2^{*'}(r)}{1+r}-M_1^{'}(r)>0$$

- Multiplying through by (1+r)(1+r):

$$-M_2^*(r)+(1+r)M_2^{*'}(r)-(1+r)^2M_1^{'}(r)>0$$

- Dividing through by second period foreign exports:

$$-1+\frac{(1+r)M_2^{*'}(r)}{M_2^*(r)}-\frac{(1+r)^2M_1^{'}(r)}{M_2^*(r)}>0$$

Large Open Economies: 2 Period Model (V)

- Substituting for second period foreign exports:

$$M_2^*(r) = (1+r)X_1^* = (1+r)M_1$$

- This leaves us with:

$$-1 + \frac{(1+r)M_2^{*'}(r)}{M_2^*(r)} - \frac{(1+r)M_1^{'}(r)}{M_1(r)} > 0$$

- Finally, substituting the elasticity expressions and rearranging: $\zeta^* + \zeta > 1$

Optimal Control Theory: Certainty (I)

- Maximization Problem:

$$\max_{c(t),k(t)} \int_0^\infty e^{-\delta t} U[c(t),k(t)]dt$$

- Subject to:

$$\dot{k}(t) = G[c(t),k(t)]$$

$$k(0) = \bar{k}$$

Note: we are omitting the transversality condition for the sake of convenience; obviously, we still need it to get the optimal solution

Optimal Control Theory: Certainty (II)

- Optimal Control Definitions:

$$c^*(t) = control$$

$$k^*(t) = state$$

- Value Function Definition:

$$J[k(s)] = \int_s^\infty e^{-\delta(t-s)} U\left[c^*(t), k^*(t)\right]$$

- Bellman's Principle (when does it hold?):

$$J[k(s)] = \int_s^T e^{-\delta(t-s)} U\left[c^*(t), k^*(t)\right] + e^{-\delta(T-s)} J[k(T)]$$

Optimal Control Theory: Certainty (III)

- Theorem
  - Let $c^*(t)$ solve the problem of maximizing:

$$\max_{c(t),k(t)} \int_0^\infty e^{-\delta t} U[c(t),k(t)]dt$$

  - Subject to $\dot{k}(t) = G[c(t),k(t)]$ given $k(0)$

  - Then there exists a costate variable $\lambda(t)$ such that the Hamiltonian is defined by:

$$H[c,k(t),\lambda(t)] = U[c,k(t)] + \lambda(t)G[c,k(t)]$$

Optimal Control Theory: Certainty (IV)

- Theorem (cont.)
  - is maximized at $c = c^*(t)$ given $\lambda(t), k(t)$
  - in other words:

$$\frac{\partial H}{\partial c}\left(c^*, k, \lambda\right) = U_c\left(c^*, k\right) + \lambda G_c\left(c^*, k\right) = 0$$

$$\dot{\lambda} = \lambda\delta - \frac{\partial H\left[c^*, k, \lambda\right]}{\partial k} = \lambda\delta - U_k\left[c^*, k\right] + \lambda G_k\left[c^*, k\right]$$

Optimal Control Theory: Certainty (V)

- Why is this true? Take a discrete analogue to:

$$\max_{c(t),k(t)} \int_0^\infty e^{-\delta t} U[c(t), k(t)]dt \text{ s.t. } \dot{\mathrm{k}} = \mathrm{G}(\mathrm{c(t), k(t)})$$

- Defining each period as length h (where the continuous case would result in the limit as h goes to zero), it would be

$$\max_{c(t),k(t)} \sum_{t=0}^{\infty} e^{-\delta th} U[c(t), k(t)]h$$

- subject to:

$$\mathrm{k}(t+h) - k(t) = h\mathrm{G}(\mathrm{c(t), k(t)})$$

Optimal Control Theory: Certainty (VI)

- Then, from Bellman's principle, we get:

$$J\left[k(t)\right] = \max_{c(t)} \left\{ U\left[c(t), k(t)\right] h + e^{-\delta h} J\left[k(t+h)\right] \right\}$$

- Subtracting J from each side:

$$0 = \max_{c(t)} \left\{ U\left[c(t), k(t)\right] h + e^{-\delta h} J\left[k(t+h)\right] - J\left[k(t)\right] \right\}$$

- Replace the future capital stock using the capital accumulation equation:

$$0 = \max_{c(t)} \left\{ U\left[c(t), k(t)\right] h + e^{-\delta h} J\left[k(t) + hG\left[c(t), k(t)\right]\right] - J\left[k(t)\right] \right\}$$

Optimal Control Theory: Certainty (VII)

- Take the power series representation of the discount factor:

$$e^{-\delta h} = 1 - \delta h + \frac{(\delta h)^2}{2} + \sum_{i=3}^{\infty} (-1)^i \frac{(\delta h)^i}{i!}$$

$$0 = \max_{c(t)} \left\{ \begin{array}{l} U[c(t),k(t)]h - \left[\delta h - \frac{(\delta h)^2}{2} + \ldots \right] J[k(t) + hG[c(t),k(t)]] + \\ J[k(t) + hG[c(t),k(t)]] - J[k(t)] \end{array} \right\}$$

- Divide by h:

$$0 = \max_{c(t)} \left\{ \begin{array}{l} U[c(t),k(t)] - \left[\delta - \frac{\delta^2 h}{2} + \ldots \right] J[k(t) + hG[c(t),k(t)]] + \\ \frac{J[k(t) + hG[c(t),k(t)]] - J[k(t)]}{h} \end{array} \right\}$$

Optimal Control Theory: Certainty (VIII)

- Take the limit as h goes to zero:

$$0=\lim_{h\to 0}\max_{c(t)}\left\{\begin{array}{l} U\left[c(t),k(t)\right]-\left[\delta-\dfrac{\delta^{2}h}{2}+\ldots\right]J\left[k(t)+hG\left[c(t),k(t)\right]\right]+ \\ \dfrac{J\left[k(t)+hG\left[c(t),k(t)\right]\right]-J\left[k(t)\right]}{h} \end{array}\right\}$$

- Multiply and divide by G:

$$0=\max_{c(t)}\left\{\begin{array}{l} U\left[c(t),k(t)\right]-\delta J\left[k(t)\right]+ \\ \lim\limits_{h\to 0}\dfrac{\left(J\left[k(t)+hG\left[c(t),k(t)\right]\right]-J\left[k(t)\right]\right)G\left[c(t),k(t)\right]}{hG\left[c(t),k(t)\right]} \end{array}\right\}$$

Optimal Control Theory: Certainty (IX)

- Take the limit of the last term as h goes to zero and supress the time subscripts:

$$0 = \max_{c}\{U[c,k] - \delta J[k] + J'[k]G[c,k]\}$$

$$\delta J[k] = \max_{c}\{U[c,k] + J'[k]G[c,k]\}$$

- Interpretation:
  - (1.) Maximizing life-time utility is the same as maximizing current utility given the stock of wealth (k) plus the change in the capital stock (G) multiplied by the marginal impact of an increase in the capital stock on wealth (J'(k)).
  - (2.) The maximized net flow of utility in a given period is a fraction (depending upon the discount factor) of the total. Reminiscent of???

Optimal Control Theory: Certainty (X)

- We can now take derivatives of the un-maximized value function to get the Pontryagin necessary conditions. We start with maximizing with respect to c:

$$U'[c,k] = -J'[k]G_c[c,k]$$

- Now, we maximize with respect to k. But c is a function of the current stock of k. So, we know there will be a policy function: c(k)

$$U[c(k),k] - \delta J[k] + J'[k]G[c(k),k]$$

Optimal Control Theory: Certainty (XI)

- Maximizing this function with respect to k, we obtain:

$$U_c\left[c(k),k\right]c'\left(k\right)+U_k\left[c(k),k\right]-\delta J'\left[k\right]+J''\left[k\right]G\left[c(k),k\right]+$$
$$J'\left[k\right]G_c\left[c(k),k\right]c'(k)+J'\left[k\right]G_k\left[c(k),k\right]=0$$

- Combining like terms:

$$\left\{U_c\left[c(k),k\right]+J'\left[k\right]G_c\left[c(k),k\right]\right\}c'(k)+U_k\left[c(k),k\right]$$
$$-\delta J'\left[k\right]+J''\left[k\right]G\left[c(k),k\right]+J'\left[k\right]G_k\left[c(k),k\right]=0$$

- From the optimal conditions for c (i.e. using the envelope theorem):

$$U_k\left[c(k),k\right]+J''\left[k\right]G\left[c(k),k\right]+J'\left[k\right]\left(G_k\left[c(k),k\right]-\delta\right)=0$$

Optimal Control Theory: Certainty (XII)

- Remember that J'(k) is the marginal value of wealth, like a lagrange multiplier. Lets give it a name:

$$J'[k] = \lambda$$

- Then:

$$\dot{\lambda} = \frac{d\lambda}{dt} = \frac{d\lambda}{dk}\frac{dk}{dt} = J''(k)\frac{dk}{dt} = J''(k)G(c,k)$$

- So:

$$U_k\left[c^*(k),k\right] + \dot{\lambda} + \lambda\left(G_k\left[c^*(k),k\right] - \delta\right) = 0$$

Optimal Control Theory: Certainty (XIII)

- Finally, we can write:

$$\frac{U_k\left[c^*(k),k\right]+G_k\left[c^*(k),k\right]+\dot{\lambda}}{\lambda}=\delta$$

- Asset Pricing Interpretation: the benefits of holding additional k has two components: dividends and capital gains. The dividend portion is represented by the first two terms where the asset (k) pays direct utility benefits as well as additional benefit in generating future wealth, multiplied by the marginal value of wealth. The capital gains leads to an increase in the marginal value of k, which is given by lambda. The above equation, then, says that, in equilibrium, the dividend plus the capital gains must equal the rate of time discounting (delta).

Optimal Control Theory: Certainty (XIV)

- Finally, we come to our big theorem:
  - Let $c^*(t)$ solve the utility maximization problem which we have been analyzing. Then, there exists so-called co-state variables $\lambda(t)$ and a Hamiltonian given by:

$$H(c, k(t), \lambda(t)) = U[c, k(t)] + \lambda(t) G[c, k(t)]$$

  - Which is maximized $c = c^*(t)$ at all times and for which the costate variable obeys the equation:

$$\dot{\lambda} = \delta\lambda - \lambda G_k[c^*(k), k] - U_k[c^*(k), k]$$

Optimal Control Theory: Certainty (XV)

- In other words, you can set up a Hamiltonian which will leave you with 3 FOCs and a transversality condition:

$$\dot{k} = G\left(c, k(t)\right) \text{ given } \mathrm{k}(0)$$

$$U'\left[c, k\right] = -\lambda G_c\left[c, k\right]$$

$$\dot{\lambda} = \delta\lambda - \lambda G_k\left[c^*(k), k\right] - U_k\left[c^*(k), k\right]$$

$$\lim_{t \to \infty} e^{-\delta t} \lambda\left(t\right)$$

Stochastic Calculus (I)

- We now turn to stochastic continuous time maximization. This is used as opposed to discrete time because some times, it is easier (leads to more tractible solutions). However, there is a bit of machinery to learn before you can do continuous time stochastic maximization.

- The main problem with stochastic continuous time maximization is that shocks occur at every instant of time, changing the slope of the realized process. Therefore, whereas the paths are continuous, they are in general nowhere differentiable and thus standard calculus does not work. We will show more precisely how this occurs and then what can be done to rectify it. The results will be a new stochastic calculus, relying on a new chain rule called Ito’s Lemma.

Stochastic Calculus (II)

- We start with a discrete analogy:

$$\max_{c(t),k(t)} E_s \sum_{t=s}^{\infty} e^{-\delta(t-s)h} U\big(c(t),k(t)\big)h$$

- subject to:

$$k(t+h)-k(t) = G\big(c,k(t),\theta\big(t+h\big),h\big) \quad \text{given} \quad \text{k(0)}$$

  - again, here, we will supress transversality conditions
  - note that theta is a first-order markov stochastic process

Stochastic Calculus (III)

- What are we solving for here? Not just c or c(k) but c as a function of k and the realized random variable theta; in other words, we are solving for state contingent plans.

- We now form the value function (where have we made the Markov assumption?):

$$J\left[k(t);\theta(t)\right] = \max_{c(t)} \left\{ U\left(c(t), k(t)\right)h + E_s e^{-\delta(t-s)} J\left[k(t+h);\theta(t+h)\right] \right\}$$

Stochastic Calculus (IV)

- For discrete time, our value function works. For continuous time, we follow the process of rearranging the value function and deriving an equation for the maximized Hamiltonian as we did before. However, when we do, we end up with:

$$\lim_{h \to 0} \frac{J\left[k(t+h);\theta\left(t+h\right)\right] - J\left[k(t);\theta(t)\right]}{h}$$

- At first, the above expression looks innocuous. On more careful expression, we realize that the above need not converge to a well defined random variable and the sample path is often not differentiable.

Stochastic Calculus (V)

- First we start by defining a continuous time random process. Since in any given interval, there will be an uncountably infinite number of realizations of the random variable, the law of large numbers should apply (given that enough moments of the distribution exist), and the random variable should be normally distributed over any finite time interval.

- We now define a random variable over a discrete time interval:

$$\Delta X = X(t) - X(t-1)$$

- We want the random variable to be normally distributed over finite time intervals:

$$\Delta X \to N(\mu, \sigma^2)$$

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Stochastic Calculus (VI)

- Now, we divide up the time period into equal intervals. Let there be n such time periods of length h. Then we can rewrite our random variable as:

$$\Delta X(t) = \sum_{i=1}^{n} \left[\mu h + \sigma h^{\frac{1}{2}} v(i) \right]$$

- Where v(i) is iid and distributed:

$$v(i) \rightarrow N(0,1)$$

- Then the expectation of this variable is:

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$$E\Delta X(t) = \sum_{i=1}^{n} \mu h + \sigma h^{\frac{1}{2}} v(i) = \mu$$

Stochastic Calculus (VII)

- Calculating the variance of the random variable, we get:

$$V\Delta X\left(t\right)=\sigma^2 hE\sum_{i=1}^{n}\sum_{j=1}^{n}v(j)v(i)=\frac{\sigma^2}{n}E\sum_{i=1}^{n}\left[v(i)\right]^2=\sigma^2$$

- So, this random variable is normally distributed with the desired mean and variance as the sum of independent random variables. We now take, of course, the limit as the time interval, h, goes to zero. We represent this as:

$$dX\left(t\right)=\mu dt+\sigma dz\left(t\right)$$

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Stochastic Calculus (VIII)

- X(t) is called a continuous random walk with drift mu and instantaneous rate of variance sigma-squared. It is also called a Gaussian diffusion process. A more general diffusion process can be written with the drift and variance both depending upon time as well as the level of the random variable:

$$dX(t) = \mu(X,t)dt + \sigma(X,t)dz(t)$$

- Now we will develop rules for multiplication of differentials using the stochastic calculus. There will essentially be two types of differentials, stochastic (i.e. dz) and non-stochastic (i.e. dt). Remember from differential calculus that terms which are going to zero of order higher than h drop out:

$$(y+h)^2 - y^2 = 2yh + h^2 \rightarrow 2y$$

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Stochastic Calculus (IX)

- This same principle will hold for stochastic calculus. Heuristically, dt is of order h and dz is of order h to the one-half times v. So:

$$dt^k = 0, \mathrm{k} > 1$$

$$dzdt = h^{\frac{3}{2}} v \rightarrow 0 \Rightarrow dzdt = 0$$

- Finally, we come to the strange case. (intuition: variance is constant over intervals):

$$dzdz = hv^2 \rightarrow h \Rightarrow dz^2 = dt$$

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Ito's Lemma (I)

- Now we are ready to show Ito's lemma (basically, the chain rule for stochastic calculus). Ito's lemma states:

$$df(X) = f'(X)dX + \frac{1}{2} f''(X)dX^2$$

- In particular if X follows a diffusion process, then

$$dX = \mu dt + \sigma dz$$

$$dX^2 = \mu^2 dt^2 + \sigma^2 dz^2 + 2\mu\sigma dt dz = \sigma^2 dt$$

$$\Rightarrow df(X) = (\mu dt + \sigma dz) f'(X) + \frac{\sigma^2}{2} f''(X)dt$$

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$$= \left[\mu f'(X) + \frac{\sigma^2}{2} f''(X)\right]dt + \sigma f'(X)dz$$

Ito’s Lemma (II)

- Now we are ready to show Ito’s lemma (basically, the chain rule for stochastic calculus). Ito’s lemma states:

$$df(X) = f'(X)dX + \frac{1}{2} f''(X)dX^2$$

- Use the multivariate Taylor Rule:

$$df(X) = \frac{\partial f(X)}{\partial t}dt + \frac{\partial f(X)}{\partial z}dz + \frac{1}{2}\frac{\partial^2 f(X)}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 f(X)}{\partial z^2}dz^2 + \frac{\partial^2 f(X)}{\partial z \partial t}dzdt + \ldots$$

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Ito's Lemma (III)

- All higher order terms disappear – they are equal to zero by the rules of the calculus and we are left with exactly:

$$df(X) = \frac{\partial f(X)}{\partial t}dt + \frac{\partial f(X)}{\partial z}dz + \frac{1}{2}\frac{\partial^2 f(X)}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 f(X)}{\partial z^2}dz^2 + \frac{\partial^2 f(X)}{\partial z \partial t}dzdt$$

- Which can be re-expressed as:

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)dX^2$$

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Continuous time stochastic maximization (I)

- We now return to our Bellman equation:

$$J[k(t);\theta(t)] = \max_{c(t)}\left\{U\left(c(t),k(t);\theta(t)\right)h + e^{-\delta h}E_t J[k(t+h);\theta(t+h)]\right\}$$

- Doing what we did in the non-stochastic case, we obtain:

$$0 = \max_{c(t)}\left\{U\left(c(t),k(t);\theta(t)\right)h + (1-\delta h)E_t J[k(t+h);\theta(t+h)] - J[k(t);\theta(t)]\right\}$$

- Rewriting:

$$0 = \max_{c(t)}\left\{U\left(c(t),k(t)\right)h + E_t J[k(t+h);\theta(t+h)] - J[k(t);\theta(t)] - \delta h E_t J[k(t+h);\theta(t+h)]\right\}$$

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Continuous time stochastic maximization (II)

- Now we divide by h:

$$0 = \max_{c(t)} \left\{ \frac{U\left(c(t), k(t)\right)h + E_t J\left[k(t+h); \theta\left(t+h\right)\right] - J\left[k(t); \theta\left(t\right)\right] - \delta h E_t J\left[k(t+h); \theta\left(t+h\right)\right]}{h} \right\}$$

- and take the limit as h goes to zero:

$$0 = \max_{c(t)} \left\{ U\left(c(t), k(t)\right) - \delta E_t J\left[k(t+h); \theta\left(t+h\right)\right] + \lim_{h \to 0} \frac{E_t J\left[k(t+h); \theta\left(t+h\right)\right] - J\left[k(t); \theta\left(t\right)\right]}{h} \right\}$$

- Now, using Ito’s lemma for the chain rule, we derive:

$$0 = \max_{c(t)} \left\{ U\left(c(t), k(t)\right) - \delta J\left[k(t)\right] dt + J'\left(k\right) E_t dk + \frac{1}{2} J''(k) E_t dk^2 \right\}$$

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Continuous time stochastic maximization (III)

- We can now state our theorem: If we are maximizing

$$E_s \int_s^\infty e^{-\delta t} U(c,k)\,dt$$

- subject to:

$$dk = G(c, k, dX, dt)$$

- And given an initial condition, k(s), at each point in time, the optimal control, c, is given by:

$$0 = \max_{c(t)} \left\{ U\big(c(t), k(t)\big) - \delta J[k]dt + J'(k)E_t dk + \frac{1}{2} J''(k) E_t dk^2 \right\}$$

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Continuous time stochastic maximization (IV)

- There is a Hamiltonian form and a separate theorem which I will not go over. However, it is often referred to as the risk-adjusted Hamiltonian. For our purposes, the Bellman Equation will suffice.

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Feldstein-Horioka Puzzle

- Feldstein-Horioka (1980, Economic Journal) ran the following regression for the 21 OECD countries (at that time) from 1960-1974:

$$\left(\frac{I}{Y}\right)_i = \alpha + \beta\left(\frac{NS}{Y}\right)_i + \varepsilon_i$$

- They found very high correlation (ranges between .85 and .95) depending upon specification.

- Is this what we would expect? Why or why not? Would you expect it to be more or less true for developing countries?

Table 1
Feldstein-Horioka Regressions, 1990-1997[a]

$$\frac{I}{Y} = \alpha + \beta \frac{NS}{Y} + \varepsilon$$

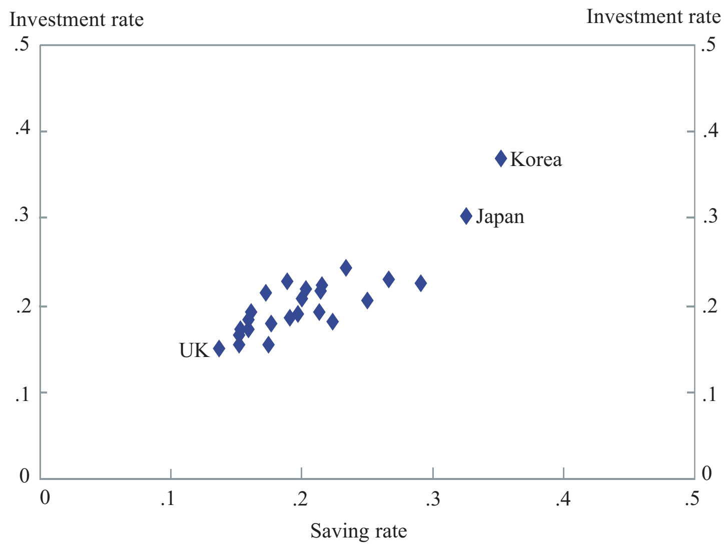
	No. of Obsvs.	α	β	R^2
All countries[b]	55	.13 (.02)	.49 (.07)	.46
Countries with GNP/cap > $2000	41	.07 (.02)	.70 (.09)	.62
OECD countries[c]	24	.08 (.02)	.60 (.09)	.68

[a] OLS regression. Standard errors in parenthesis.
[b] Israel and Jordan are excluded from all regressions in this table. The inclusion of both countries in the first regression decreases the estimate of β to .39. The inclusion of Israel in the second regression decreases it to .63.
[c] If one adds Korea to the OECD sample, the estimate for β rises to .76. Korea is included in the larger samples.

"mother of all puzzles." Table 1, which looks at cross-country regressions of average investment and savings rates, gives an update of the Feldstein-Horioka results, and extends them to look at a broader sample of countries. We do find that the coefficient on saving has fallen, from 0.89 in Feldstein and Horioka's 1960-1974 sample, to 0.60 for our 1990-1997 sample of OECD countries, excluding Korea. But this is not to say that the puzzle has gone away. First, even 0.60 is still quite a bit larger than one would expect in world of perfectly integrated markets. Second, if one includes Korea, the coefficient rises to 0.76.

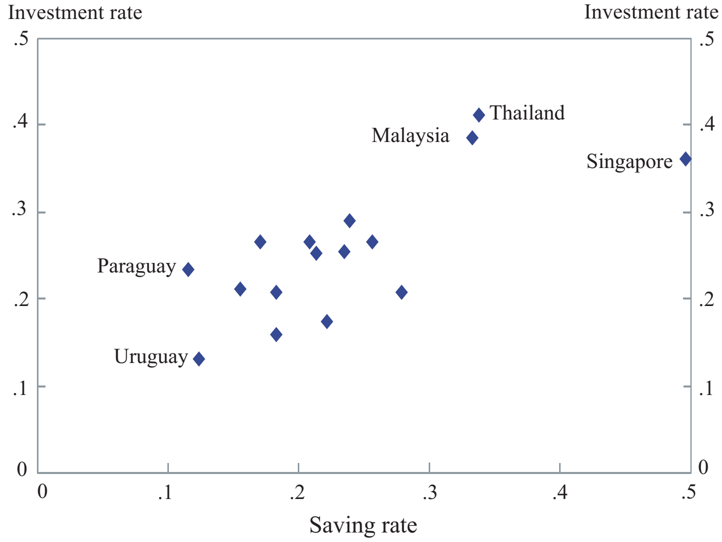
Indeed, the popular wisdom, based on Dooley, Frankel, and Mathieson (1987) and Summers (1988), is that the Feldstein- Horioka results are sharply diluted when one adds developing countries to their original OECD sample. But while the first row in Table 1 confirms

Chart 2
Saving-Investment Puzzle: 1990-1997
OECD Countries



this view (the coefficient falls to 0.49 in the extended fifty-five-country sample), it is quite misleading. In fact, as Table 1 also shows, the Feldstein-Horioka paradox holds as strongly for middle income countries (>$2000 per capita) as it does for rich countries. It is really only when the poorest countries are added that the coefficient drops (and here one might also question whether this is partly an artifact of much poorer data). Chart 2 graphs saving versus investment for the OECD countries. Chart 3 gives a sample of middle income countries (those for which the necessary data are available in the IMF's IFS database). As one can see from the figures, and as the regressions in Table 1 confirm, the slope coefficients are actually quite similar for the two groups. Finally, since the folk wisdom is also that the Feldstein-Horioka coefficient can be quite sensitive to outliers, we test robustness in Table 2 by checking how the estimated F-H relationship changes when various outliers are excluded. It is notable that when Japan is excluded from our OECD sample (sans Korea),

Chart 3
Saving-Investment Puzzle: 1990-1997
Non-OECD Countries with GNP Per Capita Greater Than $2000



the coefficient drops from 0.60 to 0.49. On the other hand, when Singapore is dropped from the extended sample that includes middle-income countries, the coefficient rises from 0.70 to 0.82.

All in all, we may conclude that the past twenty-five years of capital-market integration have slightly tempered the Feldstein-Horioka results, but overall the paradox is still alive and well.

Home bias in trade 2.3

Recent research has also documented the remarkable extent of home bias in international trade patterns. Of course, one would expect that states within a country should trade more with each other than states across national boundaries. Geographical proximity, language, a common legal and regulatory system, and the like, all help to promote intranational trade relative to international trade. But

Kraay and Ventura (I)

- Summary:

  – Traditional Rule: Temporary positive (negative) shocks lead to increases (decreases) in national savings and equivalent increases (decreases) in holdings of net foreign assets (i.e. the current account).

  – Kraay and Ventura (New) Rule: Temporary positive (negative) shocks lead to increases (decreases) in national savings which get invested in domestic versus foreign assets in proportion to current net foreign assets. Deficit countries experiencing positive shocks (for instance) then increase their deficits and surplus countries experiencing positive shocks then increase their surpluses. Similary, negative shocks will attenuate pre-existing patterns.

Kraay and Ventura (II)

- Intuition of the new rule:
  - A small open economy has a temporary positive income shock. It consumes only the dividends on the shock, which essentially are zero. Essentially, it invests all the additional income. If there are no decreasing returns in aggregate production, then it chooses to invest based upon the risk characteristics of investment which shouldnt change from a transitory income shock. In this case, the country will invest in foreign versus domestic assets in proportion to their current allocation. If the country is running deficits, they will invest more in domestic assets and increase their deficit. If the country is running a surplus, they will invest more in foreign assets and increase their surplus.

Kraay and Ventura (III)

- The small country solves the following decision-making problem:

$$\max_{c(t)} E_0 \int_0^{\infty} e^{-\delta t} \ln c(t) dt$$

- subject to:

$$da = \left[\pi k + \pi^* k^* + \rho\left(a - k - k^*\right) - c\right]dt + k\sigma d\omega + k^* \sigma^* d\omega^*$$

$$d\pi = \mu dt + \chi d\omega + \chi^* d\omega^*$$

$$\lim_{t \to \infty} e^{-\delta t} \lambda(t)$$

- and assuming:

$$\delta > 0$$

Kraay and Ventura (IV)

- Where
  - a is the value of assets
  - k is domestic ownership of the domestic capital stock
  - k^* is domestic ownership of the foreign capital stock
  - $\pi(k)$ is the rate of return on domestic capital
  - π^* is the rate of return on foreign capital (which is constant due to the small country assumption)
  - ρ is the rate of return on a risk-free domestic bond
  - $d\omega, d\omega^*$ are Wiener processes which are normally distributed and for which:

    $$Ed\omega = Ed\omega^* = 0, Ed\omega^2 = Ed\omega^{*2} = dt, E\left[d\omega d\omega^*\right] = \eta dt$$

    (in other words, η is the covariance of returns across countries)

Kraay and Ventura (V)

- Note that:

$$Ed\pi = \mu$$

$$Eda = \pi k + \pi^* k^* + \rho\left(a - k - k^*\right) - c$$

$$E\left[da^2\right] = k^2\sigma^2 + k^{*2}\sigma^{*2} + 2\eta k\sigma k^*\sigma^*$$

$$E\left[d\pi^2\right] = \chi^2 + \chi^{*2} + 2\eta\chi\chi^*$$

$$E\left[dad\pi\right] = E\left[k\sigma d\omega + k^*\sigma^* d\omega^*\right]\left[\chi d\omega + \chi^* d\omega^* + \mu dt\right] =$$

$$\chi k\sigma + \eta\chi^* k\sigma + \eta\chi k^*\sigma^* + \chi^* k^*\sigma^* = k\sigma\left(\chi + \eta\chi^*\right) + k^*\sigma^*\left(\chi^* + \eta\chi\right)$$

Kraay and Ventura (VI)

- Then the Bellman equation is given by:

$$\delta V = \max_{c,k,k^*} \left\{ \begin{array}{l} \ln c + \frac{\partial V}{\partial a}\left[\pi k + \pi^* k^* + \rho\left(a - k - k^*\right) - c\right] + \\ \frac{\partial V}{\partial \pi}\mu + \frac{1}{2}\frac{\partial^2 V}{\partial a^2}\left[k^2\sigma^2 + k^{*2}\sigma^{*2} + 2\eta k\sigma k^*\sigma^*\right] + \\ \frac{1}{2}\frac{\partial^2 V}{\partial \pi^2}\left[\chi^2 + \chi^{*2} + 2\eta\chi\chi^*\right] + \\ \frac{\partial^2 V}{\partial a \partial \pi}\left[k\sigma\left(\chi + \eta\chi^*\right) + k^*\sigma^*\left(\chi^* + \eta\chi\right)\right] \end{array} \right\}$$

Kraay and Ventura (VII)

- Now we take our first order conditions with respect to c, k, and k^*:

$$c: \qquad 0 = \frac{1}{c} - \frac{\partial V}{\partial a}$$

$$k: \qquad 0 = \frac{\partial \mathrm{V}}{\partial \mathrm{a}}(\pi - \rho) + \frac{\partial^2 \mathrm{V}}{\partial \mathrm{a}^2}\sigma\left(k\sigma + \eta k^* \sigma^*\right) + \frac{\partial^2 \mathrm{V}}{\partial \mathrm{a} \partial \pi}\sigma\left(\chi + \eta \chi^*\right)$$

$$k^*: \qquad 0 = \frac{\partial \mathrm{V}}{\partial \mathrm{a}}(\pi^* - \rho) + \frac{\partial^2 \mathrm{V}}{\partial \mathrm{a}^2}\sigma^*\left(k^*\sigma^* + \eta k \sigma\right) + \frac{\partial^2 \mathrm{V}}{\partial \mathrm{a} \partial \pi}\sigma^*\left(\chi^* + \eta \chi\right)$$

Kraay and Ventura (VIII)

- Now, using the guess and check method (the guess coming from intuition in the discrete time case), we conjecture that the value function is equal to:

$$V = \frac{1}{\delta}\ln a + f(\pi)$$

- This holds for some function f which does not need to be more specified.

- In order to check, just plug this into the bellman equation and see that it is satisfied.

Kraay and Ventura (IX)

- Now we take our first order conditions with respect to c, k, and k^*:

$$c: \qquad 0 = \frac{1}{c} - \frac{1}{\delta a} \Rightarrow c^* = \delta a$$

$$k: \qquad 0 = \frac{1}{\delta \mathrm{a}}(\pi - \rho) - \frac{1}{\delta \mathrm{a}^2}\sigma\left(k\sigma + \eta k^* \sigma^*\right) \Rightarrow$$

$$\pi - \rho = \frac{\sigma^2 k}{\mathrm{a}} + \frac{\eta \sigma \sigma^* k^*}{\mathrm{a}}$$

$$k^*: \qquad \pi^* - \rho = \frac{\sigma^{*2} k^*}{\mathrm{a}} + \frac{\eta \sigma \sigma^* k}{\mathrm{a}}$$

Kraay and Ventura (X)

- Expressing things cleanly:

$$c: \qquad c^* = \delta a$$

- What is the interpretation of this first equation? What assumption gets this result?

$$k: \qquad \pi - \rho = \frac{\sigma^2 k}{\mathrm{a}} + \frac{\eta\sigma\sigma^* k^*}{\mathrm{a}}$$

$$k^*: \qquad \pi^* - \rho = \frac{\sigma^{*2} k^*}{\mathrm{a}} + \frac{\eta\sigma\sigma^* k}{\mathrm{a}}$$

Kraay and Ventura (XI)

- Relating the two capital equations, we get:

$$\frac{\sigma\eta}{\sigma^*}\left(\pi^* - \rho\right) = \frac{\sigma\eta}{\sigma^*}\left[\frac{\sigma^{*2}k^*}{\mathrm{a}} + \frac{\eta\sigma\sigma^* k}{\mathrm{a}}\right] =$$

$$\frac{\eta\sigma\sigma^* k}{\mathrm{a}} + \frac{\eta^2\sigma^2 k}{\mathrm{a}} = \pi - \rho - \left(\eta^2 - 1\right)\frac{\sigma^2 k}{a}$$

$$\Rightarrow \pi - \rho = \frac{\sigma\eta}{\sigma^*}\left(\pi^* - \rho\right) + \left(\eta^2 - 1\right)\frac{\sigma^2 k}{a}$$

Kraay and Ventura (XII)

- Implicitly Differentiating the Capital Equation:

$$\frac{\partial \pi}{\partial k}\frac{\partial k}{\partial a} = \frac{\sigma^2(1-\eta^2)}{a}\frac{\partial k}{\partial a} - \frac{\sigma^2(1-\eta^2)k}{a^2}$$

$$\Rightarrow \frac{\sigma^2(1-\eta^2)k}{a^2} = \frac{\sigma^2(1-\eta^2)}{a}\frac{\partial k}{\partial a} - \frac{\partial \pi}{\partial k}\frac{\partial k}{\partial a}$$

$$\Rightarrow \frac{\frac{\sigma^2(1-\eta^2)k}{a^2}}{\frac{\sigma^2(1-\eta^2)}{a} - \frac{\partial \pi}{\partial k}} = \frac{\partial k}{\partial a}$$

$$\Rightarrow \frac{\sigma^2(1-\eta^2)}{\sigma^2(1-\eta^2) - a\frac{\partial \pi}{\partial k}}\frac{k}{a} = \frac{\partial k}{\partial a}$$

Kraay and Ventura (XIII)

- Here is our final equation:

$$\frac{\partial k}{\partial a} = \frac{\sigma^2(1-\eta^2)}{\sigma^2(1-\eta^2) - a\dfrac{\partial \pi}{\partial k}} \frac{k}{a}$$

- How do we interpret it when diminshing returns are small (or zero)? What happens when we increase diminishing returns?

- What happens when the domestic capital asset ratio is negative? positive? Interpretation?

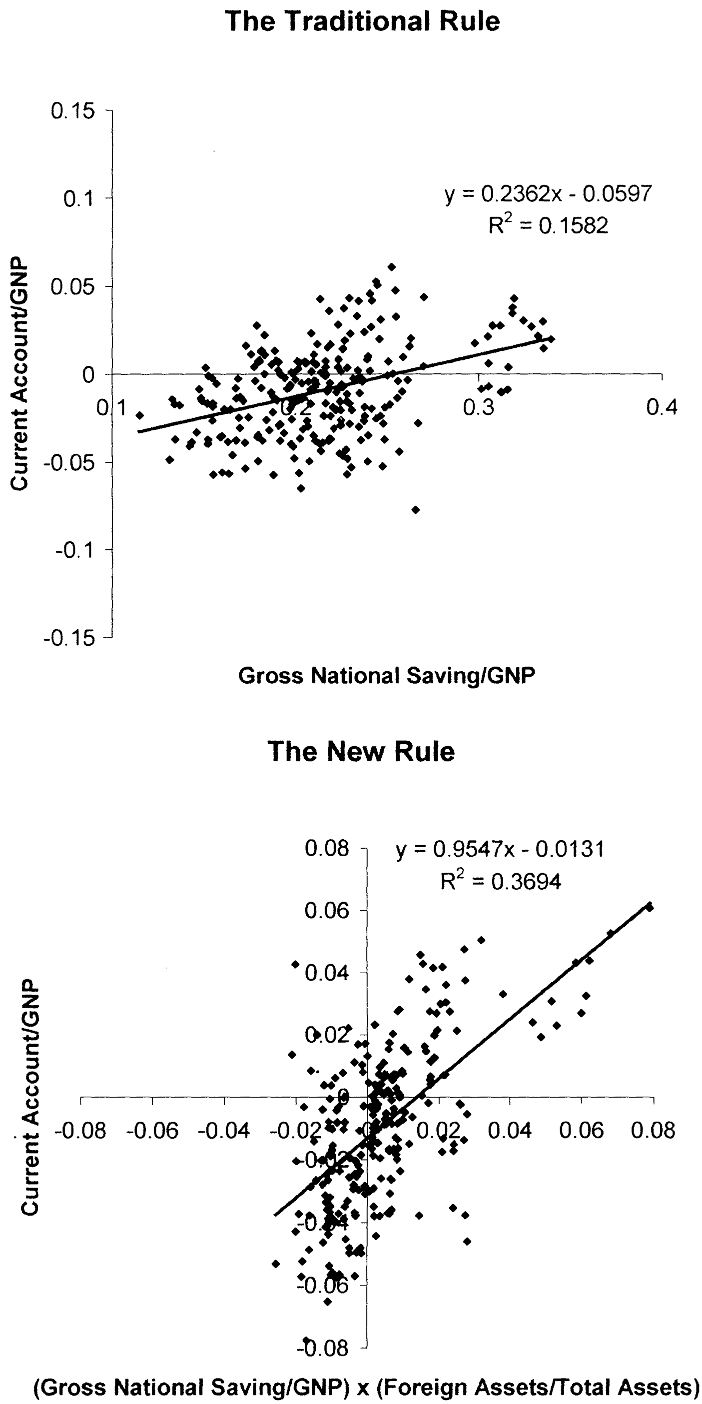


FIGURE I
Saving and the Current Account in Thirteen OECD Economies, 1973–1995
See Appendix 2 for variable definitions and data sources.

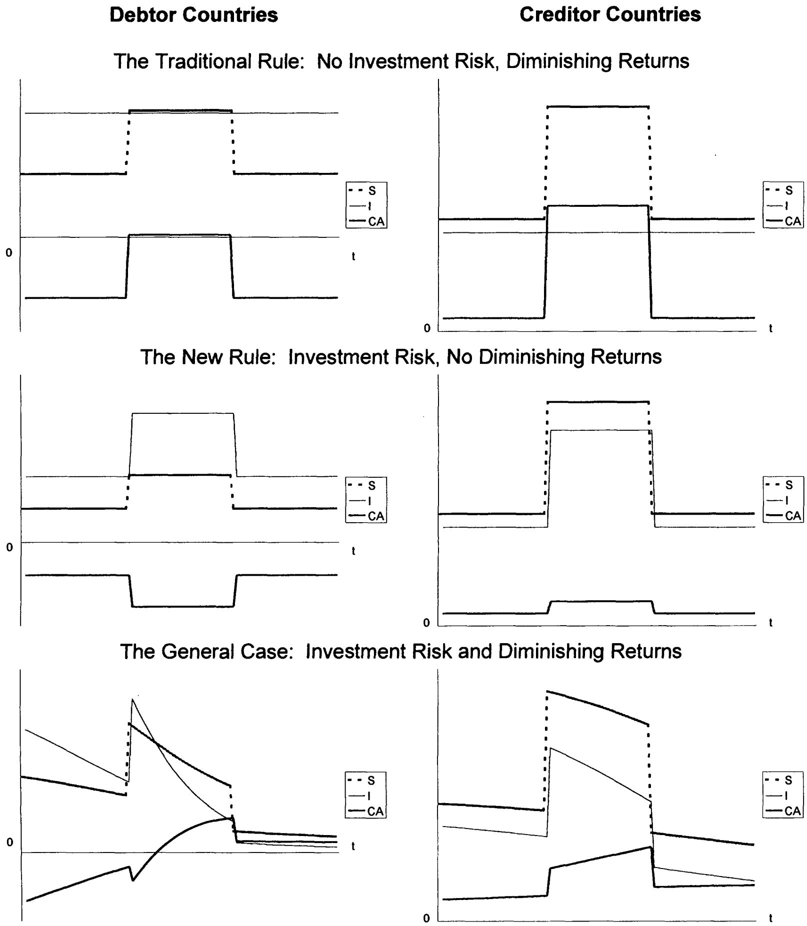


FIGURE III
Saving, Investment and the Current Account during Booms

These figures are generated under the following assumptions: (i) no foreign investment, $k^* = 0$; (ii) $\pi(k) = \alpha - \beta \cdot k$, with $\alpha = 0.04$ and $\beta = 0$ ($\beta = 0.001$) for the case of no diminishing (diminishing returns); (iii) $\sigma = 0.10$ ($\sigma = 0.15$) for debtor (creditor) countries; (iv) $\rho = \delta = 0.02$; (v) initial wealth $a_0 = 1$; and (vi) the shock $\epsilon = 0.02$.

back to its original level afterward. Weak diminishing returns ensure that new investment remains as attractive as existing investment, and so there is no incentive to change the portfolio composition. In addition, high investment risk makes investors reluctant to change the composition of their portfolios. In the