Correlated Choice

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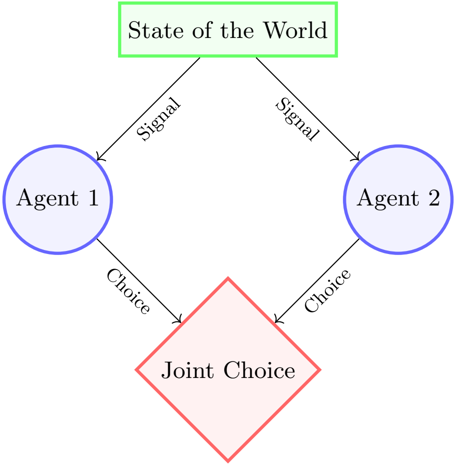
Motivation

- Random Joint Choice Data
  - Peer effects
  - Dynamic discrete choice
  - Choices in different markets
- Stochastic Separability
  - (Correlated) private signals
  - Naive voting
  - Lack of Influence

Example - Private Signals (No Influence)



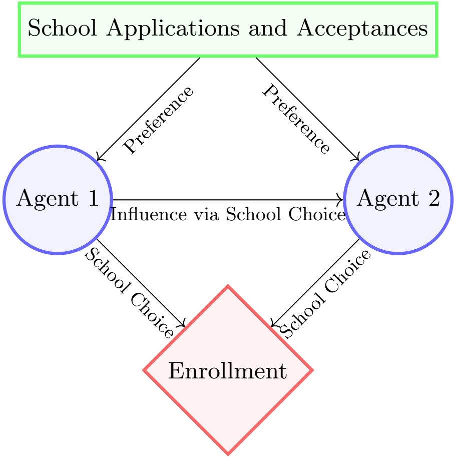
Example - Private Signals (No Influence)

- Two states of the world: $\{h, l\}$.
- Agents want to take actions that match the state of the world: $\{h, l\}$.
- Agents receive private but correlated signals.

	h	l
h	0.4	0.1
l	0.1	0.4

	h
h	0.5
l	0.5

Example - School Choice (Influence)



Example - School Choice (Influence)

- Two types of schools: $\{public, private\}$.
- Agents have varying preferences which may depend on their peer's choice.

	$public$	$private$
$public$	0.4	0.1
$private$	0.1	0.4

	$public$
$public$	0.7
$private$	0.3

An Observation

- Private Signals:
  - When one agent's choice set varied, the other agent's marginal choice probabilities were constant.
  - $(0.5, 0.5) \to (0.5, 0.5)$
- School Choice:
  - When one agent's set of feasible schools changes, the other agent's marginal choice probabilities varied.
  - $(0.5, 0.5) \to (0.7, 0.3)$
- Is there a connection between a lack of influence (stochastic separability) and marginal choice probabilities being well-defined (marginality)?

Research Questions

- What is the connection between stochastic separability and marginality?
- Does marginality characterize stochastic separability?
- How can we test for separable random utility?

Preview of Main Results

- We offer two generating processes which are characterized by marginality.
- We show that marginality is necessary but insufficient for stochastic separability.
- We characterize separable random utility when each agent has a unique random utility representation.
- We develop a tool kit for analyzing random joint choice rules.

Our Model

- Let X and Y be finite sets of alternatives.
  - $x \in A \subseteq X$
  - $y \in B \subseteq Y$
- Let $\mathcal{L}(S)$ denote the set of linear orders over S.
  - $\succ \in \mathcal{L}(X)$
  - $\succ' \in \mathcal{L}(Y)$
- Let $\mathcal{C}(S)$ denote the set of choice functions of S.
  - $c_X \in \mathcal{C}(X)$
  - $c_Y \in \mathcal{C}(Y)$
- Let $\Delta(S)$ denote the set of probability distributions over finite set S.
  - $\nu_+ \in \Delta(S)$
- Let $\Sigma(S)$ denote the set of signed measures over finite set S.
  - $\nu \in \Sigma(S)$

Our Data

- Let $\mathcal{X}$ be the collection of each non-empty subset of X.
- Let $\mathcal{Y}$ be the collection of each non-empty subset of Y.
- We observe joint choice probabilities on product sets.
  - For each $A \times B \in \mathcal{X} \times \mathcal{Y}$, we observe how frequently the pair $(x, y) \in A \times B$ is chosen.
- $p(x, y|A, B)$ denotes how frequently the pair (x, y) is chosen from the choice set $A \times B$.
  - We call p a *random joint choice rule.*

Interpretations of Our Data

- Repeated choice by two agents
  - Voting history of two senators
- Population level choice data by two groups
  - Choice of major among roommates, choice of education level among twins
- Repeated choice by a single agent
  - Choice in two markets, choice of cereal and shampoo
- Population level choice data across time
  - Voting data in 2016 and 2020, dynamic discrete choice

Stochastic Separability

- We call a function $c : \mathcal{X} \times \mathcal{Y} \to X \times Y$ a joint choice function if $c(A, B) \in A \times B$.
- We say that a joint choice function is *separable* if $c(\cdot, \cdot) = (c_X(\cdot), c_Y(\cdot))$.

Definition

A random joint choice rule p is *stochastically separable* if there exists $\nu_+ \in \Delta(\mathcal{C}(X) \times \mathcal{C}(Y))$ such that the following holds for all $A \in \mathcal{X}$, $B \in \mathcal{Y}$, $x \in A$, and $y \in B$.

$$p(x, y|A, B) = \sum_{c \in \mathcal{C}(X) \times \mathcal{C}(Y)} \nu_+(c) \mathbf{1}\{c(A, B) = (x, y)\}$$

Marginality

Definition

We say that a random joint choice rule p satisfies *marginality* if the following holds for all $A, A' \in \mathcal{X}$, $B, B' \in \mathcal{Y}$, $x \in A$, and $y \in B$.

- $\sum_{y \in B} p(x, y|A, B) = \sum_{y' \in B'} p(x, y'|A, B')$
- $\sum_{x \in A} p(x, y|A, B) = \sum_{x' \in A'} p(x', y|A', B)$

- We can define marginal choice probabilities.
  - $p_1(x, A) = \sum_{y \in Y} p(x, y|A, Y)$
  - $p_2(y, B) = \sum_{x \in X} p(x, y|X, B)$

Marginality vs Stochastic Separability

Theorem 1

1. A random joint choice rule p satisfies marginality if and only if there exists a signed measure ν over $\mathcal{C}(X) \times \mathcal{C}(Y)$ such that for all $A \in \mathcal{X}$, $B \in \mathcal{Y}$, $x \in A$, and $y \in B$ we have the following.

$$p(x, y|A, B) = \sum_{c \in \mathcal{C}(X) \times \mathcal{C}(Y)} \nu(c) \mathbf{1}\{c(A, B) = (x, y)\}$$

2. There exist random joint choice rules which satisfy marginality but are not stochastically separable.

Counterexample

	w	x		y	z
a	0.5	0	a	0.5	0
b	0	0.5	b	0	0.5
	w	x		y	z
c	0.5	0	c	0	0.5
d	0	0.5	d	0.5	0

Counterexample

	w	x		y	z
a	0.5	0	a	0.5	0
b	0	0.5	b	0	0.5
	w	x		y	z
c	0.5	0	c	0	0.5
d	0	0.5	d	0.5	0

Counterexample

	w	x		y	z
a	0.5	0	a	0.5	0
b	0	0.5	b	0	0.5
	w	x		y	z
c	0.5	0	c	0	0.5
d	0	0.5	d	0.5	0

Counterexample

	w	x		y	z
a	0.5	0	a	0.5	0
b	0	0.5	b	0	0.5
	w	x		y	z
c	0.5	0	c	0	0.5
d	0	0.5	d	0.5	0

Counterexample

	w	x		y	z
a	0.5	0	a	0.5	0
b	0	0.5	b	0	0.5
	w	x		y	z
c	0.5	0	c	0	0.5
d	0	0.5	d	0.5	0

Proof Sketch - Necessity

- Separable choice functions satisfy marginality.
- The linear combination of vectors which satisfy marginality also satisfies margianlity.
- This is the easy direction. Sufficiency is hard.

Proof Sketch - Sufficiency

	w	x		y	z
a	0.2	0.3	a	0.5	0
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	0.15	0.35	c	0.3	0.2
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0.2	0.3	a	0.5	0
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	0.15	0.35	c	0.3	0.2
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0.3	a	0.3	0
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	-0.05	0.35	c	0.1	0.2
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0.3	a	0.3	0
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	-0.05	0.35	c	0.1	0.2
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	-0.3
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	-0.05	0.05	c	0.1	-0.1
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	-0.3
b	0.4	0.1	b	0.1	0.4
	w	x		y	z
c	-0.05	0.05	c	0.1	-0.1
d	0.45	0.05	d	0.3	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	-0.3
b	0	0.1	b	-0.3	0.4
	w	x		y	z
c	-0.05	0.05	c	0.1	-0.1
d	0.05	0.05	d	-0.1	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	-0.3
b	0	0.1	b	-0.3	0.4
	w	x		y	z
c	-0.05	0.05	c	0.1	-0.1
d	0.05	0.05	d	-0.1	0.2

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	−0.3
b	0	0	b	−0.3	0.3
	w	x		y	z
c	−0.05	0.05	c	0.1	−0.1
d	0.05	−0.05	d	−0.1	0.1

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0.3	−0.3
b	0	0	b	−0.3	0.3
	w	x		y	z
c	−0.05	0.05	c	0.1	−0.1
d	0.05	−0.05	d	−0.1	0.1

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0	0
b	0	0	b	-0.3	0.3
	w	x		y	z
c	-0.05	-0.25	c	-0.2	-0.1
d	0.05	0.25	d	-0.1	0.4

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0	0
b	0	0	b	-0.3	0.3
	w	x		y	z
c	-0.05	-0.25	c	-0.2	-0.1
d	0.05	0.25	d	-0.1	0.4

Proof Sketch - Sufficiency

	w	x		y	z
a	0	0	a	0	0
b	0	0	b	0	0
	w	x		y	z
c	-0.05	0.05	c	0.1	-0.1
d	0.05	-0.05	d	-0.1	0.1

- Repeat this process for the remaining choice sets.

Marginality Revisited

- Marginality fails to be sufficient for stochastic separability.
- We can characterize marginality via the linear span of separable choice functions.
- What about separable utility functions?

Marginality vs Separable Random Utility

Theorem 2

A random joint choice rule p satisfies marginality if and only if there exists a signed measure ν over $\mathcal{L}(X) \times \mathcal{L}(Y)$ such that for all $A \in \mathcal{X}$, $B \in \mathcal{Y}$, $x \in A$, and $y \in B$ we have the following.

$$p(x, y|A, B) = \sum_{(\succ, \succ') \in \mathcal{L}(X) \times \mathcal{L}(Y)} \nu(\succ, \succ') \mathbf{1}\{x \succ A \setminus \{x\}, y \succ' B \setminus \{y\}\}$$

Some Additional Technology

- Block-Marschak polynomials
  - For multiple agents:

$$p(x, y|A, B) = \sum_{A':A\subseteq A'} \sum_{B':B\subseteq B'} q(x, y|A, B)$$

  - For a single agent:

$$p_1(x, A) = \sum_{A':A\subseteq A'} q_1(x, A')$$

$$p_2(y, B) = \sum_{B':B\subseteq B'} q_2(y, B')$$

- These keep track of the change in the choice probability of x not already explained by the supersets of A.

$$q_1(x, A) = p_1(x, A) - \sum_{A':A\subsetneq A'} q_1(x, A')$$

Block-Marschak Polynomials and Random Utility

Proposition 1

A signed measure ν over $\mathcal{L}(X)$ induces a marginal random choice rule p_1 if and only if the following holds.

$$\nu(\{\succ |X \setminus A \succ x \succ A \setminus \{x\}\}) = q_1(x, A)$$

Proposition 2

A signed measure ν over $\mathcal{L}(X) \times \mathcal{L}(Y)$ induces a random joint choice rule p if and only if the following holds.

$$\nu(\{(\succ, \succ')|X \setminus A \succ x \succ A \setminus \{x\}, Y \setminus B \succ' y \succ' B \setminus \{y\}\}) = q(x, y|A, B)$$

Marginal Graph

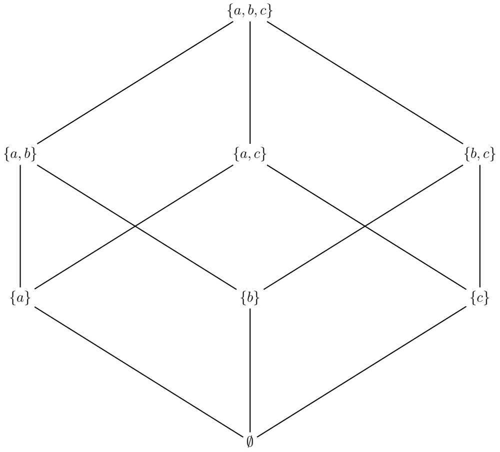
$\{a, b, c\}$

$\{a, b\}$ $\{a, c\}$ $\{b, c\}$

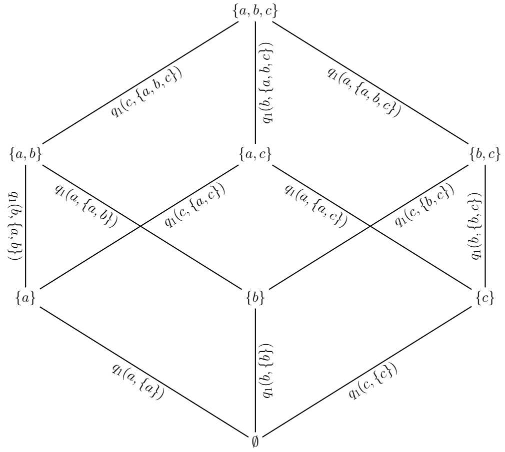
$\{a\}$ $\{b\}$ $\{c\}$

$\emptyset$

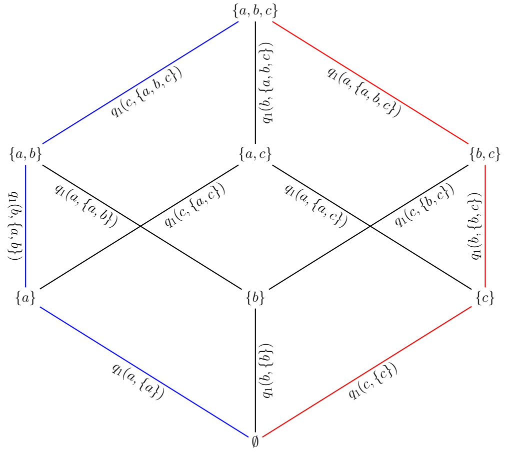
Marginal Graph



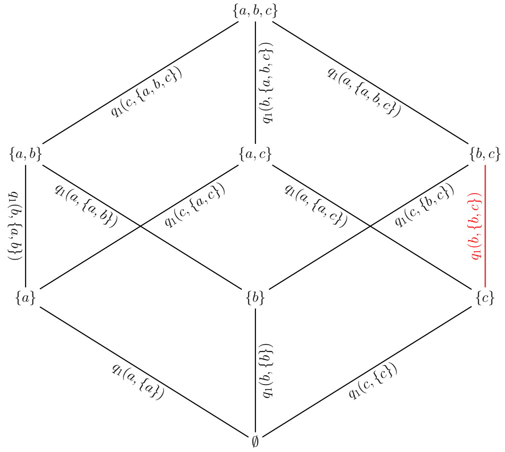
Marginal Graph



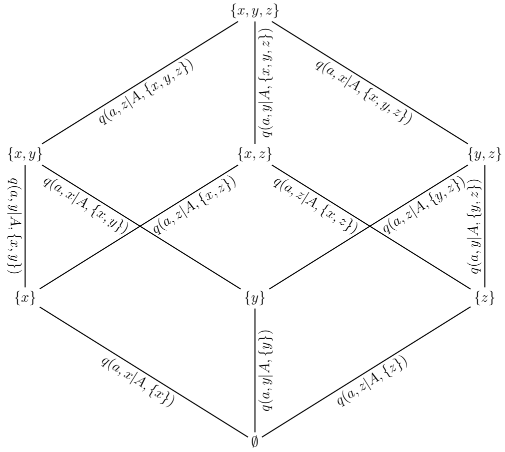
Marginal Graph



Marginal Graph



Conditional Graph



Linear Order Pairs and Our Graphs

- A path on the marginal graph corresponds to a linear order of X.
- A path on the conditional graph corresponds to a linear order of Y.
- What does a linear order pair look like using our graphs?
  - A path on the marginal graph: $\succ \in \mathcal{L}(X)$.
  - For each conditional graph along that path, a common path on each conditional graph: $\succ' \in \mathcal{L}(Y)$.

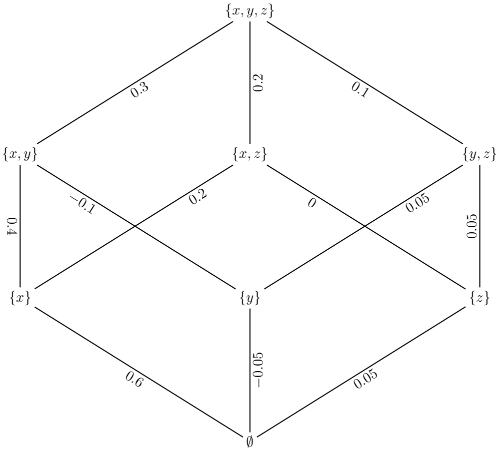
Some Preliminary Results

- Inflow equals outflow on the marginal graph.
  - $\sum_{x \in A} q_1(x, A) = \sum_{z \notin A} q_1(z, A \cup \{z\})$
  - This is a result of probabilities summing to one.
- Inflow equals outflow on the conditional graph.
  - $\sum_{y \in B} q(x, y|A, B) = \sum_{z \notin B} q(x, z|A, B \cup \{z\})$
  - This is equivalent to marginality and is a result of $p(x, A|B) = p(x, A|B')$.
- Inflow equals outflow between conditional graphs for each (y, B).
  - $\sum_{x \in A} q(x, y|A, B) = \sum_{z \notin A} q(z, y|A \cup \{z\}, B)$
  - This is equivalent to marginality and is a result of $p(x, A|B) = p(x, A|B')$.

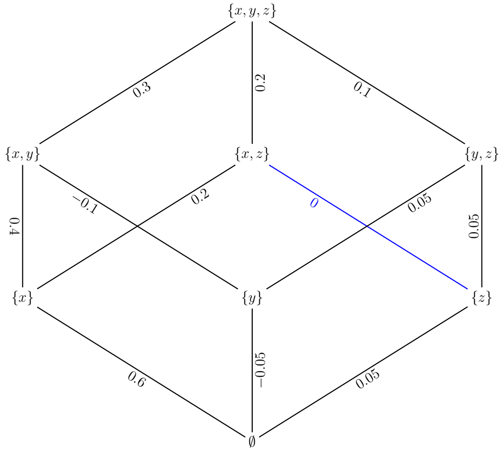
Proof Sketch

- As before, necessity is easy.
- Maximization of a pair of linear orders induces a separable choice function.
- The linear span of separable choice functions satisfies marginality.
- Sufficiency is hard and proceeds in steps.
  1. Show how we can decompose any conditional graph if we have marginality.
  2. Decompose every conditional graph on one "layer" of the marginal graph.
  3. Adapt the marginality trick from the proof of Theorem 1 to this collection of graphs.

Proof Sketch - Conditional Graph



Proof Sketch - Conditional Graph



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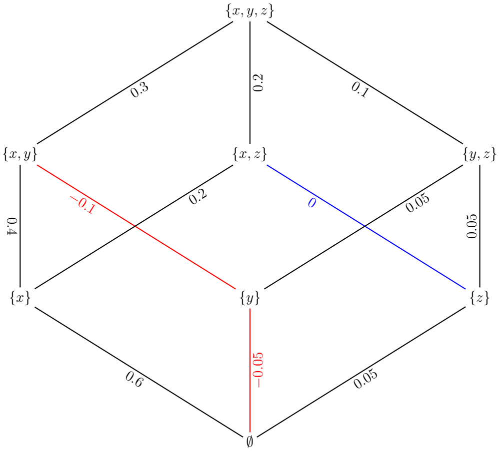

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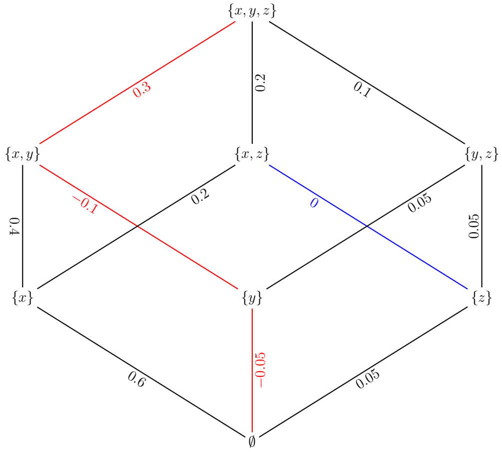

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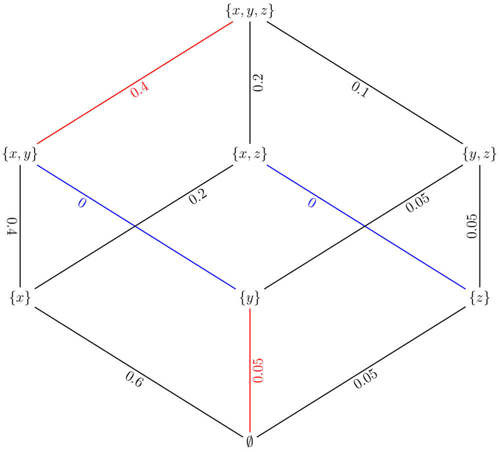

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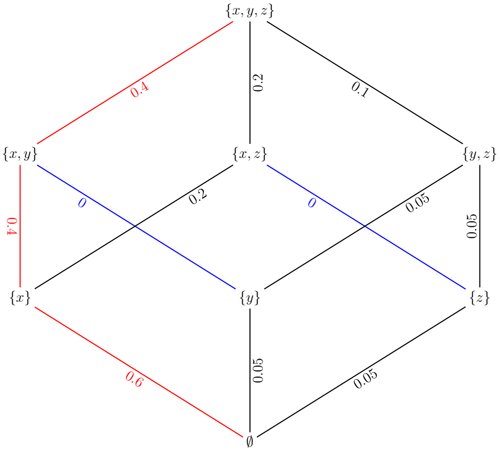
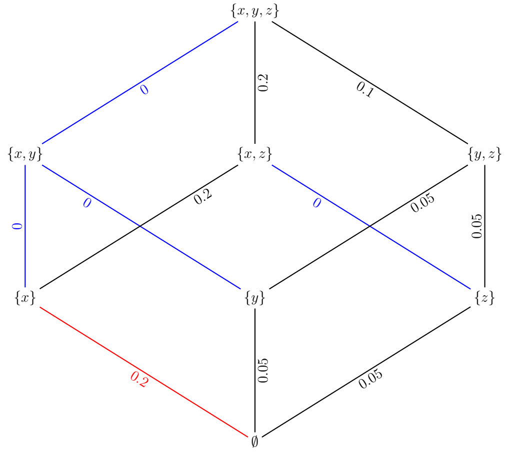


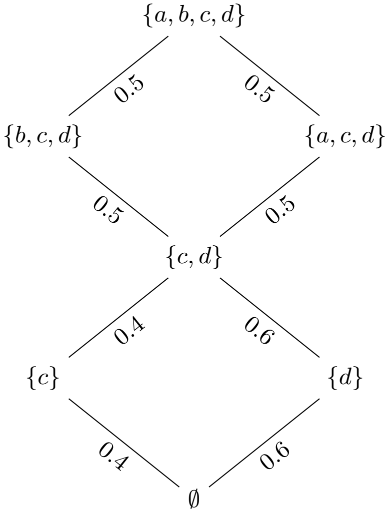
Proof Sketch - Conditional Graph



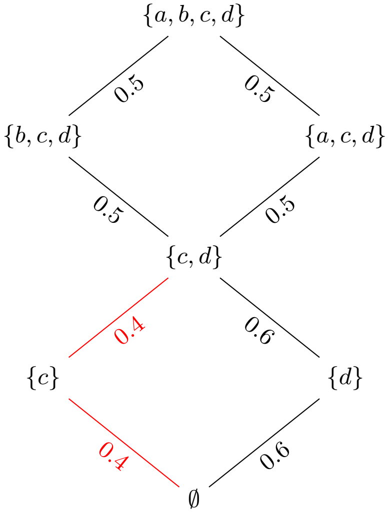
Proof Sketch - Marginal Graph

- Decomposing a conditional graph leaves us with a signed measure over linear orders of Y (that sums to $q_1(x, A)$).
- We'll use a similar process to decompose the marginal graph.
- Whenever we "subtract out" an edge on the marginal graph, we are decomposing the conditional graph associated with that edge.
  - Decomposing the marginal graph gives us the marginal distribution over linear orders of X.
  - Decomposing the conditional graphs gives us the distribution over linear orders of Y conditional on a linear order of X.
  - Note that the marginal and conditional distributions are not unique.

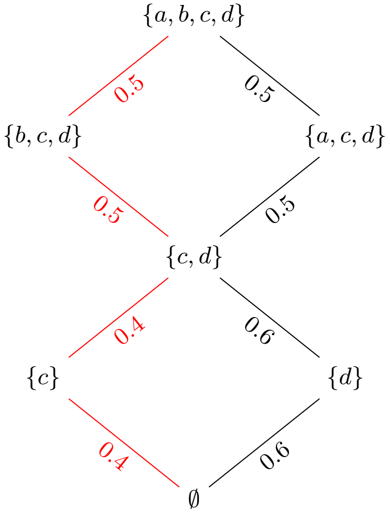
Proof Sketch - Marginal Graph



Proof Sketch - Marginal Graph



Proof Sketch - Marginal Graph



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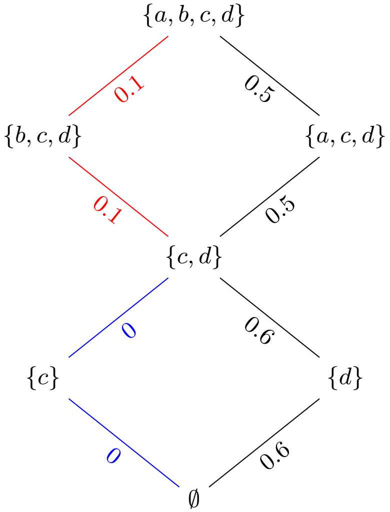
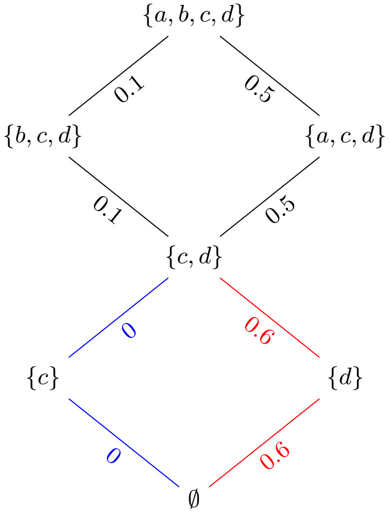


Proof Sketch - Marginal Graph



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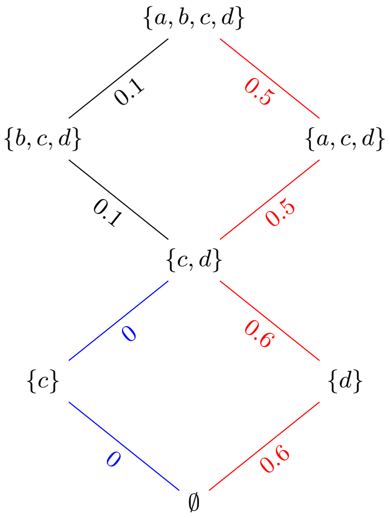
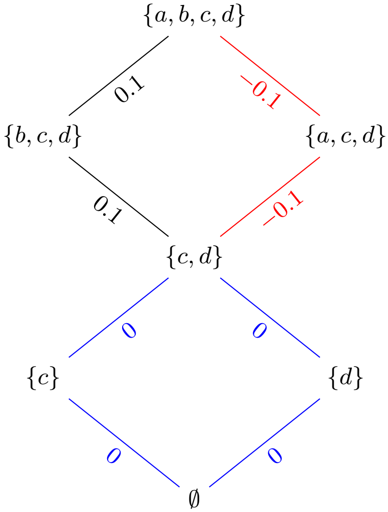


Proof Sketch - Marginal Graph

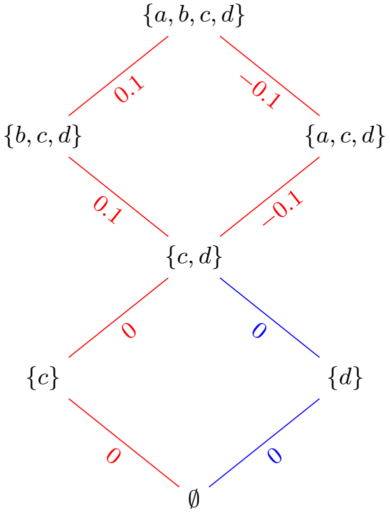


Proof Sketch - Marginal Graph

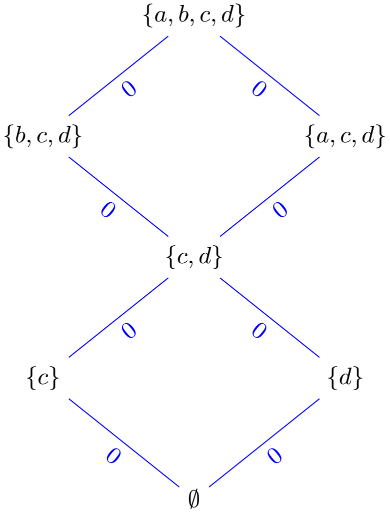
- Recall that inflow equals outflow between conditional graphs for each (y, B).
- $\sum_{x \in A} q(x, y|A, B) = \sum_{z \notin A} q(z, y|A \cup \{z\}, B)$
- This tells us the following for all $y \in B \subseteq Y$.

$$q(a, y|\{a, c, d\}, B) + q(b, y|\{b, c, d\}, B) = 0$$

Proof Sketch - Marginal Graph



Proof Sketch - Marginal Graph



Proof Sketch - Sufficiency

- We combine our marginal distribution with our conditional distributions to create one joint measure.
- Since our decomposition leaves every graph with zero weight everywhere, the joint measure we found satisfies Proposition 2, so we are done.

Proposition 2

A signed measure ν over $\mathcal{L}(X) \times \mathcal{L}(Y)$ induces a random joint choice rule p if and only if the following holds.

$$\nu(\{(\succ, \succ')|X \setminus A \succ x \succ A \setminus \{x\}, Y \setminus B \succ' y \succ' B \setminus \{y\}\}) = q(x, y|A, B)$$

Separable Random Utility

Definition

A random joint choice rule p is *rationalizable* by separable random utility if there exists $\nu_+ \in \Delta(\mathcal{L}(X) \times \mathcal{L}(Y))$ such that the following holds for all $A \in \mathcal{X}$, $B \in \mathcal{Y}$, $x \in A$, and $y \in B$.

$$p(x, y|A, B) = \sum_{(\succ,\succ') \in \mathcal{L}(X) \times \mathcal{L}(Y)} \nu_+(\succ, \succ') \mathbf{1}\{x \succ A \setminus \{x\}, y \succ' B \setminus \{y\}\}$$

Non-Negativity

Proposition 2

A signed measure ν over $\mathcal{L}(X) \times \mathcal{L}(Y)$ induces a random joint choice rule p if and only if the following holds.

$$\nu(\{(\succ, \succ')|X \setminus A \succ x \succ A \setminus \{x\}, Y \setminus B \succ' y \succ' B \setminus \{y\}\}) = q(x, y|A, B)$$

- We're looking for a probability distribution, so $\nu \geq 0$.
- This means $q \geq 0$.

Non-negativity

For each $x \in A \subseteq X$ and $y \in B \subseteq Y$, $q(x, y|A, B) \geq 0$.

Marginality Still Too Weak

- Our counterexample to Theorem 1 still works.
- When the choice set is not a (weak) subset of $\{c, d\} \times \{y, z\}$:

$$\nu_1(\succ, \succ') = \begin{cases} \frac{1}{2} & \text{if } (\succ, \succ') = (a \succ b \succ c \succ d, w \succ x \succ y \succ z) \\ \frac{1}{2} & \text{if } (\succ, \succ') = (b \succ a \succ d \succ c, x \succ w \succ z \succ y) \\ 0 & \text{otherwise} \end{cases}$$

- When the choice set is a (weak) subset of $\{c, d\} \times \{y, z\}$:

$$\nu_2(\succ, \succ') = \begin{cases} \frac{1}{2} & \text{if } (\succ, \succ') = (d \succ c, y \succ z) \\ \frac{1}{2} & \text{if } (\succ, \succ') = (c \succ d, z \succ y) \\ 0 & \text{otherwise} \end{cases}$$

Failures of Uniqueness and Marginality

- The marginal choice probabilities for each agent fail to have a unique RUM representation.

$$\nu_1(\succ) = \begin{cases} \frac{1}{2} & \text{if } \succ \in \{a \succ b \succ c \succ d, b \succ a \succ d \succ c\} \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_2(\succ) = \begin{cases} \frac{1}{2} & \text{if } \succ \in \{a \succ b \succ d \succ c, b \succ a \succ c \succ d\} \\ 0 & \text{otherwise} \end{cases}$$

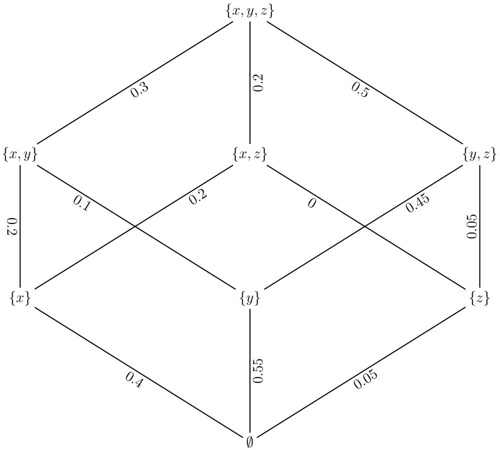
- Marginality only fails to be sufficient if both agents' marginal choice probabilities fail to have a unique RUM representation.

Separable Random Utility with Unique Marginals

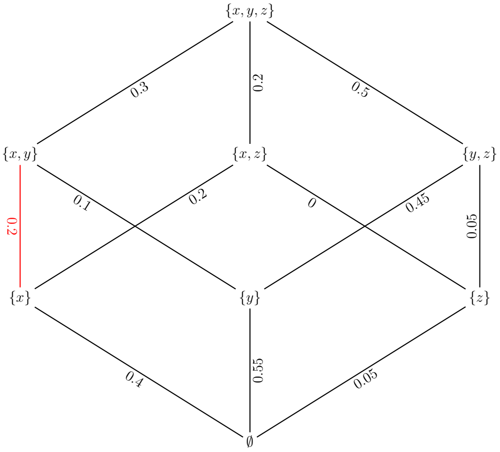
Theorem 3

Suppose that a random joint choice rule p satisfies marginality and at least one marginal random joint choice rule has a unique random utility representation. p is rationalizable by separable random utility if and only if it satisfies non-negativity.

Proof Sketch



Proof Sketch



Proof Sketch

- When a marginal random choice rule has a unique random utility representation, each linear order in the support of the representation has some edge unique to that linear order among linear orders in the support. (Turansick (2022))
- Recall that inflow equals outflow between conditional graphs for each (y, B).
  - $\sum_{x \in A} q(x, y|A, B) = \sum_{z \notin A} q(z, y|A \cup \{z\}, B)$
- This means, when we decompose the conditional graph at that edge, we can always subtract out that decomposition at every conditional graph along the path.

Conclusion

- We study stochastic choice data that captures the joint choice of multiple agents.
- We consider the extension of the latent variable hypothesis to multiple agents.
- Without imposing rationality, the latent variable hypothesis has no content in the single agent case.
- With multiple agents, the latent variable hypothesis has testable content beyond marginality.
- Without joint choice data, we may frequently fail to reject separable stochastic choice theories.