

Analytic Bias Correction for Dynamic Panel Models with Fixed Effects: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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1 Bias Correction using Higher Order Expansions

In Section 2 we explain the bias of the GMM estimator as a result of the biases of the 2SLS estimators. In this appendix, we attempt eliminating the bias (i) by adopting the second order Taylor type approximation as by Nagar (1959), and Rothenberg (1983); and (ii) by adopting a “long difference” perspective.

For this purpose we first analyze the second order bias of a general minimization estimator b of a single parameter $\beta \in \mathbb{R}$ defined by

$$b = \underset{c \in C}{\operatorname{argmin}} Q_n(c) \tag{1}$$

for some $C \subset \mathbb{R}$. The score for the minimization problem is denoted by $S_n(c) \equiv \partial Q_n(c) / \partial c$. The criterion function is assumed to be of the form $Q_n(c) = g(c)' G(c)^{-1} g(c)$ where $g(c)$ and $G(c)$ are defined in Condition 4. The criterion function depends on primitive functions $\delta(w_i, c)$ and $\psi(w_i, c)$ mapping $\mathbb{R}^k \times \mathbb{R}$ into \mathbb{R}^d for $d \geq 1$, where w_i are i.i.d. observations. We assume that $E[\delta(w_i, \beta)] = 0$. We impose the following additional conditions on w_i, δ , and ψ .

Condition 1 *The random variables w_i are i.i.d.*

Condition 2 *The functions $\delta(w, c)$ and $\psi(w, c)$ are three times differentiable in c for $c \in C$ where $C \subset \mathbb{R}$ is a compact set such that $\beta \in \operatorname{int} C$. Assume that $\delta(w_i, c)$ and $\psi(w_i, c)$ satisfy a Lipschitz condition $\|\delta(w_i, c_1) - \delta(w_i, c_2)\| \leq M_\delta(w_i) |c_1 - c_2|$ for some function $M_\delta(\cdot) : \mathbb{R}^k \rightarrow \mathbb{R}$ and $c_1, c_2 \in C$ with the same statement holding for ψ . The functions $M_\delta(\cdot)$ and $M_\psi(\cdot)$ satisfy $E[M_\delta(w_i)] < \infty$ and $E[|M_\psi(w_i)|^2] < \infty$.*

Condition 3 *Let $\delta_j(w_i, c) \equiv \partial^j \delta(w_i, c) / \partial c^j$, $\Psi(w_i, c) \equiv \psi(w_i, c) \psi(w_i, c)'$ and $\Psi_j(w_i, c) \equiv \partial^j \Psi(w_i, c) / \partial c^j$. Then, $\lambda_j(c) \equiv E[\delta_j(w_i, c)]$, and $\Lambda_j(c) \equiv E[\Psi_j(w_i, c)]$ all exist and are finite for $j = 0, \dots, 3$. For simplicity, we use the notation $\lambda_j \equiv \lambda_j(\beta)$, $\Lambda_j \equiv \Lambda_j(\beta)$, $\lambda(c) \equiv \lambda_0(c)$ and $\Lambda(c) \equiv \Lambda_0(c)$.*

Condition 4 *Let $g(c) \equiv \frac{1}{n} \sum_{i=1}^n \delta(w_i, c)$, $g_j(c) \equiv \frac{1}{n} \sum_{i=1}^n \delta_j(w_i, c)$, $G(c) \equiv \frac{1}{n} \sum_{i=1}^n \psi(w_i, c) \psi(w_i, c)'$ and $G_j(c) \equiv \frac{1}{n} \sum_{i=1}^n \Psi_j(w_i, c)$. Then $g(c) \xrightarrow{p} E[\delta(w_i, c)]$, $g_j(c) \xrightarrow{p} E[\delta_j(w_i, c)]$, $G(c) \xrightarrow{p} E[\Psi(w_i, c)]$, and $G_j(c) \xrightarrow{p} E[\Psi_j(w_i, c)]$ for all $c \in C$.*

Our asymptotic approximation of the second order bias of b is based on an approximate estimator \tilde{b} such that $b - \tilde{b} = o_p(n^{-1})$. The approximate bias of b is then defined as $E[\tilde{b}] - \beta$ while the original estimator b need not necessarily possess moments of any order. In order to justify our approximation we need to establish that b is \sqrt{n} -consistent and that $S_n(b) = 0$ with probability tending to one. For this purpose we introduce the following additional conditions.

Condition 5 (i) *There exists some finite $0 < \mathbf{M} < \infty$ such that the eigenvalues of $E[\Psi(w_i, c)]$ are contained in the compact interval $[\mathbf{M}^{-1}, \mathbf{M}]$ for all $c \in C$; (ii) the vector $E[\delta(w_i, c)] = 0$ if and only if $c = \beta$; (iii) $\lambda_1 \neq 0$.*

Condition 6 *There exists some $\eta > 0$ such that $E[M_\delta(w_i)^{2+\eta}] < \infty$, $E[M_\psi(w_i)^{2+\eta}] < \infty$, $E[\sup_{c \in C} \|\delta(w_i, c)\|^{2+\eta}] < \infty$, and $E[\sup_{c \in C} \|\psi(w_i, c)\|^{4+\eta}] < \infty$.*

Condition 5 is an identification condition that guarantees the existence of a unique interior minimum of the limiting criterion function. Condition 6 corresponds to Assumption B of Andrews (1994) and is used to establish a stochastic equicontinuity property of the criterion function.

Theorem 1 *Under conditions 1 - 6, b defined in (1) satisfies $\sqrt{n}(b - \beta) = O_p(1)$ and $S_n(b) = 0$ with probability tending to 1.*

Proof. See Section 2. ■

Based on Theorem 1 the first order condition for (1) can be characterized by

$$0 = 2g_1(b)' G(b)^{-1} g(b) - g(b)' G(b)^{-1} G_1(b) G(b)^{-1} g(b) \text{ wp } \rightarrow 1. \quad (2)$$

A second order Taylor expansion of (2) around β leads to a representation of $b - \beta$ up to terms of order $o_p(n^{-1})$:

Definition 1

$$\begin{aligned} \Psi &\equiv 3\lambda_1' \Lambda^{-1} \lambda_2 - 3\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1, \\ \Upsilon &\equiv 2\lambda_1 \Lambda^{-1} \lambda_1, \quad \frac{1}{\sqrt{n}} \Phi \equiv 2\lambda_1' \Lambda^{-1} g, \\ \frac{1}{\sqrt{n}} \Xi &\equiv 4(g_1 - \lambda_1)' \Lambda^{-1} \lambda_1 - 2\lambda_1' \Lambda^{-1} (G - \Lambda) \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g + 2\lambda_2' \Lambda^{-1} g, \\ \frac{1}{n} \Gamma &\equiv 2(g_1 - \lambda_1)' \Lambda^{-1} g - 2\lambda_1' \Lambda^{-1} (G - \Lambda) \Lambda^{-1} g - g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g. \end{aligned}$$

In Section 3, it is shown that

$$\sqrt{n}(b - \beta) = -\frac{1}{\Upsilon} \Phi + \frac{1}{\sqrt{n}} \left(-\frac{1}{\Upsilon} \Gamma + \frac{1}{\Upsilon^2} \Phi \Xi - \frac{\Psi}{\Upsilon^3} \Phi^2 \right) + o_p\left(\frac{1}{\sqrt{n}}\right). \quad (3)$$

Ignoring the $o_p\left(\frac{1}{\sqrt{n}}\right)$ term in (3), and taking expectations, we obtain the ‘‘approximate mean’’ of $\sqrt{n}(b - \beta)$. We present the second order bias of b in the next Theorem.

Theorem 2 *Under Conditions 1-6, the second order bias of b is equal to*

$$-\frac{1}{\sqrt{n}} \frac{1}{\Upsilon} E[\Phi] - \frac{1}{n} \frac{1}{\Upsilon} E[\Gamma] + \frac{1}{n} \frac{1}{\Upsilon^2} E[\Phi \Xi] - \frac{\Psi}{n \Upsilon^3} E[\Phi^2]. \quad (4)$$

where

$$E[\Phi] = 0,$$

$$E[\Gamma] = 2 \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta'_i}{\partial \beta} \right] \right) - 2\lambda'_1 \Lambda^{-1} E [\psi_i \psi'_i \Lambda^{-1} \delta_i] - \text{trace} (\Lambda^{-1} \Lambda_1 \Lambda^{-1} E [\delta_i \delta'_i]),$$

and

$$\begin{aligned} E[\Phi \Xi] &= 8\lambda'_1 \Lambda^{-1} E \left[\delta_i \frac{\partial \delta'_i}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda'_1 \Lambda^{-1} E [\delta_i \lambda'_1 \Lambda^{-1} \psi_i \psi'_i] \Lambda^{-1} \lambda_1 \\ &\quad - 8\lambda'_1 \Lambda^{-1} E [\delta_i \delta'_i] \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 + 4\lambda'_1 \Lambda^{-1} E [\delta_i \delta'_i] \Lambda^{-1} \lambda_2, \end{aligned}$$

and

$$E[\Phi^2] = 4\lambda'_1 \Lambda^{-1} E [\delta_i \delta'_i] \Lambda^{-1} \lambda_1,$$

Proof. See Section 3. ■

Remark 1 For the particular case where $\psi_i = \delta_i$, i.e. when b is a CUE, the bias formula (4) exactly coincides with Newey and Smith's (2000).

We now apply these general results to the GMM estimator of the dynamic panel model. The GMM estimator \hat{b}_{GMM} can be understood to be a solution to the minimization problem

$$\min_c \left(\frac{1}{n} \sum_{i=1}^n m_i(c) \right)' V_n^{-1} \left(\frac{1}{n} \sum_{i=1}^n m_i(c) \right)$$

for $c \in C$ where C is some closed interval on the real line containing the true parameter value and

$$m_i(c) = \begin{pmatrix} z_{i1} (y_{i1}^* - c \cdot x_{i1}^*) \\ \vdots \\ z_{i,T-1} (y_{i,T-1}^* - c \cdot x_{i,T-1}^*) \end{pmatrix}, \quad V_n = \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} z_{i1} z'_{i1} & & 0 \\ & \ddots & \\ 0 & & z_{i,T-1} z'_{i,T-1} \end{bmatrix}.$$

We now characterize the finite sample bias of the GMM estimator \hat{b}_{GMM} of the dynamic panel model using Theorem 2. Recall that we assume

Condition 7 $\varepsilon_{it} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$ over i and t .

Condition 8 $y_{i0} | \alpha_i \sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right)$ and $\alpha_i \sim N(0, \sigma_\alpha^2)$.

It can be shown that:

Theorem 3 Under Conditions 7-8 the second order bias of \hat{b}_{GMM} is equal to

$$\frac{B_1 + B_2 + B_3}{n} + o\left(\frac{1}{n}\right), \quad (5)$$

where

$$\begin{aligned} B_1 &\equiv \Upsilon_1^{-1} \sum_{t=1}^{T-1} \text{trace} \left((\Gamma_t^{zz})^{-1} \Gamma_{t,t}^{\varepsilon x z z} \right) \\ B_2 &\equiv -2\Upsilon_1^{-2} \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{z x'} (\Gamma_t^{zz})^{-1} \Gamma_{t,s}^{\varepsilon x z z} (\Gamma_s^{zz})^{-1} \Gamma_s^{z x} \\ B_3 &\equiv \Upsilon_1^{-2} \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{z x'} (\Gamma_t^{zz})^{-1} B_{3,1}(t, s) (\Gamma_s^{zz})^{-1} \Gamma_s^{z x}. \end{aligned}$$

where $\Gamma_t^{zz} \equiv E[z_{it} z'_{it}]$, $\Gamma_t^{zx} \equiv E[z_{it} x'_{it}]$, $\Gamma_{t,s}^{\varepsilon x z z} \equiv E[\varepsilon_{it}^* x_{is}^* z_{it} z'_{is}]$, $B_{3,1}(t, s) \equiv E[\varepsilon_{it}^* z_{it} \Gamma_s^{z x'} (\Gamma_s^{zz})^{-1} z_{is} z'_{is}]$

and $\Upsilon_1 \equiv \sum_{t=1}^{T-1} \Gamma_t^{z x'} (\Gamma_t^{zz})^{-1} \Gamma_t^{z x}$.

Proof. See Section 4. ■

Because the summand

$$E \left[E [z_{it} x_{it}^*]' E [z_{it} z_{it}']^{-1} \varepsilon_{it}^* z_{it} \lambda_s' \Lambda_s^{-1} z_{is} z_{is}' E [z_{is} z_{is}']^{-1} E [z_{is} x_{is}^*] \right]$$

in the numerator of B_3 is equal to zero for $s < t$, we may also write

$$\begin{aligned} B_1 &\equiv \Upsilon_1^{-1} \sum_{t=1}^{T-1} \text{trace} \left((\Gamma_t^{zz})^{-1} \Gamma_{t,t}^{\varepsilon x z z} \right) \\ B_2 &\equiv -2 \Upsilon_1^{-2} \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{z x'} (\Gamma_t^{zz})^{-1} \Gamma_{t,s}^{\varepsilon x z z} (\Gamma_s^{zz})^{-1} \Gamma_s^{z x} \\ B_3 &\equiv \Upsilon_1^{-2} \sum_{t=1}^{T-1} \sum_{s=t}^{T-1} \Gamma_t^{z x'} (\Gamma_t^{zz})^{-1} B_{3,1}(t, s) (\Gamma_s^{zz})^{-1} \Gamma_s^{z x}. \end{aligned}$$

2 Proof of Theorem 1

By Lemma 2 (a) and Theorem 1(a) of Andrews (1992) and Conditions 1-4 it follows that $\sup_{c \in C} |Q_n(c) - Q(c)| = o_p(1)$, where $Q(c) \equiv \lambda(c)' \Lambda(c)^{-1} \lambda(c)$. Let $B(\beta, \epsilon)$ be an open interval of length ϵ centered at β . By Condition 5 it follows that $\inf_{c \notin B(\beta, \epsilon)} Q(c) > Q(\beta) = 0$ for all $\epsilon > 0$. It then follows from standard arguments that $b - \beta = o_p(1)$. It therefore follows that $\Pr(S_n(b) \neq 0) \leq \Pr(b \in \partial C) = 1 - \Pr(b \in \text{int } C) \leq 1 - \Pr(b \in B(\beta, \epsilon)) \rightarrow 0$ for any $\epsilon > 0$ where ∂C denotes the boundary of C .

Using similar arguments as in Pakes and Pollard (1989) we write

$$\begin{aligned} |Q(b)| &= \left| \lambda(b)' \Lambda(b)^{-1} \lambda(b) \right| \\ &\leq \left| g(b)' G(b)^{-1} g(b) - \lambda(b)' \Lambda(b)^{-1} \lambda(b) - g(\beta)' G(\beta)^{-1} g(\beta) \right| \\ &\quad + \left| g(b)' G(b)^{-1} g(b) \right| + \left| g(\beta)' G(\beta)^{-1} g(\beta) \right| \\ &\leq \left| g(b)' G(b)^{-1} g(b) - \lambda(b)' G(b)^{-1} \lambda(b) - g(\beta)' G(\beta)^{-1} g(\beta) \right| \\ &\quad + \left| \lambda(b)' G(b)^{-1} \lambda(b) - \lambda(b)' \Lambda(b)^{-1} \lambda(b) \right| \\ &\quad + \left| g(b)' G(b)^{-1} g(b) \right| + \left| g(\beta)' G(\beta)^{-1} g(\beta) \right|, \end{aligned}$$

where $\left| g(b)' G(b)^{-1} g(b) \right| \leq \left| g(\beta)' G(\beta)^{-1} g(\beta) \right| = O_p(n^{-1})$ by the definition of b and Condition 6. We have $G(b)^{-1} = O_p(1)$ by consistency of b and the uniform law of large numbers, from which we obtain $\left| g(\beta)' G(\beta)^{-1} g(\beta) \right| = O_p(n^{-1})$. We also have

$$\begin{aligned} &g(b)' G(b)^{-1} g(b) - \lambda(b)' G(b)^{-1} \lambda(b) - g(\beta)' G(\beta)^{-1} g(\beta) \\ &= g(b)' G(b)^{-1} (g(b) - g(\beta) - \lambda(b)) + (g(b) - g(\beta) - \lambda(b))' G(b)^{-1} \lambda(b) \\ &\quad + 2g(\beta)' G(\beta)^{-1} \lambda(b) + (g(b) - g(\beta) - \lambda(b))' G(b)^{-1} g(\beta). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |Q(b)| &\leq \left| \lambda(b)' G(b)^{-1} \lambda(b) - \lambda(b)' \Lambda(b)^{-1} \lambda(b) \right| \\ &\quad + \left| g(b)' G(b)^{-1} (g(b) - g(\beta) - \lambda(b)) \right| \\ &\quad + \left| (g(b) - g(\beta) - \lambda(b))' G(b)^{-1} \lambda(b) \right| \\ &\quad + 2 \left| g(\beta)' G(\beta)^{-1} \lambda(b) \right| + \left| (g(b) - g(\beta) - \lambda(b))' G(b)^{-1} g(\beta) \right| \\ &\quad + O_p(n^{-1}). \end{aligned}$$

The terms $G(\beta)^{-1}$, and $\Lambda(b)^{-1}$ are $O_p(1)$ by consistency of b and the uniform law of large numbers. Also, the terms $g(b)$ and $\lambda(b)$ are $o_p(1)$ by consistency of b and the uniform law of large numbers. From Theorems 1 and 2 in Andrews (1994) and Conditions 1-6 it follows that $g(b) - \lambda(b) - g(\beta) = o_p(n^{-1/2})$. From a standard CLT and consistency of b it follows that $(g(b) - \lambda(b))' G(b)^{-1} g(\beta) = o_p(n^{-1})$, and $g(\beta)' G(b)^{-1} = O_p(n^{-1/2})$. These results show that

$$\begin{aligned} |Q(b)| &\leq \left\| G(b)^{-1} - \Lambda(b)^{-1} \right\| \|\lambda(b)\|^2 + O_p(n^{-1/2}) \|\lambda(b)\| + o_p(n^{-1}) \\ &= o_p(1) \|\lambda(b)\|^2 + O_p(n^{-1/2}) \|\lambda(b)\| + o_p(n^{-1}). \end{aligned}$$

Because $|Q(b)| = \left| \lambda(b)' \Lambda(b)^{-1} \lambda(b) \right| \geq \frac{1}{\mathbf{M}} \|\lambda(b)\|^2$, we conclude that

$$\left(\frac{1}{\mathbf{M}} - o_p(1) \right) \|\lambda(b)\|^2 - O_p(n^{-1/2}) \|\lambda(b)\| \leq o_p(n^{-1})$$

or

$$\|\lambda(b)\| = O_p(n^{-1/2}),$$

which implies that $b - \beta = O_p(n^{-1/2})$.

3 Proof of Theorem 2

Note that we have

$$\begin{aligned} g_1(b) &= g_1 + \frac{1}{\sqrt{n}} g_2 \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} g_3 \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right), \\ g(b) &= g + \frac{1}{\sqrt{n}} g_1 \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} g_2 \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right), \end{aligned}$$

and

$$\begin{aligned} G(b)^{-1} &= G^{-1} - \frac{1}{\sqrt{n}} G^{-1} G_1 G^{-1} \cdot \sqrt{n}(b - \beta) \\ &\quad + \frac{1}{2n} (2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}) (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right), \\ G_1(b) &= G_1 + \frac{1}{\sqrt{n}} G_2 \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} G_3 \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right) \end{aligned}$$

where g, g_j, G, G_j and $\sqrt{n}(b - \beta)$ are $O_p(1)$ by Conditions 3 and 4 and Lemma 1. Therefore, we have

$$g_1(b)' G(b)^{-1} g(b) = g_1' G^{-1} g + \frac{1}{\sqrt{n}} h_1 \cdot \sqrt{n}(b - \beta) + \frac{1}{n} h_2 \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right),$$

and

$$\begin{aligned} g(b)' G(b)^{-1} G_1(b) G(b)^{-1} g(b) &= g' G^{-1} G_1 G^{-1} g + \frac{1}{\sqrt{n}} h_3 \cdot \sqrt{n}(b - \beta) \\ &\quad + \frac{1}{n} h_4 \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right), \end{aligned}$$

where

$$\begin{aligned}
h_1 &= g'_2 G^{-1} g - g'_1 G^{-1} G_1 G^{-1} g + g'_1 G^{-1} g_1, \\
h_2 &= \frac{1}{2} g'_3 G^{-1} g + \frac{1}{2} g'_1 (2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}) g + \frac{1}{2} g'_1 G^{-1} g_2 \\
&\quad - g'_2 G^{-1} G_1 G^{-1} g - g'_1 G^{-1} G_1 G^{-1} g_1 + g'_2 G^{-1} g_1 \\
&= \frac{1}{2} g'_3 G^{-1} g + g'_1 G^{-1} G_1 G^{-1} G_1 G^{-1} g - \frac{1}{2} g'_1 G^{-1} G_2 G^{-1} g + \frac{3}{2} g'_1 G^{-1} g_2 \\
&\quad - g'_2 G^{-1} G_1 G^{-1} g - g'_1 G^{-1} G_1 G^{-1} g_1,
\end{aligned}$$

and

$$\begin{aligned}
h_3 &= g'_1 G^{-1} G_1 G^{-1} g - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_2 G^{-1} g \\
&\quad - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_1 G^{-1} g_1 \\
&= 2g'_1 G^{-1} G_1 G^{-1} g - 2g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_2 G^{-1} g, \\
h_4 &= \frac{1}{2} g'_2 G^{-1} G_1 G^{-1} g + \frac{1}{2} g' (2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}) G_1 G^{-1} g \\
&\quad + \frac{1}{2} g' G^{-1} G_3 G^{-1} g + \frac{1}{2} g' G^{-1} G_1 (2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}) g \\
&\quad + \frac{1}{2} g' G^{-1} G_1 G^{-1} g_2 \\
&\quad - g'_1 G^{-1} G_1 G^{-1} G_1 G^{-1} g + g'_1 G^{-1} G_2 G^{-1} g - g'_1 G^{-1} G_1 G^{-1} G_1 G^{-1} g \\
&\quad + g'_1 G^{-1} G_1 G^{-1} g_1 - g' G^{-1} G_1 G^{-1} G_2 G^{-1} g \\
&\quad + g' G^{-1} G_1 G^{-1} G_1 G^{-1} G_1 G^{-1} g - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g_1 - g' G^{-1} G_2 G^{-1} G_1 G^{-1} g \\
&\quad + g' G^{-1} G_2 G^{-1} g_1 - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g_1
\end{aligned}$$

We may therefore rewrite the score $S_n(b)$ as

$$\begin{aligned}
S_n(b) &= (2g'_1 G^{-1} g - g' G^{-1} G_1 G^{-1} g) + \frac{1}{\sqrt{n}} (2h_1 - h_3) \sqrt{n} (b - \beta) \\
&\quad + \frac{1}{n} (2h_2 - h_4) (\sqrt{n} (b - \beta))^2 + o_p\left(\frac{1}{n}\right). \tag{6}
\end{aligned}$$

Next note that $P(|S_n(b)| > \epsilon) = P(|S_n(b)| > \epsilon/n^{-1})$ for any $\epsilon > 0$ because of Lemma 1. Thus $S_n(b) = o_p(n^{-1})$ and we can subsume this error into the $o_p(n^{-1})$ term of 6. Using these arguments and Lemmas 1, 2, and 3 below, we may rewrite the first order condition (6) as

$$0 = \frac{1}{\sqrt{n}} \Phi + \frac{1}{n} \Gamma + \frac{1}{\sqrt{n}} \left(\Upsilon + \frac{1}{\sqrt{n}} \Xi \right) \sqrt{n} (b - \beta) + \frac{1}{n} \Psi (\sqrt{n} (b - \beta))^2 + o_p\left(\frac{1}{n}\right)$$

or

$$0 = \Phi + \frac{1}{\sqrt{n}} \Gamma + \left(\Upsilon + \frac{1}{\sqrt{n}} \Xi \right) \sqrt{n} (b - \beta) + \frac{1}{\sqrt{n}} \Psi (\sqrt{n} (b - \beta))^2 + o_p\left(\frac{1}{\sqrt{n}}\right),$$

based on which we obtain (3). Noting that

$$E[\Phi] = 2\sqrt{n} \lambda'_1 \Lambda^{-1} E[g] = 0, \tag{7}$$

$$\begin{aligned}
E[\Gamma] &= 2nE[(g_1 - \lambda_1)' \Lambda^{-1} g] - 2n\lambda'_1 \Lambda^{-1} E[(G - \Lambda) \Lambda^{-1} g] - nE[g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g] \\
&= 2 \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta'_i}{\partial \beta} \right] \right) - 2\lambda'_1 \Lambda^{-1} E[\psi_i \psi'_i \Lambda^{-1} \delta_i] - \text{trace}(\Lambda^{-1} \Lambda_1 \Lambda^{-1} E[\delta_i \delta'_i]), \tag{8}
\end{aligned}$$

$$\begin{aligned}
E[\Phi\Xi] &= 8n\lambda_1'\Lambda^{-1}E[g(g_1 - \lambda_1)']\Lambda^{-1}\lambda_1 - 4nE[\lambda_1'\Lambda^{-1}g\lambda_1'\Lambda^{-1}(G - \Lambda)\Lambda^{-1}\lambda_1] \\
&\quad - 8n\lambda_1'\Lambda^{-1}E[gg']\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + 4n\lambda_1'\Lambda^{-1}E[gg']\Lambda^{-1}\lambda_2 \\
&= 8\lambda_1'\Lambda^{-1}E\left[\delta_i\frac{\partial\delta_i'}{\partial\beta}\right]\Lambda^{-1}\lambda_1 - 4\lambda_1'\Lambda^{-1}E[\delta_i\lambda_1'\Lambda^{-1}\psi_i\psi_i']\Lambda^{-1}\lambda_1 \\
&\quad - 8\lambda_1'\Lambda^{-1}E[\delta_i\delta_i']\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + 4\lambda_1'\Lambda^{-1}E[\delta_i\delta_i']\Lambda^{-1}\lambda_2,
\end{aligned} \tag{9}$$

and

$$E[\Phi^2] = 4\lambda_1'\Lambda^{-1}E[\delta_i\delta_i']\Lambda^{-1}\lambda_1, \tag{10}$$

we obtain the desired conclusion.

Lemma 1 *Under Conditions 3 and 4*

$$\begin{aligned}
h_2 &= \frac{3}{2}\lambda_1'\Lambda^{-1}\lambda_2 - \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + o_p(1), \\
h_4 &= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + o_p(1).
\end{aligned}$$

Proof. Follows from $\text{plim } g = 0$. ■

Lemma 2 *Under Conditions 3 and 4*

$$2h_1 - h_3 = \Upsilon + \frac{1}{\sqrt{n}}\Xi + o_p\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Because

$$\begin{aligned}
g_2'G^{-1}g &= \lambda_2'\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right), \\
g_1'G^{-1}G_1G^{-1}g &= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
g_1'G^{-1}g_1 &= (\lambda_1 + (g_1 - \lambda_1))' \left(\Lambda^{-1} - \Lambda^{-1}(G - \Lambda)\Lambda^{-1} + o_p\left(\frac{1}{\sqrt{n}}\right) \right) (\lambda_1 + (g_1 - \lambda_1)) \\
&= \lambda_1'\Lambda^{-1}\lambda_1 + 2(g_1 - \lambda_1)'\Lambda^{-1}\lambda_1 - \lambda_1'\Lambda^{-1}(G - \Lambda)\Lambda^{-1}\lambda_1 + o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned} \tag{11}$$

we obtain

$$\begin{aligned}
h_1 &= \lambda_2'\Lambda^{-1}g - \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g \\
&\quad + \lambda_1'\Lambda^{-1}\lambda_1 + 2(g_1 - \lambda_1)'\Lambda^{-1}\lambda_1 - \lambda_1'\Lambda^{-1}(G - \Lambda)\Lambda^{-1}\lambda_1 + o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Similarly, we obtain

$$h_3 = 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right).$$

The conclusion follows. ■

Lemma 3 *Under Conditions 3 and 4*

$$2g_1'G^{-1}g - g_1'G^{-1}G_1G^{-1}g = \frac{1}{\sqrt{n}}\Phi + \frac{1}{n}\Gamma + o_p\left(\frac{1}{n}\right).$$

Proof. We have

$$\begin{aligned} g_1' G^{-1} g &= (\lambda_1 + (g_1 - \lambda_1))' \left(\Lambda^{-1} - \Lambda^{-1} (G - \Lambda) \Lambda^{-1} + o_p \left(\frac{1}{\sqrt{n}} \right) \right) g \\ &= \lambda_1' \Lambda^{-1} g + (g_1 - \lambda_1)' \Lambda^{-1} g - \lambda_1' \Lambda^{-1} (G - \Lambda) \Lambda^{-1} g + o_p \left(\frac{1}{n} \right) \end{aligned}$$

and

$$g' G^{-1} G_1 G^{-1} g = g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g + o_p \left(\frac{1}{n} \right).$$

from which the conclusion follows. ■

4 Proof of Theorem 3

The second order bias is computed using Theorem 3. Because the “weight matrix” here does not involve the parameter of interest, we have $\Lambda_1 = 0$. Also, because the moment restriction is linear in the parameter of interest, we have $\lambda_2 = 0$. It therefore follows that

$$\Psi = 3\lambda_1' \Lambda^{-1} \lambda_2 - 3\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 = 0,$$

and the second order bias is equal to

$$-\frac{1}{n} \frac{1}{\Upsilon} E[\Gamma] + \frac{1}{n} \frac{1}{\Upsilon^2} E[\Phi \Xi],$$

where

$$\begin{aligned} E[\Gamma] &= 2 \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) - 2\lambda_1' \Lambda^{-1} E[\psi_i \psi_i' \Lambda^{-1} \delta_i] \\ E[\Phi \Xi] &= 8\lambda_1' \Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E[\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i'] \Lambda^{-1} \lambda_1 \end{aligned}$$

Furthermore, because $E[z_{it} z_{it}' E[z_{it} z_{it}']^{-1} z_{it} \varepsilon_{it}^*] = 0$ under conditional symmetry of ε_{it}^* ,

$$\lambda_1' \Lambda^{-1} E[\psi_i \psi_i' \Lambda^{-1} \delta_i] = - \sum_{t=1}^{T-1} E[z_{it} x_{it}^*]' E[z_{it} z_{it}']^{-1} E[z_{it} z_{it}' E[z_{it} z_{it}']^{-1} z_{it} \varepsilon_{it}^*] = 0$$

and therefore, we have further simplification

$$\begin{aligned} E[\Gamma] &= 2 \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) \\ E[\Phi \Xi] &= 8\lambda_1' \Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E[\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i'] \Lambda^{-1} \lambda_1 \end{aligned}$$

the second term should be equal to zero. Because

$$\begin{aligned} \lambda_1 \Lambda^{-1} \lambda_1 &= \sum_{t=1}^{T-1} E[z_{it} x_{it}^*]' E[z_{it} z_{it}']^{-1} E[z_{it} x_{it}^*], \\ \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) &= - \sum_{t=1}^{T-1} \text{trace} \left(E[z_{it} z_{it}']^{-1} E[\varepsilon_{it}^* x_{it}^* z_{it} z_{it}'] \right), \\ \lambda_1' \Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 &= - \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} E[z_{it} x_{it}^*]' E[z_{it} z_{it}']^{-1} E[\varepsilon_{it}^* x_{it}^* z_{it} z_{it}'] E[z_{is} z_{is}']^{-1} E[z_{is} x_{is}^*], \end{aligned}$$

and

$$\begin{aligned} & \lambda_1' \Lambda^{-1} E [\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i'] \Lambda^{-1} \lambda_1 \\ &= - \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} E [z_{it} x_{it}^*]' E [z_{it} z_{it}']^{-1} E [\varepsilon_{it}^* z_{it} E [z_{is} x_{is}^*]' E [z_{is} z_{is}']^{-1} z_{is} z_{is}'] E [z_{is} z_{is}']^{-1} E [z_{is} x_{is}^*], \end{aligned}$$

we now obtain

$$\begin{aligned} \Upsilon &= 2\lambda_1 \Lambda^{-1} \lambda_1 \\ &= 2 \sum_{t=1}^{T-1} E [z_{it} x_{it}^*]' E [z_{it} z_{it}']^{-1} E [z_{it} x_{it}^*] \\ &= 2\Upsilon_1, \end{aligned}$$

$$\begin{aligned} E[\Gamma] &= 2 \text{trace} \left(\Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) \\ &= -2 \sum_{t=1}^{T-1} \text{trace} \left(E [z_{it} z_{it}']^{-1} E [\varepsilon_{it}^* x_{it}^* z_{it} z_{it}'] \right) \\ &= -2 \sum_{t=1}^{T-1} \text{trace} \left((\Gamma_t^{zz})^{-1} \Gamma_{t,t}^{\varepsilon x z z} \right), \end{aligned}$$

$$\begin{aligned} E[\Phi \Xi] &= 8\lambda_1' \Lambda^{-1} E \left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E [\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i'] \Lambda^{-1} \lambda_1 \\ &= -8 \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} E [z_{it} x_{it}^*]' E [z_{it} z_{it}']^{-1} E [\varepsilon_{it}^* x_{is}^* z_{it} z_{is}'] E [z_{is} z_{is}']^{-1} E [z_{is} x_{is}^*] \\ &\quad + 4 \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} E [z_{it} x_{it}^*]' E [z_{it} z_{it}']^{-1} E [\varepsilon_{it}^* z_{it} E [z_{is} x_{is}^*]' E [z_{is} z_{is}']^{-1} z_{is} z_{is}'] E [z_{is} z_{is}']^{-1} E [z_{is} x_{is}^*] \\ &= -8 \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{zx'} (\Gamma_t^{zz})^{-1} \Gamma_{t,s}^{\varepsilon x z z} (\Gamma_s^{zz})^{-1} \Gamma_s^{zx} \\ &\quad + 4 \sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{zx'} (\Gamma_t^{zz})^{-1} B_{3,1}(t, s) (\Gamma_s^{zz})^{-1} \Gamma_s^{zx} \end{aligned}$$

from which we obtain the desired conclusion.

Higher Order Bias and MSE of GMM: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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1 Higher Order Expansion of GMM

Consider a general minimization estimator b of a single parameter $\beta \in \mathbb{R}$ defined by

$$b = \underset{c \in C}{\operatorname{argmin}} g_n(c)' G_n(\tilde{\beta})^{-1} g_n(c) \tag{1}$$

where

$$g_n(c) \equiv \frac{1}{n} \sum_{i=1}^n \psi(z_i, c)$$

and

$$G_n(\tilde{\beta}) \equiv \frac{1}{n} \sum_{i=1}^n \psi(z_i, \tilde{\beta}) \psi(z_i, \tilde{\beta})'$$

Here, $\tilde{\beta}$ is an initial \sqrt{n} -consistent estimator for β such that

$$\sqrt{n}(\tilde{\beta} - \beta) = L_n + \frac{1}{\sqrt{n}}Q_n + \frac{1}{n}S_n + o_p\left(\frac{1}{n}\right)$$

where

$$L_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i$$

f_i is i.i.d. mean zero, and $Q_n = O_p(1)$, $S_n = O_p(1)$.

Definition 1

$$\psi_j(z_i, c) \equiv \partial^j \psi(z_i, c) / \partial c^j$$

$$\lambda_j(c) \equiv E[\psi_j(z_i, c)]$$

$$\Psi(z_i, c) \equiv \psi(z_i, c) \psi(z_i, c)'$$

$$\Psi_j(z_i, c) \equiv \partial^j \Psi(z_i, c) / \partial c^j$$

$$\Lambda_j(c) \equiv E[\Psi_j(z_i, c)]$$

For simplicity, we use the notation

$$\begin{aligned}\lambda_j &\equiv \lambda_j(\beta) \\ \Lambda_j &\equiv \Lambda_j(\beta) \\ \lambda(c) &\equiv \lambda_0(c) \\ \Lambda(c) &\equiv \Lambda_0(c)\end{aligned}$$

Definition 2

$$\begin{aligned}g_n(c) &\equiv \frac{1}{n} \sum_{i=1}^n \psi(z_i, c) \\ g_{j,n}(c) &\equiv \frac{1}{n} \sum_{i=1}^n \psi_j(z_i, c) \\ G_n(c) &\equiv \frac{1}{n} \sum_{i=1}^n \Psi(z_i, c) = \frac{1}{n} \sum_{i=1}^n \psi(z_i, c) \psi'(z_i, c) \\ G_{j,n}(c) &\equiv \frac{1}{n} \sum_{i=1}^n \Psi_j(z_i, c)\end{aligned}$$

Definition 3

$$\begin{aligned}w_n &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi(z_i, \beta) - \lambda) = \sqrt{n} g_n(\beta) \\ w_{1,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi_1(z_i, \beta) - \lambda_1) \\ w_{2,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi_2(z_i, \beta) - \lambda_2) \\ w_{3,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi_3(z_i, \beta) - \lambda_3) \\ w_{4,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi_4(z_i, \beta) - \lambda_4) \\ W_n &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\Psi(z_i, \beta) - \Lambda) \\ \tilde{W}_{1,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\Psi_1(z_i, \beta) - \Lambda_1) \\ W_{2,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\Psi_2(z_i, \beta) - \Lambda_2) \\ W_{3,n} &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (\Psi_3(z_i, \beta) - \Lambda_3)\end{aligned}$$

Definition 4

$$\begin{aligned}\tilde{\Lambda}_1 &\equiv W_n + \Lambda_1 L_n \\ \tilde{\Lambda}_2 &\equiv 2\Lambda_1 Q_n + 2W_{1,n} L_n + \Lambda_2 L_n^2 \\ \tilde{\Lambda}_3 &\equiv 6\Lambda_1 S_n + 6W_{1,n} Q_n + 3W_{2,n} L_n^2 + 6\Lambda_2 L_n Q_n + \Lambda_3 L_n^3\end{aligned}$$

Lemma 1

$$G_n(\tilde{\beta}) = \Lambda + \frac{1}{\sqrt{n}}\tilde{\Lambda}_1 + \frac{1}{2n}\tilde{\Lambda}_2 + \frac{1}{6n\sqrt{n}}\tilde{\Lambda}_3 + o_p\left(\frac{1}{n\sqrt{n}}\right).$$

Proof. Write

$$\begin{aligned} G_n(\tilde{\beta}) &= \frac{1}{n} \sum_{i=1}^n \Psi(z_i, \tilde{\beta}) \\ &= \frac{1}{n} \sum_{i=1}^n \Psi(z_i, \beta) \\ &\quad + \frac{1}{\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n \Psi_1(z_i, \beta) \right) \cdot \sqrt{n}(\tilde{\beta} - \beta) \\ &\quad + \frac{1}{2n} \left(\frac{1}{n} \sum_{i=1}^n \Psi_2(z_i, \beta) \right) \cdot (\sqrt{n}(\tilde{\beta} - \beta))^2 \\ &\quad + \frac{1}{6n\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n \Psi_3(z_i, \beta) \right) \cdot (\sqrt{n}(\tilde{\beta} - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right) \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \psi(z_i, \beta) \psi(z_i, \beta)' &= \Lambda + \frac{1}{\sqrt{n}} W_n, \\ \frac{1}{\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n \Psi_1(z_i, \beta) \right) \cdot \sqrt{n}(\tilde{\beta} - \beta) &= \frac{1}{\sqrt{n}} \left(\Lambda_1 + \frac{1}{\sqrt{n}} W_{1,n} \right) \cdot \left(L_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right) \\ &= \frac{1}{\sqrt{n}} \Lambda_1 L_n \\ &\quad + \frac{1}{n} (\Lambda_1 Q_n + W_{1,n} L_n) \\ &\quad + \frac{1}{n\sqrt{n}} (\Lambda_1 S_n + W_{1,n} Q_n) + o_p\left(\frac{1}{n\sqrt{n}}\right), \\ \frac{1}{2n} \left(\frac{1}{n} \sum_{i=1}^n \Psi_2(z_i, \beta) \right) \cdot (\sqrt{n}(\tilde{\beta} - \beta))^2 &= \frac{1}{2n} \left(\Lambda_2 + \frac{1}{\sqrt{n}} W_{2,n} \right) \cdot \left(L_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right)^2 \\ &= \frac{1}{2n} \Lambda_2 L_n^2 + \frac{1}{2n\sqrt{n}} (W_{2,n} L_n^2 + 2\Lambda_2 L_n Q_n) + o_p\left(\frac{1}{n\sqrt{n}}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{6n\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^n \Psi_3(z_i, \beta) \right) \cdot (\sqrt{n}(\tilde{\beta} - \beta))^3 &= \frac{1}{6n\sqrt{n}} \left(\Lambda_3 + \frac{1}{\sqrt{n}} W_{3,n} \right) \left(L_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right)^3 \\ &= \frac{1}{6n\sqrt{n}} \Lambda_3 L_n^3 + o_p\left(\frac{1}{n\sqrt{n}}\right). \end{aligned}$$

It follows that

$$\begin{aligned}
G_n(\tilde{\beta}) &= \Lambda + \frac{1}{\sqrt{n}}W_n + \frac{1}{\sqrt{n}}\Lambda_1L_n \\
&\quad + \frac{1}{n}(\Lambda_1Q_n + W_{1,n}L_n) + \frac{1}{2n}\Lambda_2L_n^2 \\
&\quad + \frac{1}{n\sqrt{n}}(\Lambda_1S_n + W_{1,n}Q_n) + \frac{1}{2n\sqrt{n}}(W_{2,n}L_n^2 + 2\Lambda_2L_nQ_n) + \frac{1}{6n\sqrt{n}}\Lambda_3L_n^3 \\
&\quad + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \Lambda + \frac{1}{\sqrt{n}}\tilde{\Lambda}_1 + \frac{1}{2n}\tilde{\Lambda}_2 + \frac{1}{6n\sqrt{n}}\tilde{\Lambda}_3 + o_p\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

■

Definition 5

$$\begin{aligned}
H_1 &\equiv -\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} \\
H_2 &\equiv 2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1} \\
H_3 &\equiv -6\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1} - \Lambda^{-1}\tilde{\Lambda}_3\Lambda^{-1}
\end{aligned}$$

Lemma 2

$$G_n(\tilde{\beta})^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}}H_1 + \frac{1}{2n}H_2 + \frac{1}{6n\sqrt{n}}H_3 + o_p\left(\frac{1}{n\sqrt{n}}\right)$$

Proof. Using Lemma 1, we obtain

$$\begin{aligned}
G_n(\tilde{\beta})^{-1} &= \Lambda^{-1} - \frac{1}{\sqrt{n}}\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} \\
&\quad + \frac{1}{2n}\left(2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1}\right) \\
&\quad - \frac{1}{6n\sqrt{n}}\left(6\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} - 3\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} - 3\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1} + \Lambda^{-1}\tilde{\Lambda}_3\Lambda^{-1}\right) \\
&\quad + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}$$

■

Lemma 3

$$H_1 = H_{1,0} + L_n \cdot H_{1,1}$$

where

$$\begin{aligned}
H_{1,0} &\equiv -\Lambda^{-1}W_n\Lambda^{-1} \\
H_{1,1} &\equiv -\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}$$

Proof.

$$\begin{aligned}
H_1 &= -\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} = -\Lambda^{-1}(W_n + \Lambda_1L_n)\Lambda^{-1} \\
&= -\Lambda^{-1}W_n\Lambda^{-1} - L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}$$

■

Lemma 4

$$H_2 = H_{2,0} + L_n \cdot H_{2,1} + L_n^2 \cdot H_{2,2} + Q_n \cdot H_{2,3}$$

where

$$\begin{aligned} H_{2,0} &\equiv 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\ H_{2,1} &\equiv 2\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 2\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} - 2\Lambda^{-1}W_{1,n}\Lambda^{-1} \\ H_{2,2} &\equiv 2\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_2\Lambda^{-1} \\ H_{2,3} &\equiv -2\Lambda^{-1}\Lambda_1\Lambda^{-1} \end{aligned}$$

Proof.

$$\begin{aligned} H_2 &= 2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1} \\ &= 2\Lambda^{-1}(W_n + \Lambda_1L_n)\Lambda^{-1}(W_n + \Lambda_1L_n)\Lambda^{-1} - \Lambda^{-1}(2\Lambda_1Q_n + 2W_{1,n}L_n + \Lambda_2L_n^2)\Lambda^{-1} \\ &= 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} + 2L_n \cdot \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} + 2L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 2L_n^2 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} \\ &\quad - 2Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1} - 2L_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1} - L_n^2 \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1} \\ &= 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\ &\quad + L_n \cdot (2\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} + 2\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 2\Lambda^{-1}W_{1,n}\Lambda^{-1}) \\ &\quad + L_n^2 \cdot (2\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_2\Lambda^{-1}) \\ &\quad - 2Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1} \end{aligned}$$

■

Lemma 5

$$H_3 = H_{3,0} + L_n \cdot H_{3,1} + L_n^2 \cdot H_{3,2} + L_n^3 \cdot H_{3,3} + Q_n \cdot H_{3,4} + Q_nL_n \cdot H_{3,5} + S_n \cdot H_{3,6}$$

where

$$\begin{aligned} H_{3,0} &\equiv -6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\ H_{3,1} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\ &\quad + 6\Lambda^{-1}W_{1,n}\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_n\Lambda^{-1}W_{1,n}\Lambda^{-1} \\ H_{3,2} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\ &\quad + 3\Lambda^{-1}\Lambda_2\Lambda^{-1}W_n\Lambda^{-1} + 3\Lambda^{-1}W_n\Lambda^{-1}\Lambda_2\Lambda^{-1} \\ &\quad + 6\Lambda^{-1}W_{1,n}\Lambda^{-1}\Lambda_1\Lambda^{-1} + 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_{1,n}\Lambda^{-1} \\ &\quad - 3\Lambda^{-1}W_{2,n}\Lambda^{-1} \\ H_{3,3} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} + 3\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_2\Lambda^{-1} + 3\Lambda^{-1}\Lambda_2\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_3\Lambda^{-1} \\ H_{3,4} &\equiv 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}W_{1,n}\Lambda^{-1} \\ H_{3,5} &\equiv 12\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}\Lambda_2\Lambda^{-1} \\ H_{3,6} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1} \end{aligned}$$

Proof. We have

$$\begin{aligned}
H_3 &= -6\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\tilde{\Lambda}_1\Lambda^{-1}\tilde{\Lambda}_2\Lambda^{-1} - \Lambda^{-1}\tilde{\Lambda}_3\Lambda^{-1} \\
&= -6\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1} \\
&\quad + 3\Lambda^{-1}(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1} \\
&\quad + 3\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2)\Lambda^{-1} \\
&\quad - \Lambda^{-1}(6\Lambda_1 S_n + 6W_{1,n}Q_n + 3W_{2,n}L_n^2 + 6\Lambda_2 L_n Q_n + \Lambda_3 L_n^3)\Lambda^{-1}
\end{aligned}$$

Note that the terms on the RHS can be expanded as follows:

$$\begin{aligned}
&-6\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1} \\
= &-6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\
&-6L_n \cdot (\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}) \\
&-6L_n^2 \cdot (\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} + \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}) \\
&-6L_n^3 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&3\Lambda^{-1}(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2)\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1} \\
= &6L_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1}W_n\Lambda^{-1} \\
&+ L_n^2 \cdot (3\Lambda^{-1}\Lambda_2\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_{1,n}\Lambda^{-1}\Lambda_1\Lambda^{-1}) \\
&+ 3L_n^3 \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&+ 6Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} \\
&+ 6Q_n L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&3\Lambda^{-1}(W_n + \Lambda_1 L_n)\Lambda^{-1}(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2)\Lambda^{-1} \\
= &6L_n \cdot \Lambda^{-1}W_n\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
&+ L_n^2 \cdot (3\Lambda^{-1}W_n\Lambda^{-1}\Lambda_2\Lambda^{-1} + 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_{1,n}\Lambda^{-1}) \\
&+ 3L_n^3 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_2\Lambda^{-1} \\
&+ 6Q_n \cdot \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&+ 6Q_n L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&-\Lambda^{-1}(6\Lambda_1 S_n + 6W_{1,n}Q_n + 3W_{2,n}L_n^2 + 6\Lambda_2 L_n Q_n + \Lambda_3 L_n^3)\Lambda^{-1} \\
= &-6S_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1} - 6Q_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1} - 3L_n^2 \cdot \Lambda^{-1}W_{2,n}\Lambda^{-1} - 6Q_n L_n \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1} - L_n^3 \cdot \Lambda^{-1}\Lambda_3\Lambda^{-1}
\end{aligned}$$

■

Definition 6

$$\begin{aligned}
h_{0,1} &\equiv \lambda'_1\Lambda^{-1}w_n \\
h_{0,2} &\equiv w'_{1,n}\Lambda^{-1}w_n + \lambda'_1 H_1 w_n \\
h_{0,3} &\equiv w'_{1,n} H_1 w_n + \frac{1}{2}\lambda'_1 H_2 w_n
\end{aligned}$$

$$\begin{aligned}
h_{1,1} &\equiv \lambda'_1 \Lambda^{-1} \lambda_1 \\
h_{1,2} &\equiv 2\lambda'_1 \Lambda^{-1} w_{1,n} + \lambda'_2 \Lambda^{-1} w_n + \lambda'_1 H_1 \lambda_1 \\
h_{1,3} &\equiv w'_{1,n} \Lambda^{-1} w_{1,n} + w'_{2,n} \Lambda^{-1} w_n + 2\lambda'_1 H_1 w_{1,n} + \lambda'_2 H_1 w_n + \frac{1}{2} \lambda'_1 H_2 \lambda_1 \\
\\
h_{2,2} &\equiv 3\lambda'_1 \Lambda^{-1} \lambda_2 \\
h_{2,3} &\equiv 3\lambda'_2 \Lambda^{-1} w_{1,n} + 3\lambda'_1 \Lambda^{-1} w_{2,n} + \lambda'_3 \Lambda^{-1} w_n + 3\lambda'_1 H_1 \lambda_2 \\
h_{3,3} &\equiv 3\lambda'_2 \Lambda^{-1} \lambda_2 + 4\lambda'_1 \Lambda^{-1} \lambda_3
\end{aligned}$$

Lemma 6

$$g_1(b)' G(\tilde{\beta})^{-1} g(b) = h + h_1 \cdot \sqrt{n}(b - \beta) + \frac{1}{2} h_2 \cdot (\sqrt{n}(b - \beta))^2 + \frac{1}{6} h_3 \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right),$$

where

$$\begin{aligned}
h &\equiv \frac{1}{\sqrt{n}} h_{0,1} + \frac{1}{n} h_{0,2} + \frac{1}{n\sqrt{n}} h_{0,3} \\
h_1 &\equiv \frac{1}{\sqrt{n}} h_{1,1} + \frac{1}{n} h_{1,2} + \frac{1}{n\sqrt{n}} h_{1,3} \\
h_2 &\equiv \frac{1}{n} h_{2,2} + \frac{1}{n\sqrt{n}} h_{2,3} \\
h_3 &\equiv \frac{1}{n\sqrt{n}} h_{3,3}
\end{aligned}$$

Proof. Note that we have

$$\begin{aligned}
g_1(b) &= g_1 + \frac{1}{\sqrt{n}} g_2 \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} g_3 \cdot (\sqrt{n}(b - \beta))^2 + \frac{1}{6n\sqrt{n}} g_4 \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \lambda_1 + \frac{1}{\sqrt{n}} w_{1,n} + \frac{1}{\sqrt{n}} \left(\lambda_2 + \frac{1}{\sqrt{n}} w_{2,n} \right) \sqrt{n}(b - \beta) + \frac{1}{2n} \left(\lambda_3 + \frac{1}{\sqrt{n}} w_{3,n} \right) \cdot (\sqrt{n}(b - \beta))^2 \\
&\quad + \frac{1}{6n\sqrt{n}} \left(\lambda_4 + \frac{1}{\sqrt{n}} w_{4,n} \right) \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}$$

and

$$\begin{aligned}
g(b) &= \frac{1}{\sqrt{n}} w_n + \frac{1}{\sqrt{n}} g_1 \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} g_2 \cdot (\sqrt{n}(b - \beta))^2 + \frac{1}{6n\sqrt{n}} g_3 \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{n}} w_n + \frac{1}{\sqrt{n}} \left(\lambda_1 + \frac{1}{\sqrt{n}} w_{1,n} \right) \cdot \sqrt{n}(b - \beta) + \frac{1}{2n} \left(\lambda_2 + \frac{1}{\sqrt{n}} w_{2,n} \right) \cdot (\sqrt{n}(b - \beta))^2 \\
&\quad + \frac{1}{6n\sqrt{n}} \left(\lambda_3 + \frac{1}{\sqrt{n}} w_{3,n} \right) \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}$$

By Lemma 2, we also have

$$G(\tilde{\beta})^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}} H_1 + \frac{1}{2n} H_2 + \frac{1}{6n\sqrt{n}} H_3 + o_p\left(\frac{1}{n\sqrt{n}}\right)$$

■

Lemma 7

$$\begin{aligned}
h_{0,1} &= \lambda'_1 \Lambda^{-1} w_n \\
h_{0,2} &= w'_{1,n} \Lambda^{-1} w_n + \lambda'_1 H_{1,0} w_n + L_n \cdot \lambda'_1 H_{1,1} w_n \\
h_{0,3} &= w'_{1,n} H_{1,0} w_n + \frac{1}{2} \lambda'_1 H_{2,0} w_n \\
&\quad + L_n \cdot w'_{1,n} H_{1,1} w_n + \frac{1}{2} L_n \cdot \lambda'_1 H_{2,1} w_n + \frac{1}{2} L_n^2 \cdot \lambda'_1 H_{2,2} w_n + \frac{1}{2} Q_n \cdot \lambda'_1 H_{2,3} w_n \\
\\
h_{1,1} &= \lambda'_1 \Lambda^{-1} \lambda_1 \\
h_{1,2} &= 2\lambda'_1 \Lambda^{-1} w_{1,n} + \lambda'_2 \Lambda^{-1} w_n + \lambda'_1 H_{1,0} \lambda_1 + L_n \cdot \lambda'_1 H_{1,1} \lambda_1 \\
h_{1,3} &= w'_{1,n} \Lambda^{-1} w_{1,n} + w'_{2,n} \Lambda^{-1} w_n + 2\lambda'_1 H_{1,0} w_{1,n} + 2L_n \cdot \lambda'_1 H_{1,1} w_{1,n} \\
&\quad + \lambda'_2 H_{1,0} w_n + L_n \cdot \lambda'_2 H_{1,1} w_n \\
&\quad + \frac{1}{2} \lambda'_1 H_{2,0} \lambda_1 + \frac{1}{2} L_n \cdot \lambda'_1 H_{2,1} \lambda_1 + \frac{1}{2} L_n^2 \cdot \lambda'_1 H_{2,2} \lambda_1 + \frac{1}{2} Q_n \cdot \lambda'_1 H_{2,3} \lambda_1 \\
\\
h_{2,2} &= 3\lambda'_1 \Lambda^{-1} \lambda_2 \\
h_{2,3} &= 3\lambda'_2 \Lambda^{-1} w_{1,n} + 3\lambda'_1 \Lambda^{-1} w_{2,n} + \lambda'_3 \Lambda^{-1} w_n + 3\lambda'_1 H_{1,0} \lambda_2 + 3L_n \cdot \lambda'_1 H_{1,1} \lambda_2 \\
\\
h_{3,3} &= 3\lambda'_2 \Lambda^{-1} \lambda_2 + 4\lambda'_1 \Lambda^{-1} \lambda_3
\end{aligned}$$

Proof. Follows from Lemmas 3, 4, and 5. ■

Lemma 8 *Let*

$$b_1 \equiv \operatorname{argmin}_{c \in C} g(c)' G(\tilde{\beta})^{-1} g(c)$$

Then,

$$\sqrt{n}(b - \beta) = T_1 + \frac{1}{2\sqrt{n}} T_2 + \frac{1}{6n} T_3 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
T_1 &\equiv -\frac{1}{h_{1,1}} h_{0,1} \\
T_2 &\equiv -\frac{2}{h_{1,1}} h_{0,2} + \frac{2}{h_{1,1}^2} h_{1,2} h_{0,1} - \frac{1}{h_{1,1}^3} h_{2,2} h_{0,1}^2 \\
T_3 &\equiv -\frac{6}{h_{1,1}} h_{0,3} + \frac{6h_{1,2} h_{0,2} + 6h_{1,3} h_{0,1}}{h_{1,1}^2} + \frac{-6h_{1,2}^2 h_{0,1} - 3h_{2,3} h_{0,1}^2 - 6h_{2,2} h_{0,1} h_{0,2}}{h_{1,1}^3} \\
&\quad + \frac{9h_{1,2} h_{2,2} h_{0,1}^2 + h_{3,3} h_{0,1}^3}{h_{1,1}^4} - \frac{3}{h_{1,1}^5} h_{2,2}^2 h_{0,1}^3
\end{aligned}$$

Proof. We now solve for $\sqrt{n}(b - \beta)$ by iterative substitution. Using Lemma 6, we first write the FOC as

$$0 = \frac{1}{\sqrt{n}} h_{0,1} + \left(\frac{1}{\sqrt{n}} h_{1,1}\right) \cdot \sqrt{n}(b - \beta) + o_p\left(\frac{1}{\sqrt{n}}\right),$$

from which we obtain

$$\sqrt{n}(b - \beta) = -\frac{1}{h_{1,1}}h_{0,1} + o_p(1) = T_1 + o_p(1)$$

We now write the FOC as

$$0 = \frac{1}{\sqrt{n}}h_{0,1} + \frac{1}{n}h_{0,2} + \left(\frac{1}{\sqrt{n}}h_{1,1} + \frac{1}{n}h_{1,2}\right) \cdot (\sqrt{n}(b - \beta)) + \frac{1}{2}\left(\frac{1}{n}h_{2,2}\right) \cdot (\sqrt{n}(b - \beta))^2 + o_p\left(\frac{1}{n}\right).$$

Writing

$$\sqrt{n}(b - \beta) = -\frac{1}{h_{1,1}}h_{0,1} + \frac{x}{\sqrt{n}}$$

we can rewrite the FOC as

$$0 = \frac{1}{n}\left(h_{0,2} + \frac{1}{2}\frac{h_{2,2}}{h_{1,1}^2}h_{0,1}^2 + h_{1,1}x - \frac{h_{1,2}}{h_{1,1}}h_{0,1}\right) + o_p\left(\frac{1}{n}\right).$$

Solving for x , we obtain

$$\begin{aligned}\sqrt{n}(b - \beta) &= T_1 + \frac{1}{2\sqrt{n}}\left(-\frac{2h_{0,2}h_{1,1}^2 + h_{2,2}h_{0,1}^2 - 2h_{1,2}h_{0,1}h_{1,1}}{h_{1,1}^3}\right) + o_p\left(\frac{1}{\sqrt{n}}\right) \\ &= T_1 + \frac{1}{2\sqrt{n}}T_2 + o_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

We now write the FOC as

$$\begin{aligned}0 &= \frac{1}{\sqrt{n}}h_{0,1} + \frac{1}{n}h_{0,2} + \frac{1}{n\sqrt{n}}h_{0,3} + \left(\frac{1}{\sqrt{n}}h_{1,1} + \frac{1}{n}h_{1,2} + \frac{1}{n\sqrt{n}}h_{1,3}\right) \cdot \sqrt{n}(b - \beta) \\ &\quad + \frac{1}{2}\left(\frac{1}{n}h_{2,2} + \frac{1}{n\sqrt{n}}h_{2,3}\right) \cdot (\sqrt{n}(b - \beta))^2 + \frac{1}{6}\frac{1}{n\sqrt{n}}h_{3,3} \cdot (\sqrt{n}(b - \beta))^3 + o_p\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

Writing

$$\sqrt{n}(b - \beta) = -\frac{1}{h_{1,1}}h_{0,1} - \frac{1}{2\sqrt{n}}\frac{2h_{0,2}h_{1,1}^2 + h_{2,2}h_{0,1}^2 - 2h_{1,2}h_{0,1}h_{1,1}}{h_{1,1}^3} + \frac{x}{n}$$

we can rewrite the FOC as

$$\begin{aligned}0 &= \frac{h_{1,1}x - \frac{h_{1,2}}{h_{1,1}}h_{0,2} + \frac{1}{2}\frac{h_{2,2}}{h_{1,1}^4}h_{0,1}^3 + \frac{h_{1,2}}{h_{1,1}^2}h_{0,1} + \frac{h_{2,2}}{h_{1,1}^2}h_{0,1}h_{0,2} - \frac{h_{1,3}}{h_{1,1}}h_{0,1} + h_{0,3} + \frac{1}{2}\frac{h_{2,3}}{h_{1,1}^2}h_{0,1}^2 - \frac{3}{2}\frac{h_{1,2}}{h_{1,1}^3}h_{2,2}h_{0,1}^2 - \frac{1}{6}\frac{h_{3,3}}{h_{1,1}^3}h_{0,1}^3}{n^{\frac{3}{2}}} \\ &\quad + o_p\left(\frac{1}{n\sqrt{n}}\right)\end{aligned}$$

Solving for x , we obtain

$$\sqrt{n}(b - \beta) = T_1 + \frac{1}{2\sqrt{n}}T_2 + \frac{1}{6n}T_3 + o_p\left(\frac{1}{n}\right)$$

■

Lemma 9

$$\begin{aligned}T_1 &= \bar{T}_1 \\ T_2 &= \bar{T}_2 + L_n \cdot \tilde{T}_2 \\ T_3 &= \bar{T}_3 + L_n \cdot \tilde{T}_{3,1} + L_n^2 \cdot \tilde{T}_{3,2} + Q_n \cdot \tilde{T}_{3,3}\end{aligned}$$

where

$$\begin{aligned}
\bar{T}_1 &\equiv -\frac{\lambda'_1 \Lambda^{-1} w_n}{\lambda'_1 \Lambda^{-1} \lambda_1} \\
\bar{T}_2 &\equiv -\frac{2}{\lambda'_1 \Lambda^{-1} \lambda_1} (w'_{1,n} \Lambda^{-1} w_n) \\
&\quad -\frac{2}{\lambda'_1 \Lambda^{-1} \lambda_1} (\lambda'_1 H_{1,0} w_n) \\
&\quad +\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (2\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n) \\
&\quad +\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
&\quad +\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) \\
&\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (3\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
\tilde{T}_2 &\equiv -\frac{2}{\lambda'_1 \Lambda^{-1} \lambda_1} \lambda'_1 H_{1,1} w_n + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)
\end{aligned}$$

$$\begin{aligned}
\bar{T}_3 \equiv & -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (w'_{1,n} H_{1,0} w_n) - \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (\lambda'_1 H_{2,0} w_n) \\
& + \frac{12 (\lambda'_1 \Lambda^{-1} w_{1,n}) (w'_{1,n} \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{12 (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 H_{1,0} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& + \frac{6 (\lambda'_2 \Lambda^{-1} w_n) (w'_{1,n} \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 H_{1,0} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& + \frac{6 (\lambda'_1 H_{1,0} \lambda_1) (w'_{1,n} \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 H_{1,0} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& + \frac{6 (w'_{1,n} \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (w'_{2,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& + \frac{12 (\lambda'_1 H_{1,0} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_2 H_{1,0} w_n) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& + \frac{3 (\lambda'_1 H_{2,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
& - \frac{24 (\lambda'_1 \Lambda^{-1} w_{1,n})^2 (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{6 (\lambda'_2 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& - \frac{6 (\lambda'_1 H_{1,0} \lambda_1)^2 (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{24 (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& - \frac{12 (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{24 (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& - \frac{9 (\lambda'_2 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{9 (\lambda'_1 \Lambda^{-1} w_{2,n}) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& - \frac{3 (\lambda'_3 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{9 (\lambda'_1 H_{1,0} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& - \frac{18 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n) (w'_{1,n} \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{18 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 H_{1,0} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
& + \frac{54 (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} + \frac{27 (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \\
& + \frac{27 (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \\
& + \frac{3 (\lambda'_2 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} + \frac{4 (\lambda'_1 \Lambda^{-1} \lambda_3) (\lambda'_1 \Lambda^{-1} w_n)^3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \\
& - \frac{27}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2)^2 (\lambda'_1 \Lambda^{-1} w_n)^3
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{3,1} &\equiv -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (w'_{1,n} H_{1,1} w_n) - \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (\lambda'_1 H_{2,1} w_n) \\
&+ \frac{6 (\lambda'_1 H_{1,1} \lambda_1) (w'_{1,n} \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 H_{1,0} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&+ \frac{12 (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 H_{1,1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 H_{1,1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 H_{1,1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&+ \frac{12 (\lambda'_1 H_{1,1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{6 (\lambda'_2 H_{1,1} w_n) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{3 (\lambda'_1 H_{2,1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&- \frac{24 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} - \frac{12 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
&- \frac{12 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
&- \frac{9 (\lambda'_1 H_{1,1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
&- \frac{18 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 H_{1,1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
&+ \frac{27 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \\
\tilde{T}_{3,2} &\equiv -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (\lambda'_1 H_{2,2} w_n) \\
&+ \frac{6 (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 H_{1,1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&+ \frac{3 (\lambda'_1 H_{2,2} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&+ \frac{-6 (\lambda'_1 H_{1,1} \lambda_1)^2 (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \\
\tilde{T}_{3,3} &\equiv -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)} (\lambda'_1 H_{2,3} w_n) + \frac{3 (\lambda'_1 H_{2,3} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2}
\end{aligned}$$

Proof. Follows from Lemmas 7 and 8. ■

Theorem 1 *Let*

$$b_2 \equiv \operatorname{argmin}_{c \in C} g(c)' G(b_1)^{-1} g(c)$$

and

$$b_3 \equiv \operatorname{argmin}_{c \in C} g(c)' G(b_2)^{-1} g(c)$$

Then,

$$\sqrt{n}(b_3 - \beta) = \bar{T}_1 + \frac{1}{2\sqrt{n}} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) + \frac{1}{6n} (\bar{T}_3 + \bar{T}_1 \cdot \tilde{T}_{3,1} + \bar{T}_1^2 \cdot \tilde{T}_{3,2} + (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \cdot \tilde{T}_{3,3}) + o_p\left(\frac{1}{n}\right)$$

Proof. We use Lemmas 8 and 9. With induction, we can obtain

$$\sqrt{n}(b_2 - \beta) = \bar{T}_1 + \frac{1}{2\sqrt{n}} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) + \frac{1}{6n} (\bar{T}_3 + \bar{T}_1 \cdot \tilde{T}_{3,1} + \bar{T}_1^2 \cdot \tilde{T}_{3,2} + (\bar{T}_2 + L_1 \cdot \tilde{T}_2) \cdot \tilde{T}_{3,3}) + o_p\left(\frac{1}{n}\right)$$

and

$$\sqrt{n}(b_3 - \beta) = \bar{T}_1 + \frac{1}{2\sqrt{n}} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) + \frac{1}{6n} (\bar{T}_3 + \bar{T}_1 \cdot \tilde{T}_{3,1} + \bar{T}_1^2 \cdot \tilde{T}_{3,2} + (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \cdot \tilde{T}_{3,3}) + o_p\left(\frac{1}{n}\right)$$

■

2 Higher Order Moments

For Maple/Mathematica programming purpose, it is convenient to introduce a new set of symbols:

$$u_0 \equiv \psi(z_i, \beta) - \lambda = \psi(z_i, \beta)$$

$$u_1 \equiv \psi_1(z_i, \beta) - \lambda_1$$

$$u_2 \equiv \psi_2(z_i, \beta) - \lambda_2$$

$$u_3 \equiv \psi_3(z_i, \beta) - \lambda_3$$

$$u_4 \equiv \psi_4(z_i, \beta) - \lambda_4$$

$$U_0 \equiv \Psi(z_i, \beta) - \Lambda$$

$$U_1 \equiv \Psi_1(z_i, \beta) - \Lambda_1$$

$$U_2 \equiv \Psi_2(z_i, \beta) - \Lambda_2$$

$$U_3 \equiv \Psi_3(z_i, \beta) - \Lambda_3$$

$$v_0 \equiv \psi(z_j, \beta) - \lambda = \psi(z_j, \beta) \quad i \neq j$$

$$v_1 \equiv \psi_1(z_j, \beta) - \lambda_1$$

$$v_2 \equiv \psi_2(z_j, \beta) - \lambda_2$$

$$v_3 \equiv \psi_3(z_j, \beta) - \lambda_3$$

$$v_4 \equiv \psi_4(z_j, \beta) - \lambda_4$$

$$V_0 \equiv \Psi(z_j, \beta) - \Lambda$$

$$V_1 \equiv \Psi_1(z_j, \beta) - \Lambda_1$$

$$V_2 \equiv \Psi_2(z_j, \beta) - \Lambda_2$$

$$V_3 \equiv \Psi_3(z_j, \beta) - \Lambda_3$$

2.1 Higher Order Bias

We focus on the version of GMM b_3 iterated 3 times.¹ This is convenient because its third order properties are invariant to the choice of initial consistent estimator. With Theorem 1, we can see that

Theorem 2

$$\begin{aligned} E[b_2 - \beta] &\approx \frac{1}{\sqrt{n}} E[\bar{T}_1] + \frac{1}{2n} E[\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2] \\ &= \frac{1}{2n} E[\bar{T}_2] + \frac{1}{2n} E[\bar{T}_1 \cdot \tilde{T}_2] \\ &\equiv \frac{B}{n} \end{aligned}$$

Definition 7

$$\begin{aligned} \Gamma_{0,1} &\equiv E[u_0 u_1'] = E[\psi(z_i, \beta) (\psi_1(z_i, \beta) - \lambda_1)'] \\ &= E[\psi(z_i, \beta) \psi_1(z_i, \beta)'] \\ \Gamma_{0,2} &\equiv E[u_0 u_2'] = E[\psi(z_i, \beta) (\psi_2(z_i, \beta) - \lambda_2)'] \\ &= E[\psi(z_i, \beta) \psi_2(z_i, \beta)'] \\ \Gamma_{1,1} &\equiv E[u_1 u_1'] = E[(\psi_1(z_i, \beta) - \lambda_1) (\psi_1(z_i, \beta) - \lambda_1)'] \\ &= E[\psi_1(z_i, \beta) \psi_1(z_i, \beta)'] - \lambda_1 \lambda_1' \\ \Upsilon_{0,0} &\equiv E[u_0 \lambda_1' \Lambda^{-1} U_0] \Lambda^{-1} \\ &= E[\psi(z_i, \beta) \lambda_1' \Lambda^{-1} \Psi(z_i, \beta)] \Lambda^{-1} \\ \Upsilon_{1,0} &\equiv E[u_1 \lambda_1' \Lambda^{-1} U_0] \Lambda^{-1} \\ &= E[\psi_1(z_i, \beta) \lambda_1' \Lambda^{-1} \Psi(z_i, \beta)] \Lambda^{-1} - \lambda_1 \lambda_1' \Lambda^{-1} \\ \Upsilon_{0,1} &\equiv E[u_0 \lambda_1' \Lambda^{-1} U_1] \Lambda^{-1} \\ &= E[\psi_0(z_i, \beta) \lambda_1' \Lambda^{-1} \Psi_1(z_i, \beta)] \Lambda^{-1} \\ \Upsilon_{0,0}^{(2)} &\equiv E[u_0 \lambda_2' \Lambda^{-1} U_0] \Lambda^{-1} \\ &= E[\psi(z_i, \beta) \lambda_2' \Lambda^{-1} \Psi(z_i, \beta)] \Lambda^{-1} \\ \Upsilon_{0,0}^{(3)} &\equiv E[(\lambda_1' \Lambda^{-1} u_0) \cdot U_0 \Lambda^{-1}] \\ &= E[(\lambda_1' \Lambda^{-1} \psi(z_i, \beta)) \cdot \Psi(z_i, \beta) \Lambda^{-1}] \\ \Upsilon_{0,0}^{(4)} &\equiv E[U_0 \Lambda^{-1} u_0] \\ &= E[\Psi(z_i, \beta) \Lambda^{-1} \psi(z_i, \beta)] \\ \Xi_{0,0} &\equiv E[\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\ \Xi_{0,2} &\equiv E[(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2] \end{aligned}$$

¹This means that b_3 is a four step estimator. Recall that

$$\begin{aligned} b_1 &= \underset{c \in C}{\operatorname{argmin}} g_n(c)' G_n(\tilde{\beta})^{-1} g_n(c) \\ b_2 &\equiv \underset{c \in C}{\operatorname{argmin}} g(c)' G(b_1)^{-1} g(c) \\ b_3 &\equiv \underset{c \in C}{\operatorname{argmin}} g(c)' G(b_2)^{-1} g(c) \end{aligned}$$

Note that

$$B \equiv \sum_{k=1}^8 \tilde{B}_k$$

and

$$\begin{aligned}
\tilde{B}_1 &\equiv \frac{1}{2} E \left[-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} w_{1,n}' \Lambda^{-1} w_n \right] \\
&= -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E \left[(\psi_1(z_i, \beta) - \lambda_1)' \Lambda^{-1} \psi(z_i, \beta) \right] \\
&= -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} \text{trace} \left(\Lambda^{-1} E \left[\psi(z_i, \beta) (\psi_1(z_i, \beta) - \lambda_1)' \right] \right) \\
&= -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} \text{trace} \left(\Lambda^{-1} E \left[\psi(z_i, \beta) \psi_1(z_i, \beta)' \right] \right) \\
&= -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} \frac{\text{trace}(\Lambda^{-1} \Lambda_1)}{2}
\end{aligned} \tag{2}$$

$$\begin{aligned}
\tilde{B}_2 &\equiv \frac{1}{2} E \left[-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \lambda_1' H_{1,0} w_n \right] \\
&= -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E \left[\lambda_1' (-\Lambda^{-1} W_n \Lambda^{-1}) w_n \right] \\
&= \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E \left[\lambda_1' \Lambda^{-1} (\Psi(z_i, \beta) - \Lambda) \Lambda^{-1} (\psi(z_i, \beta) - \lambda) \right] \\
&= \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E \left[\lambda_1' \Lambda^{-1} (\psi(z_i, \beta) \psi(z_i, \beta)' - \Lambda) \Lambda^{-1} \psi(z_i, \beta) \right] \\
&= \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E \left[\lambda_1' \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \psi(z_i, \beta) \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_3 &\equiv \frac{1}{2} E \left[\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} (2\lambda_1' \Lambda^{-1} w_{1,n}) (\lambda_1' \Lambda^{-1} w_n) \right] \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} w_{1,n}) (\lambda_1' \Lambda^{-1} w_n) \right] \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} (\psi_1(z_i, \beta) - \lambda_1)) (\lambda_1' \Lambda^{-1} \psi(z_i, \beta)) \right] \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} \psi(z_i, \beta)) ((\psi_1(z_i, \beta) - \lambda_1)' \Lambda^{-1} \lambda_1) \right] \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \lambda_1' \Lambda^{-1} E \left[\psi(z_i, \beta) (\psi_1(z_i, \beta) - \lambda_1)' \right] \Lambda^{-1} \lambda_1 \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \lambda_1' \Lambda^{-1} E \left[\psi(z_i, \beta) \psi_1(z_i, \beta)' \right] \Lambda^{-1} \lambda_1 \\
&= \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \frac{1}{2} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
&= \frac{\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2}
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_4 &\equiv \frac{1}{2} E \left[\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \right] \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)] \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_2 \Lambda^{-1} E [w_n w'_n] \Lambda^{-1} \lambda_1 \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_2 \Lambda^{-1} \Lambda \Lambda^{-1} \lambda_1 \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_2 \Lambda^{-1} \lambda_1
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_5 &\equiv \frac{1}{2} E \left[\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 H_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) \right] \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 (-\Lambda^{-1} W_n \Lambda^{-1}) \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)] \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} (\psi(z_i, \beta) \psi(z_i, \beta)' - \Lambda) \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))] \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))]
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_6 &\equiv \frac{1}{2} E \left[-\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (3\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 \right] \\
&= -\frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} w_n)^2] \\
&= -\frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_1 \Lambda^{-1} \Lambda \Lambda^{-1} \lambda_1 \\
&= -\frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_1 \Lambda^{-1} \lambda_1 \\
&= -\frac{3\lambda'_2 \Lambda^{-1} \lambda_1}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^2}
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_7 &\equiv \frac{1}{2} E \left[\left(-\frac{\lambda'_1 \Lambda^{-1} w_n}{\lambda'_1 \Lambda^{-1} \lambda_1} \right) \left(-\frac{2}{\lambda'_1 \Lambda^{-1} \lambda_1} \lambda'_1 H_{1,1} w_n \right) \right] \\
&= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 (-\Lambda^{-1} \Lambda_1 \Lambda^{-1}) w_n)] \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_1 \Lambda^{-1} \Lambda \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_8 &\equiv \frac{1}{2} E \left[\left(-\frac{\lambda'_1 \Lambda^{-1} w_n}{\lambda'_1 \Lambda^{-1} \lambda_1} \right) \left(\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) \right) \right] \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 H_{1,1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)] \\
&= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 (-\Lambda^{-1} \Lambda_1 \Lambda^{-1}) \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^2] \\
&= \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_1 \Lambda^{-1} \Lambda \Lambda^{-1} \lambda_1 \\
&= \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
B &= -\frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} \frac{\text{trace}(\Lambda^{-1} \Lambda_1)}{2} + \frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} E [\lambda'_1 \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \psi(z_i, \beta)] \\
&\quad + \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_2 \Lambda^{-1} \lambda_1 \\
&\quad - \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))] - \frac{3\lambda'_2 \Lambda^{-1} \lambda_1}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&\quad - \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 + \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&= -\frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} \frac{\text{trace}(\Lambda^{-1} \Lambda_1)}{2} + \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} - \frac{\lambda'_2 \Lambda^{-1} \lambda_1}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \\
&\quad + \frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} E [\lambda'_1 \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \psi(z_i, \beta)] \\
&\quad - \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [\lambda'_1 \Lambda^{-1} \psi(z_i, \beta) \psi(z_i, \beta)' \Lambda^{-1} \lambda_1 \lambda'_1 \Lambda^{-1} \psi(z_i, \beta)]
\end{aligned}$$

or

Theorem 3

$$B \equiv \sum_{k=1}^5 B_k$$

where

$$\begin{aligned}
B_1 &\equiv -\frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} \frac{\text{trace}(\Lambda^{-1} \Lambda_1)}{2} \\
B_2 &\equiv \frac{\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \\
B_3 &\equiv -\frac{\lambda_2' \Lambda^{-1} \lambda_1}{2 (\lambda_1' \Lambda^{-1} \lambda_1)^2} \\
B_4 &\equiv \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E [\lambda_1' \Lambda^{-1} u_0 u_0' \Lambda^{-1} u_0] = \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E [\lambda_1' \Lambda^{-1} (U_0 + \Lambda) \Lambda^{-1} u_0] = \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} E [\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\
&= \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1} \text{trace}(\Upsilon_{0,0}) \\
B_5 &\equiv -\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E [\lambda_1' \Lambda^{-1} u_0 u_0' \Lambda^{-1} \lambda_1 \lambda_1' \Lambda^{-1} u_0] = -\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E [\lambda_1' \Lambda^{-1} (U_0 + \Lambda) \Lambda^{-1} \lambda_1 \lambda_1' \Lambda^{-1} u_0] \\
&= -\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E [(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda_1' \Lambda^{-1} u_0)] = -\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E [(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= -\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

Remark 1 The last equality in (2) follows from

$$\begin{aligned}
\Lambda_1 &= E [\partial \Psi(z_i, \beta) / \partial c] = E [\psi(z_i, \beta) \psi_1(z_i, \beta)' + \psi_1(z_i, \beta) \psi(z_i, \beta)'] \\
&= E [\psi(z_i, \beta) \psi_1(z_i, \beta)'] + (E [\psi(z_i, \beta) \psi_1(z_i, \beta)'])'
\end{aligned}$$

and the lemma below.

Lemma 10 For an arbitrary $k \times k$ matrix A , and an arbitrary symmetric $k \times k$ matrix B , we have $\text{trace}(B^{-1}A) = \frac{1}{2} \text{trace}(B^{-1}(A + A'))$ and $\text{trace}((A + A')^{-1}A) = \frac{k}{2}$

Proof. The first claim follows from

$$\text{trace}(B^{-1}A) = \text{trace}(A'B^{-1}) = \text{trace}((A'B^{-1})') = \text{trace}(B^{-1}A)$$

The second claim follows from

$$\text{trace}((A + A')^{-1}A) = \text{trace}(A'(A + A')^{-1}) = \text{trace}((A'(A + A')^{-1})') = \text{trace}((A + A')^{-1}A)$$

and

$$\text{trace}((A + A')^{-1}A) + \text{trace}((A + A')^{-1}A') = \text{trace}((A + A')^{-1}(A + A')) = \text{trace}(I_k) = k$$

■

2.2 Higher Order MSE

Again, we focus on the version of GMM iterated 3 times, i.e., b_3 . With Theorem 1, we can see that

$$E \left[(\sqrt{n}(b_2 - \beta))^2 \right] \approx E \left[\bar{T}_1^2 \right] + \frac{1}{\sqrt{n}} E \left[\bar{T}_1 \cdot (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \right] \quad (3)$$

$$+ \frac{1}{n} E \left[\left(\frac{1}{2} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \right)^2 \right] \quad (4)$$

$$+ \frac{1}{3n} E \left[\bar{T}_1 \cdot (\bar{T}_3 + \bar{T}_1 \cdot \tilde{T}_{3,1} + \bar{T}_1^2 \cdot \tilde{T}_{3,2} + \bar{T}_2 \cdot \tilde{T}_{3,3} + \bar{T}_1 \cdot \tilde{T}_2 \cdot \tilde{T}_{3,3}) \right] \quad (5)$$

Theorem 4

$$E \left[\bar{T}_1^2 \right] = E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right)^2 \right] = \frac{\lambda_1' \Lambda^{-1} \Lambda \Lambda^{-1} \lambda_1}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} = \frac{1}{\lambda_1' \Lambda^{-1} \lambda_1}$$

2.3 Term (3)

We now provide a detailed analysis of (3).

Theorem 5

$$\frac{1}{\sqrt{n}} E \left[\bar{T}_1 \cdot (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \right] = \frac{1}{n} \sum_{k=1}^8 M_k$$

where

$$\begin{aligned} M_1 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} w_{1,n}' \Lambda^{-1} w_n \right) \right] \\ M_2 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \lambda_1' H_{1,0} w_n \right) \right] \\ M_3 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} (2\lambda_1' \Lambda^{-1} w_{1,n}) (\lambda_1' \Lambda^{-1} w_n) \right) \right] \\ M_4 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} (\lambda_2' \Lambda^{-1} w_n) (\lambda_1' \Lambda^{-1} w_n) \right) \right] \\ M_5 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} (\lambda_1' H_{1,0} \lambda_1) (\lambda_1' \Lambda^{-1} w_n) \right) \right] \\ M_6 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \left(-\frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} (3\lambda_1' \Lambda^{-1} \lambda_2) (\lambda_1' \Lambda^{-1} w_n)^2 \right) \right] \\ M_7 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right)^2 \left(-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \lambda_1' H_{1,1} w_n \right) \right] \\ M_8 &\equiv \sqrt{n} E \left[\left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right)^2 \left(\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} (\lambda_1' H_{1,1} \lambda_1) (\lambda_1' \Lambda^{-1} w_n) \right) \right] \end{aligned}$$

Lemma 11

$$M_1 = \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \text{trace}(\Psi_{1,0})$$

Proof. Note that

$$\begin{aligned}
& \sqrt{n} \left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \cdot \left(-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} w_{1,n}' \Lambda^{-1} w_n \right) \\
= & \sqrt{n} \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_1' \Lambda^{-1} (\psi(z_i, \beta) - \lambda) \right) \left(\frac{1}{n} \sum_{j=1}^n \sum_{k=1}^n (\psi_1(z_j, \beta) - \lambda_1)' \Lambda^{-1} (\psi(z_k, \beta) - \lambda) \right) \\
= & \frac{1}{n} \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \lambda_1' \Lambda^{-1} (\psi(z_i, \beta) - \lambda) (\psi_1(z_j, \beta) - \lambda_1)' \Lambda^{-1} (\psi(z_k, \beta) - \lambda)
\end{aligned}$$

Because $E[\lambda_1' \Lambda^{-1} (\psi(z_i, \beta) - \lambda) (\psi_1(z_j, \beta) - \lambda_1)' \Lambda^{-1} (\psi(z_k, \beta) - \lambda)] = 0$ unless $i = j = k$, we have

$$\begin{aligned}
& E \left[\sqrt{n} \left(-\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1} \right) \cdot \left(-\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} w_{1,n}' \Lambda^{-1} w_n \right) \right] \\
= & \frac{1}{n} \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \sum_{i=1}^n \lambda_1' \Lambda^{-1} E [(\psi(z_i, \beta) - \lambda) (\psi_1(z_i, \beta) - \lambda_1)' \Lambda^{-1} (\psi(z_i, \beta) - \lambda)] \\
= & \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \lambda_1' \Lambda^{-1} E [(\psi(z_i, \beta) - \lambda) (\psi_1(z_i, \beta) - \lambda_1)' \Lambda^{-1} (\psi(z_i, \beta) - \lambda)]
\end{aligned}$$

We therefore have

$$M_1 = \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E [(\lambda_1' \Lambda^{-1} u_0) (u_1' \Lambda^{-1} u_0)]$$

Because

$$\begin{aligned}
E [(\lambda_1' \Lambda^{-1} u_0) (u_1' \Lambda^{-1} u_0)] &= E [(u_1' \Lambda^{-1} u_0) (u_0' \Lambda^{-1} \lambda_1)] \\
&= E [u_1' \Lambda^{-1} u_0 u_0' \Lambda^{-1} \lambda_1] \\
&= E [u_1' \Lambda^{-1} \Psi(z_i, \beta) \Lambda^{-1} \lambda_1] \\
&= E [u_1' \Lambda^{-1} (U_0 + \Lambda) \Lambda^{-1} \lambda_1] \\
&= E [u_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= E [\text{trace} (u_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= E [\text{trace} (\lambda_1 u_1' \Lambda^{-1} U_0 \Lambda^{-1})] \\
&= E [\text{trace} (\Lambda^{-1} U_0 \Lambda^{-1} \lambda_1 u_1')] \\
&= E [\text{trace} ((\lambda_1 u_1')' (\Lambda^{-1} U_0 \Lambda^{-1})')] \\
&= E [\text{trace} (u_1 \lambda_1' \Lambda^{-1} U_0 \Lambda^{-1})] \\
&= \text{trace} (\Psi_{1,0})
\end{aligned}$$

we have

$$M_1 = \frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \text{trace} (\Psi_{1,0})$$

■

Lemma 12

$$M_2 = -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \Xi_{0,0}$$

Proof. As in the proof of Lemma 11, we can obtain

$$M_2 = -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} \psi(z_i, \beta)) \lambda_1' \Lambda^{-1} (\psi(z_i, \beta) \psi(z_i, \beta)' - \Lambda) \Lambda^{-1} \psi(z_i, \beta) \right]$$

and hence

$$\begin{aligned} M_2 &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} u_0) \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} u_0) (u_0' \Lambda^{-1} \lambda_1) \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} u_0 u_0' \Lambda^{-1} \lambda_1 \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} (U_0 + \Lambda) \Lambda^{-1} \lambda_1 \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} E \left[\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1 \right] \\ &= -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^2} \Xi_{0,0} \end{aligned}$$

■

Lemma 13

$$M_3 = -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} \lambda_1' \Lambda^{-1} \Upsilon_{1,0} \lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_3 &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} E \left[(\lambda_1' \Lambda^{-1} \psi(z_i, \beta))^2 \lambda_1' \Lambda^{-1} (\psi_1(z_i, \beta) - \lambda_1) \right] \\ &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} E \left[(\lambda_1' \Lambda^{-1} u_0)^2 (\lambda_1' \Lambda^{-1} u_1) \right] \\ &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} E \left[(\lambda_1' \Lambda^{-1} u_0 u_0' \Lambda^{-1} \lambda_1) (\lambda_1' \Lambda^{-1} u_1) \right] \\ &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} E \left[(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} (U_0 + \Lambda) \Lambda^{-1} \lambda_1) \right] \\ &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} E \left[(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\ &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} \lambda_1' \Lambda^{-1} \Upsilon_{1,0} \lambda_1 \end{aligned}$$

■

Lemma 14

$$M_4 = -\frac{2}{(\lambda_1' \Lambda^{-1} \lambda_1)^3} \lambda_2' \Lambda^{-1} \Upsilon_{0,0} \lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned}
M_4 &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))^2 \lambda'_2 \Lambda^{-1} \psi(z_i, \beta) \right] \\
&= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 (\lambda'_2 \Lambda^{-1} u_0) \right] \\
&= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} u_0 u'_0 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} u_0) \right] \\
&= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\
&= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_2 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

■

Lemma 15

$$M_5 = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \Xi_{0,2}$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned}
M_5 &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))^2 \lambda'_1 \Lambda^{-1} (\psi(z_i, \beta) \psi(z_i, \beta)' - \Lambda) \Lambda^{-1} \lambda_1 \right] \\
&= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\
&= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} u_0 u'_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\
&= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2 \right] \\
&=
\end{aligned}$$

■

Lemma 16

$$M_6 = 3 \frac{(\lambda'_1 \Lambda^{-1} \lambda_2)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned}
M_6 &= \frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))^3 \right] \\
&= \frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0)^3 \right] \\
&= \frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\
&= \frac{(3\lambda'_1 \Lambda^{-1} \lambda_2)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)
\end{aligned}$$

■

Lemma 17

$$M_7 = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_7 &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))^2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \psi(z_i, \beta)) \right] \\ &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) \right] \\ &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E \left[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\ &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

■

Lemma 18

$$M_8 = -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_8 &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} \psi(z_i, \beta))^3 \right] \\ &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0)^3 \right] \\ &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) \right] \\ &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \end{aligned}$$

■

2.4 Term (4)

We now provide a detailed analysis of (4). Write

$$\frac{1}{2} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) = \sum_{k=1}^7 S_k$$

where

$$\begin{aligned}
S_1 &\equiv -\frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} (w'_{1,n} \Lambda^{-1} w_n) \\
S_2 &\equiv \frac{1}{\lambda'_1 \Lambda^{-1} \lambda_1} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) \\
S_3 &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n) \\
S_4 &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
S_5 &\equiv -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) \\
S_6 &\equiv \left(-\frac{3\lambda'_1 \Lambda^{-1} \lambda_2}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^3} + \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \right) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
S_7 &\equiv -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n)
\end{aligned}$$

We then have

Theorem 6

$$E \left[\left(\frac{1}{2} (\bar{T}_2 + \bar{T}_1 \cdot \tilde{T}_2) \right)^2 \right] = \sum_{k=1}^7 E [S_k^2] + 2 \sum_{k < k'} E [S_k S_{k'}]$$

Lemma 19

$$E [S_1^2] = M_9 + O(n^{-1}),$$

where

$$M_9 = \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \left((\text{trace}(\Lambda^{-1} \Gamma_{0,1}))^2 + \text{trace}(\Lambda^{-1} \Gamma_{1,1}) + \text{trace}(\Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1}) \right)$$

Proof. As in Donald and Newey (1998), we use the fact that for a pair of double arrays, say $\{A_{ij}^1\}$ and $\{A_{ij}^2\}$, we have

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n A_{ij}^1 \sum_{i=1}^n \sum_{j=1}^n A_{ij}^2 &= \sum_{i=1}^n A_{ii}^1 A_{ii}^2 + \sum_{i=1}^n \sum_{j \neq i} A_{ii}^1 A_{jj}^2 \\
&\quad + \sum_{i=1}^n \sum_{j \neq i} A_{ij}^1 A_{ij}^2 + \sum_{i=1}^n \sum_{j \neq i} A_{ij}^1 A_{ji}^2 + \dots
\end{aligned}$$

where the omitted terms will not be needed for our calculations because they will have zero expectations in our context. It therefore follows that

$$\begin{aligned}
M_9 &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(u'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(u'_1 \Lambda^{-1} v_0) (u'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(u'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0)] &= (E [u'_1 \Lambda^{-1} u_0])^2 = (\text{trace} (\Lambda^{-1} \Gamma_{0,1}))^2 \\
E [(u'_1 \Lambda^{-1} v_0) (u'_1 \Lambda^{-1} v_0)] &= E [(u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} u_1)] = E [u'_1 \Lambda^{-1} v_0 v'_0 \Lambda^{-1} u_1] \\
&= E [u'_1 \Lambda^{-1} u_1] = \text{trace} (\Lambda^{-1} \Gamma_{1,1})
\end{aligned}$$

and

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} u_0)] &= E [u'_1 \Lambda^{-1} E [v_0 v'_1] \Lambda^{-1} u_0] = \text{trace} (\Lambda^{-1} E [u_0 u'_1] \Lambda^{-1} E [u_0 u'_1]) \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1})
\end{aligned}$$

■

Lemma 20

$$E [S_2^2] = M_{10} + O(n^{-1}),$$

where

$$M_{10} \equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \left(([\text{trace} (\Upsilon_{0,0})])^2 + \Xi_{0,0} + \text{trace} (\Upsilon_{0,0} \Upsilon_{0,0}) \right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{10} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0)] \\
&\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)]
\end{aligned}$$

Note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] &= (E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0])^2 \\
&= (E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)])^2 \\
&= (E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})])^2 \\
&= ([\text{trace} (E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1})])^2 \\
&= ([\text{trace} (\Upsilon_{0,0})])^2
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0 v'_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1]
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] &= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1} u_0] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1} u_0)] \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1})] \\
&= \text{trace} (E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1}) \\
&= \text{trace} (E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1}) \\
&= \text{trace} (\Upsilon_{0,0} \Upsilon_{0,0})
\end{aligned}$$

■

Lemma 21

$$E [S_3^2] = M_{11} + O(n^{-1}),$$

where

$$M_{11} \equiv \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} \left(2 (\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2 + (\lambda_1' \Lambda^{-1} \lambda_1) (\lambda_1' \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1) \right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{11} &\equiv \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_1) (\lambda_1' \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_1) (\lambda_1' \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_1) (\lambda_1' \Lambda^{-1} v_0)] &= (E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} u_0)])^2 = (\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2 \\ E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0)] &= E [(\lambda_1' \Lambda^{-1} u_1)^2] E [(\lambda_1' \Lambda^{-1} v_0)^2] \\ &= (\lambda_1' \Lambda^{-1} \lambda_1) \cdot E [(\lambda_1' \Lambda^{-1} u_1)^2] \\ &= (\lambda_1' \Lambda^{-1} \lambda_1) (\lambda_1' \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_1) (\lambda_1' \Lambda^{-1} u_0)] = (E [(\lambda_1' \Lambda^{-1} u_1) (\lambda_1' \Lambda^{-1} u_0)])^2 = (\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2$$

■

Lemma 22

$$E [S_4^2] = M_{12} + O(n^{-1}),$$

where

$$M_{12} \equiv \frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} \left(2 (\lambda_1' \Lambda^{-1} \lambda_2)^2 + (\lambda_1' \Lambda^{-1} \lambda_1) (\lambda_2' \Lambda^{-1} \lambda_2) \right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{12} &\equiv \frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_2' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_2' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0)] \\ &\quad + \frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_2' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_2' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0)] \\ &\quad + \frac{1}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} E [(\lambda_2' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_2' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= (E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)])^2 \\ &= (\lambda'_1 \Lambda^{-1} \lambda_2)^2 \end{aligned}$$

$$\begin{aligned} &E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_2 \Lambda^{-1} u_0)^2] E [(\lambda'_1 \Lambda^{-1} v_0)^2] = (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot E [(\lambda'_2 \Lambda^{-1} u_0)^2] \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \lambda_2) \end{aligned}$$

and

$$\begin{aligned} &E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0)] \\ &= (\lambda'_2 \Lambda^{-1} \lambda_1)^2 \end{aligned}$$

■

Lemma 23

$$E [S_5^2] = M_{13} + O(n^{-1}),$$

where

$$M_{13} \equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left(2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2 + (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \Xi_{0,2} \right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{13} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)])^2 \\ &= (E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)])^2 \\ &= (\lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1)^2 \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2 \end{aligned}$$

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2] \end{aligned}$$

and

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= (E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)])^2 \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2 \end{aligned}$$

■

Lemma 24

$$E [S_6^2] = M_{14} + O(n^{-1}),$$

where

$$M_{14} \equiv \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left(-\frac{3\lambda'_1 \Lambda^{-1} \lambda_2}{2} + \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 \right)^2$$

Proof. As in Lemma 19, we obtain

$$M_{14} \equiv 3 \left(-\frac{3\lambda'_1 \Lambda^{-1} \lambda_2}{2(\lambda'_1 \Lambda^{-1} \lambda_1)^3} + \frac{\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \right)^2 E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 (\lambda'_1 \Lambda^{-1} v_0)^2 \right]$$

Now note that

$$E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 (\lambda'_1 \Lambda^{-1} v_0)^2 \right] = \left(E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 \right] \right)^2 = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 25

$$E [S_7^2] = M_{15} + O(n^{-1}),$$

where

$$M_{15} \equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left(2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 + (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{15} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) \right] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) \right] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) \right] \end{aligned}$$

Now note that

$$\begin{aligned} &E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) \right] \\ &= \left(E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) \right] \right)^2 \\ &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 \end{aligned}$$

$$\begin{aligned} &E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) \right] \\ &= E \left[(\lambda'_1 \Lambda^{-1} u_0)^2 \right] E \left[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)^2 \right] \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) \right] \\ &= \left(E \left[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) \right] \right)^2 \\ &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 \end{aligned}$$

■

Lemma 26

$$2[S_1 S_2] = M_{16} + O(n^{-1}),$$

where

$$M_{16} \equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) \text{trace}(\Upsilon_{0,0}) + \text{trace}(\Upsilon_{1,0}) + \text{trace}(\Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1}))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{16} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] &= E[u'_1 \Lambda^{-1} u_0] E[\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) E[\text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) E[\text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) \text{trace}(E[u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1}) \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) \text{trace}(\Upsilon_{0,0}) \end{aligned}$$

$$\begin{aligned} &E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0)] \\ &= E[(u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] = E[u'_1 \Lambda^{-1} v_0 v'_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\ &= E[u'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] = E[\text{trace}(u'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] = E[\text{trace}(\lambda_1 u'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\ &= E[\text{trace}((\lambda_1 u'_1) (\Lambda^{-1} U_0 \Lambda^{-1}))] = E[\text{trace}((\Lambda^{-1} U_0 \Lambda^{-1})' (\lambda_1 u'_1)')] \\ &= E[\text{trace}((\lambda_1 u'_1)' (\Lambda^{-1} U_0 \Lambda^{-1})')] = E[\text{trace}(u_1 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\ &= \text{trace}(\Upsilon_{1,0}) \end{aligned}$$

and

$$\begin{aligned} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] &= E[\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} E[u_0 u'_1] \Lambda^{-1} v_0] \\ &= E[\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} u_0] \\ &= E[\text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} u_0)] \\ &= E[\text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1})] \\ &= \text{trace}(E[u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1}) \\ &= \text{trace}(\Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1}) \end{aligned}$$

■

Lemma 27

$$2[S_1 S_3] = M_{17} + O(n^{-1}),$$

where

$$M_{17} \equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + \lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1 + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{17} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(u'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \\ E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_1) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} v_0 v'_0 \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0 v'_1 \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 28

$$2[S_1 S_4] = M_{18} + O(n^{-1}),$$

where

$$M_{18} \equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_2) + \lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{18} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [u'_1 \Lambda^{-1} u_0] E [(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_2) \end{aligned}$$

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_2 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0 v'_0 \Lambda^{-1} \lambda_1] \\
&= \lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_2)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2
\end{aligned}$$

■

Lemma 29

$$2[S_1 S_5] = M_{19} + O(n^{-1}),$$

where

$$M_{19} \equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace} (\Lambda^{-1} \Gamma_{0,1}) \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{19} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
&E [(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \\
&E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} E [u_1 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
&E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

■

Lemma 30

$$2[S_1 S_6] = M_{20} + O(n^{-1}),$$

where

$$M_{20} \equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (3\lambda'_1 \Lambda^{-1} \lambda_2 - 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{20} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (3\lambda'_1 \Lambda^{-1} \lambda_2 - 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (3\lambda'_1 \Lambda^{-1} \lambda_2 - 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (3\lambda'_1 \Lambda^{-1} \lambda_2 - 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = E[(u'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0)^2] = \text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

$$\begin{aligned} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 31

$$2[S_1 S_7] = M_{21} + O(n^{-1}),$$

where

$$M_{21} \equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 + \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{21} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] &= E [(u'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\ &= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

$$\begin{aligned} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0 v'_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0 v'_0 \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 32

$$2[S_2 S_3] = M_{22} + O(n^{-1}),$$

where

$$M_{22} = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} ((\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \text{trace} (\Upsilon_{0,0}) + \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{22} &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] &= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} \lambda_1)] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \text{trace} (E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1}) \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \text{trace} (\Upsilon_{0,0}) \end{aligned}$$

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} E [u_1 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1 \end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

■

Lemma 33

$$2[S_2 S_4] = M_{23} + O(n^{-1}),$$

where

$$M_{23} = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} ((\lambda'_2 \Lambda^{-1} \lambda_1) \text{trace}(\Upsilon_{0,0}) + \lambda'_2 \Lambda^{-1} \Upsilon_{00} \lambda_1 + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{23} &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} U_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) \text{trace}(\Upsilon_{0,0})
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_2 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\
&= \lambda'_2 \Lambda^{-1} \Upsilon_{00} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2
\end{aligned}$$

■

Lemma 34

$$2[S_2S_5] = M_{24} + O(n^{-1}),$$

where

$$M_{24} = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Upsilon_{0,0}) \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 + \Xi_{0,2} + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{24} &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E [\text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= E [\text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\ &= \text{trace}(\Upsilon_{0,0}) \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \\ &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2] \end{aligned}$$

and

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} u_0) \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1] \\ &= E [\lambda'_1 \Lambda^{-1} u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1} \lambda_1 \\ &= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

■

Lemma 35

$$2[S_2S_6] = M_{25} + O(n^{-1}),$$

where

$$M_{25} = \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\text{trace}(\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{25} &= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
&E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_1 \Lambda^{-1} v_0)^2] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= \text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
&E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0] \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

■

Lemma 36

$$2[S_2 S_7] = M_{26} + O(n^{-1}),$$

where

$$M_{26} = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1 + \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{26} &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&- \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&- \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
&E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
&= \text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

■

Lemma 37

$$2[S_3 S_4] = M_{27} + O(n^{-1}),$$

where

$$M_{27} = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) + (\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{27} &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E [(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)
\end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_2 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)
\end{aligned}$$

■

Lemma 38

$$2[S_3S_5] = M_{28} + O(n^{-1}),$$

where

$$M_{28} = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{28} &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} E[v_0 \lambda'_1 \Lambda^{-1} V_0] \Lambda^{-1} \lambda_1) \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \\ &E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \end{aligned}$$

■

Lemma 39

$$2[S_3S_6] = M_{29} + O(n^{-1}),$$

where

$$M_{29} = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{29} &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 40

$$2[S_3 S_7] = M_{30} + O(n^{-1}),$$

where

$$M_{30} = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{30} &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 41

$$2[S_4S_5] = M_{31} + O(n^{-1}),$$

where

$$M_{31} = -\frac{2}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} (2(\lambda'_1\Lambda^{-1}\lambda_2)(\lambda'_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1) + (\lambda'_1\Lambda^{-1}\lambda_1)(\lambda'_2\Lambda^{-1}\Upsilon_{0,0}\lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{31} &= -\frac{2}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}v_0)] \\ &\quad -\frac{2}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}v_0)] \\ &\quad -\frac{2}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}v_0)] \\ &= E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)] E[(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)] \\ &= (\lambda'_1\Lambda^{-1}\lambda_2)(\lambda'_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1) \\ &E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}v_0)] \\ &= E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1)] E[(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] \\ &= (\lambda'_2\Lambda^{-1}\Upsilon_{0,0}\lambda_1)(\lambda'_1\Lambda^{-1}\lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}u_0)] \\ &= E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)] E[(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}V_0\Lambda^{-1}\lambda_1)] \\ &= (\lambda'_1\Lambda^{-1}\lambda_2)(\lambda'_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1) \end{aligned}$$

■

Lemma 42

$$2[S_4S_6] = M_{32} + O(n^{-1}),$$

where

$$M_{32} = \frac{3}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} (-3\lambda'_1\Lambda^{-1}\lambda_2 + 2\lambda'_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1)(\lambda'_1\Lambda^{-1}\lambda_2)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{32} &= \frac{1}{(\lambda'_1\Lambda^{-1}\lambda_1)^5} (-3\lambda'_1\Lambda^{-1}\lambda_2 + 2\lambda'_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1) E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] \\ &\quad + \frac{1}{(\lambda'_1\Lambda^{-1}\lambda_1)^5} (-3\lambda'_1\Lambda^{-1}\lambda_2 + 2\lambda'_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1) E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)] \\ &\quad + \frac{1}{(\lambda'_1\Lambda^{-1}\lambda_1)^5} (-3\lambda'_1\Lambda^{-1}\lambda_2 + 2\lambda'_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1) E[(\lambda'_2\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)] \end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 43

$$2[S_4 S_7] = M_{33} + O(n^{-1}),$$

where

$$M_{33} = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{33} &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 44

$$2[S_5 S_6] = M_{34} + O(n^{-1}),$$

where

$$M_{34} = -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{34} &= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 45

$$2[S_5 S_7] = M_{35} + O(n^{-1}),$$

where

$$M_{35} = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{35} &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 46

$$2[S_6 S_7] = M_{36} + O(n^{-1}),$$

where

$$M_{36} = -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{36} &= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (-3\lambda'_1 \Lambda^{-1} \lambda_2 + 2\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0)^2] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
& E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

2.5 Term (5)

We now provide a detailed analysis of (5). We need to calculate the expectation of

$$\frac{1}{3}\bar{T}_1 \cdot \bar{T}_3 + \frac{1}{3}(\bar{T}_1)^2 \cdot \tilde{T}_{3,1} + \frac{1}{3}\bar{T}_1^3 \cdot \tilde{T}_{3,2} + \frac{1}{3}\bar{T}_1 \cdot \bar{T}_2 \cdot \tilde{T}_{3,3} + \frac{1}{3}(\bar{T}_1)^2 \cdot \tilde{T}_2 \cdot \tilde{T}_{3,3}$$

We first note that

$$\frac{1}{3}\bar{T}_1 \cdot \bar{T}_3 = \sum_{k=1}^{31} A_k$$

where

$$\begin{aligned} A_1 &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (w'_{1,n} \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_2 &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_3 &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} w_{1,n}) (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_4 &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_5 &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} w_n) (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_6 &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_7 &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_8 &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_9 &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (w'_{1,n} \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\ A_{10} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (w'_{2,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \end{aligned}$$

$$\begin{aligned}
A_{11} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{12} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{13} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{14} &\equiv \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} w_{1,n})^2 (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{15} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_2 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{16} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1)^2 (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{17} &\equiv \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{18} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{19} &\equiv -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{20} &\equiv \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_2 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{21} &\equiv \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} w_{2,n}) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{22} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_3 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{23} &\equiv -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{24} &\equiv \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n) (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{25} &\equiv -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{26} &\equiv -\frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{27} &\equiv -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{28} &\equiv \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{29} &\equiv -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{30} &\equiv -\frac{4}{3 (\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_3) (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} w_n) \\
A_{31} &\equiv \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \lambda_2)^2 (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} w_n)
\end{aligned}$$

Lemma 47

$$\begin{aligned}
& E[A_1] \\
&= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \left(\text{trace}(\Upsilon_{1,0}) + \text{trace}(\Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(3)}) + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(4)} \right) \\
&\quad + O(n^{-1})
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_1] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(u'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E[(u'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(u'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] = E[u'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= E[\text{trace}(u'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] = E[\text{trace}(\lambda_1 u'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\
&= E[\text{trace}((\lambda_1 u'_1) (\Lambda^{-1} U_0 \Lambda^{-1}))] = E[\text{trace}((\Lambda^{-1} U_0 \Lambda^{-1}) (\lambda_1 u'_1))] \\
&= E[\text{trace}((\lambda_1 u'_1)' (\Lambda^{-1} U_0 \Lambda^{-1})')] = E[\text{trace}(u_1 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] \\
&= \text{trace}(\Upsilon_{1,0})
\end{aligned}$$

$$\begin{aligned}
& E[(u'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(u'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(u_0 u'_1 \Lambda^{-1} V_0 \Lambda^{-1})] \\
&= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(\Gamma_{0,1} \Lambda^{-1} V_0 \Lambda^{-1})] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) \cdot \text{trace}(\Gamma_{0,1} \Lambda^{-1} U_0 \Lambda^{-1})] \\
&= E[\text{trace}(\Gamma_{0,1} \Lambda^{-1} \cdot (\lambda'_1 \Lambda^{-1} u_0) \cdot U_0 \Lambda^{-1})] \\
&= \text{trace}(\Gamma_{0,1} \Lambda^{-1} \cdot \Upsilon_{0,0}^{(3)})
\end{aligned}$$

and

$$\begin{aligned}
E[(u'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= E[\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} V_0 \Lambda^{-1} v_0] \\
&= E[\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(4)}
\end{aligned}$$

■

Lemma 48

$$E[A_2] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} \left(\Xi_{0,0} + \text{trace}(\Upsilon_{0,0} \Upsilon_{0,0}^{(3)}) + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0}^{(4)} \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_2] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^2} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} v_0 v'_0 \Lambda^{-1} \lambda_1] \\
&= E[\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= \Xi_{0,0}
\end{aligned}$$

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1})] \\
&= E[(\lambda'_1 \Lambda^{-1} v_0) \cdot \text{trace}(\Upsilon_{0,0} V_0 \Lambda^{-1})] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) \cdot \text{trace}(\Upsilon_{0,0} U_0 \Lambda^{-1})] \\
&= \text{trace}(\Upsilon_{0,0} \Upsilon_{0,0}^{(3)})
\end{aligned}$$

and

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= E[\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) V_0 \Lambda^{-1} v_0] \\
&= E[\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} V_0 \Lambda^{-1} v_0] \\
&= E[\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} U_0 \Lambda^{-1} u_0] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0}^{(4)}
\end{aligned}$$

■

Lemma 49

$$\begin{aligned}
&E[A_3] \\
&= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1 + (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Lambda^{-1} \Gamma_{0,1})) \\
&\quad + O(n^{-1})
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_3] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} u_1) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} u_1) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} u_1) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_1) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [\lambda'_1 \Lambda^{-1} u_1 u'_1 \Lambda^{-1} \lambda_1] = \lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1 \\
E [(\lambda'_1 \Lambda^{-1} u_1) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(v'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [v'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \lambda'_1 \Lambda^{-1} v_0] \\
&= \text{trace} (\Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \lambda'_1 \Lambda^{-1} \Gamma_{0,1}) \\
&= \text{trace} ((\Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1})) \\
&= \text{trace} ((\lambda'_1 \Lambda^{-1} \Gamma_{0,1}) (\Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)) \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_1) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} v_0)] \\
&= E [\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 (v'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \cdot \text{trace} (\Lambda^{-1} \Gamma_{0,1})
\end{aligned}$$

■

Lemma 50

$$E[A_4] = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1} \lambda_1 + (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \text{trace}(\Upsilon_{0,0})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_4] &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_1) \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} E [u_1 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1}) u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} u_0) (u'_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) E [\text{trace} (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) E [\text{trace} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1})] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) \text{trace} (\Upsilon_{0,0})
\end{aligned}$$

■

Lemma 51

$$E[A_5] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2 + (\lambda'_2 \Lambda^{-1} \lambda_1) \text{trace} (\Lambda^{-1} \Gamma_{0,1})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_5] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= \lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \\
E [(\lambda'_2 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_2 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] = E [(\lambda'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} \lambda_2)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_2 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) \text{trace} (\Lambda^{-1} \Gamma_{0,1})
\end{aligned}$$

■

Lemma 52

$$E[A_6] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2 + (\lambda'_1 \Lambda^{-1} \lambda_2) \text{trace} (\Upsilon_{0,0})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_6] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_2 \Lambda^{-1} u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1] \\
&= \lambda'_2 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_2 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} \lambda_2)] \\
&= E [\lambda'_1 \Lambda^{-1} v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_2] \\
&= \lambda'_1 \Lambda^{-1} E [v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1}] \lambda_2 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) E [\text{trace} (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) E [\text{trace} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1})] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_2) \text{trace} (\Upsilon_{0,0})
\end{aligned}$$

■

Lemma 53

$$E[A_7] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1 + \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0} \lambda_1 + \text{trace} (\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_7] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (u'_1 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (v'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

and

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [v'_1 \Lambda^{-1} v_0] E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= \text{trace} (\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \end{aligned}$$

■

Lemma 54

$$E [A_8] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\Xi_{0,2} + \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0} \lambda_1 + (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \cdot \text{trace} (\Upsilon_{0,0})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E [A_8] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\ &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2] \end{aligned}$$

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1}) \Upsilon_{0,0} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0] \\ &= E [\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \lambda_1] E [\text{trace} (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \cdot E [\text{trace} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1})] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \cdot \text{trace} (\Upsilon_{0,0}) \end{aligned}$$

■

Lemma 55

$$E [A_9] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace} (\Lambda^{-1} \Gamma_{1,1}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_9] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(u'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(u'_1 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= \text{trace}(\Lambda^{-1} \Gamma_{1,1}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) \\ E[(u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_1) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E[(u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(u'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 56

$$E[A_{10}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Lambda^{-1} \Gamma_{0,2}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,2} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{10}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(u'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= \text{trace}(\Lambda^{-1} \Gamma_{0,2}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) \\ E[(u'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_2 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_2 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,2} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E[(u'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_2 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,2} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 57

$$E[A_{11}] = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Upsilon_{1,0}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) + 2(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma'_{0,1} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{11}] &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[\text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= \text{trace}(\Upsilon_{1,0}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

$$\begin{aligned} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} v_1) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma'_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Gamma'_{0,1} \Lambda^{-1} \lambda_1$$

■

Lemma 58

$$E[A_{12}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \left((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Upsilon_{0,0}^{(2)}) + 2(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0}^{(2)} \lambda_1) \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{12}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot E[\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} u_0] \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Upsilon_{0,0}^{(2)}) \end{aligned}$$

$$\begin{aligned}
E [(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} (u_0 \lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1}) \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0}^{(2)} \lambda_1
\end{aligned}$$

and

$$E [(\lambda'_2 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,0}^{(2)} \lambda_1$$

■

Lemma 59

$$E [A_{13}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\Xi_{0,0} \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda \Upsilon'_{0,0} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E [A_{13}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= \Xi_{0,0} \cdot (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
&E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} V_0 \Lambda^{-1} \lambda_1)'] \\
&= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Upsilon'_{0,0} \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} (v_0 \lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1}) \Lambda \Upsilon'_{0,0} \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda \Upsilon'_{0,0} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda \Upsilon'_{0,0} \Lambda^{-1} \lambda_1$$

■

Lemma 60

$$E [A_{14}] = \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left((\lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2 \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{14}] &= \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= (\lambda'_1 \Lambda^{-1} \Gamma_{1,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \\ E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2 \end{aligned}$$

and

$$E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)^2$$

■

Lemma 61

$$E[A_{15}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left((\lambda'_2 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_2 \Lambda^{-1} \lambda_1)^2 \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{15}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= (\lambda'_2 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1) \\ E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_2 \Lambda^{-1} \lambda_1)^2 \end{aligned}$$

and

$$E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = (\lambda'_2 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 62

$$E[A_{16}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left((\lambda'_1 \Lambda^{-1} \lambda_1) \Xi_{0,2} + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2 \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{16}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)^2] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2
\end{aligned}$$

and

$$E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)^2$$

■

Lemma 63

$$E[A_{17}] = \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{17}] &= \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_2 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 64

$$A_{18} \equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_2 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} A_{18} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_2 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\ &= (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\ &= (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \end{aligned}$$

■

Lemma 65

$$E[A_{19}] = -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{19}] &= -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{1,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_1)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 66

$$E[A_{20}] = \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{20}] &= \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_2 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 67

$$E[A_{21}] = \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Gamma_{0,2} \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{21}] &= \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_2) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_2) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_2) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_2) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_2)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,2} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 68

$$E[A_{22}] = \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \lambda_3) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{22}] &= \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_3 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_3 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&+ \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_3 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_3 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_3 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_3) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 69

$$E[A_{23}] = -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{23}] &= -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&- \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&- \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_2)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 70

$$E[A_{24}] = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \lambda_1) \text{trace} (\Lambda^{-1} \Gamma_{0,1})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{24}] &= \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} (u_0 u'_1) \Lambda^{-1} (v_0 v'_0) \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \\
E [(\lambda'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} v_1) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} (u_0 u'_0) \Lambda^{-1} (v_1 v'_0) \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_0) (v'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(v'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \text{trace} (\Lambda^{-1} \Gamma_{0,1})
\end{aligned}$$

■

Lemma 71

$$E[A_{25}] = -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + (\lambda'_1 \Lambda^{-1} \lambda_1) \text{trace} (\Upsilon_{0,0})) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{25}] &= -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [\text{trace} (\lambda'_1 \Lambda^{-1} V_0 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \text{trace} (\Upsilon_{0,0})
\end{aligned}$$

■

Lemma 72

$$E [A_{26}] = -\frac{54}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E [A_{26}] &= -\frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 73

$$E [A_{27}] = -\frac{27}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2)^2 + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E [A_{27}] &= -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(\lambda'_2 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_2)(\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 74

$$E[A_{28}] = \frac{27}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2)(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{28}] &= \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)(\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 75

$$E[A_{29}] = -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} \lambda_2) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{29}] &= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} u_0)(\lambda'_1 \Lambda^{-1} v_0)(\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 76

$$E[A_{30}] = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \lambda_3) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{30}] &= -\frac{4}{3(\lambda'_1\Lambda^{-1}\lambda_1)^5} (\lambda'_1\Lambda^{-1}\lambda_3) E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] \\
&\quad -\frac{4}{3(\lambda'_1\Lambda^{-1}\lambda_1)^5} (\lambda'_1\Lambda^{-1}\lambda_3) E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)] \\
&\quad -\frac{4}{3(\lambda'_1\Lambda^{-1}\lambda_1)^5} (\lambda'_1\Lambda^{-1}\lambda_3) E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] = (\lambda'_1\Lambda^{-1}\lambda_1)^2$$

■

Lemma 77

$$E[A_{31}] = \frac{27}{(\lambda'_1\Lambda^{-1}\lambda_1)^4} (\lambda'_1\Lambda^{-1}\lambda_2)^2 + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{31}] &= \frac{9}{(\lambda'_1\Lambda^{-1}\lambda_1)^6} (\lambda'_1\Lambda^{-1}\lambda_2)^2 E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] \\
&\quad + \frac{9}{(\lambda'_1\Lambda^{-1}\lambda_1)^6} (\lambda'_1\Lambda^{-1}\lambda_2)^2 E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)] \\
&\quad + \frac{9}{(\lambda'_1\Lambda^{-1}\lambda_1)^6} (\lambda'_1\Lambda^{-1}\lambda_2)^2 E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}u_0)(\lambda'_1\Lambda^{-1}v_0)(\lambda'_1\Lambda^{-1}v_0)] = (\lambda'_1\Lambda^{-1}\lambda_1)^2$$

■

We also note that

$$\frac{1}{3} (\bar{T}_1)^2 \cdot \tilde{T}_{3,1} = \sum_{k=32}^{51} A_k$$

Lemma 78

$$E[A_{32}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \left((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1}) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{32}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[\text{trace}(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[\text{trace}(u_0 u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1})] \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &= \text{trace}(\Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1}) \cdot (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

$$\begin{aligned} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E[(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 79

$$E[A_{33}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} \left((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0}^{(4)}) + 2(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1) \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{33}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[\text{trace}(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0}^{(4)}) \end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)'] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1$$

■

Lemma 80

$$E [A_{34}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} ((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace} (\Upsilon_{0,0} \Lambda_1 \Lambda^{-1}) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E [A_{34}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace} (\Upsilon_{0,0} \Lambda_1 \Lambda^{-1}) \\
& \\
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= E [\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \Lambda_1 \Lambda^{-1} \lambda_1] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1$$

■

Lemma 81

$$E [A_{35}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} ((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace} (\Upsilon_{0,1}) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{35}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
&E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[\text{trace}(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[\text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Upsilon_{0,1}) \\
& \\
&E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \lambda_1
\end{aligned}$$

and

$$E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \lambda_1$$

■

Lemma 82

$$E[A_{36}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) ((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Gamma_{0,1} \Lambda^{-1}) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{36}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[\text{trace}(u'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[\text{trace}(u_0 u'_1 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Gamma_{0,1} \Lambda^{-1}) \\
& \\
E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned} E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 83

$$E [A_{37}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) ((\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Upsilon_{0,0}) + 2(\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E [A_{37}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [\text{trace}(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [\text{trace}(u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot \text{trace}(\Upsilon_{0,0}) \\ &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= E [\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) (v_0 v'_0) \Lambda^{-1} \lambda_1] \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} &E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\ &= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

■

Lemma 84

$$E [A_{38}] = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E [A_{38}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (v'_0 \Lambda^{-1} \lambda_1)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 85

$$E[A_{39}] = -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{39}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 86

$$E[A_{40}] = \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left((\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{40}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \lambda_1] (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) \\ &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 87

$$E[A_{41}] = -\frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{41}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_1) (u'_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma'_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 88

$$E[A_{42}] = -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{42}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (u'_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_2 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 89

$$E[A_{43}] = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{43}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)'] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &= E[\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \Lambda_1 \Lambda^{-1} \lambda_1] (\lambda'_1 \Lambda^{-1} \lambda_1) \\ &= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 90

$$E[A_{44}] = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{44}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
&E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}) \Lambda_1 \Lambda^{-1} \lambda_1] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 91

$$E[A_{45}] = -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{45}] &= -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
&E[(\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E[\lambda'_1 \Lambda^{-1} (u_0 \lambda'_1 \Lambda^{-1} U_1 \Lambda^{-1}) \lambda_1] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= (\lambda'_1 \Lambda^{-1} \lambda_1) \cdot (\lambda'_1 \Lambda^{-1} \Upsilon_{0,1} \lambda_1)
\end{aligned}$$

■

Lemma 92

$$E[A_{46}] = \frac{24}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{46}] &= \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 93

$$A_{47} = \frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_2 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
A_{47} &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 94

$$E[A_{48}] = -\frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{48}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad - \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E [(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 95

$$E[A_{49}] = \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{49}] &= \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 96

$$E[A_{50}] = \frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{50}] &= \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 97

$$E[A_{51}] = -\frac{27}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{51}] &= -\frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{9}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

We also have

$$\frac{1}{3} \bar{T}_1^3 \cdot \tilde{T}_{3,2} = \sum_{k=52}^{57} A_k$$

where

$$\begin{aligned} A_{52} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \\ A_{53} &\equiv -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \\ A_{54} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \\ A_{55} &\equiv -\frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \\ A_{56} &\equiv \frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \\ A_{57} &\equiv \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 (\lambda'_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^3 \end{aligned}$$

Lemma 98

$$E[A_{52}] = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{52}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 99

$$E[A_{53}] = -\frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{53}] &= -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{1}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 100

$$E[A_{54}] = -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{54}] &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 101

$$E[A_{55}] = -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{55}] &= -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{2(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 102

$$E[A_{56}] = \frac{3}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{56}] &= \frac{(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{(\lambda'_1 \Lambda^{-1} \Lambda_2 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 103

$$E[A_{57}] = \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{57}] &= \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{2}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

We also have

$$\frac{1}{3} \bar{T}_1 \cdot \bar{T}_2 \cdot \tilde{T}_{3,3} = \sum_{k=58}^{69} A_k$$

where

$$\begin{aligned}
A_{58} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{59} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (w'_{1,n} \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
A_{60} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n) \\
A_{61} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
A_{62} &\equiv -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
A_{63} &\equiv \frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} w_{1,n}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^3 \\
A_{64} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} w_n)^2 \\
A_{65} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_2 \Lambda^{-1} w_n) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^3 \\
A_{66} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) \\
A_{67} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} W_n \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^3 \\
A_{68} &\equiv \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) \\
A_{69} &\equiv -\frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} w_n)^4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

Lemma 104

$$\begin{aligned}
&E[A_{58}] \\
&= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{58}] &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = \text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)$$

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E [(u'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E [(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E [(u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (u'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

■

Lemma 105

$$E[A_{59}] = -\frac{4(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2(\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{59}] &= -\frac{4(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(u'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= \text{trace}(\Lambda^{-1} \Gamma_{0,1}) (\lambda'_1 \Lambda^{-1} \lambda_1) \\
E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(u'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1
\end{aligned}$$

and

$$E[(u'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = E[(u'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] = \lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1$$

■

Lemma 106

$$E[A_{60}] = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} (\text{trace}(\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{60}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^3} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
&= \text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\
& \\
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \Lambda_1 \Lambda^{-1} \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \Lambda_1 \Lambda^{-1} \lambda_1
\end{aligned}$$

■

Lemma 107

$$E[A_{61}] = \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1))$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{61}] &= \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [\text{trace} (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} u_0)] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= E [\text{trace} (u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1})] (\lambda'_1 \Lambda^{-1} \lambda_1) \\
&= \text{trace} (\Upsilon_{0,0}) (\lambda'_1 \Lambda^{-1} \lambda_1) \\
& \\
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} E [u_0 \lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1}] \lambda_1 \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
& E [(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] \\
&= E [(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] \\
&= \lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1
\end{aligned}$$

■

Lemma 108

$$\begin{aligned}
& E[A_{62}] \\
&= -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{62}] &= -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad -\frac{8}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \\
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

and

$$\begin{aligned}
E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_1)] E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&= (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)
\end{aligned}$$

■

Lemma 109

$$E[A_{63}] = \frac{24 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{63}] &= \frac{8 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\
&\quad + \frac{8 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E [(\lambda'_1 \Lambda^{-1} u_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 110

$$E [A_{64}] = -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} ((\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E [A_{64}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

$$E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)$$

and

$$E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)$$

■

Lemma 111

$$E [A_{65}] = \frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_2)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E [A_{65}] &= \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E [(\lambda'_2 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 112

$$E[A_{66}] = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (2 (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{66}] &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \\ &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

and

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 113

$$E[A_{67}] = -\frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{67}] &= -\frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{4 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$\begin{aligned} &E[(\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1)] E[(\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &= (\lambda'_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1) \end{aligned}$$

■

Lemma 114

$$E[A_{68}] = \frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{68}] &= \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{6}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \lambda_2) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 115

$$E[A_{69}] = -\frac{18}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{69}] &= -\frac{6 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{6 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad - \frac{6 (\lambda'_1 \Lambda^{-1} \lambda_2) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1)^2$$

■

We also have

$$\frac{1}{3} (\bar{T}_1)^2 \cdot \tilde{T}_2 \cdot \tilde{T}_{3,3} = \sum_{k=70}^{73} A_k$$

where

$$\begin{aligned} A_{70} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} w_n)^2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n)^2 \\ A_{71} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) \\ A_{72} &\equiv -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} w_n)^3 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} w_n) \\ A_{73} &\equiv \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^6} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 (\lambda'_1 \Lambda^{-1} w_n)^4 \end{aligned}$$

Lemma 116

$$E[A_{70}] = \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} \left((\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) + 2 (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 \right) + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{70}] &= \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] \\ &\quad + \frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)$$

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2$$

and

$$E[(\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2$$

■

Lemma 117

$$E[A_{71}] = -\frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{71}] &= -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0)] \\ &\quad -\frac{4}{(\lambda'_1 \Lambda^{-1} \lambda_1)^5} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} u_0)] + O(n^{-1}) \end{aligned}$$

Now note that

$$E[(\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} u_0) (\lambda'_1 \Lambda^{-1} v_0) (\lambda'_1 \Lambda^{-1} v_0)] = (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda'_1 \Lambda^{-1} \lambda_1)$$

■

Lemma 118

$$E[A_{72}] = -\frac{12}{(\lambda'_1 \Lambda^{-1} \lambda_1)^4} (\lambda'_1 \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{72}] &= -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^5} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^5} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0)] \\
&\quad -\frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^5} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) E[(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0)] = (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1) (\lambda_1' \Lambda^{-1} \lambda_1)$$

■

Lemma 119

$$E[A_{73}] = \frac{12}{(\lambda_1' \Lambda^{-1} \lambda_1)^4} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 + O(n^{-1})$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E[A_{73}] &= \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^6} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^6} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0)] \\
&\quad + \frac{4}{(\lambda_1' \Lambda^{-1} \lambda_1)^6} (\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1)^2 E[(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} u_0)] + O(n^{-1})
\end{aligned}$$

Now note that

$$E[(\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} u_0) (\lambda_1' \Lambda^{-1} v_0) (\lambda_1' \Lambda^{-1} v_0)] = (\lambda_1' \Lambda^{-1} \lambda_1)^2$$

■

3 GMM Estimation and Dynamic Panel Model with Fixed Effects ($T=4$)

We now assume that our model is generated by

$$\begin{aligned}
y_{it} &= \alpha_i + \beta y_{it-1} + \varepsilon_{it} \quad t = 1, \dots, T \\
\varepsilon_{it} &\sim N(0, 1) \\
y_{i0} &\sim N\left(\frac{\alpha_i}{1-\beta}, \frac{1}{1-\beta^2}\right) \\
\alpha_i &\sim N(0, 1)
\end{aligned}$$

We will assume that

$$T = 4$$

3.1 Ahn and Schmidt

Ahn and Schmidt propose following moment restrictions

$$\begin{aligned} E[y_{is} ((y_{it} - y_{it-1}) - \beta (y_{it-1} - y_{is-1}))] &= 0 & t = 2, \dots, T : s = 0, \dots, t-2 \\ E[(y_{iT} - \beta y_{iT-1}) ((y_{it} - y_{it-1}) - \beta (y_{it-1} - y_{it-2}))] &= 0 & t = 2, \dots, T-1 \end{aligned}$$

Therefore, we have

$$\frac{T(T-1)}{2} + T - 2 = \frac{1}{2}T^2 + \frac{1}{2}T - 2 = \frac{1}{2}(4)^2 + \frac{1}{2}(4) - 2 = 8$$

moment restrictions:

$$\psi(z_i, c) = \begin{bmatrix} y_{i,0} ((y_{i,2} - y_{i,1}) - c(y_{i,1} - y_{i,0})) \\ y_{i,0} ((y_{i,3} - y_{i,2}) - c(y_{i,2} - y_{i,1})) \\ y_{i,1} ((y_{i,3} - y_{i,2}) - c(y_{i,2} - y_{i,1})) \\ y_{i,0} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ y_{i,1} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ y_{i,2} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ (y_{i,4} - cy_{i,3}) ((y_{i,2} - y_{i,1}) - c(y_{i,1} - y_{i,0})) \\ (y_{i,4} - cy_{i,3}) ((y_{i,3} - y_{i,2}) - c(y_{i,2} - y_{i,1})) \end{bmatrix}$$

3.2 Blundell and Bond

Blundell and Bond propose following moment restrictions

$$\begin{aligned} E[y_{is} ((y_{it} - y_{it-1}) - \beta (y_{it-1} - y_{is-1}))] &= 0 & t = 2, \dots, T : s = 0, \dots, t-2 \\ E[(y_{it} - \beta y_{i,t-1}) (y_{i,t-1} - y_{i,t-2})] &= 0 & t = 2, \dots, T \end{aligned}$$

Therefore, we have

$$\frac{T(T-1)}{2} + T - 1 = \frac{1}{2}T^2 + \frac{1}{2}T - 1 = \frac{1}{2}(4)^2 + \frac{1}{2}(4) - 1 = 9$$

moment restrictions:

$$\psi(z_i, c) = \begin{bmatrix} y_{i,0} ((y_{i,2} - y_{i,1}) - c(y_{i,1} - y_{i,0})) \\ y_{i,0} ((y_{i,3} - y_{i,2}) - c(y_{i,2} - y_{i,1})) \\ y_{i,1} ((y_{i,3} - y_{i,2}) - c(y_{i,2} - y_{i,1})) \\ y_{i,0} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ y_{i,1} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ y_{i,2} ((y_{i,4} - y_{i,3}) - c(y_{i,3} - y_{i,2})) \\ (y_{i,2} - cy_{i,1}) (y_{i,1} - y_{i,0}) \\ (y_{i,3} - cy_{i,2}) (y_{i,2} - y_{i,1}) \\ (y_{i,4} - cy_{i,3}) (y_{i,3} - y_{i,2}) \end{bmatrix}$$

3.3 Long Difference

Long difference note that

$$(y_{iT} - y_{i,1}) - \beta (y_{iT-1} - y_{i,0}) = \varepsilon_{iT} - \varepsilon_{i1}$$

is orthogonal to

$$\begin{aligned}
 & y_{i,0} \\
 y_{i,2} - \beta y_{i,1} &= \alpha_i + \varepsilon_{i,2} \\
 & \vdots \\
 y_{i,T-1} - \beta y_{i,T-2} &= \alpha_i + \varepsilon_{i,T-1}
 \end{aligned}$$

In other words, it considers the moment restrictions

$$\begin{aligned}
 E[y_{i,0}((y_{iT} - y_{i1}) - \beta(y_{iT-1} - y_{i0}))] &= 0 \\
 E[(y_{i,s} - \beta y_{i,s-1})((y_{iT} - y_{i1}) - \beta(y_{iT-1} - y_{i0}))] &= 0, \quad s = 2, \dots, T-1
 \end{aligned}$$

We therefore have

$$1 + T - 2 = T - 1 = 4 - 1 = 3$$

moment restrictions:

$$\psi(z_i, c) = \begin{bmatrix} y_{i,0}((y_{i,4} - y_{i,1}) - c(y_{i,3} - y_{i,0})) \\ (y_{i,2} - cy_{i,1})((y_{i,4} - y_{i,1}) - c(y_{i,3} - y_{i,0})) \\ (y_{i,3} - cy_{i,2})((y_{i,4} - y_{i,1}) - c(y_{i,3} - y_{i,0})) \end{bmatrix}$$

Higher Order Expansion of Iterated IV Estimator: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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June, 2002

1 Iterated 2SLS

We consider 2SLS

$$b = \left[\begin{pmatrix} n \\ i=1 \end{pmatrix} x_i \widehat{z}_i' \right] \left[\begin{pmatrix} n \\ i=1 \end{pmatrix} \widehat{z}_i \widehat{z}_i' \right]^{-1} \left[\begin{pmatrix} n \\ i=1 \end{pmatrix} \widehat{z}_i x_i \right]^{-1} \begin{pmatrix} n \\ i=1 \end{pmatrix} x_i \widehat{z}_i' \begin{pmatrix} n \\ i=1 \end{pmatrix} \widehat{z}_i \widehat{z}_i' \begin{pmatrix} n \\ i=1 \end{pmatrix} \widehat{z}_i y_i \quad (1)$$

applied to the single equation

$$y_i = \beta x_i + \varepsilon_i \quad (2)$$

using instrument $\widehat{z}_i = z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i$, where $\widehat{\theta} = \sqrt{n} (\widehat{\beta} - \beta)$. Here, z_i is the “proper” instrument, and $\widehat{\beta}$ is a preliminary estimator for β . We assume that

$$\widehat{\theta} = F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p \left(\frac{1}{n} \right) \quad (3)$$

where

$$F_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i$$

for some f_i i.i.d. with mean zero, $Q_n = O_p(1)$, and $S_n = O_p(1)$. Below in Theorem 1, we derive an expansion of the form

$$\sqrt{n}(b - \beta) = R_0 + \frac{1}{\sqrt{n}} R_1 + \frac{1}{n} R_2 + o_p \left(\frac{1}{n} \right) \quad (4)$$

where

$$\begin{aligned} R_0 &\equiv \frac{N_0}{D_0} \\ R_1 &\equiv -\frac{D_1 N_0}{D_0^2} + \frac{N_1}{D_0} \\ R_2 &\equiv -\frac{D_2 N_0}{D_0^2} + \frac{D_1^2 N_0}{D_0^3} + \frac{N_2}{D_0} - \frac{D_1 N_1}{D_0^2} \end{aligned}$$

where R_s depend on (F_n, Q_n, S_n) .

We therefore can see that the higher order bias of b is given by¹

$$\frac{1}{n}E[R_1]$$

which can be stochastically approximated by

$$\frac{1}{n}\widehat{E}[R_1]$$

where

$$\widehat{E}[R_1] = \frac{1}{J} \sum_{j=1}^J R_{1j}$$

for J large. We can also see that the expected value of $(b - \beta)^2$ is approximately equal to

$$\frac{1}{n}E[R_0^2] + \frac{2}{n\sqrt{n}}E[R_0R_1] + \frac{1}{n^2}E[R_1^2] + \frac{2}{n^2}E[R_0R_2]$$

which can be stochastically approximated by

$$\frac{1}{n}\widehat{E}[R_0^2] + \frac{2}{n\sqrt{n}}\widehat{E}[R_0R_1] + \frac{1}{n^2}\widehat{E}[R_1^2] + \frac{2}{n^2}\widehat{E}[R_0R_2]$$

where

$$\begin{aligned} \widehat{E}[R_0^2] &= \frac{1}{J} \sum_{j=1}^J R_{0j}^2 \\ \widehat{E}[R_0R_1] &= \frac{1}{J} \sum_{j=1}^J R_{0j}R_{1j} \\ \widehat{E}[R_1^2] &= \frac{1}{J} \sum_{j=1}^J R_{1j}^2 \\ \widehat{E}[R_0R_2] &= \frac{1}{J} \sum_{j=1}^J R_{0j}R_{2j} \end{aligned}$$

for J large.

2 Expansion

We present an expansion for 2SLS using instrument $\widehat{z}_i = z_i - \frac{1}{\sqrt{n}}\widehat{\theta}w_i$. We have

$$\sqrt{n}(b - \beta) = \frac{\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i\right)}{\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i x_i\right)} \quad (5)$$

Definition 1

$$\begin{aligned} \lambda &\equiv E[z_i x_i] \\ \Lambda &\equiv E[z_i z_i'] \\ \phi &\equiv E[w_i x_i] \\ \Delta &\equiv E[w_i z_i' + z_i w_i'] \\ \Omega &\equiv E[w_i w_i'] \\ \varphi &\equiv E[w_i \varepsilon_i] \end{aligned}$$

¹It can be verified that $E[R_0] = 0$.

Lemma 1

$$\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i = \lambda + \frac{1}{\sqrt{n}} A_1 + \frac{1}{n} A_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} A_1 &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i x_i - \lambda) - F_n \phi = \sqrt{n} \left(\frac{1}{n} Z' x - \lambda \right) - F_n \phi \\ A_2 &\equiv -F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i x_i - \phi) \right) - Q_n \phi \\ &= -\sqrt{n} F_n \left(\frac{1}{n} W' x - \phi \right) - Q_n \phi \end{aligned}$$

Proof. We have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i &= \frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) x_i \\ &= \frac{1}{n} \sum_{i=1}^n z_i x_i - \frac{1}{\sqrt{n}} \widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n w_i x_i \right) \end{aligned}$$

Because

$$\frac{1}{n} \sum_{i=1}^n z_i x_i = \lambda + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i x_i - \lambda) \right)$$

and

$$\begin{aligned} &\widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n w_i x_i \right) \\ &= \left(F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right) \left(\phi + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i x_i - \phi) \right) \right) \\ &= F_n \phi \\ &\quad + \frac{1}{\sqrt{n}} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i x_i - \phi) \right) + \frac{1}{\sqrt{n}} Q_n \phi \\ &\quad + o_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) x_i \\ &= \lambda \\ &\quad + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i x_i - \lambda) \right) - \frac{1}{\sqrt{n}} F_n \phi \\ &\quad - \frac{1}{n} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i x_i - \phi) \right) - \frac{1}{n} Q_n \phi \\ &\quad + o_p\left(\frac{1}{n}\right) \end{aligned}$$

■

Lemma 2

$$\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}'_i = \Lambda + \frac{1}{\sqrt{n}} L_1 + \frac{1}{n} L_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} L_1 &\equiv \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z'_i - \Lambda) \right) - F_n \Delta \\ &= \sqrt{n} \left(\frac{1}{n} Z' Z - \Lambda \right) - F_n \Delta \\ L_2 &\equiv -Q_n \Delta - F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i z'_i + z_i w'_i - \Delta) \right) + F_n^2 \Omega \\ &= -Q_n \Delta - \sqrt{n} F_n \left(\frac{1}{n} Z' W + \frac{1}{n} W' Z - \Delta \right) + F_n^2 \Omega \end{aligned}$$

Proof. We have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}'_i &= \frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right)' \\ &= \frac{1}{n} \sum_{i=1}^n z_i z'_i - \frac{1}{\sqrt{n}} \widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n (w_i z'_i + z_i w'_i) \right) + \frac{1}{n} \widehat{\theta}^2 \left(\frac{1}{n} \sum_{i=1}^n w_i w'_i \right) \end{aligned}$$

Because

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n z_i z'_i &= \Lambda + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z'_i - \Lambda) \right) \\ &\quad + \widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n (w_i z'_i + z_i w'_i) \right) \\ &= \left(F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right) \left(\Delta + \frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i z'_i + z_i w'_i - \Delta) \right) \\ &= F_n \Delta \\ &\quad + \frac{1}{\sqrt{n}} Q_n \Delta + \frac{1}{\sqrt{n}} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i z'_i + z_i w'_i - \Delta) \right) \\ &\quad + o_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

and

$$\begin{aligned} &\widehat{\theta}^2 \left(\frac{1}{n} \sum_{i=1}^n w_i w'_i \right) \\ &= \left(F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right) \right)^2 \left(\Omega + \frac{1}{n} \sum_{i=1}^n (w_i w'_i - \Omega) \right) \\ &= F_n^2 \Omega + o_p(1), \end{aligned}$$

we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right)' \\
&= \Lambda \\
&+ \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) - \frac{1}{\sqrt{n}} F_n \Delta \\
&- \frac{1}{n} Q_n \Delta - \frac{1}{n} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i z_i' + z_i w_i' - \Delta) \right) \\
&+ \frac{1}{n} F_n^2 \Omega \\
&+ o_p \left(\frac{1}{n} \right)
\end{aligned}$$

■

Lemma 3

$$\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i' \right)^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p \left(\frac{1}{n} \right)$$

where

$$\begin{aligned}
M_1 &\equiv -\Lambda^{-1} L_1 \Lambda^{-1} \\
&= -\Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} + F_n \Lambda^{-1} \Delta \Lambda^{-1}
\end{aligned}$$

$$\begin{aligned}
M_2 &\equiv \Lambda^{-1} L_1 \Lambda^{-1} L_1 \Lambda^{-1} - \Lambda^{-1} L_2 \Lambda^{-1} \\
&= \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} \\
&- F_n \Lambda^{-1} \Delta \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} - F_n \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} \Delta \Lambda^{-1} \\
&+ F_n^2 \Lambda^{-1} \Delta \Lambda^{-1} \Delta \Lambda^{-1} \\
&+ Q_n \Lambda^{-1} \Delta \Lambda^{-1} \\
&+ F_n \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i z_i' + z_i w_i' - \Delta) \right) \Lambda^{-1} \\
&- F_n^2 \Lambda^{-1} \Omega \Lambda^{-1}
\end{aligned}$$

Note that

$$M_1 = -\sqrt{n} \Lambda^{-1} \left(\frac{1}{n} Z' Z - \Lambda \right) \Lambda^{-1} + F_n \Lambda^{-1} \Delta \Lambda^{-1}$$

and

$$\begin{aligned}
M_2 &= n\Lambda^{-1} \left(\frac{1}{n} Z'Z - \Lambda \right) \Lambda^{-1} \left(\frac{1}{n} Z'Z - \Lambda \right) \Lambda^{-1} \\
&\quad - \sqrt{n} F_n \Lambda^{-1} \Delta \Lambda^{-1} \left(\frac{1}{n} Z'Z - \Lambda \right) \Lambda^{-1} \\
&\quad - \sqrt{n} F_n \Lambda^{-1} \left(\frac{1}{n} Z'Z - \Lambda \right) \Lambda^{-1} \Delta \Lambda^{-1} \\
&\quad + F_n^2 \Lambda^{-1} \Delta \Lambda^{-1} \Delta \Lambda^{-1} \\
&\quad + Q_n \Lambda^{-1} \Delta \Lambda^{-1} \\
&\quad + \sqrt{n} F_n \Lambda^{-1} \left(\frac{1}{n} Z'W + \frac{1}{n} W'Z - \Delta \right) \Lambda^{-1} \\
&\quad - F_n^2 \Lambda^{-1} \Omega \Lambda^{-1}
\end{aligned}$$

Proof. From Lemma 2, we obtain

$$\begin{aligned}
\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i' \right)^{-1} &= \left(\Lambda + \frac{1}{\sqrt{n}} L_1 + \frac{1}{n} L_2 + o_p \left(\frac{1}{n} \right) \right)^{-1} \\
&= \Lambda^{-1} - \frac{1}{\sqrt{n}} \Lambda^{-1} L_1 \Lambda^{-1} + \frac{1}{n} \left(\Lambda^{-1} L_1 \Lambda^{-1} L_1 \Lambda^{-1} - \Lambda^{-1} L_2 \Lambda^{-1} \right) + o_p \left(\frac{1}{n} \right)
\end{aligned}$$

■

Lemma 4

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i = C_0 + \frac{1}{\sqrt{n}} C_1 + \frac{1}{n} C_2 + o_p \left(\frac{1}{n} \right)$$

where

$$\begin{aligned}
C_0 &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i - F_n \varphi \\
&= \frac{1}{\sqrt{n}} Z' \varepsilon - F_n \varphi \\
C_1 &\equiv -F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) - Q_n \varphi \\
&= -\sqrt{n} F_n \left(\frac{1}{n} W' \varepsilon - \varphi \right) - Q_n \varphi \\
C_2 &\equiv -Q_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) - S_n \varphi \\
&= -\sqrt{n} Q_n \left(\frac{1}{n} W' \varepsilon - \varphi \right) - S_n \varphi
\end{aligned}$$

Proof. We have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) \varepsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i - \widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n w_i \varepsilon_i \right)$$

Because

$$\begin{aligned}
\widehat{\theta} \left(\frac{1}{n} \sum_{i=1}^n w_i \varepsilon_i \right) &= \left(F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p \left(\frac{1}{n} \right) \right) \left(\varphi + \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) \right) \\
&= F_n \varphi \\
&\quad + \frac{1}{\sqrt{n}} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) + \frac{1}{\sqrt{n}} Q_n \varphi \\
&\quad + \frac{1}{n} Q_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) + \frac{1}{n} S_n \varphi \\
&\quad + o_p \left(\frac{1}{n} \right)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i \right) \varepsilon_i \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i - F_n \varphi \\
&\quad - \frac{1}{\sqrt{n}} F_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) - \frac{1}{\sqrt{n}} Q_n \varphi \\
&\quad - \frac{1}{n} Q_n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (w_i \varepsilon_i - \varphi) \right) - \frac{1}{n} S_n \varphi \\
&\quad + o_p \left(\frac{1}{n} \right)
\end{aligned}$$

■

Lemma 5

$$\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i \right) = N_0 + \frac{1}{\sqrt{n}} N_1 + \frac{1}{n} N_2 + o_p \left(\frac{1}{n} \right)$$

where

$$\begin{aligned}
N_0 &\equiv \lambda' \Lambda^{-1} C_0 \\
N_1 &\equiv \lambda' \Lambda^{-1} C_1 + \lambda' M_1 C_0 + A_1' \Lambda^{-1} C_0 \\
N_2 &\equiv A_1' M_1 C_0 + \lambda' \Lambda^{-1} C_2 + \lambda' M_2 C_0 + \lambda' M_1 C_1 + A_1' \Lambda^{-1} C_1 + A_2' \Lambda^{-1} C_0
\end{aligned}$$

Proof. From Lemmas 1, 3, and 4, we obtain

$$\begin{aligned}
&\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i \right) \\
&= \left(\lambda + \frac{1}{\sqrt{n}} A_1 + \frac{1}{n} A_2 + o_p \left(\frac{1}{n} \right) \right)' \left(\Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p \left(\frac{1}{n} \right) \right) \left(C_0 + \frac{1}{\sqrt{n}} C_1 + \frac{1}{n} C_2 + o_p \left(\frac{1}{n} \right) \right) \\
&= \lambda' \Lambda C_0 \\
&\quad + \frac{1}{\sqrt{n}} (\lambda' \Lambda C_1 + \lambda' M_1 C_0 + A_1' \Lambda C_0) \\
&\quad + \frac{1}{n} (A_1' M_1 C_0 + \lambda' \Lambda C_2 + \lambda' M_2 C_0 + \lambda' M_1 C_1 + A_1' \Lambda C_1 + A_2' \Lambda C_0)
\end{aligned}$$

■

Lemma 6

$$\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i x_i\right) = D_0 + \frac{1}{\sqrt{n}} D_1 + \frac{1}{n} D_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} D_0 &\equiv \lambda' \Lambda^{-1} \lambda \\ D_1 &\equiv 2\lambda' \Lambda^{-1} A_1 + \lambda' M_1 \lambda \\ D_2 &\equiv 2\lambda' \Lambda^{-1} A_2 + 2\lambda' M_1 A_1 + \lambda' M_2 \lambda + A_1' \Lambda^{-1} A_1 \end{aligned}$$

Proof. From Lemmas 1, and 3, we obtain

$$\begin{aligned} &\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i x_i\right) \\ &= \left(\lambda + \frac{1}{\sqrt{n}} A_1 + \frac{1}{n} A_2 + o_p\left(\frac{1}{n}\right)\right)' \left(\Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p\left(\frac{1}{n}\right)\right) \left(\lambda + \frac{1}{\sqrt{n}} A_1 + \frac{1}{n} A_2 + o_p\left(\frac{1}{n}\right)\right) \\ &= \lambda' \Lambda^{-1} \lambda \\ &\quad + \frac{1}{\sqrt{n}} (\lambda' \Lambda^{-1} A_1 + \lambda' M_1 \lambda + A_1' \Lambda^{-1} \lambda) \\ &\quad + \frac{1}{n} (A_1' M_1 \lambda + \lambda' \Lambda^{-1} A_2 + \lambda' M_2 \lambda + \lambda' M_1 A_1 + A_1' \Lambda^{-1} A_1 + A_2' \Lambda^{-1} \lambda) \end{aligned}$$

■

Theorem 1

$$\sqrt{n}(b - \beta) = R_0 + \frac{1}{\sqrt{n}} R_1 + \frac{1}{n} R_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} R_0 &\equiv \frac{N_0}{D_0} \\ R_1 &\equiv -\frac{D_1 N_0}{D_0^2} + \frac{N_1}{D_0} \\ R_2 &\equiv -\frac{D_2 N_0}{D_0^2} + \frac{D_1^2 N_0}{D_0^3} + \frac{N_2}{D_0} - \frac{D_1 N_1}{D_0^2} \end{aligned}$$

Proof. From Lemmas 5 and 6, we obtain

$$\begin{aligned} \sqrt{n}(b - \beta) &= \frac{\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i \varepsilon_i\right)}{\left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i x_i\right)' \left(\frac{1}{n} \sum_{i=1}^n \widehat{z}_i \widehat{z}_i'\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{z}_i x_i\right)} \\ &= \frac{N_0 + \frac{1}{\sqrt{n}} N_1 + \frac{1}{n} N_2 + o_p\left(\frac{1}{n}\right)}{D_0 + \frac{1}{\sqrt{n}} D_1 + \frac{1}{n} D_2 + o_p\left(\frac{1}{n}\right)} \\ &= \frac{N_0}{D_0} + \frac{1}{\sqrt{n}} \left(-\frac{D_1 N_0}{D_0^2} + \frac{N_1}{D_0}\right) + \frac{1}{n} \left(-\frac{D_2 N_0}{D_0^2} + \frac{D_1^2 N_0}{D_0^3} + \frac{N_2}{D_0} - \frac{D_1 N_1}{D_0^2}\right) + o_p\left(\frac{1}{n}\right) \end{aligned}$$

■

3 Long Difference

$$y_{iT} - y_{i1} = \beta(y_{iT-1} - y_{i0}) + (\varepsilon_{iT} - \varepsilon_{i1}) \quad (6)$$

It is easy to see that the initial observation y_{i0} would serve as a valid instrument. Using similar intuition as in Hausman and Taylor (1983) or Ahn and Schmidt (1995), we can see that $y_{iT-1} - \beta y_{iT-2}, \dots, y_{i2} - \beta y_{i1}$ would be valid instruments as well. If we use a preliminary estimator $\hat{\beta}$, then we would use

$$\begin{aligned} \hat{z}_i &\equiv \begin{bmatrix} y_{i0} \\ y_{i2} - \hat{\beta}y_{i1} \\ \vdots \\ y_{iT-1} - \hat{\beta}y_{iT-2} \end{bmatrix} = \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} - (\hat{\beta} - \beta)y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} - (\hat{\beta} - \beta)\beta y_{iT-2} \end{bmatrix} \\ &= \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} \end{bmatrix} - (\hat{\beta} - \beta) \begin{bmatrix} 0 \\ y_{i1} \\ \vdots \\ y_{iT-2} \end{bmatrix} = z_i - \frac{1}{\sqrt{n}}\hat{\theta}w_i \end{aligned}$$

as instrument, where

$$z_i \equiv \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} \end{bmatrix}, \quad w_i \equiv \begin{bmatrix} 0 \\ y_{i1} \\ \vdots \\ y_{iT-2} \end{bmatrix}, \quad \hat{\theta} = \sqrt{n}(\hat{\beta} - \beta)$$

In our context,

$$\begin{aligned} y_i &= y_{iT} - y_{i1} \\ x_i &= y_{iT-1} - y_{i0} \\ \varepsilon_i &= \varepsilon_{iT} - \varepsilon_{i1} \end{aligned}$$

$$\begin{aligned} \lambda &= E[z_i x_i] \\ \Lambda &= E[z_i z_i'] \\ \phi &= E[w_i x_i] \\ \Delta &= E[w_i z_i' + z_i w_i'] \\ \Omega &= E[w_i w_i'] \\ \varphi &= E[w_i \varepsilon_i] \end{aligned}$$

Now, suppose that

$$\begin{aligned} y_{it} &= \alpha_i + \beta y_{it-1} + \varepsilon_{it} \quad t = 1, \dots, T \\ \varepsilon_{it} &\sim \mathcal{N}(0, \sigma_\varepsilon^2), \quad \alpha_i \sim \mathcal{N}(0, \sigma_\alpha^2) \\ y_{i0} &\sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta}, \frac{1}{1-\beta^2}\right) = \frac{\alpha_i}{1-\beta} + \frac{1}{\sqrt{1-\beta^2}}\mathcal{N}(0, 1) \end{aligned}$$

Further suppose that

$$\sigma_\varepsilon^2 = \sigma_\alpha^2 = 1, \quad T = 5$$

We then have the following results:

$$\lambda = \begin{bmatrix} -\beta^2 - 1 \\ \beta^2 \\ \beta \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \frac{2}{(1-\beta^2)(1-\beta)} & \frac{1}{1-\beta} & \frac{1}{1-\beta} & \frac{1}{1-\beta} \\ \frac{1}{1-\beta} & 2 & 1 & 1 \\ \frac{1}{1-\beta} & 1 & 2 & 1 \\ \frac{1}{1-\beta} & 1 & 1 & 2 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 0 \\ -\beta \\ 0 \\ \beta \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} & \frac{-2-\beta+\beta^2}{-1+\beta} \\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{-3+\beta}{-1+\beta} & -\frac{2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} \\ -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{-2-\beta+\beta^2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} & -\frac{2}{-1+\beta} \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ 0 & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} \\ 0 & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} \end{bmatrix}$$

$$\varphi = \begin{bmatrix} 0 \\ -1 \\ -\beta \\ -\beta^2 \end{bmatrix}$$

Higher Order Expansion of Arellano-Bover Estimator for Dynamic Panel Model with Fixed Effects: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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1 Arellano-Bover Estimator for Dynamic Panel Model with Fixed Effects

Consider the usual dynamic panel model with fixed effects:

$$y_{it} = \alpha_i + \beta y_{i,t-1} + \varepsilon_{it}, \quad i = 1, \dots, n; \quad t = 1, \dots, T \quad (1)$$

We consider a version of the GMM estimator developed by Arellano and Bover (1995), which simplifies the characterization of the “weight matrix” in GMM estimation. We define the innovation $u_{it} \equiv \alpha_i + \varepsilon_{it}$. Arellano and Bover (1995) eliminate the fixed effect α_i in (1) by applying Helmert’s transformation

$$u_{it}^* \equiv \sqrt{\frac{T-t}{T-t+1}} \left[u_{it} - \frac{1}{T-t} (u_{i,t+1} + \dots + u_{iT}) \right], \quad t = 1, \dots, T-1$$

instead of first differencing. The transformation produces

$$y_{it}^* = \beta x_{it}^* + \varepsilon_{it}^*, \quad t = 1, \dots, T-1$$

where $x_{i,t}^* \equiv y_{i,t-1}^*$. Let $z_{it}^* \equiv (y_{i0}, \dots, y_{i,t-1})'$. Our moment restriction is summarized by

$$E [z_{i,t}^* \varepsilon_{i,t}^*] = 0 \quad t = 1, \dots, T-1$$

It can be shown that, with the homoscedasticity assumption on ε_{it} , the optimal “weight matrix” is proportional to a block-diagonal matrix, with typical diagonal block equal to $E [z_{i,t}^* z_{i,t}^{*'}]$. Therefore, the optimal GMM estimator by Arellano-Bover is equal to

$$\hat{b}_{AB} \equiv \frac{\sum_{t=1}^{T-1} x_t^{*'} P_t^* y_t^*}{\sum_{t=1}^{T-1} x_t^{*'} P_t^* x_t^*}$$

so that

$$\sqrt{n} (\hat{b}_{AB} - \beta) \equiv \frac{\sum_{t=1}^{T-1} \left(\frac{1}{\sqrt{n}} x_t^{*'} P_t^* \varepsilon_t^* \right)}{\sum_{t=1}^{T-1} \left(\frac{1}{n} x_t^{*'} P_t^* x_t^* \right)} \quad (2)$$

where $x_t^* \equiv (x_{1,t}^*, \dots, x_{n,t}^*)'$, $y_t^* \equiv (y_{1,t}^*, \dots, y_{n,t}^*)'$, $Z_t^* \equiv (z_{1,t}^*, \dots, z_{n,t}^*)'$, and $P_t^* \equiv Z_t^* (Z_t^{*'} Z_t^*)^{-1} Z_t^{*}$. In Theorem 1 below, we provide an approximation of the form

$$\sqrt{n} (\hat{b}_{AB} - \beta) = F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p \left(\frac{1}{n} \right).$$

2 Higher Order Expansion of $\sqrt{n} (\hat{b}_{AB} - \beta)$

We now adopt obvious notations, and make a second order analysis of the right side of (2). Recall that

$$\sqrt{n} (\hat{b}_{AB} - \beta) \equiv \frac{\sum_{t=1}^{T-1} \left(\frac{1}{\sqrt{n}} x_t^{*'} P_t^* \varepsilon_t^* \right)}{\sum_{t=1}^{T-1} \left(\frac{1}{n} x_t^{*'} P_t^* x_t^* \right)}$$

where $z_{it}^* \equiv (y_{i0}, \dots, y_{it-1})'$, $x_t^* \equiv (x_{1t}^*, \dots, x_{nt}^*)'$, $y_t^* \equiv (y_{1t}^*, \dots, y_{nt}^*)'$, $Z_t^* \equiv (z_{1t}^*, \dots, z_{nt}^*)'$, and $P_t^* \equiv Z_t^* (Z_t^{*'} Z_t^*)^{-1} Z_t^{*}$. The higher order expansion for $\sqrt{n} (\hat{b}_{AB} - \beta)$ is contained in Theorem 1 below.

Definition 1

$$\begin{aligned} \lambda_t^* &\equiv E [z_{i,t}^* x_{i,t}^*] \\ \Lambda_t^* &\equiv E [z_{i,t}^* z_{i,t}^{*'}] \end{aligned}$$

Definition 2

$$\begin{aligned} A_{1,t}^* &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (z_{i,t}^* x_{i,t}^* - \lambda_t^*) \\ &= \sqrt{n} \left(\frac{1}{n} Z_t^{*'} x_t^* - \lambda_t^* \right) \quad (= O_p(1)) \end{aligned}$$

Definition 3

$$\begin{aligned} M_{1,t}^* &\equiv -\Lambda_t^{*-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_{i,t}^* z_{i,t}^{*'} - \Lambda_t^*) \right) \Lambda_t^{*-1} \\ &= -\sqrt{n} \Lambda_t^{*-1} \left(\frac{1}{n} Z_t^{*'} Z_t^* - \Lambda_t^* \right) \Lambda_t^{*-1} \quad (= O_p(1)) \end{aligned}$$

$$\begin{aligned} M_{2,t}^* &\equiv \Lambda_t^{*-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_{i,t}^* z_{i,t}^{*'} - \Lambda_t^*) \right) \Lambda_t^{*-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_{i,t}^* z_{i,t}^{*'} - \Lambda_t^*) \right) \Lambda_t^{*-1} \\ &= n \Lambda_t^{*-1} \left(\frac{1}{n} Z_t^{*'} Z_t^* - \Lambda_t^* \right) \Lambda_t^{*-1} \left(\frac{1}{n} Z_t^{*'} Z_t^* - \Lambda_t^* \right) \Lambda_t^{*-1} \quad (= O_p(1)) \end{aligned}$$

Definition 4

$$\begin{aligned} C_{0,t}^* &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{i,t}^* \varepsilon_{i,t}^* \\ &= \frac{1}{\sqrt{n}} Z_t^{*'} \varepsilon_t^* \quad (= O_p(1)) \end{aligned}$$

Lemma 1

$$\frac{1}{\sqrt{n}}x_t^{*'}P_t^*\varepsilon_t^* = N_{0,t}^* + \frac{1}{\sqrt{n}}N_{1,t}^* + \frac{1}{n}N_{2,t}^* + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} N_{0,t}^* &\equiv \lambda_t^{*'}\Lambda_t^{*-1}C_{0,t}^* && (= O_p(1)) \\ N_{1,t}^* &\equiv \lambda_t^{*'}M_{1,t}^*C_{0,t}^* + A_{1,t}^{*'}\Lambda_t^{*-1}C_{0,t}^* && (= O_p(1)) \\ N_{2,t}^* &\equiv A_{1,t}^{*'}M_{1,t}^*C_{0,t}^* + \lambda_t^{*'}M_{2,t}^*C_{0,t}^* && (= O_p(1)) \end{aligned}$$

Proof. Note that

$$\frac{1}{\sqrt{n}}x_t^{*'}P_t^*\varepsilon_t^* = \left(\frac{1}{n}\sum_{i=1}^nz_{it}^*x_{it}^*\right)' \left(\frac{1}{n}\sum_{i=1}^nz_{it}^*z_{it}^{*'}\right)^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^nz_{it}^*\varepsilon_{it}^*\right)$$

The conclusion then follows by an argument as in Lemma 7. ■

Lemma 2

$$\frac{1}{n}x_t^{*'}P_t^*x_t^* = D_{0,t}^* + \frac{1}{\sqrt{n}}D_{1,t}^* + \frac{1}{n}D_{2,t}^* + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} D_{0,t}^* &\equiv \lambda_t^{*'}\Lambda_t^{*-1}\lambda_t^* && (= O_p(1)) \\ D_{1,t}^* &\equiv 2\lambda_t^{*'}\Lambda_t^{*-1}A_{1,t}^* + \lambda_t^{*'}M_{1,t}^*\lambda_t^* && (= O_p(1)) \\ D_{2,t}^* &\equiv \lambda_t^{*'}M_{2,t}^*\lambda_t^* + 2\lambda_t^{*'}M_{1,t}^*A_{1,t}^* + A_{1,t}^{*'}\Lambda_t^{*-1}A_{1,t}^* && (= O_p(1)) \end{aligned}$$

Proof. Note that

$$\frac{1}{n}x_t^{*'}P_t^*x_t^* = \left(\frac{1}{n}\sum_{i=1}^nz_{it}^*x_{it}^*\right)' \left(\frac{1}{n}\sum_{i=1}^nz_{it}^*z_{it}^{*'}\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^nz_{it}^*x_{it}^*\right)$$

The conclusion then follows by an argument as in Lemma 8. ■

Theorem 1

$$\sqrt{n}\left(\widehat{b}_{AB} - \beta\right) = F_n + \frac{1}{\sqrt{n}}Q_n + \frac{1}{n}S_n + o_p\left(\frac{1}{n}\right),$$

where

$$\begin{aligned} F_n &\equiv \frac{N_0^*}{D_0^*} \\ Q_n &\equiv \frac{N_1^*}{D_0^*} - \frac{N_0^*D_1^*}{D_0^{*2}} \\ S_n &\equiv \frac{N_2^*}{D_0^*} - \frac{N_0^*D_2^*}{D_0^{*2}} - \frac{D_1^*N_1^*}{D_0^{*2}} + \frac{D_1^{*2}N_0^*}{D_0^{*3}} \end{aligned}$$

and

$$\begin{aligned} N_0^* &\equiv \sum_{t=1}^{T-1} N_{0,t}^* = O_p(1) & N_1^* &\equiv \sum_{t=1}^{T-1} N_{1,t}^* = O_p(1) & N_2^* &\equiv \sum_{t=1}^{T-1} N_{2,t}^* = O_p(1) \\ D_0^* &\equiv \sum_{t=1}^{T-1} D_{0,t}^* = O_p(1) & D_1^* &\equiv \sum_{t=1}^{T-1} D_{1,t}^* = O_p(1) & D_2^* &\equiv \sum_{t=1}^{T-1} D_{2,t}^* = O_p(1) \end{aligned}$$

Proof. From equation (2), and Lemmas 1 and 2, we obtain

$$\sqrt{n} \left(\widehat{b}_{AB} - \beta \right) = \frac{N_0^* + \frac{1}{\sqrt{n}}N_1^* + \frac{1}{n}N_2^* + o_p\left(\frac{1}{n}\right)}{D_0^* + \frac{1}{\sqrt{n}}D_1^* + \frac{1}{n}D_2^* + o_p\left(\frac{1}{n}\right)}$$

■

3 Some simplification for stochastic approximation

Suppose that

$$\begin{aligned} y_{it} &= \alpha_i + \beta y_{it-1} + \varepsilon_{it} & t = 1, \dots, T \\ \varepsilon_{it} &\sim \mathcal{N}(0, \sigma_\varepsilon^2), & \alpha_i \sim \mathcal{N}(0, \sigma_\alpha^2) \\ y_{i0} &\sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta}, \frac{1}{1-\beta^2}\right) = \frac{\alpha_i}{1-\beta} + \frac{1}{\sqrt{1-\beta^2}}\mathcal{N}(0, 1) \end{aligned}$$

Further suppose that

$$\sigma_\varepsilon^2 = \sigma_\alpha^2 = 1, \quad T = 5$$

We then have the following results:

$$\begin{aligned} \lambda_1^* &= \left[\frac{1}{10} (\beta^3 + 2\beta^2 + 3\beta + 4) \frac{\sqrt{5}}{\beta+1} \right] \\ \lambda_2^* &= \left[\begin{array}{c} \frac{1}{6} (\beta^2 + 2\beta + 3) \sqrt{3} \frac{\beta}{\beta+1} \\ \frac{1}{6} (\beta^2 + 2\beta + 3) \frac{\sqrt{3}}{\beta+1} \end{array} \right] \\ \lambda_3^* &= \left[\begin{array}{c} \frac{1}{6} (\beta + 2) \sqrt{6} \frac{\beta^2}{\beta+1} \\ \frac{1}{6} (\beta + 2) \sqrt{6} \frac{\beta}{\beta+1} \\ \frac{1}{6} (\beta + 2) \frac{\sqrt{6}}{\beta+1} \end{array} \right] \\ \lambda_4^* &= \left[\begin{array}{c} \frac{1}{2} \sqrt{2} \frac{\beta^3}{\beta+1} \\ \frac{1}{2} \sqrt{2} \frac{\beta^2}{\beta+1} \\ \frac{1}{2} \sqrt{2} \frac{\beta}{\beta+1} \\ \frac{1}{2} \frac{\sqrt{2}}{\beta+1} \end{array} \right] \\ \Lambda_1^* &= \left[\frac{2}{(-1+\beta^2)(-1+\beta)} \right] \\ \Lambda_2^* &= \left[\begin{array}{cc} \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} \end{array} \right] \\ \Lambda_3^* &= \left[\begin{array}{ccc} \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} \end{array} \right] \\ \Lambda_4^* &= \left[\begin{array}{cccc} \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} \end{array} \right] \end{aligned}$$

A Digression: Some Useful Results Related to Higher Order Properties of 2SLS

We first make a digression to the discussion of single equation model,

$$\begin{aligned} y_i &= \beta x_i + \varepsilon_i, \\ x_i &= z_i' \pi + u_i \end{aligned}$$

The only reason we consider such analysis is because all the analysis we found in the literature are conditional analysis given instruments: They all assume that the instruments are nonstochastic. Our purpose is to make a “marginal” higher order analysis, which is more natural in the dynamic panel model context.

Definition 5

$$\begin{aligned} \lambda &\equiv E[z_i x_i] \\ \Lambda &\equiv E[z_i z_i'] \end{aligned}$$

Lemma 3

$$\frac{1}{n} \sum_{i=1}^n z_i x_i = \lambda + \frac{1}{\sqrt{n}} A_1$$

where

$$A_1 \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i x_i - \lambda) = O_p(1)$$

Lemma 4

$$\frac{1}{n} \sum_{i=1}^n z_i z_i' = \Lambda + \frac{1}{\sqrt{n}} L_1$$

where

$$L_1 \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) = O_p(1)$$

Lemma 5

$$\left(\frac{1}{n} \sum_{i=1}^n z_i z_i' \right)^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} M_1 &\equiv -\Lambda^{-1} L_1 \Lambda^{-1} \\ &= -\Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} = O_p(1) \end{aligned}$$

$$\begin{aligned} M_2 &\equiv \Lambda^{-1} L_1 \Lambda^{-1} L_1 \Lambda^{-1} \\ &= \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (z_i z_i' - \Lambda) \right) \Lambda^{-1} = O_p(1) \end{aligned}$$

Lemma 6

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i = C_0$$

where

$$C_0 \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i = O_p(1)$$

Lemma 7

$$\left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n z_i z_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i \right) = \mathbb{N}_0 + \frac{1}{\sqrt{n}} \mathbb{N}_1 + \frac{1}{n} \mathbb{N}_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} \mathbb{N}_0 &\equiv \lambda' \Lambda^{-1} C_0 = O_p(1) \\ \mathbb{N}_1 &\equiv \lambda' M_1 C_0 + A_1' \Lambda^{-1} C_0 = O_p(1) \\ \mathbb{N}_2 &\equiv A_1' M_1 C_0 + \lambda' M_2 C_0 = O_p(1) \end{aligned}$$

Proof. From Lemmas 3, 5, and 6, we obtain

$$\begin{aligned} &\left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n z_i z_i' \right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n z_i \varepsilon_i \right) \\ &= \left(\lambda + \frac{1}{\sqrt{n}} A_1 \right)' \left(\Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p\left(\frac{1}{n}\right) \right) C_0 \\ &= \lambda' \Lambda^{-1} C_0 + \frac{1}{\sqrt{n}} (\lambda' M_1 C_0 + A_1' \Lambda^{-1} C_0) + \frac{1}{n} (A_1' M_1 C_0 + \lambda' M_2 C_0) \end{aligned}$$

■

Lemma 8

$$\left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n z_i z_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right) = \mathbb{D}_0 + \frac{1}{\sqrt{n}} \mathbb{D}_1 + \frac{1}{n} \mathbb{D}_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} \mathbb{D}_0 &\equiv \lambda' \Lambda^{-1} \lambda = O_p(1) \\ \mathbb{D}_1 &\equiv 2\lambda' \Lambda^{-1} A_1 + \lambda' M_1 \lambda = O_p(1) \\ \mathbb{D}_2 &\equiv \lambda' M_2 \lambda + 2\lambda' M_1 A_1 + A_1' \Lambda^{-1} A_1 = O_p(1) \end{aligned}$$

Proof. From Lemmas 3, and 5, we obtain

$$\begin{aligned} &\left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right)' \left(\frac{1}{n} \sum_{i=1}^n z_i z_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n z_i x_i \right) \\ &= \left(\lambda + \frac{1}{\sqrt{n}} A_1 \right)' \left(\Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p\left(\frac{1}{n}\right) \right) \left(\lambda + \frac{1}{\sqrt{n}} A_1 \right) \\ &= \lambda' \Lambda^{-1} \lambda \\ &\quad + \frac{1}{\sqrt{n}} (2\lambda' \Lambda^{-1} A_1 + \lambda' M_1 \lambda) \\ &\quad + \frac{1}{n} (\lambda' M_2 \lambda + 2\lambda' M_1 A_1 + A_1' \Lambda^{-1} A_1) \end{aligned}$$

■

Alternative Parametrization of the Stationarity Property in Dynamic Panel Models with Weak Identification: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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June, 2002

We consider the dynamic panel model

$$\begin{aligned}
 y_{it} &= (1 - \beta_n) \alpha_i + \beta_n y_{it-1} + \varepsilon_{it} \\
 \varepsilon_{it} &\sim N(0, \sigma_\varepsilon^2) \\
 \alpha_i &\sim N(0, \sigma_\alpha^2) \\
 y_{i0} &\sim N\left(\alpha_i, \frac{\sigma_y^2}{1 - \beta_n^2}\right)
 \end{aligned} \tag{1}$$

where we assume that $t = 1, \dots, T$ and $i = 1, \dots, n$ with T being fixed and $n \rightarrow \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/n)$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{n} + o(n^{-1})$ for any constant k . We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = y_{i0} - \alpha_i$ such that

$$\begin{aligned}
 \Delta y_{it} &= -(\beta_n - 1) \beta_n^{t-1} \alpha_i + (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + (\beta_n - 1) \beta_n^{t-1} y_{i0} + \varepsilon_{it} \\
 &= (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + (\beta_n - 1) \beta_n^{t-1} \eta_{i0} + \varepsilon_{it}.
 \end{aligned}$$

We define $g_{i1}(\beta) = [u_{it} \Delta u_{is}(\beta), u_{iT} \Delta u_{ij}(\beta), \bar{u}_i \Delta u_{ik}(\beta)]'$ and $g_{i2}(\beta) = [\Delta u_i(\beta)' y_{i0}]$ such that

$$g_n(\beta) = n^{-3/2} \sum_{i=1}^n [g_{i1}(\beta)', g_{i2}(\beta)']'$$

with the optimal weight matrix $\Omega = \lim_n E g_n(\beta_n) g_n(\beta_n)'$. The infeasible 2SLS of a possibly transformed set of moment conditions then solves

$$\beta_{2SLS} = \arg \min_{\beta} g_n(\beta)' C (C' \Omega C)^+ C' g_n(\beta)$$

where C is $n \times r$ matrix for $1 \leq r \leq G$ such that C is of full column rank and $\text{rank}(C(C' \Omega C)^+ C') \geq 1$. We use Ω^+ to denote the Moore-Penrose inverse (see Magnus and Neudecker p. 32, 1988). We thus allow

the use of a singular weight matrix. Choosing r less than G allows to exclude certain moment conditions. We define the vectors

$$\begin{aligned} f_{i,1} &= [u_{it}\Delta y_{is-1}, u_{iT}\Delta y_{ij-1}, \bar{u}_i\Delta y_{ik-1}]' \\ f_{i,2} &= [\Delta y_{i1}y_{i0}, \dots, \Delta y_{iT-1}y_{i0}]' \end{aligned}$$

for $s = 2, \dots, T; t = 1, \dots, s-2; j = 2, \dots, T-1; k = 2, \dots, T$ and let $f_n = n^{-3/2} \sum_{i=1}^n [f'_{i,1}, f'_{i,2}]'$ such that the infeasible 2SLS estimator can be written as

$$\beta_{2SLS} - \beta_{n0} = (f'_n C (C' \Omega C)^+ C' f_n)^{-1} f'_n C (C' \Omega C)^+ C' g_n(\beta_{n0}). \quad (2)$$

Lemma 1 Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \rightarrow \infty$

$$n^{-1} \sum_{i=1}^n f_{i,1} \xrightarrow{P} \mu_1, n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0) \xrightarrow{P} 0$$

with $\mu'_1 = [0, \dots, 0, \sigma_\varepsilon^2 T^{-1} \mathbf{1}'_{T-1}]$ and

$$n^{-1} \sum_{i=1}^n [f'_{i,2}, g'_{i,2}(\beta_0)]' \xrightarrow{d} [\xi'_x, \xi'_y]'$$

where $[\xi'_x, \xi'_y] \sim N(\mu_2, \Sigma)$ with $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\mu'_2 = [\mathbf{1}'_{T-1} \sigma_y^2 / 2, 0, \dots, 0]$ and $\Sigma_{11} = \delta I, \Sigma_{12} = \delta M_1, \Sigma_{22} = \delta M_2$ where $\delta = \frac{\sigma_y^2 \sigma_\varepsilon^2}{c}$,

$$M_1 = \begin{bmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & \ddots & 1 \\ & & & & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and $\Sigma_{12} = \Sigma'_{21}$.

An immediate consequence of this result is that an estimator based solely on the condition

$$E \bar{u}_i \Delta u_{ik}(\beta_0) = 0 \quad (3)$$

is consistent. If we consider estimators that do use all the moment conditions except (3) then we are in a situation where again the moment conditions in $g_{i1}(\beta)$ become asymptotically redundant. Now, however the limiting distribution of the non-redundant conditions has a noncentrality parameter.

More explicitly, if β_{2SLS} is the GMM estimator based on $g_{i2}(\beta)$ then it has the following nonstandard limiting distribution under the weak identification asymptotics adopted here

$$\beta_{2SLS} - \beta_0 \xrightarrow{d} \frac{(\mu + \xi_x)' C (C' \Omega C)^{-1} C' \xi_y}{(\mu + \xi_x)' C (C' \Omega C)^{-1} C' (\mu + \xi_x)} = X$$

where $\mu = -\mathbf{1}_{T-1} \sigma_y^2 / 2$. It turns out that this limiting distribution is the same as the one obtained when $\beta = \exp(-c/\sqrt{n})$ as is shown in a separate Appendix. There we carry out some numerical calculations which indicate that this limiting distribution leads to small biases and does not capture the type of issues we are most concerned with.

Proof of Lemma (1). Consider $Ef_{i,1}$ with

$$\begin{aligned} Eu_{it}\Delta y_{is-1} &= Eu_{it} \left[(\beta_n - 1) \beta_n^{s-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right] \\ &= (\beta_n - 1) \beta_n^{t-s-1} \sigma_\varepsilon^2 = O(n^{-1}) \end{aligned}$$

and $\text{Var}(u_{it}\Delta y_{is-1}) = Eu_{it}^2 (\Delta y_{is-1})^2 - (\beta_n - 1)^2 \beta_n^{2(s-t)} \sigma_\varepsilon^4$ where

$$\begin{aligned} Eu_{it}^2 (\Delta y_{is-1})^2 &= \beta_n^{2(s-2)} (\sigma_\varepsilon^2 + (1 - \beta_n)^2 \sigma_\alpha^2) \frac{\sigma_y^2 (\beta_n - 1)^2}{1 - \beta_n^2} + (\sigma_\varepsilon^2 + (1 - \beta_n)^2 \sigma_\alpha^2) \sigma_\varepsilon^2 \\ &\quad + (\beta_n - 1)^2 (\sigma_\varepsilon^2 + (1 - \beta_n)^2 \sigma_\alpha^2) \sigma_\varepsilon^2 \sum_{r=1}^{s-2} \beta_n^{2(r-1)} + 2(\beta_n - 1)^2 \sigma_\varepsilon^4 \beta_n^{2(s+1-t)} \end{aligned}$$

such that $\text{Var}(u_{it}\Delta y_{is-1}) = \sigma_\varepsilon^4 + O(n^{-1})$. By independence of $u_{it}\Delta y_{is-1}$ across i it follows by the Markov inequality that $P\left(|n^{-1} \sum_{i=1}^n u_{it}\Delta y_{is-1} - Eu_{it}\Delta y_{is-1}| > \epsilon\right) \leq \frac{n^{-1}(\sigma_\varepsilon^2 + \sigma_\alpha^2)\sigma_\varepsilon^2 + o(n^{-1})}{\epsilon}$ for any $\epsilon > 0$ which shows that $n^{-1} \sum_{i=1}^n (u_{it}\Delta y_{is-1} - Eu_{it}\Delta y_{is-1}) = o_p(1)$. Since $n^{-1} \sum_{i=1}^n Eu_{it}\Delta y_{is-1} = O(n^{-1/2})$ it follows that $n^{-1} \sum_{i=1}^n u_{it}\Delta y_{is-1} = o_p(1)$. For $\bar{u}_i\Delta y_{is-1}$ note that

$$\frac{1}{T} E \left[\sum_{t=1}^T u_{it} \left((\beta_n - 1) \beta_n^{s-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right] = \frac{\sigma_\varepsilon^2}{T} + \frac{\sigma_\varepsilon^2 (\beta_n - 1)}{T} \sum_{t=1}^{s-1} \beta_n^{r-1}$$

and

$$\begin{aligned} &E \left[T^{-1} \sum_{t=1}^T u_{it} \left((\beta_n - 1) \beta_n^{s-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right]^2 \\ &= (\beta_n - 1)^2 \beta_n^{2(s-2)} T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \eta_{i0}^2 + T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \varepsilon_{is-1}^2 \\ &\quad + 2(\beta_n - 1) T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \varepsilon_{is-1} \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \\ &\quad + \frac{(\beta_n - 1)^2}{T^2} E \sum_{t,s=1}^T u_{it} u_{is} \sum_{q,r=1}^{k-2} \beta_n^{r-1} \beta_n^{q-1} \varepsilon_{ik-1-q} \varepsilon_{ik-1-r} \\ &= \beta_n^{2(s-2)} T^{-2} (T^2 (1 - \beta_n)^2 \sigma_\alpha^2 + T \sigma_\varepsilon^2) \frac{\sigma_y^2 (\beta_n - 1)^2}{1 - \beta_n^2} + \frac{(1 - \beta_n)^2 \sigma_\alpha^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^4}{T} + \frac{2\sigma_\varepsilon^4}{T^2} \\ &\quad + (\beta_n - 1) \frac{\sigma_\varepsilon^4}{T^2} \sum_{r=1}^{s-2} \beta_n^{r-2} + \frac{(\beta_n - 1)^2}{T^2} (T^2 (1 - \beta_n)^2 \sigma_\alpha^2 \sigma_\varepsilon^2 + (T - 1) \sigma_\varepsilon^4) \sum_{r=1}^{s-2} \beta_n^{2r-2} \\ &\quad + (\beta_n - 1)^2 \frac{\sigma_\varepsilon^4}{T^2} \left(\sum_{t=1}^{k-1} \beta_n^{k-1-t} \right)^2 + (\beta_n - 1)^2 \frac{2\sigma_\varepsilon^4}{T^2} \sum_{r=1}^{s-2} \beta_n^{2r-2} = \frac{(T+2)\sigma_\varepsilon^4}{T^2} + O(n^{-1}). \end{aligned}$$

such that by a law of large numbers $n^{-1} \sum_{i=1}^n \bar{u}_i \Delta y_{ik-1} \xrightarrow{p} \sigma_\varepsilon^2/T$ using the same arguments as before.

Next consider

$$\begin{aligned} E\Delta y_{it} y_{i0} &= E y_{i0} \left((\beta_n - 1) \beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \\ &= \beta_n^{t-2} \sigma_y^2 (\beta_n - 1) / (1 - \beta_n^2) = -\frac{\sigma_y^2}{2} + O(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned}
E(\Delta y_{it} y_{i0})^2 &= E y_{i0}^2 \left((\beta_n - 1) \beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right)^2 \\
&= \beta_n^{2(t-2)} \frac{(\beta_n - 1)^2 \sigma_y^2}{(1 - \beta_n^2)} \sigma_\alpha^2 + 3 \beta_n^{2(t-2)} \frac{\sigma_y^4 (\beta_n - 1)^2}{(1 - \beta_n^2)^2} \\
&\quad + \left[\sigma_\varepsilon^2 + (\beta_n - 1)^2 \sigma_\varepsilon^2 \sum_{r=1}^{s-2} \beta_n^{2(r-1)} \right] (\sigma_\alpha^2 + \sigma_y^2 (1 - \beta_n^2)^{-1}) \\
&= \frac{\sigma_\varepsilon^2 \sigma_y^2}{2c} n + o(n).
\end{aligned}$$

such that $\text{Var}(n^{-1} \sum_{i=1}^n \Delta y_{it} y_{i0}) = O(1)$.

For $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0)$ consider $E g_{i,1}(\beta_0) = 0$ which is implied by the orthogonality conditions and $E(u_{it} \Delta u_{is})^2 = 2((1 - \beta_n)^2 \sigma_\alpha^2 + \sigma_\varepsilon^2) \sigma_\varepsilon^2 = 2\sigma_\varepsilon^2 + o(n^{-1})$ by independence. For $E(\bar{u}_i \Delta u_{ik}) = T^{-1} \sum_{t=1}^T E u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) = 0$ while

$$E(\bar{u}_i \Delta u_{ik})^2 = T^{-2} E \left(\sum_{t=1}^T u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) \right)^2 = 2(1 - \beta_n)^2 \sigma_\alpha^2 \sigma_\varepsilon^2 + T^{-1} \sigma_\varepsilon^4 + 2T^{-2} \sigma_\varepsilon^2$$

such that by the independence of $g_{i,1}(\beta_0)$ across i and a standard WLLN it follows that $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0) \xrightarrow{p} 0$. For $n^{-1} \sum_{i=1}^n g_{i,2}(\beta_0)$ we have from the moment conditions that $E g_{i,2}(\beta_0) = 0$ and $\text{Var}(\Delta u_i(\beta_0) y_{i0}) = 2\sigma_\varepsilon^2 (\sigma_\alpha^2 + \sigma_y^2 / (1 - \beta_n^2)) = \frac{\sigma_\varepsilon^2 \sigma_y^2}{c} n + O(1)$.

The joint limiting distribution of $n^{-1} \sum_{i=1}^n [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)']'$ can now be obtained from a triangular array CLT. By previous arguments

$$E [f'_{i,2}, g_{i,2}(\beta_0)'] = \begin{bmatrix} \mu' & 0 & \cdots & 0 \end{bmatrix}$$

with $\mu = \mathbf{1}_{T-1} \sigma_y^2 / 2 + O(n^{-1})$ where $\mathbf{1}_{T-1}$ is the $T - 1$ dimensional vector with elements 1. Then $E [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)']' [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)'] = \Sigma_n$ where

$$\Sigma_n = \begin{bmatrix} \Sigma_{11,n} & \Sigma_{12,n} \\ \Sigma_{21,n} & \Sigma_{22,n} \end{bmatrix}$$

By previous calculations we have found the diagonal elements of $\Sigma_{11,n}$ and $\Sigma_{22,n}$ to be $\frac{\sigma_\varepsilon^2 \sigma_y^2}{2c} n$ and $\frac{\sigma_\varepsilon^2 \sigma_y^2}{c} n$. The off-diagonal elements are found to be

$$\begin{aligned}
E \Delta y_{it} \Delta y_{is} y_{i0}^2 &= E \left[y_{i0}^2 \left((\beta_n - 1) \beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right. \\
&\quad \left. \times \left((\beta_n - 1) \beta_n^{t-2} \eta_{i0} + \varepsilon_{it-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{it-1-r} \right) \right] \\
&= \beta_n^{t-2} \beta_n^{s-2} \frac{(\beta_n - 1)^2}{(1 - \beta_n^2)} (\sigma_\alpha^2 \sigma_y^2 + 3\sigma_y^4 (1 - \beta_n^2)^{-1}) + O(n^{-1}) = -\frac{3\sigma_y^4}{2} + O(1)
\end{aligned}$$

which is of lower order of magnitude while $n^{-1} (E \Delta y_{it} y_{i0})^2 = O(1)$. Thus $n^{-1} \Sigma_{11,n} \rightarrow \text{diag}(\frac{\sigma_\varepsilon^2 \sigma_y^2}{2c}, \dots, \frac{\sigma_\varepsilon^2 \sigma_y^2}{2c})$. The off-diagonal elements of $\Sigma_{22,n}$ are obtained from

$$E \Delta u_{it} \Delta u_{is} y_{i0}^2 = \begin{cases} \frac{-\sigma_\varepsilon^2 \sigma_y^2}{2c} n + O(1) & t = s + 1 \text{ or } t = s - 1 \\ 0 & \text{otherwise} \end{cases}$$

zero and for $\Sigma_{12,n}$ we consider

$$E\Delta y_{it}\Delta u_{is}y_{i0}^2 = \begin{cases} \frac{\sigma_\varepsilon^2\sigma_y^2}{c}n + O(1) & \text{if } t = s \\ -\frac{\sigma_\varepsilon^2\sigma_y^2}{c}n + O(1) & \text{if } t = s - 1 \\ 0 & \text{otherwise} \end{cases}$$

It then follows that for $\ell \in \mathbb{R}^{T(T+1)/+2T-6}$ such that $\ell'\ell = 1$ $n^{-1} \sum_{i=1}^n \ell' \Sigma_n^{-1/2} [f'_{i,2} - E f_{i,2}, g_{i,2}(\beta_0)]' \xrightarrow{d} N(0, 1)$ by the Lindeberg-Feller CLT for triangular arrays. It then follows from a straightforward application of the Cramer-Wold theorem and the continuous mapping theorem that $n^{-1} \sum_{i=1}^n [f'_{i,2}, g_{i,2}(\beta_0)]' \xrightarrow{d} [\xi'_x, \xi'_y]'$ where $[\xi'_x, \xi'_y]' \sim N((\mu', 0)', \Sigma)$. ■

Note on Near Unit Root Asymptotics in Dynamic Panel Models
when Unity is Approached at $n^{-1/2}$: Supplementary Appendix
for “Long Difference Instrumental Variables Estimation for
Dynamic Panel Models with Fixed Effects”

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We consider the dynamic panel model

$$\begin{aligned}
 y_{it} &= \alpha_i + \beta_n y_{it-1} + \varepsilon_{it} & (1) \\
 \varepsilon_{it} &\sim N(0, \sigma_\varepsilon^2) \\
 \alpha_i &\sim N(0, \sigma_\alpha^2) \\
 y_{i0} &\sim N\left(\frac{\alpha_i}{1 - \beta_n}, \frac{\sigma_\varepsilon^2}{1 - \beta_n^2}\right)
 \end{aligned}$$

where we assume that $t = 1, \dots, T$ and $i = 1, \dots, n$ with T being fixed and $n \rightarrow \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/\sqrt{n})$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{\sqrt{n}} + o(n^{-1/2})$ for any constant k . We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = \alpha_i - (1 - \beta_n) y_{i0}$ such that

$$\begin{aligned}
 \Delta y_{it} &= \beta_n^{t-1} \alpha_i + (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + (\beta_n - 1) \beta_n^{t-1} y_{i0} + \varepsilon_{it} \\
 &= (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + \beta_n^{t-1} \eta_{i0} + \varepsilon_{it}.
 \end{aligned}$$

We define $g_{i1}(\beta) = [u_{it} \Delta u_{is}(\beta), u_{iT} \Delta u_{ij}(\beta), \bar{u}_i \Delta u_{ik}(\beta)]'$ and $g_{i2}(\beta) = [\Delta u_i(\beta)' y_{i0}]$ such that

$$g_n(\beta) = n^{-1} \sum_{i=1}^n [g_{i1}(\beta)', g_{i2}(\beta)']'$$

with the optimal weight matrix $\Omega = \lim_n E g_n(\beta_n) g_n(\beta_n)'$. The infeasible 2SLS of a possibly transformed set of moment conditions then solves

$$\beta_{2SLS} = \arg \min_{\beta} g_n(\beta)' C (C' \Omega C)^+ C' g_n(\beta)$$

where C is $n \times r$ matrix for $1 \leq r \leq G$ such that $C' C = I$ and $\text{rank}(C (C' \Omega C)^+ C') \geq 1$. We use Ω^+ to denote the Moore-Penrose inverse (see Magnus and Neudecker p. 32, 1988). We thus allow the use of a

singular weight matrix. Choosing r less than G allows to exclude certain moment conditions. We define the vectors

$$\begin{aligned} f_{i,1} &= [u_{it}\Delta y_{is-1}, u_{iT}\Delta y_{ij-1}, \bar{u}_i\Delta y_{ik-1}]' \\ f_{i,2} &= [\Delta y_{i1}y_{i0}, \dots, \Delta y_{iT-1}y_{i0}]' \end{aligned}$$

for $s = 2, \dots, T; t = 1, \dots, s-2; j = 2, \dots, T-1; k = 2, \dots, T$ and let $f_n = n^{-1} \sum_{i=1}^n [f'_{i,1}, f'_{i,2}]'$ such that the infeasible 2SLS estimator can be written as

$$\beta_{2SLS} - 1 = (f'_n C (C' \Omega C)^+ C' f_n)^{-1} f'_n C (C' \Omega C)^+ C' g_n(\beta_{n0}). \quad (2)$$

Lemma 1 Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \rightarrow \infty$

$$n^{-1} \sum_{i=1}^n f_{i,1} \xrightarrow{p} \mu_1, n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0) \xrightarrow{p} 0$$

with $\mu'_1 = [0, \dots, 0, \sigma_\varepsilon^2 T^{-1} \mathbf{1}'_{T-1}]$ and

$$n^{-1} \sum_{i=1}^n [f'_{i,2}, g'_{i,2}(\beta_0)]' \xrightarrow{d} [\xi'_x, \xi'_y]'$$

where $[\xi'_x, \xi'_y]' \sim N(\mu_2, \Sigma)$ with $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\mu'_2 = [-\mathbf{1}'_{T-1} \sigma_\varepsilon^2 / 2, 0, \dots, 0]$ and $\Sigma_{11} = \delta I, \Sigma_{12} = \delta M_1, \Sigma_{22} = \delta M_2$ where $\delta = \frac{\sigma_\alpha^2 \sigma_\varepsilon^2}{c}$,

$$M_1 = \begin{bmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & \ddots & 1 \\ & & & & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and $\Sigma_{12} = \Sigma'_{21}$.

An immediate consequence of this result is that an estimator based solely on the condition

$$E \bar{u}_i \Delta u_{ik}(\beta_0) = 0 \quad (3)$$

is consistent. If we consider estimators that do use all the moment conditions except (3) then we are in a situation where again the moment conditions in $g_{i1}(\beta)$ become asymptotically redundant. Now, however the limiting distribution of the non-redundant conditions has a noncentrality parameter.

The limiting normal distribution of Lemma (1) is singular. An alternative representation of the limiting distribution can be given by eliminating redundant dimensions from $[\xi'_x, \xi'_y]'$. Let $\xi_y = [\xi_{y0}, \xi_{y1}]'$ where ξ_{y1} is the last element of ξ_y and $\xi_u = [\xi'_x, \xi_{y1}]'$. Then $\xi_y = H \xi_z$ where H is a $T-1 \times T$ matrix defined as

$$H = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ & \ddots & \ddots & \vdots & \\ & & 0 & -1 & 0 \\ 0 & \dots & & 0 & 1 \end{bmatrix}.$$

It can be checked easily that the covariance matrix of vector $[\xi'_x, \xi'_u H']'$ is Σ as required. Using Lemma (1) the limiting distribution of $\beta_{2SLS} - \beta_n$ is stated in the next corollary. For this purpose we define the augmented vectors $\xi_x^\# = [0, \dots, 0, \xi'_x]'$ and $\xi_y^\# = [0, \dots, 0, \xi_y]$ and partition $C = [C'_0, C'_1]'$ such that $C'\xi_x^\# = C'_1\xi_x$. Let r_1 denote the rank of C_1 .

More explicitly, if β_{2SLS} is the GMM estimator based on $g_{i2}(\beta)$ then it has the following nonstandard limiting distribution under the weak identification asymptotics adopted here

$$\beta_{2SLS} - \beta_0 \xrightarrow{d} \frac{(\mu + \xi_x)' C_1 (C'_1 \Omega C_1)^{-1} C'_1 H \xi_u}{(\mu + \xi_x)' C_1 (C'_1 \Omega C_1)^{-1} C'_1 (\mu + \xi_x)} = X$$

where $\mu = -\mathbf{1}_{T-1} \sigma_\varepsilon^2 / 2$.

We report bias and MSE for the long difference (LD) and Arellano and Bond/Ahn and Schmidt estimators in Tables 1 and 2. As is immediately obvious from the tables bias and MSE are much lower than what we observe in actual Monte Carlo experiments.

Proof of Lemma (1). Consider $Ef_{i,1}$ with

$$\begin{aligned} Eu_{it}\Delta y_{is-1} &= Eu_{it} \left[\beta_n^{s-2}\eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right] \\ &= (\beta_n - 1) \beta_n^{t-s-1} \sigma_\varepsilon^2 = O(n^{-1/2}) \end{aligned}$$

and $\text{Var}(u_{it}\Delta y_{is-1}) = Eu_{it}^2 (\Delta y_{is-1})^2 - (\beta_n - 1)^2 \beta_n^{2(s-t)} \sigma_\varepsilon^4$ where

$$\begin{aligned} Eu_{it}^2 (\Delta y_{is-1})^2 &= \beta_n^{2(s-t)} (\sigma_\varepsilon^2 + \sigma_\alpha^2) \frac{\sigma_\varepsilon^2 (\beta_n - 1)^2}{1 - \beta_n^2} + (\sigma_\varepsilon^2 + \sigma_\alpha^2) \sigma_\varepsilon^2 \\ &\quad + (\beta_n - 1)^2 (\sigma_\varepsilon^2 + \sigma_\alpha^2) \sigma_\varepsilon^2 \sum_{r=1}^{s-2} \beta_n^{2(r-1)} \end{aligned}$$

such that $\text{Var}(u_{it}\Delta y_{is-1}) = \sigma_\varepsilon^4 + O(n^{-1/2})$. By independence of $u_{it}\Delta y_{is-1}$ across i it follows by the Markov inequality that $P(|n^{-1}(\sum_{i=1}^n u_{it}\Delta y_{is-1}) - Eu_{it}\Delta y_{is-1})| > \epsilon) \leq \frac{n^{-1}(\sigma_\varepsilon^2 + \sigma_\alpha^2)\sigma_\varepsilon^2 + o(n^{-1/2})}{\epsilon}$ for any $\epsilon > 0$ which shows that $n^{-1} \sum_{i=1}^n (u_{it}\Delta y_{is-1} - Eu_{it}\Delta y_{is-1}) = o_p(1)$. Since $n^{-1} \sum_{i=1}^n Eu_{it}\Delta y_{is-1} = O(n^{-1/2})$ it follows that $n^{-1} \sum_{i=1}^n u_{it}\Delta y_{is-1} = o_p(1)$. For $\bar{u}_i\Delta y_{is-1}$ note that

$$\frac{1}{T} E \left[\sum_{t=1}^T u_{it} \left(\beta_n^{s-2}\eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right] = \frac{\sigma_\varepsilon^2}{T} + \frac{\sigma_\varepsilon^2 (\beta_n - 1)}{T} \sum_{t=1}^{s-1} \beta_n^{r-1}$$

and

$$\begin{aligned} E \left[T^{-1} \sum_{t=1}^T u_{it} \left(\beta_n^{s-2}\eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right]^2 \\ &= \beta_n^{2(s-2)} T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \eta_{i0}^2 + T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \varepsilon_{is-1}^2 \\ &\quad + 2(\beta_n - 1) T^{-2} E \left(\sum_{t=1}^T u_{it} \right)^2 \varepsilon_{is-1} \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \\ &\quad + \frac{(\beta_n - 1)^2}{T^2} E \sum_{t,s=1}^T u_{it} u_{is} \sum_{q,r=1}^{k-2} \beta_n^{r-1} \beta_n^{q-1} \varepsilon_{ik-1-q} \varepsilon_{ik-1-r} \\ &= \beta_n^{2(s-2)} T^{-2} (T^2 \sigma_\alpha^2 + T \sigma_\varepsilon^2) \frac{\sigma_\varepsilon^2 (\beta_n - 1)^2}{1 - \beta_n^2} + \sigma_\alpha^2 \sigma_\varepsilon^2 + \frac{3\sigma_\varepsilon^4}{T} \\ &\quad + 4(\beta_n - 1) T^{-2} \sigma_\varepsilon^4 \sum_{r=1}^{s-2} \beta_n^{r-1} \\ &\quad + (\beta_n - 1)^2 \left(\sigma_\alpha^2 \sigma_\varepsilon^2 + \frac{3\sigma_\varepsilon^4}{T} \right) \sum_{r=1}^{s-2} \beta_n^{2r-4} + (\beta_n - 1)^2 \frac{\sigma_\varepsilon^4}{T^2} \left(\sum_{t=1}^{k-1} \beta_n^{k-1-r} \right)^2 \\ &= \sigma_\alpha^2 \sigma_\varepsilon^2 + \frac{3\sigma_\varepsilon^4}{T} + O(n^{-1/2}). \end{aligned}$$

such that by a law of large numbers $n^{-1} \sum_{i=1}^n \bar{u}_i \Delta y_{ik-1} \xrightarrow{p} \sigma_\varepsilon^2 / T$ using the same arguments as before.

Next consider

$$\begin{aligned} E\Delta y_{it} y_{i0} &= Ey_{i0} \left(\beta_n^{t-2}\eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \\ &= \beta_n^{t-2} \sigma_\varepsilon^2 (\beta_n - 1) / (1 - \beta_n^2) = -\frac{\sigma_\varepsilon^2}{2} + O(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned}
E(\Delta y_{it} y_{i0})^2 &= E y_{i0}^2 \left(\beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right)^2 \\
&= \beta_n^{2(t-2)} \frac{(\beta_n - 1)^2 \sigma_\varepsilon^2}{(1 - \beta_n^2)(1 - \beta_n)^2} \sigma_\alpha^2 + 3\beta_n^{2(t-2)} \frac{\sigma_y^4 (\beta_n - 1)^2}{(1 - \beta_n^2)^2} \\
&\quad + \left[\sigma_\varepsilon^2 + (\beta_n - 1)^2 \sigma_\varepsilon^2 \sum_{r=1}^{s-2} \beta_n^{2(r-1)} \right] (\sigma_\alpha^2 (1 - \beta_n)^{-2} + \sigma_\varepsilon^2 (1 - \beta_n^2)^{-1}) \\
&= \frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{c} n + o(n).
\end{aligned}$$

such that $\text{Var}(n^{-1} \sum_{i=1}^n \Delta y_{it} y_{i0}) = O(1)$.

For $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0)$ consider $E g_{i,1}(\beta_0) = 0$ which is implied by the orthogonality conditions and $E(u_{it} \Delta u_{is})^2 = 2(\sigma_\alpha^2 + \sigma_\varepsilon^2) \sigma_\varepsilon^2$ by independence. For $E(\bar{u}_i \Delta u_{ik}) = T^{-1} \sum_{t=1}^T E u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) = 0$ while

$$E(\bar{u}_i \Delta u_{ik})^2 = T^{-2} E \left(\sum_{t=1}^T u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) \right)^2 = 2\sigma_\alpha^2 \sigma_\varepsilon^2 + T^{-1} \sigma_\varepsilon^4 + 2T^{-2} \sigma_\varepsilon^2$$

such that by the independence of $g_{i,1}(\beta_0)$ across i and a standard WLLN it follows that $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0) \xrightarrow{p} 0$. For $n^{-1} \sum_{i=1}^n g_{i,2}(\beta_0)$ we have from the moment conditions that $E g_{i,2}(\beta_0) = 0$ and $\text{Var}(\Delta u_i(\beta_0) y_{i0}) = 2\sigma_\varepsilon^2 (\sigma_\alpha^2 / (1 - \beta_n)^2 + \sigma_\varepsilon^2 / (1 - \beta_n^2)) = \frac{2\sigma_\varepsilon^2 \sigma_\alpha^2}{c} n + O(1)$.

The joint limiting distribution of $n^{-1} \sum_{i=1}^n [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)']'$ can now be obtained from a triangular array CLT. By previous arguments

$$E [f'_{i,2}, g_{i,2}(\beta_0)'] = \begin{bmatrix} \mu' & 0 & \cdots & 0 \end{bmatrix}$$

with $\mu = -\mathbf{1}_{T-1} \sigma_\varepsilon^2 / 2 + O(n^{-1})$ where $\mathbf{1}_{T-1}$ is the $T - 1$ dimensional vector with elements 1. Then $E [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)']' [f'_{i,2} - E f'_{i,2}, g_{i,2}(\beta_0)'] = \Sigma_n$ where

$$\Sigma_n = \begin{bmatrix} \Sigma_{11,n} & \Sigma_{12,n} \\ \Sigma_{21,n} & \Sigma_{22,n} \end{bmatrix}$$

By previous calculations we have found the diagonal elements of $\Sigma_{11,n}$ and $\Sigma_{22,n}$ to be $\frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{c} n$. The off-diagonal elements are found to be

$$\begin{aligned}
E \Delta y_{it} \Delta y_{is} y_{i0}^2 &= E \left[y_{i0}^2 \left(\beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right. \\
&\quad \times \left. \left(\beta_n^{t-2} \eta_{i0} + \varepsilon_{it-1} + (\beta_n - 1) \sum_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{it-1-r} \right) \right] \\
&= \beta_n^{t-2} \beta_n^{s-2} \frac{(\beta_n - 1)^2 \sigma_\varepsilon^2}{(1 - \beta_n^2)(1 - \beta_n)^2} \left(\frac{\sigma_\alpha^2}{(1 - \beta_n)^2} + 3\sigma_\varepsilon^2 (1 - \beta_n^2)^{-1} \right) + O(n^{-1}) = -\frac{3\sigma_\varepsilon^2 \sigma_\alpha^2}{2} n^{1/2} + O(1)
\end{aligned}$$

which is of lower order of magnitude while $n^{-1} (E \Delta y_{it} y_{i0})^2 = O(1)$. Thus $n^{-1} \Sigma_{11,n} \rightarrow \text{diag}(\frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{c}, \dots, \frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{c})$. The off-diagonal elements of $\Sigma_{22,n}$ are obtained from

$$E \Delta u_{it} \Delta u_{is} y_{i0}^2 = \begin{cases} \frac{-\sigma_\varepsilon^2 \sigma_\alpha^2}{c} n + O(1) & t = s + 1 \text{ or } t = s - 1 \\ 0 & \text{otherwise} \end{cases}$$

zero and for $\Sigma_{12,n}$ we consider

$$E\Delta y_{it}\Delta u_{is}y_{i0}^2 = \begin{cases} \frac{\sigma_x^2\sigma_\alpha^2}{c}n + O(1) & \text{if } t = s \\ -\frac{\sigma_x^2\sigma_\alpha^2}{c}n + O(1) & \text{if } t = s - 1 \\ 0 & \text{otherwise} \end{cases}$$

It then follows that for $\ell \in \mathbb{R}^{T(T+1)/+2T-6}$ such that $\ell'\ell = 1$ $n^{-1} \sum_{i=1}^n \ell' \Sigma_n^{-1/2} [f'_{i,2} - E f_{i,2}, g_{i,2}(\beta_0)]' \xrightarrow{d} N(0, 1)$ by the Lindeberg-Feller CLT for triangular arrays. It then follows from a straightforward application of the Cramer-Wold theorem and the continuous mapping theorem that $n^{-1} \sum_{i=1}^n [f'_{i,2}, g_{i,2}(\beta_0)]' \xrightarrow{d} [\xi'_x, \xi'_y]'$ where $[\xi'_x, \xi'_y] \sim N((\mu', 0)', \Sigma)$. ■

Table 1: Bias of root-n Near Unit Root Approximations

T	AS/AB	LD
5	-0.016404	0.010327
6	-0.01391	0.0061784
7	-0.011778	0.0043041
8	-0.0094988	0.0041043
9	-0.0081541	0.0020319
10	-0.006843	0.0014522

Table 2: MSE of root-n Near Unit Root Approximations

T	AS/AB	LD
5	0.017079	0.022672
6	0.0095719	0.013495
7	0.0058787	0.0095064
8	0.003924	0.0068071
9	0.0027829	0.0052526
10	0.0019628	0.0040317

Note on Near Unit Root Asymptotics for the Blundell and Bond Estimator: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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June, 2002

We consider the dynamic panel model

$$\begin{aligned}
 y_{it} &= \alpha_i + \beta_n y_{it-1} + \varepsilon_{it} & (1) \\
 \varepsilon_{it} &\sim N(0, \sigma_\varepsilon^2) \\
 \alpha_i &\sim N(0, \sigma_\alpha^2) \\
 y_{i0} &\sim N\left(\frac{\alpha_i}{1-\beta_n}, \frac{\sigma_\varepsilon^2}{1-\beta_n^2}\right)
 \end{aligned}$$

where we assume that $t = 1, \dots, T$ and $i = 1, \dots, n$ with T being fixed and $n \rightarrow \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/n)$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{\sqrt{n}} + o(n^{-1/2})$ for any constant k . We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = \alpha_i - (1 - \beta_n) y_{i0}$ such that

$$\begin{aligned}
 \Delta y_{it} &= \beta_n^{t-1} \alpha_i + (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + (\beta_n - 1) \beta_n^{t-1} y_{i0} + \varepsilon_{it} \\
 &= (\beta_n - 1) \sum_{s=1}^{t-1} \beta_n^{s-1} \varepsilon_{it-s} + \beta_n^{t-1} \eta_{i0} + \varepsilon_{it}.
 \end{aligned}$$

In particular it holds that

$$\Delta y_{i1} = \eta_{i0} + \varepsilon_{i1}.$$

In this note we are interested in the behavior of the additional moment condition (4.4) proposed by Blundell and Bond (1998). As Blundell and Bond point out this moment condition can be combined with the other moment conditions proposed by Ahn and Schmidt (1995) to construct a first order asymptotically efficient estimator. Here, we are however interested in the behavior of the Blundell and Bond estimator under weak instrument conditions and thus do not need to focus on the remaining conditions which do not produce consistent estimators.

We define

$$g_{i,AB} = [y_{i,0} \Delta u_{i2}, y_{i,0} \Delta u_{i3}, y_{i1} \Delta u_{i3}, y_{i,0} \Delta u_{i4}, \dots, y_{i2} \Delta u_{i4}, \dots, y_{i0} \Delta u_{iT}, \dots, y_{iT-2} \Delta u_{iT}]'$$

and

$$g_{i,BB}(\beta) = [u_{i2}(\beta) \Delta y_{i1}, \dots, u_{iT}(\beta) \Delta y_{iT-1}]$$

We define

$$f_{i,BB}(\beta) = [y_{i1}(\beta) \Delta y_{i1}, \dots, y_{iT-1}(\beta) \Delta y_{iT-1}]'$$

Lemma 1 Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \rightarrow \infty$

$$n^{-2} \sum_{i=1}^n f_{i,BB} \xrightarrow{p} \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{c} \mathbf{1}_{T-1}, n^{-1} \sum_{i=1}^n g_{i,BB}(\beta_n) \xrightarrow{p} 0.$$

and

$$E [g_{i,AB} g'_{i,BB}] = O(1).$$

Define the Blundell and Bond (1998) estimator by letting $f_n = n^{-2} \sum_{i=1}^n [f'_{i,AB}, f'_{i,BB}]'$, $g_n = n^{-2} \sum_{i=1}^n [g'_{i,AB}, g'_{i,BB}]'$ and the weight matrix $\Omega_n = E [g_n g'_n]$ such that the infeasible 2SLS estimator can be written as

$$b_{BB} - 1 = (f'_n \Omega_n^+ f_n)^{-1} f'_n \Omega_n^+ g_n(\beta_n). \quad (2)$$

It follows from earlier results and the Lemma that for $q_3 = T(T+1)/2 - 2$

$$n\Omega_n \rightarrow \begin{bmatrix} 0_{q_3, q_3} & \cdots & 0_{q_3, T-1} \\ \vdots & \Sigma_{22} & \vdots \\ 0_{T-1, q_3} & \cdots & 0_{T-1, T-1} \end{bmatrix}$$

This suggests that the optimally weighted Blundell and Bond estimator becomes ill-defined under the weak instrument asymptotic approximation because $n^{-1} f'_n \Omega_n^+ f_n \xrightarrow{p} 0$. An inefficient version of the estimator is better behaved. Define

$$b_{IBB} - 1 = (f'_n f_n)^{-1} f'_n g_n(\beta_n).$$

Then it follows that

$$b_{IBB} - 1 \xrightarrow{p} 0$$

from Lemma (1) and results in the main part of the paper. In other words, exploiting stationarity leads to consistent estimators under weak instrument asymptotics. This result confirms the analysis in Blundell and Bond (1998).

Proof of Lemma (1). Note that

$$\begin{aligned} y_{it} &= \frac{(1 - \beta_n^t) \alpha_i}{(1 - \beta_n)} + \beta_n^t y_{i0} + \sum_{r=1}^{t-1} \beta_n^{r-1} \varepsilon_{it-r} + \varepsilon_{it} \\ &= \frac{\alpha_i}{(1 - \beta_n)} + \frac{\beta_n^t \eta_{i0}}{(1 - \beta_n)} + \varepsilon_{it} + \sum_{r=1}^{t-1} \beta_n^{r-1} \varepsilon_{it-r}. \end{aligned}$$

Consider $E f_{i,BB}$ with

$$\begin{aligned} E y_{it} \Delta y_{it} &= E y_{it} \left[\beta_n^t \eta_{i0} + \varepsilon_{it} + (\beta_n - 1) \sum_{r=1}^{t-1} \beta_n^{r-1} \varepsilon_{it-r} \right] \\ &= \frac{\sigma_\alpha^2}{(1 - \beta_n)} + \frac{\beta_n^{2t} \sigma_\varepsilon^2}{(1 - \beta_n)} + (\beta_n - 1) \sum_{r=1}^{t-1} \beta_n^{2(r-1)} \sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{c} n + O(n^{-1}) \end{aligned}$$

and $\text{Var}(y_{it} \Delta y_{it}) = E(y_{it}^2) E(\Delta y_{is-1})^2 + (E(y_{it} \Delta y_{is-1}))^2$ where

$$\begin{aligned} E(y_{it}^2) E(\Delta y_{is-1})^2 &= \left(\frac{\sigma_\alpha^2}{(1 - \beta_n)} + \frac{\beta_n^{2t} \sigma_\varepsilon^2}{(1 - \beta_n)} + \sum_{r=0}^{t-1} \beta_n^{r-1} \sigma_\varepsilon^2 \right) \left(\frac{\beta_n^{2t} \sigma_\varepsilon^2}{(1 - \beta_n)} + \sigma_\varepsilon^2 + (\beta_n - 1)^2 \sum_{r=0}^{t-1} \beta_n^{2(r-1)} \sigma_\varepsilon^2 \right) \\ &= \frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{c^2} n^2 + O(n). \end{aligned}$$

such that $\text{Var}(y_{it} \Delta y_{is-1}) = \frac{\sigma_\varepsilon^2 \sigma_\alpha^2 + (\sigma_\varepsilon^2 + \sigma_\alpha^2)^2}{c^2} n^2 + O(n)$. It then follows that $n^{-2} \sum_{i=1}^n E y_{it} \Delta y_{is-1} = O(1)$ and $n^{-2} \sum_{i=1}^n y_{it} \Delta y_{is-1} = \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{c} + o_p(1)$.

For $g_{i,BB}(\beta)$ consider

$$E(u_{it} \Delta y_{it-1}) = E(\alpha_i + \varepsilon_{it}) \left[\beta_n^{t-1} \eta_{i0} + \varepsilon_{it-1} + (\beta_n - 1) \sum_{r=1}^{t-2} \beta_n^{r-1} \varepsilon_{it-1-r} \right] = 0$$

and

$$\begin{aligned} \text{Var}(u_{it} \Delta y_{it-1}) &= E(u_{it} \Delta y_{it-1})^2 = E u_{it}^2 E(\Delta y_{it-1})^2 \\ &= (\sigma_\alpha^2 + \sigma_\varepsilon^2) \left(\beta_n^{t-1} \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + (\beta_n - 1)^2 \sum_{r=1}^{t-2} \beta_n^{2(r-1)} \sigma_\varepsilon^2 \right) \\ &= 2(\sigma_\alpha^2 + \sigma_\varepsilon^2) \sigma_\varepsilon^2 + o(1). \end{aligned}$$

such that

$$n^{-1} \sum_{i=1}^n g_{i,BB}(\beta) \xrightarrow{p} 0.$$

For the correlation between $g_{i,BB}$ and $g_{i,AB}$ consider

$$\begin{aligned} E[u_{it} \Delta y_{it-1} \Delta u_{is} y_{i0}] &= E[u_{it} \Delta u_{is}] E[\Delta y_{it-1} y_{i0}] + E[\Delta u_{is} \Delta y_{it-1}] E[y_{i0} u_{it}] \\ &= \begin{cases} (\beta_n - 1) (\beta_n^{t-1-s} - \beta_n^{t-s}) \sigma_\varepsilon^2 \frac{\sigma_\alpha^2}{1 - \beta_n} & t \geq s + 1 \\ \sigma_\varepsilon^2 \beta_n^{t-1} \frac{\sigma_\varepsilon^2 (\beta_n - 1)}{1 - \beta_n^2} - \frac{(1 - \beta_n) \sigma_\varepsilon^2 \sigma_\alpha^2}{1 - \beta_n} & s = t \\ -\sigma_\varepsilon^2 \beta_n^{t-1} \frac{\sigma_\varepsilon^2 (\beta_n - 1)}{1 - \beta_n^2} & t = s - 1 \end{cases} = O(1) \end{aligned}$$

and

$$E[u_{it} \Delta y_{it-1} \Delta u_{is} u_{ik}] = E[u_{it} \Delta u_{is}] E[\Delta y_{it-1} u_{ik}] + E[\Delta u_{is} \Delta y_{it-1}] E[u_{ik} u_{it}] = O(1)$$

■