Economics 326: Expected Utility and the Economics of Uncertainty

Ethan Kaplan

October 3, 2012

Outline

1. Probability Theory and Expected Value

2. The Saint Petersburg Paradox

3. Expected Utility

4. Demand for Assets

   (a) Demand for Stocks

   (b) Demand for Insurance

1 Probability Theory and Expected

Value

- A random variable X takes on a value of 1 with probability 0.5 and 0 with probability 0.5. There are two states then $X = 0$ and $X = 1$ and each occurs with probability 0.5.

- Example: Lottery ticket that pays off a dollar with probability 0.5.

- Expected Value: The probability weighted average of a random variable's outcomes.

- What is the expected value of the random variable from above?

$$\mu = 0.5 * 1 + 0.5 * 0 = 0.5$$

- Another Example. You have an apple tree. With a probability of 0.3 the tree produces 4 apples and with a probability of 0.7, it produces 10. What is the expected value of tree production?

$$\mu = 0.3 * 4 + 0.7 * 10 = 1.2 + 7.0 = 8.2$$

- One Last Example. A stock can go down by 1 dollar, go up by 1 dollar, or stay at the same value. The probability of these three states are 0.2, 0.4, and 0.4. What is the expected value of holding on to the stock?

$$\mu = 0.2*(-1)+0.4*1+0.4*0 = -0.2+0.4 = 0.2$$

2 Saint Petersburg Paradox

- Should we be willing to pay the expected value of a lottery in order to participate?

- Suppose I was raffling off a lottery which paid off \$100 with probability 0.5 and \$0 probability 0.5. What is the expected value of this lottery?

$$\mu = 0.5 * 100 + 0.5 * 0 = 50$$

- Would you be willing to pay \$50 to participate in this lottery (where you get \$100 with a probability 0.5 and \$0 with a probability 0.5)?

- Let me describe another lottery. A coin is flipped until a head appears. If on the N^{th} flip, the first head appears, then the participant gets 2^N dollars. How much would you be willing to pay to be able to play this game?

- What is the expected value?

$$\begin{aligned} & \frac{1}{2}2^1 + \frac{1}{2}\frac{1}{2}2^2 + \frac{1}{2}\frac{1}{2}\frac{1}{2}2^3 + ... \\ = \; & 1 + 1 + 1 + 1... = \infty \end{aligned}$$

- The expected value is infinity! Would you be willing to pay an infinite amount of money to play this game? If not, why not? Risk aversion!

3 Expected Utility

- We have evaluated utility over different commodity bundles. Now we investigate utility over allocations across future states.

- To make things simple, we consider an underlying utility function which is only a function of wealth.

- We should of a consumer allocating funds across different states as opposed to different commodities. What products can allocate funds across states?

- Insurance: from states where the consumer is rich to states where the consumer is poort
  * Car Insurance (States where the consumer is rich due to no car accident to states where the consumer is poor due to a car accident).
  * Health Insurance (States where the consumer is rich due to absence of sickness to states where the consumer is poor due to sickness).
  * Annuities (States where the consumer is rich because she doesn't live long to states where the consumer is poor due to long life).
- Risky Assets
  * Stocks (States where the consumer is poor to states where the consumer is rich)
  * Bonds (States where inflation is high to states where inflation is low)

- We can ask whether utility is higher with a gamble or with the expected value of a gamble.

- Suppose $U = \sqrt{W}$ where $W = wealth$. Suppose a consumer has \$50. However, with probability 0.5, she will gain \$50 and with probability 0.5, she will lose \$50. We now can write the expected utility function which is the expected utility across states:

$$\begin{aligned} EU &= 0.5U\left(State = Win\right) + 0.5U\left(State = Lose\right) \\ &= 0.5U\left(50+50\right) + 0.5U\left(50-50\right) \\ &= 0.5 * \sqrt{100} + 0.5\sqrt{0} \\ &= 0.5 * 10 \\ &= 5 \end{aligned}$$

- Now suppose this person faces a gamble but can buy insurance at the expected value. In other words, the insurance premium is the expected value. This is called actuarially fair insurance. What would be the cost?

$$0.5 * 50 + 0.5 * (-50) = 0$$

- Actuarially fair insurance would allow the person to fully insure for free. In other words, get \$50 for sure. What would be the expected utility then?

$$0.5\sqrt{50} + 0.5\sqrt{50} = 0.5 * 7.1 + 0.5 * 7.1 = 7.1$$

- So with square root utility, the consumer prefers to fully insure at actuarially fair prices.

- We can also ask what the certainty equivalent of the gamble would be. In other words what level of wealth would make the consumer indifferent between the gamble and insurance? Remember that the utility of the gamble (from above) is 5. What level of wealth indifferent

$$\begin{aligned} 5 &= \sqrt{W} \\ &\implies W = 25 \end{aligned}$$

- In other words, this consumer would be willing to pay a \$25 premium to fully insure.

- It turns out that the demand for insurance is positive (i.e. the certainty equivalent is below the expected value) when consumer are risk averse and this happens when utility in wealth is concave. What is the first derivate of the underlying utility function?

$$\frac{dW^{0.5}}{dW} = 0.5W^{-0.5} > 0$$

This says that the marginal utility of wealth is positive.

- Now we take the second derivative:

$$\begin{aligned} \frac{d^2W^{0.5}}{dW^2} &= \frac{d}{dW}0.5W^{-0.5} \\ &= -0.25W^{-1.5} < 0 \end{aligned}$$

- This says that the marginal utility of wealth is decreasing. In other words, as I get more money, I care about money less. What does this mean? I care more about losing money than I do about gaining money. In other words, I will reject actuarially fair gambles. In other words, I am risk averse!

- So:

  Risk Aversion = Concave Utility in Wealth $\Longrightarrow$

  Utility of Certainty Equivalent < Utility of Expected Value

  $\Longrightarrow$ Positive Demand for Insurance

- Show graph with concave and convex utility.

4 Demand for Risky Assets

4.1 Stock Demand

- We assume a utility function over money:

$$U(W) = LnW \tag{1}$$

- The consumer has an opportunity to buy stocks. Total amount of money depends upon the amount of stocks bought and the realization of the stocks. With probability p, the stock goes up by \$1 and with probability $(1-p)$, it goes down by \$1. Therefore, with probability p, the amount of money, W,that the consumer has is:

$$W+\alpha$$

where α is the number of stocks purchased and with probability $1-p$, the amount of money the consumer has is:

$$W-\alpha$$

- Writing the expected utility function, we get:

$$EU = pU\left(W+\alpha\right) + \left(1-p\right)U\left(W-\alpha\right)$$

wrting with our particular choice of utility function (1) :

$$EU = pLn\left(W+\alpha\right) + \left(1-p\right)Ln\left(W-\alpha\right)$$

- How many stocks (α) should the consumer purchase? To answer this, we use calculus. What are the endogenous variables and exogeneous parameters?

- Taking first order conditions, we get:

$$\begin{aligned} \frac{dEU}{d\alpha} &= \\ p\frac{dLn\left(W+\alpha\right)}{d\alpha}-\left(1-p\right)\frac{dLn\left(W-\alpha\right)}{d\alpha} &= 0 \implies \end{aligned}$$

$$p\frac{dLn\left(W+\alpha\right)}{d\alpha}=\left(1-p\right)\frac{dLn\left(W-\alpha\right)}{d\alpha} \qquad (2)$$

- Remembering from calculus that $\frac{dLnW}{dW}=\frac{1}{W}$, we can rewrite equation (2) as:

$$\begin{aligned} p\frac{1}{W+\alpha} &= \left(1-p\right)\frac{1}{W-\alpha} \\ &\implies p\left(W-\alpha\right)=\left(1-p\right)\left(W+\alpha\right) \\ &\implies pW-\left(1-p\right)W=\left(1-p\right)\alpha+p\alpha \\ &\implies \alpha^{*}=\left(2p-1\right)W \end{aligned}$$

- Note that when $p < 1$, α^* is negative. In other words, when $p\alpha + (1-p)(-\alpha) = (2p-1)\alpha < 0$, then the expected value of the stock is negative $\left(p < \frac{1}{2}\right)$. In that case, the optimal amount of stock purchased would be negative as well. If the expected value of the stock is zero $\left(p = \frac{1}{2}\right)$, then the optimal amount of stock purchased is zero. If the expected value is positive $\left(p > \frac{1}{2}\right)$, then the optimal amount of stock purchased is positive.

4.2 Insurance Demand

- We assume a utility function over money:

$$U(W) = LnW$$

- In one state of the world 1, the consumer is healthy and has wealth W_1. This occurs with probability p. In a second state of the world, the consumer is sick and must spend money for health services, after which she is left with wealth $W_2 < W_1$. There is an opportunity for the consumer to buy insurance. The premium is one dollar per unit of insurance and pays off $R > 1$ dollars if the consumer is sick.

- If the consumer buys α units of insurance, the amount of money the consumer has left if she is healthy is:

$$W_1 - \alpha$$

- If the consumer buys α units of insurance, the amount of money the consumer has left if she is sicks is:

$$W_2 - \alpha + \alpha R = W_2 + (R - 1)\,\alpha$$

- Writing the expected utility function, we get:

$$pU\,(W_1 - \alpha) + (1 - p)\,U\,(W_2 + (R - 1)\,\alpha)$$

- How much insurance (α) should the consumer purhcase?

- What are the exogeneous variables? What are the parameters?

- Taking first order conditions, we get:

$$\frac{dEU}{d\alpha} = p\frac{dLn\left(W_1 - \alpha\right)}{d\alpha} +$$

$$\left(1-p\right)\left(R-1\right)\frac{dLn\left(W_2 + \alpha\left(R-1\right)\right)}{d\alpha} = 0$$

$$\begin{aligned} \Longrightarrow \quad & -p\frac{dLn\left(W_1 + \alpha\right)}{d\alpha} = \\ & -\left(R-1\right)\left(1-p\right)\frac{dLnUW_2 + \alpha\left(R-1\right)}{d\alpha} \end{aligned} \tag{3}$$

- Again, remembering that $\frac{dLnW}{dW} = \frac{1}{W}$, we can rewrite (3):

$$-p\frac{1}{W_1 - \alpha} = -(R-1)(1-p)\frac{1}{W_2 + \alpha(R-1)}$$

- Dividing by -1 on both sides, we get:

$$p[W_2 + \alpha(R-1)] = (1-p)(R-1)[W_1 - \alpha]$$

- Bring the exogenous variable to one side and the terms only involving parameters on the other, we get:

$$\begin{aligned} & \alpha(1-p)(R-1) + \alpha p(R-1) \\ = \; & (1-p)(R-1)W_1 - pW_2 \end{aligned}$$

- Solving for α^*, we finally get:

$$\alpha^* = \frac{(1-p)(R-1)W_1 - pW_2}{R-1} = (1-p)W_1 - \frac{pW_2}{R-1}$$

- Comparative statics:

$$\begin{aligned}\frac{\partial \alpha^*}{\partial W_1} &= 1 - p > 0 \\ \frac{\partial \alpha^*}{\partial W_2} &= -\frac{p}{R-1} < 0 \\ \frac{\partial \alpha^*}{\partial p} &= -W_1 - \frac{W_2}{R-1} < 0\end{aligned}$$

- Do these comparative statics make sense?