# Economics 326: Marshallian Demand and Comparative Statics

Ethan Kaplan

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Outline

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3. Homogeneity of Degree Zero of Marshallian Demand
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# 1 Utility Maximization: General Formulation

- General maximization problem with two goods:

$$\max_{C_X, C_Y} U(C_X, C_Y)$$

$$s.t.\ P_{C_X} C_X + P_{C_Y} C_Y \leq I$$

- Lagrangian:

$$\begin{aligned} L &= U(C_X, C_Y) + \\ &\quad \lambda \left[ I - P_{C_X} C_X - P_{C_Y} C_Y \right] \end{aligned}$$

- FOCS:

$$\frac{\partial L}{\partial C_X} = \frac{\partial U(C_X, C_Y)}{\partial C_X} - \lambda P_{C_X} = 0 \quad (1)$$

$$\frac{\partial L}{\partial C_Y} = \frac{\partial U(C_X, C_Y)}{\partial C_Y} - \lambda P_{C_Y} = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = I - P_{C_X} C_X - P_{C_Y} Y = 0$$

- Solve for $\lambda$ using equations (1) and (2)

$$\lambda = \frac{\frac{\partial U(C_X, C_Y)}{\partial C_X}}{P_{C_X}}$$

and

$$\lambda = \frac{\frac{\partial U(C_X, C_Y)}{\partial C_Y}}{P_{C_Y}}$$

- Equation the two equations for $\lambda$, we get

$$\frac{\frac{\partial U(C_X, C_Y)}{\partial C_X}}{P_{C_X}} = \frac{\frac{\partial U(C_X, C_Y)}{\partial C_Y}}{P_{C_Y}}$$

- Interpretation of the above equation? Equate marginal utility per dollar accross goods.

- Another formulation:

$$MRS = -\frac{\frac{\partial U(C_X, C_Y)}{\partial C_X}}{\frac{\partial U(C_X, C_Y)}{\partial C_X}} = -\frac{P_{C_X}}{P_{C_Y}}$$

- Interpretation? Equating the MRS with the negative of the price ratio or the ratio of the marginal utilities with the price ratio. Intuitively, if the price for a good is high, then the marginal utility must also be high because its an expensive good.

# 2 Marshallian Demand

- We now go back to our previous example from the previous lecture:

$$\begin{aligned}\max_{C_X, C_Y} L(C_X, C_Y, \lambda) &= C_X^{0.5} C_Y^{0.5} + \\ &\lambda \left[ I - P_{C_X} C_X + P_{C_Y} C_Y \right]\end{aligned}$$

- From this, we derived:

$$C_X^* = \frac{I}{2P_{C_X}}$$

- What is this?

  1. Mathematically: The optimal choice of $C_X$ as a function of parameters $I$ and $P_{C_X}$
  2. Intuitively: It tells the amount purchased as a function of $P_{C_X}$.
  3. It's name: Marshallian Demand Function

- When you see a graph of $C_X$ on $P_{C_X}$, what you are really seeing is a graph of $C_X^*$ on $P_{C_X}$ holding $I$ and other parameters constant (i.e. for a given value of $I$ and other prices). In other words, you see a two dimensional slice of the demand function for $C_X$. (show graph)

- More generally, what is a demand function: it is the optimal consumer choice of a good (or service) as a function of parameters (income and prices).

- What else we can we do with Marshallian Demand mathematically?

  - Comparative Statics! Take the Derivative with respect to parameters. Our problem has three parameters: $P_{C_X}, P_{C_Y}, I$.

    * Own price effect: $\frac{\partial C_X^*}{\partial P_{C_X}} = -\frac{I}{2\left(P_{C_X}\right)^2} < 0$

      · Note that own price effect is negative. Why?

      · Note that the own price effect is larger in magnitude when $I$ is larger in magnitude when $I$ is larger. Why?

    * Cross-price effect: $\frac{\partial C_X^*}{\partial P_{C_Y}} = 0$

  - Why is the cross-price effect equal to zero? What does it mean? Is that always true?

* Income effect: $\frac{\partial C_X^*}{\partial I} = \frac{1}{2P_{C_X}} > 0$
  - Why is the income effect positive?
  - Is that always true?

# 3 Homogeneity of Degree Zero

- Remember: $C_X^* = \frac{I}{2P_{C_X}}$

- Suppose the price of good $C_X$ doubled, what would happen to demand with our example?

- Suppose all prices and income doubled at the same time, what would happen to demand?

- This is a general property of demand functions called homogeneity of degree zero. Marshallian demand is homogeneous of degree zero in money and prices.

- In general, a function is called homogeneous of degree $k$ in a variable $X$ if $F(\lambda X) = \lambda^K X$. Note that the particular case where $F(\lambda X) = X$ is just the case where $k = 0$ so this is homogeneity of degree zero.

- Is $C_X^*$ homogeneous of degree zero in
  - the price of $C_X$?
  - the price of $C_Y$?
  - Income?
  - Income and the price of $C_X$?
  - Income and all prices?

- Interpret homogeneity of degree zero economically.

# 4 Engel Curves, Normal Goods, Luxury Goods, Giffen Goods

- Three types of comparative statics:
  - Own price effects
  - Cross price effects
  - Income effects
- Can graph optimal quantity chosen (quantity DEMANDED) against
  - Income: Engel Curve

- Own Price: Demand Curve
- Cross Price: No name for this

- Can classify goods by derivative of demand with respect to income
  - Positive: Normal Good
  - Negative: Inferior Good
  - Examples?

- Can classify goods by derivative of demand with respect to own price
  - Positive: Normal Good
  - Negative: Giffen Good

# 5 Corner Solutions

- Really we should have a more complicated maximization problem. What if the optimal demand for a good given from utility maximization is negative?

- How could this happen?

- Graph examples

- How do we deal with this problem practivally?

  1. Check that solutions for demand functions are positive given parameters. If they are, don't worry about it. If not, follow #2 below.

  2. Constrain the optimal choice to be positive.

- Lagrangian with additional constraints

$$\begin{aligned}\max_{C_X, C_Y, \lambda, \mu_{C_X}, \mu_{C_Y}} L &= U\left(C_X, C_Y\right) + \\ & \mu_{C_X} C_X + \mu_{C_Y}\left[C_Y\right] + \\ & \lambda\left[I - P_{C_X} C_X - P_{C_Y} C_Y\right]\end{aligned}$$

- Note: really the constraints are: $\mu_{C_X}\left[C_X - 0\right]$

- In other words, we add two extra Lagrange multipliers, one to constrain each of the two variables to be at least zero.

- This forces consumption of each good not to go below zero even if the marginal utility per good is smaller at zero than that for the other good:

$$\frac{\frac{\partial U}{\partial C_X}\left(0, C_Y^*\right)}{P_{C_X}} < \frac{\frac{\partial U}{\partial C_Y}\left(0, C_Y^*\right)}{P_{C_Y}}$$

# 6 Indirect Utility Function

- Definition: Plug in the demand functions back into the utility function.

- Then the utility function is a function of parameters (prices and income) rather than variables

- Another name for this is the maximized utility function: $V\left(P_{C_X}, P_{C_Y}, I\right)$

- Lets construct it for our example with utility function $U(C_X, C_Y) = C_X^{0.5} C_Y^{0.5}$

- Remember the demand functions are

  - $C_X^* = \frac{I}{2P_{C_X}}$

– $C_Y^* = \frac{I}{2P_{C_Y}}$

- Therefore the indirect utility function is

$$V\left(P_{C_X}, P_{C_Y}, I\right) = \left(\frac{I}{2P_{C_X}}\right)^{0.5}\left(\frac{I}{2P_{C_Y}}\right)^{0.5}$$
$$= \frac{I}{2}\left(\frac{1}{P_{C_X}P_{C_Y}}\right)^{0.5}$$

- Now lets compute the marginal utility of money:

$$\frac{\partial V}{\partial I} = \frac{1}{2}\left(\frac{1}{P_{C_X}P_{C_Y}}\right)^{0.5}$$

- Now lets compute marginal utility per dollar:

$$\frac{\frac{\partial U(C_X, C_Y)}{\partial C_X}}{P_{C_X}} = \frac{C_Y^{0.5}}{P_{C_X}C_X^{0.5}}$$

at the maximum, the value is:

$$\frac{\left(\frac{I}{2P_{C_Y}}\right)^{0.5}}{P_{C_X}\left(\frac{I}{2P_{C_X}}\right)^{0.5}} = \frac{1}{2}\left(\frac{1}{P_{C_X}P_{C_Y}}\right)^{0.5}$$

- But remember from the solution of the general form of the utility maximization problem that generally speaking, the marginal utility of money per dollar is the Lagrange multiplier on income: $\lambda$.

- So: we have an interpretation of the Lagrange multiplier as the marginal utility of income. It turns out that this is general to all utility maximization problems (not specific to the utility function we are using).

- What is the intuition? $\lambda$ represents the value of relaxing the income constraint by a dollar. When money is optimally spent (utility is maximized), the relaxation of constraint by a dollar represents the utility value of relaxing getting a dollar more income or the marginal utility of income!