Supplementary Appendix:

Asymptotically Unbiased Inference for a Dynamic Panel Model with Fixed Effects When Both n and T are Large

Jinyong Hahn
University of Michigan

Guido Kuersteiner
MIT

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A Proof of Theorem 1

Theorem 1 is established by combining Lemmas 6 and 7 below. Note that

$$y_{it} = \theta_0^t y_{i0} + (I - \theta_0)^{-1} \left(1 - \theta_0^t\right) \alpha_i + \theta_0^{t-1} \varepsilon_{i1} + \theta_0^{t-2} \varepsilon_{i2} + \cdots + \varepsilon_{it}. \tag{1}$$

In the stationary case where $\lim_n \theta_0^n = 0$, we work with the stationary approximation to y_{it} which is given by

$$u_{it}^* \equiv \sum_{j=0}^{\infty} \theta_0^j \varepsilon_{it-j}, \quad t \geq 1 \tag{2}$$

$$y_{it}^* \equiv (I - \theta_0)^{-1} \alpha_i + u_{it}^*, \quad t \geq 0. \tag{3}$$

The vectorized representation of the OLS estimator for θ_0' is given by

$$\operatorname{vec}\left(\widehat{\theta}' - \theta_0'\right) = \left[I \otimes \left(\sum_{i=1}^n \sum_{t=1}^T \left(y_{it-1} - \overline{y}_{i-}\right)\left(y_{it-1} - \overline{y}_{i-}\right)' \right)^{-1} \right] \left[\sum_{i=1}^n \sum_{t=1}^T \left(I \otimes \left(y_{it-1} - \overline{y}_{i-}\right)\right)\left(\varepsilon_{it} - \overline{\varepsilon}_i\right) \right]$$

where $\bar{\varepsilon}_i \equiv \frac{1}{T} \sum_t^T \varepsilon_{it}$. We define the joint cumulant next. Let $\xi = (\xi_1, ..., \xi_k) \in \mathbb{R}^k$ and $\varepsilon = \left(\varepsilon_{t_1}^{(j_1)}, ..., \varepsilon_{t_k}^{(j_k)}\right)$ where $\varepsilon_t^{(j)}$ is the j-th element of ε_t, then $\phi_{j_1,...j_k,t_1,...,t_k}(\xi) \equiv E\left[e^{i\xi'\varepsilon}\right]$ is the joint characteristic function with corresponding cumulant generating function $\ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi)$. The joint v-th order cross-cumulant function is

$$\operatorname{cum}_{j_1,...j_k}\left(\varepsilon_{t_1}, ..., \varepsilon_{t_k}\right) \equiv \left. \frac{\partial^{v_1 + ... + v_k}}{\partial \xi_1^{v_1} \cdots \partial \xi_k^{v_k}} \ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi) \right|_{\xi=0}$$

where v_i are nonnegative integers $v_1 + ... + v_k = v$.

Lemma 1 *Let ε_{it} be iid across i and t and $E\left[\left|\varepsilon_{it}^{(j)}\right|^8\right] < \infty$. Assume Conditions 4 (ii) - (iv) hold. Then Conditions 2 and 3 hold.*

Proof. If $E\left[\left|\varepsilon_{it}^{(j)}\right|^8\right] < \infty$, then $\text{cum}_{j_1,...j_4}(\varepsilon_{t_1},...,\varepsilon_{t_4})$ is well defined. Let $\theta_{lk,j}$ be the l,k-th element of the matrix θ_0^j. Then $u_{it}^{*(l)} = \sum_{k=1}^m \sum_{j=0}^\infty \theta_{lk,j}\varepsilon_{it-j}^{(k)}$ and

$$\text{cum}_{l_1,...,l_4}\left(u_{it_1}^*, \varepsilon_{it_2}, u_{it_3}^*, \varepsilon_{i0}\right) = \sum_{k_1,k_2=1}^m \sum_{j_1,j_2=0}^\infty \theta_{l_1k_1,j_1}\theta_{l_3k_2,j_2}\,\text{cum}_{k_1,l_2,k_3,l_4}\left(\varepsilon_{it_1-j_1}, \varepsilon_{it_2}, \varepsilon_{it_3-j_2}, \varepsilon_{i0}\right).$$

Since $\text{cum}_{k_1,l_2,k_3,l_4}(\varepsilon_{it_1-j_1}, \varepsilon_{it_2}, \varepsilon_{it_3-j_2}, \varepsilon_{i0})$ is zero unless $t_1 - j_1 = t_2 = t_3 - j_2 = 0$ by Lemma (4), it follows immediately that Condition (2) is satisfied. In order to establish Condition (3), note that $z_{it} = (I\otimes u_{it-1}^*)\varepsilon_{it} = \sum_{l=0}^\infty \left(I\otimes\theta_0^l\right)(I\otimes\varepsilon_{it-1-l})\varepsilon_{it}$ has typical element $z_{it}^{(j)} = u_{it-1}^{*(r)}\varepsilon_{it}^{(s)} = \sum_{k=1}^m\sum_{l=0}^\infty \theta_{rk,l}\varepsilon_{it-l-1}^{(k)}\varepsilon_{it}^{(s)}$ with $r = j - m\cdot[(j-1)\bmod m]$ and $s = [(j-1)\bmod m]+1$. Defining $w_{t,l} = (I\otimes\varepsilon_{it-1-l})\varepsilon_{it}$, it then follows that

$$\text{cum}_{j_1,...,j_4}\left(z_{it_1}, z_{it_2}, z_{it_3}, z_{i0}\right) = \sum_{k_1,...,k_4=1}^m \sum_{l_1,...,l_4=0}^\infty \left(\prod_{i=1}^4 \theta_{r_ik_i,l_i}\right)\text{cum}_{[sm+k]}\left(w_{t_1,l_1}, w_{t_2,l_2}, w_{t_3,l_3}, w_{0,l_4}\right)$$

where we use the short hand notation $[sm+k] \equiv s_1m+k_1,...,s_4m+k_4$ for the four indices and $s_i = [(j_i-1)\bmod m]+1$ and $k_i = j_i - m\cdot[(j_i-1)\bmod m]$. From Brillinger (1981, Theorem 2.3.1 (iii)), it follows that $\text{cum}_{k^*}(w_{t_1,l_1}, w_{t_2,l_2}, w_{t_3,l_3}, w_{0,l_4}) = 0$ if any group of w_{t_i,l_1} variables is independent of the remaining w_{t_i,l_1} variables. Now fix $l_1,...,l_4$ arbitrary. Then one can pick at most 11,520 different combinations of t_1, t_2, t_3 such that none of the w_{t_i,l_1} are independent of each other. This leads to an upper bound for

$$\sum_{t_1,t_2,t_3=-\infty}^\infty \left|\text{cum}_{j_1,...,j_4}\left(z_{it_1}, z_{it_2}, z_{it_3}, z_{i0}\right)\right|$$
$$\leq 11,520m^4 \sup_{s,k,t,l}\left|\text{cum}_{[sm+k]}\left(w_{t_1,l_1}, w_{t_2,l_2}, w_{t_3,l_3}, w_{0,l_4}\right)\right| \sum_{l_1,...,l_4=0}^\infty \left(\prod_{i=1}^4 \left|\theta_{r_ik_i,l_i}\right|\right) < \infty.$$

■

Lemma 2 *Let y_{it} be generated by (1). Also, let*

$$S_{nT}^* \equiv \frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=1}^T (I\otimes u_{it}^*)(\varepsilon_{it}-\overline{\varepsilon}_i)$$

Then under Conditions 1, 2 and 3

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=1}^T \left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)(\varepsilon_{it}-\overline{\varepsilon}_i) = S_{nT}^* + o_p(1) \tag{4}$$

Proof. Because $\sum_{t=1}^T\left(I\otimes\overline{y}_{i-}\right)(\varepsilon_{it}-\overline{\varepsilon}_i) = 0$, and

$$\sum_{t=1}^T\left(I\otimes y_{it-1}^*\right)(\varepsilon_{it}-\overline{\varepsilon}_i) = \sum_{t=1}^T\left(I\otimes\left((I-\theta_0)^{-1}\alpha_i + u_{it}^*\right)\right)(\varepsilon_{it}-\overline{\varepsilon}_i) = \sum_{t=1}^T (I\otimes u_{it}^*)(\varepsilon_{it}-\overline{\varepsilon}_i),$$

it suffices to prove that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=1}^T (I\otimes y_{it-1})(\varepsilon_{it}-\overline{\varepsilon}_i) = \frac{1}{\sqrt{nT}}\sum_{i=1}^n\sum_{t=1}^T \left(I\otimes y_{it-1}^*\right)(\varepsilon_{it}-\overline{\varepsilon}_i) + o_p(1).$$

From (1), (2), and (3), we obtain $y_{it} = y_{it}^{*} + \theta_0^t \left(y_{i0} - u_{i0}^{*}\right) - \left(I - \theta_0\right)^{-1} \theta_0^t \alpha_i$. We may therefore write

$$\begin{aligned}
&\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes y_{it-1}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right) - \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes y_{it-1}^{*}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right) \\
&\quad = \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes \theta_0^t y_{i0}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right) && (5) \\
&\quad - \left(I - \theta_0\right)^{-1} \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes \theta_0^t \alpha_i\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right) && (6) \\
&\quad - \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes \theta_0^t u_{i0}^{*}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right). && (7)
\end{aligned}$$

We analyze (5) first. By assumption, its expectation is zero. Because of independence of ε_{it} and y_{i0}, we have

$$\operatorname{Var}\left(\sum_{t=1}^{T} \left(I \otimes \theta_0^t y_{i0}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right)\right) = \sum_{t=1}^{T} \left(I \otimes \theta_0^t y_{i0}\right) \Omega \left(I \otimes y_{i0}' \theta_0'^t\right) - \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(I \otimes \theta_0^t y_{i0}\right) \Omega \left(I \otimes y_{i0}' \theta_0'^s\right),$$

from which it follows that

$$\begin{aligned}
&\operatorname{vec}\left[\operatorname{Var}\left(\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(I \otimes \theta_0^t y_{i0}\right) \left(\varepsilon_{it} - \overline{\varepsilon}_i\right)\right)\right] \\
&= T^{-1} \left(I \otimes I \otimes I \otimes I - \left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right)\right)^{-1} \left(\left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right) - \left(\left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right)\right)^{T+1}\right) \\
&\qquad \times \operatorname{vec}\left(n^{-1} \sum_{i=1}^{n} \left(I \otimes y_{i0}\right) \Omega \left(I \otimes y_{i0}'\right)\right) \\
&- T^{-2} \left(I \otimes \left(I - \theta_0\right)^{-1} \left(\theta_0 - \theta_0^{T+1}\right)\right) \otimes \left(I \otimes \left(I - \theta_0\right)^{-1} \left(\theta_0 - \theta_0^{T+1}\right)\right) \operatorname{vec}\left(n^{-1} \sum_{i=1}^{n} \left(I \otimes y_{i0}\right) \Omega \left(I \otimes y_{i0}'\right)\right) \\
&= o(1).
\end{aligned}$$

It therefore follows that (5) is $o_p(1)$. In the same way it follows that (6) is $o_p(1)$. We turn to the analysis of (7). Because of independence of ε_{it} and u_{i0}^{*}, we can see that it has a mean equal to zero, and

$$\begin{aligned}
\operatorname{vec}\left[\operatorname{Var}\left(\sum_{t=1}^{T} \left(I \otimes \theta_0^t u_{i0}^{*}\right) \varepsilon_{it}\right)\right] &= \sum_{t_1, t_2 = 1}^{T} \left(\left(I \otimes \theta_0^{t_1}\right) \otimes \left(I \otimes \theta_0^{t_2}\right)\right) \operatorname{vec}\left(E\left[\left(I \otimes u_{i0}^{*}\right) \varepsilon_{it_1} \varepsilon_{it_2}' \left(I \otimes u_{i0}^{*\prime}\right)\right]\right) \\
&= \sum_{t_1, t_2 = 1}^{T} \left(\left(I \otimes \theta_0^{t_1}\right) \otimes \left(I \otimes \theta_0^{t_2}\right)\right) \left(E\left[\varepsilon_{it_1} \varepsilon_{it_2}'\right] \otimes E\left[u_{i0}^{*} u_{i0}^{*\prime}\right] + \mathcal{K}_0\left(t_1, t_2\right)\right),
\end{aligned}$$

where the matrix $\mathcal{K}_0(t_1, t_2)$ contains elements of the form $\operatorname{cum}_{j_1, \ldots, j_4}\left(u_{i0}^{*}, u_{i0}^{*}, \varepsilon_{it_1}, \varepsilon_{it_2}\right)$. The sum over the first term then can be expressed as

$$\begin{aligned}
\sum_{t=1}^{T} \left(\left(I \otimes \theta_0^t\right) \otimes \left(I \otimes \theta_0^t\right)\right) \operatorname{vec}\left[\Omega \otimes E(u_{i0}^{*} u_{i0}^{*\prime})\right] &= \left(I \otimes I \otimes I \otimes I - \left(\left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right)\right)\right)^{-1} \\
&\quad \times \left(\left(\left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right)\right) - \left(\left(I \otimes \theta_0\right) \otimes \left(I \otimes \theta_0\right)\right)^{T+1}\right) \operatorname{vec}\left[\Omega \otimes E\left[u_{i0}^{*} u_{i0}^{*\prime}\right]\right].
\end{aligned}$$

The second term is bounded by

$$\sum_{t_1,t_2=1}^{T}\left\|\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right\|\left\|\operatorname{vec}\mathcal{K}_0\left(t_1,t_2\right)\right\|\leq\sup_{t_1,t_2}\left\|\operatorname{vec}\mathcal{K}_0\left(t_1,t_2\right)\right\|\sum_{t_1,t_2=1}^{T}\left\|\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right\|<\infty$$

Together these results imply that $\operatorname{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\varepsilon_{it}\right)=o\left(1\right)$. Next consider

$$\begin{aligned}\left\|\operatorname{Var}\left(\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\overline{\varepsilon}_i\right)\right\| &= \left\|T^{-2}\sum_{t_1,\dots,t_4=1}^{T}\left(\left(I\otimes\theta_0^{t_1}\right)\otimes\left(I\otimes\theta_0^{t_2}\right)\right)\operatorname{vec}E\left[\left(I\otimes u_{i0}^*\right)\varepsilon_{it_3}\varepsilon_{it_4}'\left(I\otimes u_{i0}^{*\prime}\right)\right]\right\| \\ &\leq\left\|\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\otimes\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\right\| \\ &\quad\times\left\|T^{-1}\operatorname{vec}\left[\Omega\otimes E\left[u_{i0}^*u_{i0}^{*\prime}\right]\right]+T^{-2}\sum_{t_3,t_4=1}^{T}\operatorname{vec}\mathcal{K}_0\left(t_3,t_4\right)\right\| \\ &\leq\left\|\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\otimes\left(I\otimes\left(I-\theta_0^{-1}\right)\left(\theta_0-\theta_0^{T+1}\right)\right)\right\| \\ &\quad\times\left(\left\|T^{-1}\operatorname{vec}\left[\Omega\otimes E\left[u_{i0}^*u_{i0}^{*\prime}\right]\right]\right\|+T^{-2}\sum_{t_3,t_4=1}^{T}\left\|\operatorname{vec}\mathcal{K}_0\left(t_3,t_4\right)\right\|\right) \\ &=O\left(T^{-1}\right),\end{aligned}$$

which shows that $\operatorname{Var}\left(\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\theta_0^t u_{i0}^*\right)\overline{\varepsilon}_i\right)=o\left(1\right)$. It therefore follows that (7) is $o_p\left(1\right)$. ■

Lemma 3 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3,*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it}^*\right)\overline{\varepsilon}_i=\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\operatorname{vec}\left(\Omega\right)+o_p\left(1\right).$$

Proof. We have

$$\begin{aligned}E\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it}^*\right)\overline{\varepsilon}_i\right] &= \frac{n}{\sqrt{nT}}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\left(I\otimes u_{it}^*\right)\varepsilon_{is}\right] \\ &=\sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{t}E\left[\left(I\otimes u_{it}^*\right)\varepsilon_{is}\right]=\sqrt{\frac{n}{T}}\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}\left(I\otimes\theta_0^j\right)\operatorname{vec}\left(\Omega\right)\end{aligned}\qquad(8)$$

By the usual result on Cesàro averages, we have $\frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}(I\otimes\theta_0^j)=\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}+o\left(1\right)$. Therefore,

$$E\left[\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}(I\otimes u_{it}^*)\overline{\varepsilon}_i\right]=\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\operatorname{vec}\left(\Omega\right)+o\left(1\right).\qquad(9)$$

We also have

$$\begin{aligned}
&\operatorname{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(I\otimes u_{it}^{*}\right)\bar{\varepsilon}_{i}\right)\\
&= E\left[\frac{1}{T^{3}}\sum_{t_1,..,t_4=1}^{T}\left(\left(I\otimes u_{it_1}^{*}\right)\varepsilon_{it_2}-E\left[\left(I\otimes u_{it_1}^{*}\right)\varepsilon_{it_2}\right]\right)\left(\left(I\otimes u_{it_3}^{*}\right)\varepsilon_{it_4}-E\left[\left(I\otimes u_{it_3}^{*}\right)\varepsilon_{it_4}\right]\right)'\right]\\
&= \frac{1}{T^{3}}\sum_{t_1,..,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(I\otimes\theta_0^{j_1}\right)\left(\operatorname{vec}\left(\operatorname{Cov}\left(\varepsilon_{it_1-j_1},\varepsilon_{it_4}\right)\right)\operatorname{vec}\left(\operatorname{Cov}\left(\varepsilon_{it_3-j_2},\varepsilon_{it_2}\right)\right)'\right)\left(I\otimes\theta_0^{\prime j_2}\right)\\
&+\frac{1}{T^{3}}\sum_{t_1,..,t_4=1}^{T}\sum_{j_1,j_2=0}^{\infty}\left(I\otimes\theta_0^{j_1}\right)\left[\operatorname{Cov}\left(\varepsilon_{it_2},\varepsilon_{it_4}\right)\otimes\operatorname{Cov}\left(\varepsilon_{it_1-j_1},\varepsilon_{it_3-j_2}\right)\right]\left(I\otimes\theta_0^{\prime j_2}\right)\\
&+\frac{1}{T^{3}}\sum_{t_1,..,t_4=1}^{T}\left(I\otimes\theta_0^{j_1}\right)\mathcal{K}\left(t_1,t_2,t_3,t_4\right)\left(I\otimes\theta_0^{\prime j_2}\right)
\end{aligned}$$

where $\mathcal{K}\left(t_1,t_2,t_3,t_4\right)$ is a matrix containing elements of the form $\operatorname{cum}_{j_1,\ldots,j_4}\left(u_{it_1}^{*},\varepsilon_{it_2},u_{it_3}^{*},\varepsilon_{it_4}\right)$. This leads to

$$\begin{aligned}
\operatorname{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(I\otimes u_{it}^{*}\right)\bar{\varepsilon}_{i}\right) &= \frac{1}{T^{3}}\sum_{t_1,t_3=1}^{T}\sum_{t_2=1}^{t_3}\sum_{t_4=1}^{t_1}\left(I\otimes\theta_0^{t_1-t_4}\right)\operatorname{vec}\left(\Omega\right)\operatorname{vec}\left(\Omega\right)'\left(I\otimes\theta_0^{t_3-t_2}\right)\\
&+\frac{1}{T^{2}}\sum_{t_3=1}^{T}\sum_{j_1=0}^{\infty}\sum_{t_1=1}^{\min(T,t_3-j_1)}\left(I\otimes\theta_0^{j_1}\right)\left(\Omega\otimes\Omega'\right)\left(I\otimes\theta_0^{t_3+j_1-t_1}\right)\\
&+\frac{1}{T^{3}}\sum_{t_1,..,t_4=1}^{T}\left(I\otimes\theta_0^{t_1}\right)\mathcal{K}\left(t_1,t_2,t_3,t_4\right)\left(I\otimes\theta_0^{t_4}\right)'.
\end{aligned}$$

The first term on the right is $O\left(T^{-1}\right)$ because

$$\sum_{t_1=1}^{T}\sum_{t_4=1}^{t_1}\left(I\otimes\theta_0^{t_1-t_4}\right)=\sum_{t_1=1}^{T}\left(I\otimes I-I\otimes\theta_0\right)^{-1}\left(I\otimes I-I\otimes\theta_0^{t_1}\right)=O(T).$$

The second term is $O\left(T^{-1}\right)$ because

$$\begin{aligned}
&\left\|\operatorname{vec}\left[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\min(T,t_3-j_1+1)}\left(I\otimes\theta_0^{j_1}\right)\left(\Omega\otimes\Omega'\right)\left(I\otimes\theta_0^{j_2}\right)\right]\right\|\\
&\qquad\le\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left\|\left(I\otimes\theta_0^{j_1}\right)\otimes\left(I\otimes I\right)\right\|\left\|\left(I\otimes I\right)\otimes\left(I\otimes\theta_0^{j_2}\right)\right\|\left\|\operatorname{vec}(\Omega\otimes\Omega')\right\|=O\left(1\right)
\end{aligned}$$

Finally, the third term is $O\left(T^{-2}\right)$ because $\frac{1}{T^3}\sum_{t_1,..,t_4=1}^{T}\operatorname{cum}_{j_1,\ldots,j_4}\left(u_{it_1}^{*},\varepsilon_{it_2},u_{it_3}^{*},\varepsilon_{it_4}\right)=O\left(T^{-2}\right)$ by the cumulant summability assumption. ■

Lemma 4 *Assume ε_t is a sequence of independent, identically distributed random vectors with $E\left[\varepsilon_t\right]=0$ for all t. Then $\operatorname{cum}_{j_1,\ldots j_4}\left(\varepsilon_{t_1},\ldots,\varepsilon_{t_4}\right)=0$ unless $t_1=t_2=\cdots=t_4$. In this case we define $\operatorname{cum}\left(j_1,\ldots,j_4\right)\equiv\operatorname{cum}_{j_1,\ldots j_4}\left(\varepsilon_t,\ldots,\varepsilon_t\right)$.*

Lemma 5 *Let Conditions 1, 2 and 3 be satisfied. Then*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\rightarrow\mathcal{N}\left(0,\Omega\otimes\Upsilon+\mathcal{K}\right)$$

where $\mathcal{K}=\sum_{t=-\infty}^{\infty}\mathcal{K}\left(t,0\right)$ *and* $\mathcal{K}\left(t_1,t_2\right)\equiv E\left[\left(I\otimes u_{it_1-1}^{*}\right)\varepsilon_{it_1}\varepsilon_{it_2}'\left(I\otimes u_{it_2-1}^{*\prime}\right)\right]-E\left[\varepsilon_{it_1}\varepsilon_{it_2}'\right]\otimes E\left[u_{i0}^{*}u_{i0}^{*\prime}\right]$. *If in addition all the innovations* ε_{it} *are independent for all* i *and* t *then*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\rightarrow\mathcal{N}\left(0,\Omega\otimes\Upsilon\right).$$

Proof. We need to check the generalized Lindeberg Feller condition for joint asymptotic normality as in Theorem 2 of Phillips and Moon (1999). A sufficient condition for the theorem to hold is that for all $\ell\in\mathbb{R}^{m^2}$ such that $\ell'\ell=1$ it follows $E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ell'(I\otimes u_{it-1}^{*})\varepsilon_{it}\right)^4\right]<\infty$ uniformly in i and T. Letting $\mathbf{z}_{it}=\ell'\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}$ and noting that $E\left[\mathbf{z}_{it}\right]=0$ we consider

$$\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right]=\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\left[\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2}\right)\operatorname{Cov}\left(\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)+\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_3}\right)\operatorname{Cov}\left(\mathbf{z}_{it_2},\mathbf{z}_{it_4}\right)\right]$$
$$+\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\left[\operatorname{Cov}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_4}\right)\operatorname{Cov}\left(\mathbf{z}_{it_2},\mathbf{z}_{it_3}\right)+\operatorname{cum}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2},\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)\right],$$

where

$$\begin{aligned}\operatorname{Cov}\left(\mathbf{z}_{it},\mathbf{z}_{is}\right)&=\ell'E\left[\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\varepsilon_{is}'\left(I\otimes u_{is-1}^{\prime *}\right)\right]\ell\\&=\ell'\operatorname{vec}\left(E\left[u_{it-1}^{*}\varepsilon_{is}'\right]\right)\operatorname{vec}\left(E\left[u_{is-1}^{*}\varepsilon_{it}'\right]\right)'\ell+\ell'E\left[\varepsilon_{it}\varepsilon_{is}'\right]\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right]\ell\\&+\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\\&=0+\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right]\right)\ell\cdot1\left\{t=s\right\}\\&+\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right).\end{aligned}$$

Using these results we deduce that

$$\begin{aligned}&\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right]\\&=3\left(\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{it-1}^{*\prime}\right]\right)\ell\right)^2\\&+6\left(\ell'\left(\Omega\otimes E\left[u_{it-1}^{*}u_{it-1}^{*\prime}\right]\right)\ell\right)\left(\frac{1}{T}\sum_{t,s=1}^{T}\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\right)\\&+3\left(\frac{1}{T}\sum_{t,s=1}^{T}\sum_{j_1,\dots,j_4=0}^{m-1}\ell_{j_3m+j_1+1}\ell_{j_4m+j_2+1}\operatorname{cum}_{j_1,\dots,j_4}\left(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}\right)\right)^2\\&+\frac{1}{T^2}\sum_{t_1,\dots t_4=1}^{T}\sum_{j_1,\dots,j_4=1}^{m^2}\left(\prod_{k=1}^{4}\ell_{j_k}\right)\operatorname{cum}_{j_1,\dots,j_4}\left(\mathbf{z}_{it_1},\mathbf{z}_{it_2},\mathbf{z}_{it_3},\mathbf{z}_{it_4}\right)\end{aligned}$$

where the terms involving higher order cumulants are $\frac{1}{T}\sum_{t,s=1}^{T}\operatorname{cum}_{j_1,\ldots,j_4}(u_{it-1}^{*},\varepsilon_{it},u_{is-1}^{*},\varepsilon_{is}) = O(1)$ and $\sum_{t_1,\ldots t_4=1}^{T}\operatorname{cum}_{j_1,\ldots,j_4}(\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}) = O(T)$ independent of $i, j_1, \ldots, j_4$. This shows that

$$T^{-2}\sum_{t_1,\ldots t_4=1}^{T} E\left[\mathbf{z}_{it_1}\mathbf{z}_{it_2}\mathbf{z}_{it_3}\mathbf{z}_{it_4}\right] < \infty$$

uniformly in i and T. Finally consider

$$\begin{aligned}
&E\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\right)^{2}\right] \\
&= \frac{1}{T}\sum_{t,s=1}^{T} E\left[\left(I\otimes u_{it-1}^{*}\right)\varepsilon_{it}\varepsilon_{is}'\left(I\otimes u_{is-1}^{*\prime}\right)\right] \\
&= \frac{1}{T}\sum_{t,s=1}^{T}\operatorname{vec}\left(E\left[u_{it-1}^{*}\varepsilon_{is}'\right]\right)\operatorname{vec}\left(E\left[u_{is-1}^{*}\varepsilon_{it}'\right]\right) + \frac{1}{T}\sum_{t,s=1}^{T} E\left[\varepsilon_{it}\varepsilon_{is}'\right]\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right] + \frac{1}{T}\sum_{t,s=1}^{T}\mathcal{K}\left(t,s\right) \\
&= \Omega\otimes\Upsilon + \mathcal{K} + o(1).
\end{aligned}$$

where $\mathcal{K} = \sum_{t_1=-\infty}^{\infty}\mathcal{K}(t_1,0)$. Note that $\operatorname{vec}\left(E\left[u_{it-1}^{*}\varepsilon_{is}'\right]\right)\operatorname{vec}\left(E\left[u_{is-1}^{*}\varepsilon_{it}'\right]\right) = 0$ for all t and s and that $\frac{1}{T}\sum_{t,s=1}^{T} E\left[\varepsilon_{it}\varepsilon_{is}'\right]\otimes E\left[u_{it-1}^{*}u_{is-1}^{*\prime}\right] = \frac{1}{T}\sum_{t=1}^{T} E\left[\varepsilon_{it}\varepsilon_{it}'\right]\otimes E\left[u_{it-1}^{*}u_{it-1}^{*\prime}\right] = \Omega\otimes\Upsilon$ by strict stationarity. The last line of the display follows by Cesàro summability and stationarity. The second part of the theorem follows from Lemma (4) which implies that $\mathcal{K}\left(t_1,t_2\right) = 0$ for all t_1 and t_2. ■

Lemma 6 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3,*

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_{i}\right)\to\mathcal{N}\left(-\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\operatorname{vec}\left(\Omega\right),\Omega\otimes\Upsilon+\mathcal{K}\right).$$

Proof. Note that $\sum_{t=1}^{T}\left(I\otimes\overline{y}_{i-}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_{i}\right) = 0$, and that

$$\sum_{t=1}^{T}\left(I\otimes y_{it-1}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_{i}\right) = \sum_{t=1}^{T}\left(I\otimes\left(\left(I-\theta_0\right)^{-1}\alpha_i+u_{it}^{*}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_{i}\right) = \sum_{t=1}^{T}\left(I\otimes u_{it}^{*}\right)\left(\varepsilon_{it}-\overline{\varepsilon}_{i}\right).$$

The result then follows from Lemmas 2, 3, and 5. ■

Lemma 7 *Let y_{it} be generated by (1). Under Conditions 1, 2 and 3,*

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)' = E\left[\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)'\right] + o_p\left(1\right) = \Upsilon + o_p\left(1\right).$$

Proof. Recall $y_{it} = y_{it}^{*} + \theta_0^{t}(y_{i0}-y_{i0}^{*}) - \left(I-\theta_0\right)^{-1}\theta_0^{t}\alpha_i$, and hence

$$y_{it-1}-\overline{y}_{i-} = \left(y_{it-1}^{*}-\overline{y}_{i-}^{*}\right) + \left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\left(y_{i0}-u_{i0}^{*}-\alpha_i\right).$$

Therefore,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)' = \varphi_1+\varphi_2+\varphi_2'+\varphi_3+\varphi_3'+\varphi_4+\varphi_4'+\varphi_5+\varphi_6,$$

where

$$\varphi_1 \equiv \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}^{*}-\overline{y}_{i-}^{*}\right)\left(y_{it-1}^{*}-\overline{y}_{i-}^{*}\right)',$$

$$\varphi_2 \equiv \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}^{*}-\overline{y}_{i-}^{*}\right)\left(y_{i0}-\alpha_i\right)'\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)',$$

$$\varphi_3 \equiv \frac{-2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}^{*}-\overline{y}_{i-}^{*}\right)u_{i0}^{*\prime}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)',$$

$$\varphi_4 = \frac{-2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\left(y_{i0}-\alpha_i\right)u_{i0}^{*\prime}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)'\right],$$

$$\varphi_5 \equiv \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\left(y_{i0}-\alpha_i\right)\left(y_{i0}-\alpha_i\right)'\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)',$$

$$\varphi_6 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)u_{i0}^{*}u_{i0}^{*\prime}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)'$$

First note that $\varphi_1 = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)'-\frac{1}{n}\sum_{i=1}^{n}\left(\overline{y}_{i-}^{*}-Ey_{it-1}^{*}\right)\left(\overline{y}_{i-}^{*}-Ey_{it-1}^{*}\right)'$. It follows immediately that $\varphi_1 = E\left[\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)\left(y_{it-1}^{*}-Ey_{it-1}^{*}\right)'\right]+o_p\left(1\right)=\Upsilon+o_p\left(1\right)$. For φ_5 consider

$$\varphi_5 \equiv \frac{1}{nT}\sum_{t=1}^{T}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\otimes\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\sum_{i=1}^{n}\operatorname{vec}\left(y_{i0}-\alpha_i\right)\left(y_{i0}-\alpha_i\right)'$$

note that $\frac{1}{n}\sum_{i=1}^{n}\left(y_{i0}-\alpha_i\right)\left(y_{i0}-\alpha_i\right)'=O(1)$ by assumption and $T^{-1}\sum_{t=1}^{T}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\otimes\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)=O\left(T^{-1}\right)$ thus $\varphi_5=O(T^{-1})$. Next

$$E\left[\varphi_6\right]=\frac{1}{T}\sum_{t=1}^{T}\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)\Upsilon\left(\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right)'$$

implying that $E\varphi_6\to 0$ and thus by the Markov inequality $\varphi_6=o_p(1)$. Then we have

$$\left\|\varphi_4\right\|\leq\frac{1}{nT}\sum_{i=1}^{n}\left\|y_{i0}-\alpha_i\right\|\sum_{t=1}^{T}\left[\left\|\theta_0^{t-1}-\frac{1}{T}\left(I-\theta_0\right)^{-1}\left(I-\theta_0^{T}\right)\right\|^2\left\|u_{i0}^{*}\right\|\right]$$

and we can use the Markov inequality again to show that $\varphi_4=o_p(1)$. Similar arguments show that φ_2 and φ_3 are $o_p(1)$. ■

We finally provide the proof of Theorem 1 using Lemmas 6 and 7, which show that

$$\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(I\otimes\left(y_{it-1}-\overline{y}_{i-}\right)\right)\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\to\mathcal{N}\left(-\sqrt{\rho}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\operatorname{vec}\left(\Omega\right),\Omega\otimes\Upsilon+\mathcal{K}\right),$$

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it-1}-\overline{y}_{i-}\right)\left(y_{it-1}-\overline{y}_{i-}\right)'=\Upsilon+o_p\left(1\right).$$

It therefore follows that

$$\sqrt{nT}\operatorname{vec}(\widehat{\theta}'-\theta_0')\xrightarrow{d}\mathcal{N}\left(-\sqrt{\rho}(I\otimes\Upsilon)^{-1}\left(I\otimes I-\left(I\otimes\theta_0\right)\right)^{-1}\operatorname{vec}\left(\Omega\right),\left(I\otimes\Upsilon\right)^{-1}\left(\Omega\otimes\Upsilon+\mathcal{K}\right)\left(I\otimes\Upsilon\right)^{-1}\right)$$

B Auxiliary Results and Definitions

We first recall a few well established results on higher order cross cumulants to introduce notation. A reference for this material is Brillinger (1981).

Definition 1 *Let $\varepsilon_t \in \mathbb{R}^p$ be a strictly stationary vector process with elements $\varepsilon_t^{(j)}$ such that $E\left[\varepsilon_t^{(j)}\right] = 0$ and $E\left|\varepsilon_t^{(j)}\right|^v < \infty$ for some integer v such that $0 < v < \infty$. Let $\xi = (\xi_1, ..., \xi_k) \in \mathbb{R}^k$ and $\varepsilon = \left(\varepsilon_{t_1}^{(j_1)}, ..., \varepsilon_{t_k}^{(j_k)}\right)$, then $\phi_{j_1,...j_k,t_1,...,t_k}(\xi) \equiv E\left[e^{i\xi'\varepsilon}\right]$ is the joint characteristic function with corresponding cumulant generating function $\ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi)$. The joint v-th order cross-cumulant function is*

$$\mathrm{cum}_{j_1,...j_k}\left(\varepsilon_{t_1}, ..., \varepsilon_{t_k}\right) \equiv \left.\frac{\partial^{v_1+...+v_k}}{\partial\xi_1^{v_1}\cdots\partial\xi_k^{v_k}} \ln \phi_{j_1,...j_k,t_1,...,t_k}(\xi)\right|_{\xi=0}$$

where v_i are nonnegative integers $v_1 + ... + v_k = v$. When $v = 2$ we have

$$\mathrm{Cov}\left(\varepsilon_{t_1}^{(j_1)}, \varepsilon_{t_2}^{(j_2)}\right) = \mathrm{Cov}_{j_1,j_2}\left(\varepsilon_{t_1}, \varepsilon_{t_2}\right) \equiv \mathrm{cum}_{j_1,j_2}\left(\varepsilon_{t_1}, \varepsilon_{t_2}\right) \equiv \left.\frac{\partial^2}{\partial\xi_1\partial\xi_2} \ln \phi_{j_1,j_2,t_1,t_2}(\xi)\right|_{\xi=0}.$$

Lemma 8 *Let a, b, c, d be random vectors with zero mean and denote the j-th element of a by $a^{(j)}$ and similarly for the other vectors. Then*

$$E\left[a^{(j_1)}b^{(j_2)}c^{(j_3)}d^{(j_4)}\right] = \mathrm{Cov}_{j_1,j_2}(a,b)\,\mathrm{Cov}_{j_3,j_4}(c,d) + \mathrm{Cov}_{j_1,j_3}(a,c)\,\mathrm{Cov}_{j_2,j_4}(b,d)$$
$$+ \mathrm{Cov}_{j_1,j_4}(a,d)\,\mathrm{Cov}_{j_2,j_3}(b,c) + \mathrm{cum}_{j_1,j_2,j_3,j_4}(a,b,c,d),$$

and

$$\mathrm{Cov}\left(a^{(j_1)}b^{(j_2)}, c^{(j_3)}d^{(j_4)}\right) = \mathrm{Cov}_{j_1,j_3}(a,c)\,\mathrm{Cov}_{j_2,j_4}(b,d) + \mathrm{Cov}_{j_1,j_4}(a,d)\,\mathrm{Cov}_{j_2,j_3}(b,c) + \mathrm{cum}_{j_1,j_2,j_3,j_4}(a,b,c,d).$$

Proof. *The 4-th order cumulant $\mathrm{Cov}\left(a^{(j_1)}b^{(j_2)}, c^{(j_3)}d^{(j_4)}\right)$ is analyzed by considering the following matrix*

$$X = \begin{bmatrix} a^{(j_1)} & b^{(j_2)} \\ c^{(j_3)} & d^{(j_4)} \end{bmatrix}.$$

with typical element $X_{i,j}$. Then from Brillinger (1981, Theorem 2.3.2),

$$\mathrm{Cov}\left(\prod_{j=1}^{2} X_{1,j}, \prod_{j=1}^{2} X_{2,j}\right) = \sum_{\upsilon} \prod_{\upsilon_s \in \upsilon} \mathrm{cum}\left(X_{i,j}, (i,j) \in \upsilon_s\right),$$

where $\mathrm{cum}\left(X_{i,j}, (i,j) \in \upsilon_s\right)$ is the joint cumulant of all the $X_{i,j}$ with indices $i, j \in \upsilon_s$ and the sum is over all indecomposable partitions υ of the table

$$\begin{matrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{matrix}.$$

A definition of indecomposable partitions is given in Brillinger (1981, p.20). In this case there are only three indecomposable partitions which lead to nonzero cumulants: $\upsilon = \{\{(1,1),(2,2)\},\{(2,1),(1,2)\}\}$ or $\upsilon = \{\{(1,1),(1,2)\},\{(2,1),(2,2)\}\}$ or $\upsilon = \{(1,1),(1,2),(2,1),(2,2)\}$. Other indecomposable partitions contain a single element but in that case $\mathrm{cum}\left(a^{(j_1)}\right) = E\left[a^{(j_1)}\right] = 0$, for example, so all these terms are zero. This establishes the second statement of the Lemma. The first statement then follows directly from $\mathrm{Cov}\left(a^{(j_1)}b^{(j_2)}, c^{(j_3)}d^{(j_4)}\right) = E\left[a^{(j_1)}b^{(j_2)}c^{(j_3)}d^{(j_4)}\right] - E\left[a^{(j_1)}b^{(j_2)}\right]E\left[c^{(j_3)}d^{(j_4)}\right]$ after rearranging. ■

Lemma 9 *Assume* ε_t *is a sequence of independent, identically distributed random vectors with* $E\varepsilon_t = 0$ *for all* t*. Let* $\mathrm{cum}_{j_1,\dots j_4}(\varepsilon_{t_1}, ..., \varepsilon_{t_4})$ *be defined as in Definition (1). Then* $\mathrm{cum}_{j_1,\dots j_4}(\varepsilon_{t_1}, ..., \varepsilon_{t_4}) = 0$ *unless* $t_1 = t_2 = \cdots = t_4$*. In this case we define* $\mathrm{cum}(j_1, ..., j_4) \equiv \mathrm{cum}_{j_1,\dots j_4}(\varepsilon_t, ..., \varepsilon_t)$*.*

Proof. See Property (iii) in Brillinger (1981, p.19). ■