Economics 326: Duality and Supply

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1 Cost Minimization

- The Dual approach to profit maximization is a two step approach called cost minimization.

- Stage 1: Minimize costs for a given amount of production by choosing input mixes subject to a quantity constraint.

- Stage 2: Choose the level of production by maximizing profits as a function only of quantity produced.

- First stage:
  - Endogenous Variables: K, L
  - Exogenous Parameters: $w, r, \bar{q}$
  - Solve for:
    - $K^* (w, r, \bar{q})$
    - $L^* (w, r, \bar{q})$

- Second stage:

- Endogenous Variable: $\bar{q}$
- Exogenous parameters: w, r, p
- Solve for: $\bar{q}^* (w, r, p)$

- What is the relationship between these two approaches:
  - They solve for different things
    - The profit maximization approach solves for input demands as a function of input and output prices
    - The cost minimization approach solves for input demands as a function of quantity of output and input prices
    - To recover supply
      - Profit maximization: plug in the input demand functions into the production function

- Cost minimization: plug input demands into profit function and perform second stage maximization

2 Cost Minimization Example

- Minimize costs to produce q units of output for the production function

$$q = K^{\alpha} L^{\beta}$$

- If we minimized costs, what would choose? Economically? Mathematically?

$$\min_{K,L} rK + wL$$

- So, we minimize costs subect to the constraint that output is at least $\bar{q}$.

$$\begin{aligned} & \min_{K,L} rK + wL \\ \text{subject to } K^\alpha L^\beta &= \bar{q} \end{aligned}$$

- Writing a Lagrangian:

$$\min_{K,L,\lambda} rK + wL + \lambda\left[\bar{q} - K^\alpha L^\beta\right]$$

- In theory, we should check our second order conditions. The Lagrangian, $rK + wL + \lambda\left[\bar{q} - K^\alpha L^\beta\right]$, should be convex. This is the same thing as saying that the production function, $K^\alpha L^\beta$, should be concave.

- What are the endogenous variables? Parameters?

- Solving for FOCs:

$$\frac{\partial C}{\partial K} = r - \lambda\alpha K^{\alpha-1}L^{\beta} = 0 \quad (1)$$
$$\frac{\partial C}{\partial L} = w - \lambda\beta K^{\alpha}L^{\beta-1} = 0 \quad (2)$$
$$\frac{\partial C}{\partial \lambda} = \bar{q} - K^{\alpha}L^{\beta} = 0 \quad (3)$$

- Solving for λ in equation (1) and equation (2), we get:

$$\lambda = \frac{r}{\alpha K^{\alpha-1}L^{\beta}} = \frac{w}{\beta K^{\alpha}L^{\beta-1}}$$

or

$$\beta r K = \alpha w L$$

or

$$K = \frac{\alpha w}{\beta r}L \quad (4)$$

- Now comining with the first order condition for λ in equation (3) , we get:

$$
\begin{aligned}
\bar{q} &= K^{\alpha} L^{\beta} \\
&= \left(\frac{\alpha w}{\beta r} L\right)^{\alpha} L^{\beta} \\
\bar{q} &= \left(\frac{\alpha w}{\beta r}\right)^{\alpha} L^{\alpha+\beta} \\
L^{\alpha+\beta} &= \left(\frac{\beta r}{\alpha w}\right)^{\alpha} \bar{q} \\
L^{*} &= \left(\frac{\beta r}{\alpha w}\right)^{\frac{\alpha}{\alpha+\beta}} \bar{q}^{\frac{1}{\alpha+\beta}}
\end{aligned}
$$

- Now using equation (4) , we can solve for K^{*} as well:

$$
\begin{aligned}
K^{*} &= \frac{\alpha w}{\beta r} L^{*} \\
&= \left(\frac{\beta r}{\alpha w}\right)^{-1} \left(\frac{\beta r}{\alpha w}\right)^{\frac{\alpha}{\alpha+\beta}} \bar{q}^{\frac{1}{\alpha+\beta}} \\
&= \left(\frac{\alpha w}{\beta r}\right)^{\frac{\beta}{\alpha+\beta}} \bar{q}^{\frac{1}{\alpha+\beta}}
\end{aligned}
$$

- So, we now know how much capital and labor the firm uses as a function of output. Therefore, we can construct cost functions which are functions of output:

$$C(r,w,q) = rK^*(r,w,q) + wL^*(r,w,q)$$

we can now rewrite this as:

$$\begin{aligned} & r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\bar{q}^{\frac{1}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\bar{q}^{\frac{1}{\alpha+\beta}} \\ = & \left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1}{\alpha+\beta}} \end{aligned}$$

- This is called the cost curve (or total cost curve). It gives the cost of production as a function of output and factor prices.

3 Marginal, Average and Average Variable Costs Curves

- From the total cost curve, we can derive marginal, average and average variable cost curves.

- What is marginal cost? The marginal cost incurred from an extra unit of production:

$$\frac{\partial C}{\partial \bar{q}}$$

- Computing with our computed cost curve for Cobb-Douglas:

$$C(\bar{q}) = \left[r^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\alpha w}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}} \left(\frac{\beta r}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \right] \bar{q}^{\frac{1}{\alpha+\beta}}$$

Thus marginal cost is equal to $\frac{dC}{d\bar{q}}$:

$$\frac{1}{\alpha+\beta}\left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1-\alpha-\beta}{\alpha+\beta}}$$

- Notice that marginal cost is positive.

- When is marginal cost increasing? When the derivative of marginal cost (or the second derivative of total cost is increasing):

$$\begin{aligned}\frac{\partial^2 C}{\partial \bar{q}^2} &= \frac{1}{\alpha+\beta}\left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)* \\ &\quad \left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}+w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1-2\alpha-2\beta}{\alpha+\beta}}\end{aligned}$$

- There are 3 cases:

$$\begin{aligned}1-\alpha-\beta &> 0: \text{ Increasing Marginal Costs}\\ 1-\alpha-\beta &= 0: \text{ Constant Marginal Costs}\\ 1-\alpha-\beta &< 1: \text{ Decreasing Marginal Costs}\end{aligned}$$

- This is the same as returns to scale!

  Increasing Marginal Costs = Decreasing Returns to Scale
  Constant Marginal Costs = Constant Returns to Scale
  Decreasing Marginal Costs = Increasing Returns to Scale

- Average cost is the average cost per unit of production and is given by:

$$\frac{C}{\bar{q}}$$

- In the Cobb-Douglas case, it is equal to:

$$\begin{aligned}
\frac{C}{\bar{q}} &= \frac{\left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1}{\alpha+\beta}}}{\bar{q}} \\
&= \left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1-\alpha-\beta}{\alpha+\beta}}
\end{aligned}$$

- Graphs of marginal and average cost curves. Why does marginal cost intersect average cost at its bottom?

- What is the difference between average cost and average variable cost?

4 Supply

- In the second stage of cost minimization, we can figure out how much the firm wants to produce.

- Lets try our Cobb-Douglas example:

$$\max_{\bar{q}} p\bar{q} - \left[r^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}} \left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right] \bar{q}^{\frac{1}{\alpha+\beta}}$$

- Computing the first order conditions - $\frac{d\Pi}{d\bar{q}}$

$$p - \frac{1}{\alpha+\beta}\left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]\bar{q}^{\frac{1-\alpha-\beta}{\alpha+\beta}}$$

and we set this equal to zero. This then leads to:

$$\frac{p(\alpha+\beta)}{\left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]} = q^{\frac{1-\alpha-\beta}{\alpha+\beta}}$$

and finally solving for $\bar{q}^*$:

$$\bar{q}^* = \left(\frac{p(\alpha+\beta)}{\left[r^{\frac{\alpha}{\alpha+\beta}}\left(\frac{\alpha w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + w^{\frac{\beta}{\alpha+\beta}}\left(\frac{\beta r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]}\right)^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

- This is the supply curve: how much the firm wants to produce as a function of the price. Notice that as long as $1-\alpha-\beta > 0$, the exponent on price is positive and thus the function is upward sloping. In other words, if we have decreasing returns to scale or inreasing costs, the firm will always want to supply more as the price goes up.

- What is the problem if $1 - \alpha - \beta < 0$?

- In general, maximizing profits, we maximize:

$$\max_{\bar{q}} \Pi(\bar{q}) = p\bar{q} - c(\bar{q})$$

- - What is the difference between this profit maximization problem and the original one?

- Maximizing profits by choosing quantity of output, we get a famous result:

$$\begin{aligned} \frac{d\Pi}{d\bar{q}} &= p - c'(\bar{q}) = 0 \\ &\Longrightarrow p = c'(\bar{q}) \end{aligned}$$

This says that the firm sets price equal to marginal cost. Intuition?

- Suppose we wanted to graph supply. We could graph quantity on price or quantity on marginal cost. That is why the supply curve is sometimes called the marginal cost curve.

- One note: supply is the marginal cost curve when price is above average cost? Why?