Economics 326: Duality and the Slutsky Decomposition

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Outline

1. Convexity and Declining MRS

2. Duality and Hicksian Demand

3. Slutsky Decomposition

4. Net and Gross Substitutes and Complements

5. Elasticities

1 Convexity and Declining MRS

- Economists usually assume that preferences are strictly convex:

– If a consumer is indifferent between two bundles X and Y, then she strictly prefers any convex combination of X and Y :

$$\begin{array}{rcl} X & \sim & Y \implies tX + (1-t)Y \succ X \\ & & \text{and} \\ tX + (1-t)Y & \succ & Y \text{ for all } t \in (0,1) \end{array}$$

- This says that consumers prefer combinations of commodites to specializing in commodities.

- What does convexity imply for utility functions?

  – Quasi-concavity of the utility function (or positive semi-definite Hessian matrix)

  – Indifference curves are concave up

  – Declining MRS

  – Show the graph!

- What does declining MRS mean?
  - As I consume more of commodity X my MRS between X and Y goes down: $\frac{\partial \left| -\frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} \right|}{\partial X} < 0$
  - In other words, as I consume more pasta, my marginal utility for pasta declines relative to other goods

2 Duality and Hicksian Demand

- There is another approach to consumer choice which is called the dual approach to utility maximization. It is called expenditure minimization. It is much less intuitive but is helpful in in understanding price changes.

- Instead of maximizing utility subject to a budget constraint (i.e. picking the highest indifference curve given a budget line), you could minimize expenditure subject to a utility constraint (i.e. picking the lowest budget line given an indifference curve).

- Formulation

$$\min_{X,Y} P_X X + P_Y Y \text{ subject to } \bar{U} \leq U(X,Y)$$

- What are the endogenous variables in this problem? Exogenous parameters?

- Writing as a Lagrangian:

$$\min_{X,Y} P_X X + P_Y Y + \lambda \left[\bar{U} - U(X,Y)\right]$$

- Example with Cobb Douglass

$$\min_{X,Y} P_X X + P_Y Y + \lambda \left[\bar{U} - X^{0.5} Y^{0.5}\right]$$

- Take FOCS:

$$\begin{aligned}
\frac{\partial L}{\partial X} &= P_X - 0.5\lambda \left(\frac{Y}{X}\right)^{0.5} = 0 \\
\frac{\partial L}{\partial Y} &= P_Y - 0.5\lambda \left(\frac{X}{Y}\right)^{0.5} = 0 \\
\frac{\partial L}{\partial \lambda} &= \bar{U} - X^{0.5}Y^{0.5} = 0
\end{aligned}$$

- So we can solve for Lambda in each of the first two FOCs and equate the two Lambda solutions as with the utility maximization problem:

$$\lambda = \frac{P_X}{\left(\frac{Y}{X}\right)^{0.5}} = \frac{P_Y}{\left(\frac{X}{Y}\right)^{0.5}}$$

- Simplifying the above equation by mupltipying both sides by $X^{0.5}Y^{0.5}$, we get:

$$P_X X = P_Y Y \tag{1}$$

- Does this look familiar?

- Now we take the FOC for λ :

$$\bar{U} = X^{0.5}Y^{0.5}$$

- We can solve for one of X and Y and then plus back in to equation (1) :

$$X = \frac{\bar{U}^2}{Y} \tag{2}$$

- Substituting equation (2) into (1), we get:

$$P_X \frac{\bar{U}^2}{Y} = P_Y Y$$

- Solving for Y, we get the Hicksian Demand for Y:

$$Y_H^* = \bar{U}\left(\frac{P_X}{P_Y}\right)^{0.5}$$

- This tells me how much I demand of good Y give prices P_X and P_Y in order to acheive utility $\bar{U}$ in the lowest cost way possible.

- Again, the Hickisian demand is a function... of parameters.

- Hicksian demand is also considered compensated demand because if you take the derivative of the Hicksian with respect to price, you are asking how demand changes holding utility constant. The thought experiment is that relative prices change but you get compensated for the loss in real income so that your utility remains the same.

- Lets do comparative statics:

  – Own Price Effect

$$\frac{\partial Y_H^*}{\partial P_Y} = -0.5\bar{U}P_X^{0.5}P_Y^{-1.5} < 0$$

  – Cross Price Effect

$$\frac{\partial Y_H^*}{\partial P_X} = 0.5\bar{U}P_X^{-0.5}P_Y^{-0.5} > 0$$

- Utility Effect

$$\frac{\partial Y_H^*}{\partial \bar{U}} = \left(\frac{P_X}{P_Y}\right)^{0.5} > 0$$

- Do the signs of these effects make sense?

• Why a consumer ever want to know this? The answer is they wouldn't. A consumer would want to know how much to buy of good Y given their income in order to maximize utility not how much to buy of Good Y to buy in order to minimize their expenditure so as to acheive a level of utility.

• However, this exercise is useful for understanding (decomposing) a price change.

3 Slutsky Decomposition

- We won't prove this but it can be shown that you can decompose the impact of a price change into two different effects:

$$\begin{aligned}\frac{\partial Y^*\left(P_X, P_Y, I\right)}{\partial P_j} &= \frac{\partial Y_H^*\left(P_X, P_Y, \bar{U}\right)}{\partial P_j} - \\ &\quad \frac{\partial Y^*\left(P_X, P_Y, I\right)}{\partial I} Y^*\left(P_X, P_Y, I\right)\end{aligned}$$

- Maybe its easier to read like this

$$\frac{\partial Y^*}{\partial P_j} = \frac{\partial Y_H^*}{\partial P_j} - \frac{\partial Y^*}{\partial I} Y^*$$

- Remember: what is the difference between Y^* and Y_H^*?

- Why is the change with respect to P_j? What is j? Is it X or Y?

- The point is that there are two terms: $\frac{\partial Y_H^*}{\partial P_j}$ and $-\frac{\partial Y^*}{\partial I}Y^*$. The first of these is called the substitution effect. The second is called the income effect.

  - Substitution Effect: This considers the impact on demand for good Y of an increase in price P_j holding utility constant. If the price of a good goes up holding utility constant, income must increase in order to prevent real income deterioration. In other words, the consumer is 'compensated' for the increase in price. The effect of the price change on product demand, then, is only through the change in relative prices and not through the effective loss in real income.

  - Income Effect: This considers the impact on demand for good Y of an increase in price P_j through the effective drop in real income. The effect is equal to $-\frac{\partial Y^*}{\partial I}Y^*$. The first term of the income effect is the effect of a dollar's loss in income on demand for good Y : $-\frac{\partial Y^*}{\partial I}$. The second term is then equal to the amount of lost dollars from the price change (which is equal to the amount of the good being purchased).

- The best way to see this is graphically.

4 Net and Gross Substitutes and Complements

- Two goods can be complements or substitutes. If they are substitutes, you tend to want one good or the other. If they are complements, you tend to want to consume them together.

- More formally, two goods are considered gross substitutes if the rise in price of one good (which will lead to a decline in demand for that good) increases demand for the other good:

$$\frac{\partial X^*}{\partial P_Y} > 0$$

- Two goods are gross complements if the rise in price of one good decreases demand for the other good:

$$\frac{\partial X^*}{\partial P_Y} < 0$$

- Two goods are net substitutes if the rise in price of one good holding utility constant leads to an increase in demand for the other good:

$$\frac{\partial X_H^*}{\partial P_Y} > 0$$

- Two goods are net substitutes if the rise in price of one good holding utility constant leads to an decrease in demand for the other good:

$$\frac{\partial X_H^*}{\partial P_Y} < 0$$

- It turns out that if the rise in price of good X leads to a decline in demand for good Y then an increase in the price of good Y leads to a decline in demand for good X.

5 Elasticities

- A demand elasticity tells you how responsive the quantity demanded is to the price. We can compute it as the percentage change in quantity demand from a 1 percent change in price:

- $$\frac{\frac{X_2 - X_1}{X_1}}{\frac{P_2 - P_1}{P_1}} = \frac{X_2 - X_1}{P_2 - P_1}\frac{P_1}{X_1} = \frac{\partial X^*}{\partial P_X}\frac{P_X}{X^*}$$

- So we define the demand elasticity for a good X as:

$$\frac{\partial X^*}{\partial P_X}\frac{P_X}{X^*}$$

- If this number is high, it says that a small change in the price of X leads to a large change in demand for X. If a good has a high elasticity, we say it is very **elastic**.

- If this number is low, it says that a small change in the price of X leads to a small change in demand for X. If a good has a low elasticity, we say it is very **inelastic.**

- Note that the elasticity of a good can change with how much of it is consumed. If I am consuming a lot of pizza, my demand for pizza may become more or less elastic.

- Show graphs!