Economics 326: Budget Constraints and Utility Maximization

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Outline

1. Budget Constraint

2. Utility Maximization

1 Budget Constraint

- Two standard assumptions on utility:
  - Non-satiation: $\frac{\partial U(C_x, C_y)}{\partial C_x} > 0$ for all values of $C_x, C_y > 0$
  - Convexity: Let C_1, C_2 and C_3 be commodity bundles such that $C_1 \succeq C_3$ and $C_2 \succeq C_3$. Then any convex combination of C_1 and C_2 is also weakly preferred to C_3 : $tC_1 + (1-t)\, C_2 \succeq C_3$ for all $t \in [0, 1]$.

- Suppose you have a utility function that satisfies non-satiation: $U(C_X, C_Y)$. If you wanted to choose values of C_X and C_Y that maximized your utility, what would you choose?

- What stops the consumer from choosing her maximum utility?
  - Income! (i.e. the costs of consumption)

- We now introduce a budget constraint.
  - Note we aren't going to need a constraint on the producers side because their, the costs of production can be directly subtracted from revenues. Profits is equal to revenues minus costs. However, utility is a different unit than dollars and so you can't maximize utility net of costs like you can with revenues.

- A budget constraint for a consumer choosing between two goods has through components:
  - Amount spend on good C_X (the price of C_X: P_{C_X} mulitplied by the quantity of C_X consumed): $P_{C_X}C_X$
  - Amount spend on good C_Y (the price of C_Y: P_{C_Y} mulitplied by the quantity of C_Y consumed): $P_{C_Y}C_Y$
  - Income: I
- General Form of Budget Constraint: $P_{C_X}C_X + P_{C_Y}C_Y \leq I$
  - We leave open the possibility that the consumer might not spend all her money.
- With non-satiation, the consumer can always get higher utility from consuming more and therefore will always spend every cent of her income:

- $P_{C_X} C_X + P_{C_Y} C_Y = I$

2 Utility Maximization

- We now discuss utility maximization: $U(C_X, C_Y)$
- Components:
  - Objective function: $U(C_X, C_Y)$
  - Constraint: $P_{C_X} C_X + P_{C_Y} C_Y \leq I$
  - Endogenous Variables: ?
  - Parameters: ?
- What do we want to do?

– Maximize utility subject to budget constraint and solve for endogenous variables as a function of the parameters.

- Example with Cobb-Douglass utility function:

$$\max_{C_X, C_Y} C_X^{0.5} C_Y^{0.5}$$

$$s.t.\ P_{C_X} C_X + P_{C_Y} C_Y \leq I$$

- We solve using two different methods.

2.1 Solution by Langrangian

- Step 1: Write the Lagrangian

$$L = C_X^{0.5} C_Y^{0.5} + \lambda \left[I - P_{C_X} C_X - P_{C_Y} C_Y \right]$$

– Note that λ is the Lagrange multiplier and L is the maximand. The objective function is still: $C_X^{0.5} C_Y^{0.5}$.

- Step 2: Write down the endogenous variables: C_X, C_Y, and λ.

- Step 3: Take the derivatives (First Order Conditions or FOCs) for the endogenous variables:

$$\frac{\partial L}{\partial C_X} = .5 C_X^{-0.5} C_Y^{0.5} - \lambda P_{C_X} = 0 \quad (1)$$

$$\frac{\partial L}{\partial C_Y} = .5 C_X^{0.5} C_Y^{-0.5} - \lambda P_{C_Y} = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = I - P_{C_X} C_X - P_{C_Y} C_X = 0 \quad (3)$$

- Step 4: We have 3 equations and 3 unknowns. We can solve! Combine (1) and (3) :

$$\lambda = \frac{.5 C_X^{-0.5} C_Y^{0.5}}{P_{C_X}} = \frac{.5 C_X^{0.5} C_Y^{-0.5}}{P_{C_Y}} \quad (4)$$

- Simplifying (4) , we get:

$$P_{C_Y} C_Y = P_{C_X} C_X \tag{5}$$

  – Suppose we stopped here and solve for C_Y :

$$C_Y = \frac{P_{C_X} C_X}{P_{C_Y}}$$

  – Are we finished? Have we expressed the endogenous variable as a function of exogenous parameters?

- Finally, since $\lambda > 0$, we know that the FOC must hold:

$$I = P_{C_X} C_X + P_{C_Y} C_Y$$

or

$$P_{C_X} C_X = I - P_{C_Y} C_Y \tag{6}$$

- We can then replace (6) into (5) to get:

$$P_{C_Y} C_Y = I - P_{C_Y} C_Y$$

or

$$C_Y^* = \frac{I}{2P_{C_Y}}$$

- We similarly can solve for C_X^* :

$$C_X^* = \frac{I}{2P_{C_X}}$$

- How does C_X^* differ from C_X?
  - C_X^* is the maximized value of the variable C_X given the parameters.
  - Is C_X^* a variables?
  - What is $C_X^*\left(I, P_{C_X}, P_{C_Y}\right)$?

- Step 5: Check to see that the solution is a maximum by checking second order conditions.

- We mostly won't be doing this step for this course.
- What can go wrong?
- The derivate of a function is zero at a local optimum
  * Could be a minimum
  * Could be a local maximum
  * But we want a GLOBAL MAXIMUM! Highest utility possible given the budget constraint!
  * What would be the interpretation of a local maximum?
- Example: try solving for FOCs for $F(X) = X^3 - 9X = (X+3)(X-3)X$

$$\begin{aligned} \frac{dF(X)}{dX} &= 3X^2 - 9 = 0 \\ X^* &= \pm\sqrt{3} \end{aligned}$$

2.2 Solution by Plugging In

- Step 1: Show non-satiation of the objective function:

$$\frac{\partial U(C_X, C_Y)}{\partial C_X} = .5C_X^{-0.5}C_Y^{0.5} > 0$$
$$\frac{\partial U(C_X, C_Y)}{\partial C_Y} = .5C_X^{0.5}C_Y^{-0.5} > 0$$

  - Since the consumer's utility function represents preferences that are non-satiated, she will always spend all her money which means that the budget constraint is an equality

$$I = P_{C_X}C_X + P_Y C_Y$$

- Step 2: Write down the endogenous variables: C_X, C_Y

- Step 3: Plug in the budget constraint into the objective function by choosing one of the two endogenous variables to replace:

$$C_X = \frac{I - P_{C_Y}C_Y}{P_{C_X}}$$

– The new objective function is thus:

$$\left(\frac{I - P_{C_Y} C_Y}{P_{C_X}}\right)^{0.5} C_Y^{0.5}$$

- Step 4: Take the derivatives (First Order Conditions or FOCs) for the endogenous variable (note that the objective function is now a function of one variable and we do not need the constraint any more):

$$\max \left(\frac{I C_Y - P_{C_Y} C_Y^2}{P_{C_X}}\right)^{0.5}$$

  – Now remember that we can use a monotonic transformation of the utility function and since C_X and C_Y are always positive, we can $G = U^2$:

$$\max \frac{I C_Y - P_{C_Y} C_Y^2}{P_{C_X}}$$

– Maximizing, we get:

$$\frac{I}{P_{C_X}} - \frac{2P_{C_Y}C_Y}{P_{C_X}} = 0$$

– Solving for the endogenous variable C_Y, we get:

$$C_Y\frac{2P_{C_Y}}{P_{C_X}} = \frac{I}{P_{C_X}}$$

or

$$C_Y^* = \frac{I}{2P_{C_Y}}$$

– Look familiar?

- What are the parameters of this problem?

- Why is C_Y^* not a function of all the parameters? What's the interpretation?

- Last question: What happens to the consumption of Y when income increases? the price of Y increases? the price of X increases?