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Central limit theorems and uniform laws of large numbers for arrays of random fields

Nazgul Jenish[a,1], Ingmar R. Prucha[b,*]

[a] Department of Economics, New York University, 19 W. 4th Street, 6FL, NY 10012, United States

[b] Department of Economics, University of Maryland, College Park, MD 20742, United States

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ABSTRACT

Over the last decades, spatial-interaction models have been increasingly used in economics. However, the development of a sufficiently general asymptotic theory for nonlinear spatial models has been hampered by a lack of relevant central limit theorems (CLTs), uniform laws of large numbers (ULLNs) and pointwise laws of large numbers (LLNs). These limit theorems form the essential building blocks towards developing the asymptotic theory of M-estimators, including maximum likelihood and generalized method of moments estimators. The paper establishes a CLT, ULLN, and LLN for spatial processes or random fields that should be applicable to a broad range of data processes.

1. Introduction

Spatial-interaction models have a long tradition in geography, regional science and urban economics. For the last two decades, spatial-interaction models have also been increasingly considered in economics and the social sciences, in general. Applications range from their traditional use in agricultural, environmental, urban and regional economics to other branches of economics including international economics, industrial organization, labor and public economics, political economy, and macroeconomics.

The proliferation of spatial-interaction models in economics was accompanied by an upsurge in contributions to a rigorous theory of estimation and testing in spatial models. However, most of those contributions have focused on linear models of the Cliff–Ord type, cp. Cliff and Ord (1973, 1981), and Baltagi et al. (2007) for recent contributions. The development of a general asymptotic estimation theory for nonlinear spatial models under sets of assumptions that are both general and accessible for interpretation by applied researchers has been hampered by a lack of relevant central limit theorems (CLTs), uniform laws of large numbers (ULLNs), and pointwise laws of large numbers (LLNs) for dependent nonstationary spatial processes, also referred to as random fields. These limit theorems are the fundamental building blocks for the asymptotic theory of nonlinear spatial M-estimators, e.g., maximum likelihood and generalized method of moments estimators, and test statistics.[2]

Against this background, the aim of the paper is to establish a set of limit theorems under assumptions that are sufficiently general to accommodate a number of critical features frequently exhibited by processes in economic applications. Consequently, these limit theorems should allow for the development of a general asymptotic theory for parametric and non-parametric estimators of a wide range of linear and nonlinear spatial economic models. Many spatial processes in economics are nonstationary in that they

* Corresponding address: Department of Economics, University of Maryland, Tydings Hall, Room 3147A, College Park, MD 20742, United States. Tel.: +1 301 405 3499; fax: +1 301 405 3542.

E-mail addresses: nazgul.jenish@nyu.edu (N. Jenish), prucha@econ.umd.edu (I.R. Prucha).

1 Tel.: +1 212 998 3891; fax: +1 212 995 4186.

2 Conley (1999) makes an important contribution towards developing an asymptotic theory of GMM estimators for spatial processes. However, in deriving the limiting distribution of his estimator, he assumes stationarity which allows him to utilize Bolthausen's (1982) CLT for stationary random fields on $\mathbb{Z}^d$.

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are heteroskedastic and/or that the extent of dependence between variables may vary with locations. Furthermore, the processes may have asymptotically unbounded moments, in analogy with trending moments in the times series literature. For example, real estate prices often shoot up as one moves from the periphery towards the center of a megapolis; see, e.g., Bera and Simlai (2005) who report on sharp spikes in the variances of housing prices in Boston; for more examples, see also Cressie (1993). Spatial processes in economics are also typically not located on $\mathbb{Z}^d$, but on unevenly spaced lattices. Additionally, to cover some important classes of processes (e.g., Cliff–Ord type processes) where the random variables are also indexed by the sample size, it is critical to allow for a triangular array nature of the random field.

Towards these objectives, the paper derives a CLT, ULLN, and, for completeness, also an exemplary LLN for dependent random fields that (i) allow the random field to be nonstationary and even to exhibit asymptotically unbounded moments, (ii) allow for unevenly spaced locations and for general forms of sample regions, and (iii) allow the random variables to depend on the sample, i.e., to form a triangular array.

There exists an extensive literature on CLTs for mixing random fields. A comprehensive survey of this literature is provided in the important monographs by Bulinskii (1989), Nahapetian (1991), Doukhan (1994), Guyon (1995), Rio (2000) and Bradley (2007). As it turns out, none of the existing CLTs accommodates all of the above features essential for economic applications.

Our CLT for α-mixing random fields extends the Bolthausen (1982) and Guyon (1995) CLTs using Rio's (1993) covariance inequality. In the time series literature, Rio's inequality was employed by Doukhan et al. (1994) to derive a CLT for stationary α-mixing processes under an optimal set of moment and mixing conditions. Using the same inequality, Dedecker (1998) obtained a CLT for stationary α-mixing random fields on $\mathbb{Z}^d$. Building on these results, we establish a CLT for nonstationary α-mixing random fields on unevenly spaced lattices. As in Doukhan et al. (1994) and Dedecker (1998), Rio's (1993) inequality enables us to prove a CLT from a mild set of moment and mixing conditions. A detailed comparison of the proposed CLT with the existing results is given in Section 3.

ULLNs are the key tools for establishing consistency of nonlinear estimators; cp., e.g., Gallant and White (1988), p. 19, and Pötscher and Prucha (1997), p. 17. Generic ULLNs for time series processes have been introduced by Andrews (1987, 1992), Newey (1991) and Pötscher and Prucha (1989, 1994a,b). These ULLNs are generic in the sense that they transform pointwise LLNs into ULLNs given some form of stochastic equicontinuity of the summands. ULLNs for time series processes, by their nature, assume evenly spaced observations on a line. They are not suitable for fields on unevenly spaced lattices. The generic ULLN for random fields introduced in this paper extends the one-dimensional ULLNs given in Pötscher and Prucha (1994a) and Andrews (1992). In addition to the generic ULLN, we provide low level sufficient conditions for stochastic equicontinuity that are relatively easy to check.[3]

For completeness, we also give a pointwise weak LLN, which is based on a subset of the assumptions maintained in our CLT. Thus, the trio of the results established in this paper can be used jointly in the proof of consistency and asymptotic normality of spatial estimators. Of course, the generic ULLN can also be combined with other LLNs.

The remainder of the paper is organized as follows. Section 2 introduces the requisite notation and definitions. The CLT for arrays of nonstationary α- and ϕ-mixing random fields on irregular lattices is presented in Section 3. The generic ULLN, pointwise LLN and various sufficient conditions are discussed in Section 4. All proofs are relegated to the Appendices A–C. A longer version of the paper with additional discussions and more detailed proofs is available on the authors' web pages.

2. Weak dependence concepts and mixing inequalities

In this section, we introduce the notation and definitions used throughout the paper. We consider spatial processes located on a (possibly) unevenly spaced lattice $D \subseteq \mathbb{R}^d$, $d \geq 1$. It proves convenient to consider $\mathbb{R}^d$ as endowed with the metric $\rho(i,j) = \max_{1\leq l\leq d} |j_l - i_l|$, and the corresponding norm $|i| = \max_{1\leq l\leq d} |i_l|$, where i_l denotes the l-th component of i. The distance between any subsets $U, V \in D$ is defined as $\rho(U,V) = \inf\{\rho(i,j) : i \in U \text{ and } j \in V\}$. Furthermore, let $|U|$ denote the cardinality of a finite subset $U \in D$.

The two basic asymptotic methods commonly used in the spatial literature are the so-called increasing domain and infill asymptotics, see, e.g., Cressie (1993), p. 480. Under increasing domain asymptotics, the growth of the sample is ensured by an unbounded expansion of the sample region. In contrast, under infill asymptotics, the sample region remains fixed, and the growth of the sample size is achieved by sampling points arbitrarily dense in the given region. In this paper, we employ increasing domain asymptotics, which is ensured by the following assumption on the lattice D.

Assumption 1. The lattice $D \subset \mathbb{R}^d$, $d \geq 1$, is infinite countable. All elements in D are located at distances of at least $\rho_0 > 0$ from each other, i.e., $\forall i, j \in D : \rho(i,j) \geq \rho_0$; w.l.o.g. we assume that $\rho_0 > 1$.

The assumption of a minimum distance has also been used by Conley (1999). It turns out that this *single* restriction on irregular lattices also provides sufficient structure for the index sets to permit the derivation of our limit results. In contrast to many CLTs in the literature (e.g., Neaderhouser (1978a,b), Nahapetian (1987), Bolthausen (1982) and McElroy and Politis (2000)) we do not impose any restrictions on the configuration and growth behavior of the index sets. Based on Assumption 1, Lemma A.1 in the Appendix A gives bounds on the cardinalities of some basic sets in D that will be used in the proof of the limit theorems.

We now turn to the dependence concepts used in our theorems. Let $\{X_{i,n}; i \in D_n, n \in \mathbb{N}\}$ be a triangular array of real random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$, where D_n is a finite subset of D, and D satisfies Assumption 1. Further, let $\mathfrak{A}$ and $\mathfrak{B}$ be two sub-σ-algebras of $\mathfrak{F}$. Two common concepts of dependence between $\mathfrak{A}$ and $\mathfrak{B}$ are α- and ϕ-mixing, which have been introduced, respectively, by Rosenblatt and Ibragimov. The degree of dependence is measured in terms of the following α- and ϕ-mixing coefficients:

$$\alpha(\mathfrak{A}, \mathfrak{B}) = \sup(|P(A \cap B) - P(A)P(B)|, A \in \mathfrak{A}, B \in \mathfrak{B}),$$
$$\phi(\mathfrak{A}, \mathfrak{B}) = \sup(|P(A|B) - P(A)|, A \in \mathfrak{A}, B \in \mathfrak{B}, P(B) > 0).$$

The concepts of α- and ϕ-mixing have been used extensively in the time series literature as measures of weak dependence. Recall that a time series process $\{X_t\}_{-\infty}^{\infty}$ is α-mixing if

$$\lim_{m\to\infty} \sup_t \alpha(\mathfrak{F}_{-\infty}^t, \mathfrak{F}_{t+m}^{+\infty}) = 0$$

where $\mathfrak{F}_{-\infty}^t = \sigma(\ldots, X_{t-1}, X_t)$ and $\mathfrak{F}_{t+m}^{\infty} = \sigma(X_{t+m}, X_{t+m+1}, \ldots)$. This definition captures the basic idea of diminishing dependence between different events as the distance between them increases.

[3] The existing literature on the estimation of nonlinear spatial models has maintained high-level assumptions such as first moment continuity to imply uniform convergence; cp., e.g., Conley (1999). The results in this paper are intended to be more accessible, and in allowing, e.g., for nonstationarity, to cover larger classes of processes.

To generalize these concepts to random fields, one could resort to a direct analogy with the time-series literature and, for instance, define mixing coefficients over the σ-algebras generated by the half-spaces perpendicular to the coordinate axes. However, as demonstrated by Dobrushin (1968a,b), the resulting mixing conditions are generally restrictive for $d > 1$. They are violated even for simple two-state Markov chains on $\mathbb{Z}^2$. The problem with definitions of this ilk is that they neglect potential accumulation of dependence between σ-algebras $\sigma(X_i; i \in V_1)$ and $\sigma(X_i; i \in V_2)$ as the sets V_1 and V_2 expand while the distance between them is kept fixed. Given a fixed distance, it is natural to expect more dependence between two larger sets than between two smaller sets.

Thus, generalizing mixing concepts to random fields in a practically useful way requires accounting for the sizes of subsets on which σ-algebras reside. Mixing conditions that depend on subsets of the lattice date back to Dobrushin (1968a,b). They were further expanded by Bolthausen (1982) and Nahapetian (1987). Following these authors, we adopt the following definitions of mixing:

Definition 1. For $U \subseteq D_n$ and $V \subseteq D_n$, let $\sigma_n(U) = \sigma(X_{i,n}; i \in U)$, $\alpha_n(U, V) = \alpha(\sigma_n(U), \sigma_n(V))$ and $\phi_n(U, V) = \phi(\sigma_n(U), \sigma_n(V))$. Then, the α- and ϕ-mixing coefficients for the random field $\{X_{i,n}; i \in D_n, n \in \mathbb{N}\}$ are defined as follows:

$$\alpha_{k,l,n}(r) = \sup(\alpha_n(U, V), |U| \le k, |V| \le l, \rho(U, V) \ge r),$$
$$\phi_{k,l,n}(r) = \sup(\phi_n(U, V), |U| \le k, |V| \le l, \rho(U, V) \ge r),$$

with $k, l, r, n \in \mathbb{N}$. To account for the dependence of the random field on n, we define furthermore:

$$\overline{\alpha}_{k,l}(r) = \sup_n \alpha_{k,l,n}(r), \qquad \overline{\phi}_{k,l}(r) = \sup_n \phi_{k,l,n}(r).$$

As shown by Dobrushin (1968a,b), the weak dependence conditions based on the above mixing coefficients are satisfied by large classes of random fields including Gibbs fields. These mixing coefficients were also used by Doukhan (1994) and Guyon (1995), albeit without dependence on n. The α-mixing coefficients for arrays of random fields in McElroy and Politis (2000) are defined identically to $\overline{\alpha}_{k,l}(r)$. Doukhan (1994) provides an excellent overview of various mixing concepts.

A key role in establishing limit theorems for mixing processes is played by covariance inequalities. Early covariance inequalities for α- and ϕ-mixing fields are collected, e.g., in Hall and Heyde (1980), pp. 277–280. In these inequalities, covariances are bounded by power moments. As such, they do not allow for "lower", e.g., logarithmic, moments and are, therefore, not sharp. These inequalities were later improved by various authors including Herrndorf (1985), Bulinskii (1988), Bulinskii and Doukhan (1987), and Rio (1993). The proof of the α-mixing part of our CLT relies on Rio's (1993) inequality. Instead of moments, it is based on upper-quantile functions, which makes explicit the relationship between the mixing coefficients and tail distribution of the process. To state this result and to formulate the assumptions of our CLT, we need the following definitions.

Definition 2. (i) For a random variable X, the "upper-tail" quantile function $Q_X : (0, 1) \to [0, \infty)$ is defined as

$$Q_X(u) = \inf\{t : P(X > t) \le u\}.$$

(ii) For the non-increasing sequence of the mixing coefficients $\{\overline{\alpha}_{1,1}(m)\}_{m=1}^{\infty}$, set $\overline{\alpha}_{1,1}(0) = 1$ and define its "inverse" function $\alpha_{inv}(u)$: $(0, 1) \to \mathbb{N} \cup \{0\}$ as:

$$\alpha_{inv}(u) = \max\left\{m \ge 0 : \overline{\alpha}_{1,1}(m) > u\right\}.$$

Remark 1. For a detailed discussion of the "upper-tail" quantile function see, e.g., Bradley (2007), Vol. 1, pp. 318. It proves helpful to re-state some of those properties. For a random variable X, let $F_X(x) = P(X \le x)$ denote the cumulative distribution function and let $F_X^{-1}(u) = \inf\{x : F_X(x) \ge u\}$ be the usual quantile function of X. Then

$$Q_X(u) = F_X^{-1}(1 - u).$$

Clearly, $Q_X(u)$ is non-increasing in $(0, 1)$. Furthermore, if U is a random variable that is uniformly distributed on $[0, 1]$, then the random variable $Q_X(U)$ has the same distribution as X, and thus for any Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $E|f(X)| < \infty$, we have

$$Ef(X) = \int_0^1 f(Q_X(u))\mathrm{d}u.$$

If X and Y are random variables such that $X \le Y$ a.s., then for all $u \in (0, 1)$, $Q_X(u) \le Q_Y(u)$. If $X \ge 0$ a.s., then for all $u \in (0, 1)$, $Q_X(u) \ge 0$.

Using the upper-tail quantile function, Rio (1993) obtains a sharper covariance inequality for α-mixing variables. In deriving our CLT, we will use the following slightly weaker version of Rio's inequality given in Bradley (2007, Vol. 1, p. 320): Suppose that X and Y are two real-valued random variables such that $E|X| < \infty$, $E|Y| < \infty$ and $\int_0^1 Q_{|X|}(u)Q_{|Y|}(u)\mathrm{d}u < \infty$. Let $\alpha = \alpha(\sigma(X), \sigma(Y))$, then

$$|Cov(X, Y)| \le 4\int_0^{\alpha} Q_{|X|}(u)Q_{|Y|}(u)\mathrm{d}u. \tag{1}$$

3. Central limit theorem

In this section, we provide a CLT for random fields with (possibly) asymptotically unbounded moments. Let $\{Z_{i,n}; i \in D_n, n \in \mathbb{N}\}$ be an array of zero-mean real random fields on a probability space $(\Omega, \mathfrak{F}, P)$, where the index sets D_n are finite subsets of $D \subset \mathbb{R}^d$, $d \ge 1$, which is assumed to satisfy Assumption 1. In the following, let $S_n = \sum_{i \in D_n} Z_{i,n}$ and $\sigma_n^2 = \mathrm{Var}(S_n)$.

The CLT focuses on α- and ϕ-mixing fields which satisfy, respectively, the following sets of assumptions.

Assumption 2 (*Uniform L_2 Integrability*). There exists an array of positive real constants $\{c_{i,n}\}$ such that

$$\lim_{k\to\infty} \sup_n \sup_{i \in D_n} E[|Z_{i,n}/c_{i,n}|^2 \mathbf{1}(|Z_{i,n}/c_{i,n}| > k)] = 0,$$

where $\mathbf{1}(\cdot)$ *is the indicator function.*

To formulate the next assumption, let

$$\overline{Q}_{i,n}^{(k)} := Q_{|Z_{i,n}/c_{i,n}|\mathbf{1}(|Z_{i,n}/c_{i,n}|>k)},$$

denote the "upper-tail" quantile function of $|Z_{i,n}/c_{i,n}|\mathbf{1}(|Z_{i,n}/c_{i,n}| > k)$, and let $\alpha_{inv}(u)$ denote the inverse function of $\overline{\alpha}_{1,1}(m)$ as given in Definition 2(ii).

Assumption 3 (α-*mixing*). The α-mixing coefficients satisfy:

(a) $\lim_{k\to\infty} \sup_n \sup_{i\in D_n} \int_0^1 \alpha_{inv}^d(u)\left(\overline{Q}_{i,n}^{(k)}(u)\right)^2 \mathrm{d}u = 0$,
(b) $\sum_{m=1}^{\infty} m^{d-1}\overline{\alpha}_{k,l}(m) < \infty$ for $k + l \le 4$,
(c) $\overline{\alpha}_{1,\infty}(m) = O(m^{-d-\varepsilon})$ for some $\varepsilon > 0$.

Assumption 4 (ϕ-*mixing*). The ϕ-mixing coefficients satisfy:

(a) $\sum_{m=1}^{\infty} m^{d-1}\overline{\phi}_{1,1}^{1/2}(m) < \infty$,
(b) $\sum_{m=1}^{\infty} m^{d-1}\overline{\phi}_{k,l}(m) < \infty$ for $k + l \le 4$,
(c) $\overline{\phi}_{1,\infty}(m) = O(m^{-d-\varepsilon})$ for some $\varepsilon > 0$.

Assumption 5. $\liminf_{n\to\infty} |D_n|^{-1} M_n^{-2}\sigma_n^2 > 0$, where $M_n = \max_{i\in D_n} c_{i,n}$.

Based on the above set of assumptions, we can now state the following CLT.

Theorem 1. *Suppose $\{D_n\}$ is a sequence of arbitrary finite subsets of D, satisfying Assumption 1, with $|D_n| \to \infty$ as $n \to \infty$. Suppose further that $\{Z_{i,n}; i \in D_n, n \in \mathbb{N}\}$ is an array of zero-mean real-valued random variables satisfying Assumption 2 and where the random field is either*

(a) *α-mixing satisfying Assumption 3, or*
(b) *ϕ-mixing satisfying Assumption 4.*

If in addition Assumption 5 holds, then

$$\sigma_n^{-1} S_n \Longrightarrow N(0, 1).$$

Clearly, the CLT can be readily extended to vector-valued random fields using the standard Cramér-Wold device. Assumption 2 is a standard moment assumption seen in CLTs for time series processes with trending moments; see, e.g., De Jong (1997) and the references cited therein. It is implied by uniform L_r boundedness for some $r > 2$: $\sup_{n,i\in D_n} E\left|Z_{i,n}/c_{i,n}\right|^r < \infty$, see Billingsley (1986), p. 219.

The nonrandom constants $c_{i,n}$ in Assumption 2 are scaling numbers that allow for processes with asymptotically unbounded (trending) second moments. As remarked in the Introduction, economic data are frequently nonstationary. They may even exhibit dramatic differences in the magnitudes of their second moments. As an example, Bera and Simlai (2005) report on sharp spikes in the variances of housing prices in Boston. Of course, in this context, one could also expect household incomes, household wealth, property taxes, etc., to show similar features. Empirical researchers may be reasonably concerned in such situations as to whether an asymptotic theory that assumes uniformly bounded moments would provide a good approximation of the small sample distribution of their estimators and test statistics. Towards establishing an asymptotic theory that is reasonably robust to such "irregular" behavior of second moments, Assumption 2 avoids the assumption that the second moments are uniformly bounded.

In the case of uniformly L_2-bounded fields, the scaling numbers $c_{i,n}$ can be set to 1. In the case when there is no uniform bound on the second moments, they would typically be chosen as $c_{i,n} = \max(v_{i,n}, 1)$, where $v_{i,n}^2 = EZ_{i,n}^2$. A simple but illustrative example of a process with asymptotically unbounded second moments is the process $Z_{i,n} = c_{i,n}X_i$, where the $c_{i,n}$ are nonrandom constants increasing in $|i|$, and $\{X_i\}$ is a uniformly square integrable family. In the case $d = 1$, a CLT for this class of processes was established by Peligrad and Utev (2003), see Corollary 2.2.

Assumption 5 is a counterpart to Assumption 2, and may be viewed as an asymptotic negligibility condition, which ensures that no single summand dominates the sum. In the case of uniformly L_2-bounded fields, it reduces to $\liminf_{n\to\infty} |D_n|^{-1}\sigma_n^2 > 0$, which is the condition used by Guyon (1995); cp. also Bolthausen (1982).

To further illuminate some of the implications of Assumptions 2 and 5, we consider two special cases of the above illustrative example: Let $\{Z_i, i \in D_n \subset \mathbb{Z}^d\}$ be an independent zero-mean random field with $D_n = [1; n]^d$ and $|D_n| = n^d$. Suppose $EZ_i^2 = |i|^\gamma$ for some $\gamma > 0$. Then, as discussed above, $c_i^2 = |i|^\gamma$, $M_n^2 = n^\gamma$ and $\sigma_n^2 \sim n^{(\gamma+d)}$. Clearly, in this case Assumptions 2 and 5 are satisfied for all $\gamma > 0$. Next, suppose that $EZ_i^2 = 2^{|i|}$. Then, $c_i^2 = 2^{|i|}$, $M_n^2 = 2^n$ and $\sigma_n^2 \sim n^{d-1}2^n$, and hence, $\liminf_{n\to\infty} |D_n|^{-1}M_n^{-2}\sigma_n^2 = 0$. Thus, Assumption 5 is violated in this case.

Assumption 3(a) reflects the trade-off between the conditions on the tail functions and mixing coefficients. To connect this assumption to the literature, we note that in the case of stationary square integrable random fields, Assumption 3(a) is implied by a somewhat simpler condition. More specifically, if

$$\int_0^1 \alpha_{inv}^d(u) Q_{|Z_i|}^2(u)\mathrm{d}u < \infty, \tag{2}$$

then, $\lim_{k\to\infty} \int_0^1 \alpha_{inv}^d(u)\left(Q_{|Z_i|}^{(k)}(u)\right)^2 \mathrm{d}u = 0$, where $i \in D_n$ and $Q_{|Z_i|}^{(k)} := Q_{|Z_i|\mathbf{1}(|Z_i|>k)}$. This statement is proved in the appendix after the proof of the Corollary 1. For the case $d = 1$, condition (2) was used by Doukhan et al. (1994); cp. also Dedecker (1998). In the proof of Theorem 1, we show conversely that under the maintained assumptions

$$\sup_n \sup_{i\in D_n} \int_0^1 \alpha_{inv}^d(u) Q_{i,n}^2(u)\mathrm{d}u < \infty.$$

In the nonstationary case, a sufficient condition for Assumption 3 (a) is given in the following lemma. This condition involves uniform $L_{2+\delta}$ integrability, which may be easier to verify in applications.

Corollary 1. *Suppose the random field $\{Z_{i,n}; i \in D_n, n \in \mathbb{N}\}$ satisfies the assumptions of the α-mixing part of Theorem 1, except that Assumptions 2 and 3(a) are replaced by: For for some $\delta > 0$*

$$\lim_{k\to\infty} \sup_n \sup_{i\in D_n} E\left[\left|Z_{i,n}/c_{i,n}\right|^{2+\delta} \mathbf{1}\left(\left|Z_{i,n}/c_{i,n}\right| > k\right)\right] = 0, \tag{3}$$

and

$$\sum_{m=1}^{\infty} \overline{\alpha}_{1,1}(m) m^{[d(2+\delta)/\delta]-1} < \infty. \tag{4}$$

Then, Assumptions 2 and 3(a), and hence the conclusion of Theorem 1 hold.

Condition (3) is a typical moment assumption used in the CLTs for α-mixing processes. As shown in the proof of the Corollary 1, Condition (4) is weaker than the mixing condition $\sum_{m=1}^{\infty} m^{d-1}\overline{\alpha}_{1,1}(m)^{\delta/(2+\delta)} < \infty$, used in Bolthausen (1982).

We now relate Theorem 1 to existing results in the literature. In a seminal contribution, Bolthausen (1982) introduced a CLT for stationary α-mixing random fields on $\mathbb{Z}^d$, using Stein's (1972) lemma. The proof of Theorem 1 is also based on Stein's (1972) lemma, but Theorem 1 extends Bolthausen's 1982 CLT in the following directions: (i) it allows for nonstationary random fields with asymptotically unbounded moments, (ii) it allows for unevenly spaced locations and relaxes restrictions on index sets, and (iii) allows for triangular arrays.

As discussed in the Introduction, there exists a vast literature on CLTs for mixing random fields. However, we are not aware of a CLT for random fields that accommodates all of the above crucial features, and/or contains Theorem 1 as a special case. Seminal and important contributions include Neaderhouser (1978a,b), Nahapetian (1987), Bolthausen (1982), Guyon and Richardson (1984), Bulinskii (1988, 1989), Bradley (1992), Guyon (1995), Dedecker (1998), McElroy and Politis (2000), among others. All of these CLTs are for random fields on the evenly spaced lattice $\mathbb{Z}^d$, and the CLTs of Nahapetian (1987), Bolthausen (1982), Bradley (1992) and Dedecker (1998) furthermore maintain stationarity. The other papers permit nonstationarity but do not explicitly allow for processes with asymptotically unbounded moments. Also, most CLTs do not accommodate triangular arrays, and impose restrictions on the configuration and growth behavior of sample regions.

We next compare the moment and mixing conditions of some of the CLTs for nonstationary random fields with those maintained by Theorem 1. Guyon and Richardson (1984) and Guyon (1995), p. 11, consider random fields on $\mathbb{Z}^d$ with uniformly bounded $2+\delta$ moments, while Theorem 1 assumes uniform L_2 integrability. Moreover, Guyon and Richardson (1984) exploit mixing conditions for $\alpha_{\infty,\infty}(r)$, which is somewhat restrictive, as discussed earlier.

Bulinskii (1988) establishes a covariance inequality for real random variables in Orlicz spaces, which allows him to derive a CLT for nonstationary α-mixing fields on $\mathbb{Z}^d$ under a set of weak moment and mixing conditions. However, as shown by Rio (1993), given a moment condition, Bulinskii's inequality results in slightly stronger mixing conditions than those implied by Rio's (1993) inequality, which was used in deriving Theorem 1 and Corollary 1. For instance, in the case of finite $2+\delta$ moments, Bulinskii's (1988) CLT would involve the mixing condition $\sum_{m=1}^{\infty} m^{d-1}\overline{\alpha}_{1,1}(m)^{\delta/(2+\delta)} < \infty$. As noted above, this condition is stronger than Condition (4) postulated in Corollary 1. Finally, Neaderhouser (1978a,b) and McElroy and Politis (2000) rely on more stringent moment and mixing conditions than Theorem 1.

4. Uniform law of large numbers

Uniform laws of large numbers (ULLNs) are key tools for establishing consistency of nonlinear estimators. Suppose the true parameter of interest is $\theta_0 \in \Theta$, where Θ is the parameter space, and $\widehat{\theta}_n$ is a corresponding estimator defined as the maximizer of some real valued objective function $Q_n(\theta)$ defined on Θ, where the dependence on the data is suppressed. Suppose further that $EQ_n(\theta)$ is maximized at θ_0, and that θ_0 is identifiably unique. Then for $\widehat{\theta}_n$ to be consistent for θ_0, it suffices to show that $Q_n(\theta) - EQ_n(\theta)$ converge to zero uniformly over the parameter space; see, e.g., Gallant and White (1988), pp. 18, and Pötscher and Prucha (1997), pp. 16, for precise statements, which also allow the maximizers of $EQ_n(\theta)$ to depend on n. For many estimators the uniform convergence of $Q_n(\theta) - EQ_n(\theta)$ is established from a ULLN.

In the following, we give a generic ULLN for spatial processes. The ULLN is generic in the sense that it turns a pointwise LLN into the corresponding uniform LLN. This generic ULLN assumes (i) that the random functions are stochastically equicontinuous in the sense made precise below, and (ii) that the functions satisfy an LLN for a given parameter value. For stochastic processes this approach was taken by Newey (1991), Andrews (1992), and Pötscher and Prucha (1994a).[4] Of course, to make the approach operational for random fields, we need an LLN, and therefore we also give an LLN for random fields. This LLN is exemplary, but has the convenient feature that it holds under a subset of the conditions maintained for the CLT. We also report on two sets of sufficient conditions for stochastic equicontinuity that are fairly easy to verify.

Just as for our CLT, we consider again arrays of random fields residing on a (possibly) unevenly spaced lattice D, where $D \subset \mathbb{R}^d$, $d \geq 1$, is assumed to satisfy Assumption 1. However, for the ULLN the array is not assumed to be real-valued. More specifically, in the following let $\{Z_{i,n};\, i \in D_n, n \in \mathbb{N}\}$, with D_n a finite subset of D, denote a triangular array of random fields defined on a probability space $(\Omega, \mathfrak{F}, P)$ and taking their values in Z, where $(Z, \mathcal{Z})$ is a measurable space. In applications, Z will typically be a subset of $\mathbb{R}^s$, i.e., $Z \subseteq \mathbb{R}^s$, and $\mathcal{Z} \subseteq \mathfrak{B}^s$, where $\mathfrak{B}^s$ denotes the s-dimensional Borel σ-field. We remark, however, that it suffices for the ULLN below if $(Z, \mathcal{Z})$ is only a measurable space. Further, in the following, let $\{f_{i,n}(z,\theta), i \in D_n, n \in \mathbb{N}\}$ and $\{q_{i,n}(z,\theta), i \in D_n, n \in \mathbb{N}\}$ be doubly-indexed families of real-valued functions defined on $Z \times \Theta$, i.e., $f_{i,n}\colon Z \times \Theta \to \mathbb{R}$ and $q_{i,n}\colon Z \times \Theta \to \mathbb{R}$, where (Θ, ν) is a metric space with metric ν. Throughout the paper, the $f_{i,n}(\cdot,\theta)$ and $q_{i,n}(\cdot,\theta)$ are assumed $\mathcal{Z}/\mathfrak{B}$-measurable for each $\theta \in \Theta$ and for all $i \in D_n$, $n \geq 1$. Finally, let $B(\theta',\delta)$ be the open ball $\{\theta \in \Theta : \nu(\theta',\theta) < \delta\}$.

4.1. Generic uniform law of large numbers

The literature contains various definitions of stochastic equicontinuity. For a discussion of different stochastic equicontinuity concepts see, e.g., Andrews (1992) and Pötscher and Prucha (1994a). We note that apart from differences in the mode of convergence, the essential differences in those definitions relate to the degree of uniformity. We shall employ the following definition.[5]

Definition 3. Consider the array of random functions $\{f_{i,n}(Z_{i,n},\theta), i \in D_n, n \geq 1\}$. Then $f_{i,n}$ is said to be

(a) L_0 stochastically equicontinuous on Θ iff for every $\varepsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} P\Big(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)} |f_{i,n}(Z_{i,n},\theta) - f_{i,n}(Z_{i,n},\theta')| > \varepsilon\Big) \to 0 \quad \text{as } \delta \to 0;$$

(b) L_p stochastically equicontinuous, $p > 0$, on Θ iff

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} E\Big(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)} |f_{i,n}(Z_{i,n},\theta) - f_{i,n}(Z_{i,n},\theta')|^p\Big) \to 0 \quad \text{as } \delta \to 0;$$

(c) a.s. stochastically equicontinuous on Θ iff

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} \sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)} |f_{i,n}(Z_{i,n},\theta) - f_{i,n}(Z_{i,n},\theta')| \to 0 \quad \text{a.s. as } \delta \to 0.$$

Stochastic equicontinuity-type concepts have been used widely in the statistics and probability literature; see, e.g., Pollard (1984). Andrews (1992), within the context of one-dimensional processes, refers to L_0 stochastic equicontinuity as termwise stochastic equicontinuity. Pötscher and Prucha (1994a) refer to the stochastic equicontinuity concepts in Definition 3(a) [(b)], [[(c)]] as asymptotic Cesàro L_0 [L_p], [[a.s.]] uniform equicontinuity, and adopt the abbreviations ACL_0UEC [ACL_pUEC],[[$a.s.ACUEC$]]. The following relationships among the equicontinuity concepts are immediate: $ACL_pUEC \Longrightarrow ACL_0UEC \Longleftarrow a.s.ACUEC$.

In formulating our ULLN, we will allow again for trending moments. We will employ the following domination condition.

Assumption 6 (*Domination Condition*). There exists an array of positive real constants $\{c_{i,n}\}$ such that for some $p \geq 1$:

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} E(d_{i,n}^p \mathbf{1}(d_{i,n} > k)) \to 0 \quad \text{as } k \to \infty$$

where $d_{i,n}(\omega) = \sup_{\theta\in\Theta} |q_{i,n}(Z_{i,n}(\omega),\theta)| / c_{i,n}$.

We now have the following generic ULLN.

Theorem 2. *Suppose $\{D_n\}$ is a sequence of arbitrary finite subsets of D, satisfying Assumption 1, with $|D_n| \to \infty$ as $n \to \infty$. Let (Θ, ν) be a totally bounded metric space, and suppose $\{q_{i,n}(z,\theta), i \in D_n, n \in \mathbb{N}\}$ is a doubly-indexed family of real-valued functions defined on $Z \times \Theta$ satisfying Assumption 6. Suppose further that the $q_{i,n}(Z_{i,n},\theta)/c_{i,n}$ are L_0 stochastically equicontinuous on Θ, and that for*

[4] We note that the uniform convergence results of Bierens (1981), Andrews (1987), and Pötscher and Prucha (1989, 1994b) were obtained from closely related approach by verifying the so-called first moment continuity condition and from local laws of large numbers for certain bracketing functions. For a detailed discussion of similarities and differences, see Pötscher and Prucha (1994a).

[5] All suprema and infima over subsets of Θ of random functions used below are assumed to be P-a.s. measurable. For sufficient conditions see, e.g., Pollard (1984), Appendix C, or Pötscher and Prucha (1994b), Lemma 2.

all $\theta \in \Theta_0$, where Θ_0 is a dense subset of Θ, the stochastic functions $q_{i,n}(Z_{i,n}, \theta)$ satisfy a pointwise LLN in the sense that

$$\frac{1}{M_n |D_n|} \sum_{i \in D_n} \left[q_{i,n}(Z_{i,n}, \theta) - Eq_{i,n}(Z_{i,n}, \theta) \right] \to 0$$

$$\text{i.p. [a.s.] as } n \to \infty, \tag{5}$$

where $M_n = \max_{i \in D_n} c_{i,n}$. Let $Q_n(\theta) = [M_n |D_n|]^{-1} \sum_{i \in D_n} q_{i,n}(Z_{i,n}, \theta)$, then

(a)

$$\sup_{\theta \in \Theta} |Q_n(\theta) - EQ_n(\theta)| \to 0 \quad \text{i.p. [a.s.] as } n \to \infty \tag{6}$$

(b) *$\overline{Q}_n(\theta) = EQ_n(\theta)$ is uniformly equicontinuous in the sense that*

$$\lim_{n \to \infty} \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} \left| \overline{Q}_n(\theta) - \overline{Q}_n(\theta') \right| \to 0 \quad \text{as } \delta \to 0. \tag{7}$$

The above ULLN adapts Corollary 4.3 in Pötscher and Prucha (1994a) to arrays of random fields, and also allows for asymptotically unbounded moments. The case of uniformly bounded moments is covered as a special case with $c_{i,n} = 1$ and $M_n = 1$.

The ULLN allows for infinite-dimensional parameter spaces. It only maintains that the parameter space is totally bounded rather than compact. (Recall that a set of a metric space is totally bounded if for each $\varepsilon > 0$ it can be covered by a finite number of ε-balls). If the parameter space Θ is a finite-dimensional Euclidian space, then total boundedness is equivalent to boundedness, and compactness is equivalent to boundedness and closedness. By assuming only that the parameter space is totally bounded, the ULLN covers situations where the parameter space is not closed, as is frequently the case in applications.

Assumption 6 is implied by uniform integrability of individual terms, $d_{i,n}^p$, i.e., $\lim_{k \to \infty} \sup_n \sup_{i \in D_n} E(d_{i,n}^p \mathbf{1}(d_{i,n} > k)) = 0$, which, in turn, follows from their uniform L_r-boundedness for some $r > p$, i.e., $\sup_n \sup_{i \in D_n} \left\| d_{i,n} \right\|_r < \infty$.

Sufficient conditions for the pointwise LLN and the maintained L_0 stochastic equicontinuity of the normalized function $q_{i,n}(Z_{i,n}, \theta)/c_{i,n}$ are given in the next two subsections. The theorem only requires the pointwise LLN (5) to hold on a dense subset Θ_0, but, of course, also covers the case where $\Theta_0 = \Theta$.

As it will be seen from the proof, L_0 stochastic equicontinuity of $q_{i,n}(Z_{i,n}, \theta)/c_{i,n}$ and the Domination Assumption 6 jointly imply that $q_{i,n}(Z_{i,n}, \theta)/c_{i,n}$ is L_p stochastic equicontinuous for $p \geq 1$, which in turn implies uniform convergence of $Q_n(\theta)$ provided that a pointwise LLN is satisfied. Therefore, the weak part of the ULLN will continue to hold if L_0 stochastic equicontinuity and Assumption 6 are replaced by the single assumption of L_p stochastic equicontinuity for some $p \geq 1$.

4.2. Pointwise law of large numbers

The generic ULLN is modular in the sense that it assumes a pointwise LLN for the stochastic functions $q_{i,n}(Z_{i,n}; \theta)$ for fixed $\theta \in \Theta$. Given this feature, a ULLN can be obtained by combining the generic ULLN with available LLNs. In the following, we give an exemplary LLN for arrays of real random fields $\{Z_{i,n}; i \in D_n, n \in \mathbb{N}\}$ taking values in $Z = \mathbb{R}$ with possibly asymptotically unbounded moments, which can in turn be used to establish a LLN for $q_{i,n}(Z_{i,n}; \theta)$. The LLN below has the convenient feature that it holds under a subset of assumptions of the CLT, Theorem 1, which simplifies their joint application.

The CLT was derived under the assumption that the random field was uniformly L_2 integrable. As expected, for the LLN it suffices to assume uniform L_1 integrability.

Assumption 2* (*Uniform L_1 Integrability*). There exists an array of positive real constants $\{c_{i,n}\}$ such that

$$\lim_{k \to \infty} \sup_n \sup_{i \in D_n} E \left[\left| Z_{i,n}/c_{i,n} \right| \mathbf{1} \left(\left| Z_{i,n}/c_{i,n} \right| > k \right) \right] = 0,$$

where $\mathbf{1}(\cdot)$ is the indicator function.

A sufficient condition for Assumption 2* is $\sup_n \sup_{i \in D_n} E \left| Z_{i,n}/c_{i,n} \right|^{1+\eta} < \infty$ for some $\eta > 0$. We now have the following LLN.

Theorem 3. *Suppose $\{D_n\}$ is a sequence of arbitrary finite subsets of D, satisfying Assumption 1, with $|D_n| \to \infty$ as $n \to \infty$. Suppose further that $\{Z_{i,n}; i \in D_n, n \in \mathbb{N}\}$ is an array of real random fields satisfying Assumption 2* and where the random field is either*

(a) *α-mixing satisfying Assumption 3 (b) with $k = l = 1$, or*
(b) *ϕ-mixing satisfying Assumption 4 (b) with $k = l = 1$.*

Then

$$\frac{1}{M_n |D_n|} \sum_{i \in D_n} \left(Z_{i,n} - EZ_{i,n} \right) \xrightarrow{L_1} 0,$$

where $M_n = \max_{i \in D_n} c_{i,n}$.

The existence of first moments is assured by the uniform L_1 integrability assumption. Of course, L_1-convergence implies convergence in probability, and thus the $Z_{i,n}$ also satisfies a weak law of large numbers. Comparing the LLN with the CLT reveals that not only the moment conditions employed in the former are weaker than those in the latter, but also the dependence conditions in the LNN are only a subset of the mixing assumptions maintained for the CLT.

There is a vast literature on weak LLNs for time series processes. Most recent contributions include Andrews (1988) and Davidson (1993), among others. Andrews (1988) established an L_1-law for triangular arrays of L_1-mixingales. Davidson (1993) extended the latter result to L_1-mixingale arrays with trending moments. Both results are based on the uniform integrability condition. In fact, our moment assumption is identical to that of Davidson (1993). The mixingale concept, which exploits the natural order and structure of the time line, is formally weaker than that of mixing. It allows these authors to circumvent restrictions on the sizes of mixingale coefficients, i.e., rates at which dependence decays. In contrast, the above LLN maintains assumptions on the rates of decay of the mixing coefficients.

The above LLN can be readily used to establish a pointwise LLN for stochastic functions $q_{i,n}(Z_{i,n}; \theta)$ under the α- and ϕ-mixing conditions on $Z_{i,n}$ postulated in the theorem. For instance, suppose that $q_{i,n}(\cdot, \theta)$ is $\mathcal{Z}/\mathfrak{B}$-measurable and $\sup_n \sup_{i \in D_n} E|q_{i,n}(Z_{i,n}; \theta)/c_{i,n}|^{1+\eta} < \infty$ for each $\theta \in \Theta$ and some $\eta > 0$, then $q_{i,n}(Z_{i,n}; \theta)/c_{i,n}$ is uniformly L_1 integrable for each $\theta \in \Theta$. Recalling that the α- and ϕ-mixing conditions are preserved under measurable transformation, we see that $q_{i,n}(Z_{i,n}; \theta)$ also satisfies an LNN for a given parameter value θ.

4.3. Stochastic equicontinuity: Sufficient conditions

In the previous sections, we saw that stochastic equicontinuity is a key ingredient of a ULLN. In this section, we explore various sufficient conditions for L_0 and *a.s.* stochastic equicontinuity of functions $f_{i,n}(Z_{i,n}, \theta)$ as in Definition 3. These conditions place smoothness requirement on $f_{i,n}(Z_{i,n}, \theta)$ with respect to the parameter and/or data. In the following, we will present two sets of sufficient conditions. The first set of conditions represent Lipschitz-type conditions, and only requires smoothness of $f_{i,n}(Z_{i,n}, \theta)$ in the parameter θ. The second set requires less smoothness in the parameter, but maintains joint continuity of $f_{i,n}$ both in the parameter and data. These conditions should cover a wide range of applications and are relatively simple to verify. Lipschitz-type conditions for one-dimensional processes were proposed by Andrews (1987, 1992) and Newey (1991). Joint continuity-type conditions for one-dimensional processes were introduced by Pötscher and Prucha (1989). In the following, we adapt those conditions to random fields.

We continue to maintain the setup defined at the beginning of the section.

4.3.1. Lipschitz in parameter

Condition 1. *The array $f_{i,n}(Z_{i,n},\theta)$ satisfies for all $\theta,\theta' \in \Theta$ and $i \in D_n$, $n \geq 1$ the following condition:*

$$\left|f_{i,n}(Z_{i,n},\theta) - f_{i,n}(Z_{i,n},\theta')\right| \leq B_{i,n} h(\nu(\theta,\theta')) \text{ a.s.},$$

where h is a nonrandom function such that $h(x) \downarrow 0$ as $x \downarrow 0$, and $B_{i,n}$ are random variables that do not depend on θ such that for some $p > 0$

$$\limsup_{n\to\infty} |D_n|^{-1} \sum_{i\in D_n} EB_{i,n}^p < \infty$$

$$\left[\limsup_{n\to\infty} |D_n|^{-1} \sum_{i\in D_n} B_{i,n} < \infty \text{ a.s.}\right].$$

Clearly, each of the above conditions on the Cesàro sums of $B_{i,n}$ is implied by the respective condition on the individual terms, i.e., $\sup_n \sup_{i\in D_n} EB_{i,n}^p < \infty$ [$\sup_n \sup_{i\in D_n} B_{i,n} < \infty$ a.s.].

Proposition 1. *Under Condition 1, $f_{i,n}(Z_{i,n},\theta)$ is L_0 [a.s.] stochastically equicontinuous on Θ.*

4.3.2. Continuous in parameter and data

In this subsection, we assume additionally that Z is a metric space with metric τ and $\mathcal{Z}$ is the corresponding Borel σ-field.

We consider functions of the form:

$$f_{i,n}(Z_{i,n},\theta) = \sum_{k=1}^{K} r_{ki,n}(Z_{i,n}) s_{ki,n}(Z_{in},\theta), \tag{8}$$

where $r_{ki,n} : Z \to \mathbb{R}$ and $s_{ki,n}(\cdot,\theta) : Z \to \mathbb{R}$ are real-valued functions, which are $\mathcal{Z}/\mathfrak{B}$-measurable for all $\theta \in \Theta$, $1 \leq k \leq K$, $i \in D_n$, $n \geq 1$. We maintain the following assumptions.

Condition 2. *The random functions $f_{i,n}(Z_{i,n},\theta)$ defined in (8) satisfy the following conditions:*

(a) *For all $1 \leq k \leq K$*

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} E\left|r_{ki,n}(Z_{i,n})\right| < \infty.$$

(b) *For a sequence of sets $\{K_m\}$ with $K_m \in \mathcal{Z}$, the family of nonrandom functions $s_{ki,n}(z,\cdot)$, $1 \leq k \leq K$, satisfy the following uniform equicontinuity-type condition: For each $m \in \mathbb{N}$,*

$$\sup_n \sup_{i\in D_n} \sup_{z\in K_m} \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left|s_{ki,n}(z,\theta) - s_{ki,n}(z,\theta')\right| \to 0$$

as $\delta \to 0$.

(c) *Also, for the sequence of sets $\{K_m\}$*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} P\left(Z_{i,n} \notin K_m\right) = 0.$$

We now have the following proposition, which extends parts of Theorem 4.5 in Pötscher and Prucha (1994a) to arrays of random fields.

Proposition 2. *Under Condition 2, $f_{i,n}(Z_{i,n},\theta)$ is L_0 stochastically equicontinuous on Θ.*

We next discuss the assumptions of the above proposition and provide further sufficient conditions. We note that the $f_{i,n}$ are composed of two parts, $r_{ki,n}$ and $s_{ki,n}$, with the continuity conditions imposed only on the second part. Condition 2 allows for discontinuities in $r_{ki,n}$ with respect to the data. For example, the $r_{ki,n}$ could be indicator functions. A sufficient condition for Condition 2(a) is the uniform L_1 boundedness of $r_{ki,n}$, i.e., $\sup_n \sup_{i\in D_n} E\left|r_{ki,n}(Z_{i,n})\right| < \infty$.

Condition 2(b) requires the *non*random functions $s_{ki,n}$ to be equicontinuous with respect to θ uniformly for all $z \in K_m$. This assumption will be satisfied if the functions $s_{ki,n}(z,\theta)$, restricted to $K_m \times \Theta$, are equicontinuous jointly in z and θ. More specifically, define the distance between the points (z,θ) and (z',θ') in the product space $Z \times \Theta$ by $r((z,\theta);(z',\theta')) = \max\left\{\nu(\theta,\theta'), \tau(z,z')\right\}$. This metric induces the product topology on $Z\times\Theta$. Under this product topology, let $B((z',\theta'),\delta)$ be the open ball with center (z',θ') and radius δ in $K_m\times\Theta$. It is now easy to see that Condition 2(b) is implied by the following condition for each $1 \leq k \leq K$

$$\sup_n \sup_{i\in D_n} \sup_{(z',\theta')\in K_m\times\Theta} \sup_{(z,\theta)\in B((z',\theta'),\delta)} \left|s_{ki,n}(z,\theta) - s_{ki,n}(z',\theta')\right| \to 0$$
$$\text{as } \delta \to 0,$$

i.e., the family of nonrandom functions $\left\{s_{ki,n}(z,\theta)\right\}$, restricted to $K_m\times\Theta$, is uniformly equicontinuous on $K_m\times\Theta$. Obviously, if both Θ and K_m are compact, the uniform equicontinuity is equivalent to equicontinuity, i.e.,

$$\sup_n \sup_{i\in D_n} \sup_{(z,\theta)\in B((z',\theta'),\delta)} \left|s_{ki,n}(z,\theta) - s_{ki,n}(z',\theta')\right| \to 0 \quad \text{as } \delta \to 0.$$

Of course, if the functions furthermore do not depend on i and n, then the condition reduces to continuity on $K_m\times\Theta$. Clearly, if any of the above conditions holds on $Z\times\Theta$, then it also holds on $K_m\times\Theta$.

Finally, if the sets K_m can be chosen to be compact, then Condition 2(c) is an asymptotic tightness condition for the average of the marginal distributions of Z_{in}. Condition 2(c) can frequently be implied by a mild moment condition. In particular, the following is sufficient for Condition 2(c) in the case $Z = \mathbb{R}^s$: $K_m \uparrow \mathbb{R}^s$ is a sequence of Borel measurable convex sets (e.g., a sequence of open or closed balls), and $\limsup_{n\to\infty} |D_n|^{-1} \sum_{i\in D_n} Eh(Z_{in}) < \infty$ where $h : [0,\infty) \to [0,\infty)$ is a monotone function such that $\lim_{x\to\infty} h(x) = \infty$; for example, $h(x) = x^p$ with $p > 0$. The claim follows from Lemma A4 in Pötscher and Prucha (1994b) with obvious modification to the proof.

We note that, in contrast to Condition 1, Condition 2 will generally not cover random fields with trending moments since in this case part (c) would typically not hold.

5. Concluding remarks

The paper derives a CLT, ULLN, and, for completeness, also an exemplary LLN for spatial processes. In particular, the limit theorems (i) allow the random field to be nonstationary and to exhibit asymptotically unbounded moments, (ii) allow for locations on unevenly spaced lattices in $\mathbb{R}^d$ and for general forms of sample regions, and (iii) allow the random variables to form a triangular array.

Spatial data processes encountered in empirical work are frequently not located on evenly spaced lattices, are nonstationary, and may even exhibit spikes in their moments. Random variables generated by the important class of Cliff–Ord type spatial processes form triangular arrays. The catalogues of assumptions maintained by the limit theorems developed in this paper are intended to accommodate all these features in order to make these theorems applicable to a broad range of data processes in economics.

CLTs, ULLNs and LLNs are the fundamental building blocks for the asymptotic theory of nonlinear spatial M-estimators, e.g., maximum likelihood and generalized method of moments estimators, and test statistics. An interesting direction for future research would be to generalize the above limit theorems to random fields that are not mixing, but can be approximated by mixing fields. This could be achieved, for example, by introducing the concept of near-epoch dependent random fields similar to the one used in the time-series literature. We are currently working in this direction.

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Appendix A. Cardinalities of basic sets on irregular lattices in $\mathbb{R}^d$

The following lemma establishes bounds on the cardinalities of basic sets in D that will be used in the proof of the limit theorems. Its proof is elementary and is therefore omitted.

Lemma A.1. *Suppose that Assumption 1 holds. Let $B_i(r)$ be the closed ball of the radius r centered in $i \in \mathbb{R}^d$. Then,*

(i) *The ball $B_i(1/2)$ with $i \in \mathbb{R}^d$ contains at most one element of D, i.e., $|B_i(1/2) \cap D| \leq 1$.*

(ii) *There exists a constant $C < \infty$ such that for $h \geq 1$*

$$\sup_{i \in \mathbb{R}^d} |B_i(h) \cap D| \leq Ch^d,$$

i.e., the number of elements of D contained in a ball of radius h centered at $i \in \mathbb{R}^d$ is $O(h^d)$ uniformly in i.

(iii) *For $m \geq 1$ and $i \in \mathbb{R}^d$ let*

$$N_i(1, 1, m) = |\{j \in D : m \leq \rho(i, j) < m + 1\}|$$

be the number of all elements of D located at any distance $h \in [m, m + 1)$ from i. Then, there exists a constant $C < \infty$ such that

$$\sup_{i \in \mathbb{R}^d} N_i(1, 1, m) \leq Cm^{d-1}.$$

(iv) *Let U and V be some finite disjoint subsets of D. For $m \geq 1$ and $i \in U$ let*

$$N_i(2, 2, m) = |\{(A, B) : |A| = 2, |B| = 2, A \subseteq U \text{ with } i \in A,\ B \subseteq V \text{ and } \exists j \in B \text{ with } m \leq \rho(i, j) < m + 1\}|$$

be the number of all different combinations of subsets of U composed of two elements, one of which is i, and subsets of V composed of two elements, where for at least one of the elements, say j, we have $m \leq \rho(i, j) < m + 1$. Then there exists a constant $C < \infty$ such that

$$\sup_{i \in U} N_i(2, 2, m) \leq Cm^{d-1} |U| |V| .$$

(v) *Let V be some finite subset of D. For $m \geq 1$ and $i \in \mathbb{R}^d$ let*

$$N_i(1, 3, m) = |\{B : |B| = 3,\ B \subseteq V \text{ and } \exists j \in B \text{ with } m \leq \rho(i, j) < m + 1\}|$$

be the number of the subsets of V composed of three elements, at least one of which is located at a distance $h \in [m, m + 1)$ from i. Then there exists a constant $C < \infty$ such that

$$\sup_{i \in R^d} N_i(1, 3, m) \leq Cm^{d-1} |V|^2 .$$

Appendix B. Proofs of CLT

The proof of Theorem 1 adapts the strategy employed by Bolthausen (1982) in proving his CLT for stationary random fields on regular lattices.

B.1. Some useful Lemmata

Lemma B.1 (*Bradley, 2007, Vol. 1, pp. 326, for $q = 1$*). *Let $\alpha(m)$, $m = 1, 2, \ldots$ be a non-increasing sequence such that $0 \leq \alpha(m) \leq 1$ and $\alpha(m) \to 0$ as $m \to \infty$. Set $\alpha(0) = 1$ and define the "inverse function" $\alpha^{-1} : (0, 1) \to \mathbf{N} \cup \{0\}$ as*

$$\alpha^{-1}(u) = \max\{m \geq 0 : \alpha(m) > u\} \quad \text{for } u \in (0, 1).$$

Let $f : (0, 1) \to [0, \infty)$ be a Borel function, then for $q \geq 1$:

(a) $\sum_{m=1}^{\infty} m^{q-1} \int_0^{\alpha(m)} f(u)\mathrm{d}u \leq \int_0^1 \left[\alpha^{-1}(u)\right]^q f(u)\mathrm{d}u$,

(b) $\int_0^1 \left[\alpha^{-1}(u)\right]^q \mathrm{d}u \leq q \sum_{m=1}^{\infty} \alpha(m)m^{q-1}$, *for any $q \geq 1$.*

Proof of Lemma B.1. The proof is similar to Bradley (2007), Vol. 1, pp. 326, and is available on the authors' webpages. ■

Lemma B.2. *Let $Y \geq 0$ be some non-negative random variable, let F_Y and Q_Y be the c.d.f. and the upper-tail quantile function of Y, and for some $k > 0$ let $F_Y^{(k)}$ and $Q_Y^{(k)}$ be the c.d.f. and the upper-tail quantile function of $Y\mathbf{1}(Y > k)$. Furthermore, define*

$$u^{(k)} = P(Y > k) = 1 - F_Y(k),$$

then $Q_Y(u) \leq k$ for $u \in [u^{(k)}, 1)$, and

$$Q_Y^{(k)}(u) = \begin{cases} Q_Y(u) & \text{for } u \in (0, u^{(k)}) \\ 0 & \text{for } u \in [u^{(k)}, 1). \end{cases}$$

Proof of Lemma B.2. In light of Remark 1 for $u \in (0, 1)$:

$$Q_Y(u) = \inf\{y : F_Y(y) \geq 1 - u\},$$
$$Q_Y^{(k)}(u) = \inf\{y : F_Y^k(y) \geq 1 - u\}.$$

Furthermore, observe that

$$F_Y^{(k)}(y) = P\left(Y\mathbf{1}(Y > k) \leq y\right) = \begin{cases} 0 & y < 0 \\ F_Y(k) & 0 \leq y \leq k \\ F_Y(y) & y > k \end{cases}$$

since $P\left(Y\mathbf{1}(Y > k) = 0\right) = P(Y \leq k) = F_Y(k)$.

Let $u \in [u^{(k)}, 1)$: Then $1 - u^{(k)} \geq 1 - u$, and thus

$$Q_Y(u) = \inf\{y : F_Y(y) \geq 1 - u\} \leq \inf\{y : F_Y(y) \geq 1 - u^{(k)}\} = \inf\{y : F_Y(y) \geq F_Y(k)\} \leq k, \tag{B.1}$$

and

$$Q_Y^{(k)}(u) = \inf\{y : F_Y^{(k)}(y) \geq 1 - u\} \leq \inf\{y : F_Y^{(k)}(y) \geq 1 - u^{(k)}\} = \inf\{y : F_Y^{(k)}(y) \geq F_Y(k)\} = 0, \tag{B.2}$$

since $F_Y^{(k)}(y) \geq F_Y(k)$ for all $y \geq 0$.

Now let $u \in (0, u^{(k)})$: Then $1 - u^{(k)} < 1 - u$ and

$$\{y : F_Y^{(k)}(y) \geq 1 - u\} \subseteq \{y : F_Y^{(k)}(y) > 1 - u^{(k)}\} = \{y : F_Y^{(k)}(y) > F_Y(k)\}.$$

Consequently, $\{y : F_Y^{(k)}(y) \geq 1 - u\} \subseteq \{y : y > k\}$. Observing that for $y > k$ we have $F_Y^{(k)}(y) = F_Y(y)$, it follows that

$$Q_Y^{(k)}(u) = \inf\{y : F_Y(y) \geq 1 - u\} = Q_Y(u). \tag{B.3}$$

The claims of the lemma now follow from (B.1)–(B.3). ■

B.2. Proof of theorem and corollaries

Proof of Theorem 1. We give the proof for α-mixing fields. The argument for ϕ-mixing fields is similar. The proof is lengthy, and for readability we break it up into several steps.

1. Notation and reformulation. Define

$$X_{i,n} = Z_{i,n}/M_n$$

where $M_n = \max_{i\in D_n} c_{i,n}$ is as in Assumption 5. Let $\sigma_{n,Z}^2 = \mathrm{Var}\left[\sum_{i\in D_n} Z_{i,n}\right]$ and $\sigma_{n,X}^2 = \mathrm{Var}\left[\sum_{i\in D_n} X_{i,n}\right] = M_n^{-2}\sigma_{n,Z}^2$. Since

$$\sigma_{n,X}^{-1}\sum_{i\in D_n} X_{i,n} = \sigma_{n,Z}^{-1}\sum_{i\in D_n} Z_{i,n},$$

to prove the theorem, it suffices to show that $\sigma_{n,X}^{-1}\sum_{i\in D_n} X_{i,n} \Longrightarrow N(0,1)$. In this light, it proves convenient to switch notation from the text and to define

$$S_n = \sum_{i\in D_n} X_{i,n}, \qquad \sigma_n^2 = \mathrm{Var}\left(\sum_{i\in D_n} X_{i,n}\right).$$

That is, in the following, S_n denotes $\sum_{i\in D_n} X_{i,n}$ rather than $\sum_{i\in D_n} Z_{i,n}$, and σ_n^2 denotes the variance of $\sum_{i\in D_n} X_{i,n}$ rather than of $\sum_{i\in D_n} Z_{i,n}$.

We next establish the moment and mixing conditions for $X_{i,n}$ implied by the assumptions of the CLT. Observe that by definition of M_n, we have $|X_{i,n}| \leq |Z_{i,n}/c_{i,n}|$ and hence $E[|X_{i,n}|^2 \mathbf{1}(|X_{i,n}| > k)] \leq E[|Z_{i,n}/c_{i,n}|^2 \mathbf{1}(|Z_{i,n}/c_{i,n}| > k)]$. Thus, in light of Assumption 2,

$$\lim_{k\to\infty}\sup_n \sup_{i\in D_n} E[|X_{i,n}|^2 \mathbf{1}(|X_{i,n}| > k)] = 0, \tag{B.4}$$

i.e., the $X_{i,n}$ are uniformly L_2 integrable. This further implies that

$$\|X\|_2^2 = \sup_n \sup_{i\in D_n} E\left|X_{i,n}\right|^2 < \infty. \tag{B.5}$$

Clearly, the mixing coefficients for $X_{i,n}$ and $Z_{i,n}$ are identical, and thus the mixing conditions postulated in Assumption 3 for $Z_{i,n}$ also apply to $X_{i,n}$.

Using the new notation, Assumption 5 further implies:

$$\liminf_{n\to\infty} |D_n|^{-1}\sigma_n^2 > 0. \tag{B.6}$$

2. Truncated random variables. In the following, we will consider truncated versions of the $X_{i,n}$. For $k > 0$ we define the following random variables

$$X_{i,n}^{(k)} = X_{i,n}\mathbf{1}(|X_{i,n}| \leq k), \qquad \widetilde{X}_{i,n}^{(k)} = X_{i,n}\mathbf{1}(|X_{i,n}| > k), \tag{B.7}$$

and the corresponding variances as

$$\sigma_{n,k}^2 = \mathrm{Var}\left[\sum_{i\in D_n} X_{i,n}^{(k)}\right], \qquad \widetilde{\sigma}_{n,k}^2 = \mathrm{Var}\left[\sum_{i\in D_n} \widetilde{X}_{i,n}^{(k)}\right].$$

We note that

$$\left|\sigma_n - \sigma_{n,k}\right| \leq \widetilde{\sigma}_{n,k}. \tag{B.8}$$

To see this, define

$$S_{n,k} = \sum_{i\in D_n} X_{i,n}^{(k)} - EX_{i,n}^{(k)}, \qquad \widetilde{S}_{n,k} = \sum_{i\in D_n} \widetilde{X}_{i,n}^{(k)} - E\widetilde{X}_{i,n}^{(k)},$$

and observe that $S_n = S_{n,k} + \widetilde{S}_{n,k}$, $\sigma_n = \|S_n\|_2$, $\sigma_{n,k} = \left\|S_{n,k}\right\|_2$ and $\widetilde{\sigma}_{n,k} = \left\|\widetilde{S}_{n,k}\right\|_2$. The inequality in (B.8) is now readily established using Minkowski's inequality.

In the following, let $F_{i,n}(x)$ and $Q_{i,n}$ be the c.d.f. and the upper-tail quantile function of $|X_{i,n}|$, let $F_{i,n}^{(k)}$ and $Q_{i,n}^{(k)}$ be the c.d.f. and the upper-tail quantile function of $\left|\widetilde{X}_{i,n}^{(k)}\right|$, respectively, and let

$$u_{i,n}^{(k)} = P\left(|X_{i,n}| > k\right) = 1 - F_{i,n}(k).$$

Next, we deduce from Assumption 3 some basic properties of the upper-tail quantile function of $|X_{i,n}|$ and $\left|\widetilde{X}_{i,n}^{(k)}\right|$ that will be utilized in the proof below.

We first establish that

$$\lim_{k\to\infty}\sup_n \sup_{i\in D_n}\int_0^1 \alpha_{inv}^d(u)\left(Q_{i,n}^{(k)}(u)\right)^2 \mathrm{d}u = 0, \tag{B.9}$$

where $\alpha_{inv}(u)$ is the inverse of $\overline{\alpha}_{1,1}(m)$ given in Definition 2. Since $|X_{i,n}|\,\mathbf{1}(|X_{i,n}| > k) \leq |Z_{i,n}/c_{i,n}|\,\mathbf{1}(|Z_{i,n}/c_{i,n}| > k)$, in light of Remark 1, we have

$$Q_{i,n}^{(k)}(u) \leq Q_{|Z_{i,n}/c_{i,n}|\mathbf{1}(|Z_{i,n}/c_{i,n}|>k)}(u).$$

Proposition (B.9) now follows immediately from Assumption 3(a).

Next, we establish that

$$\sup_n \sup_{i\in D_n}\int_0^1 \alpha_{inv}^d(u) Q_{i,n}^2(u)\mathrm{d}u = K_1 < \infty. \tag{B.10}$$

In light of (B.9), for any finite $\varepsilon > 0$ there exists a $k < \infty$ (which may depend on ε) such that

$$\sup_n \sup_{i\in D_n}\int_0^1 \alpha_{inv}^d(u)\left(Q_{i,n}^{(k)}(u)\right)^2 \mathrm{d}u \leq \varepsilon.$$

By Lemma B.2,

$$Q_{i,n}^2(u) \leq \begin{cases} \left(Q_{i,n}^{(k)}(u)\right)^2 & \text{for } u \in (0; u_{i,n}^{(k)}) \\ k^2 & \text{for } u \in [u_{i,n}^{(k)}; 1). \end{cases}$$

Hence, using the above inequality and Part (b) of Lemma B.1 with $q = d$, it is readily seen that

$$\int_0^1 \alpha_{inv}^d(u) Q_{i,n}^2(u)\mathrm{d}u \leq \varepsilon + dk^2\sum_{m=1}^{\infty}\overline{\alpha}_{1,1}(m)m^{d-1} < \infty$$

since $\sum_{m=1}^{\infty}\overline{\alpha}_{1,1}(m)m^{d-1} < \infty$ by Assumption 3(b). This verifies Proposition (B.10).

3. Bounds for variances. In the following, we will use the weaker version of Rio's covariance inequality given in (1). Using this inequality gives

$$|\mathrm{cov}(X_{i,n}, X_{j,n})| \leq 4\int_0^{\overline{\alpha}_{1,1}(\rho(i,j))} Q_{i,n}(u)Q_{j,n}(u)\mathrm{d}u. \tag{B.11}$$

Also, in light of the convention $\alpha_{1,1}(0) = 1$, Remark 1, and (B.5) we have:

$$\mathrm{Var}(X_{i,n}) = \int_0^1 Q_{i,n}^2(u)\mathrm{d}u = \int_0^{\overline{\alpha}_{1,1}(0)} Q_{i,n}^2(u)\mathrm{d}u \leq \sup_n \sup_{i\in D_n} EX_{i,n}^2 = \|X\|_2^2 < \infty.$$

We next establish a bound on σ_n^2. Using Lemma A.1(iii) and (B.11) yields:

$$\begin{aligned}\sigma_n^2 &\le \sum_{i,j\in D_n}\left|\mathrm{cov}(X_{i,n},X_{j,n})\right| \le 4\sum_{i,j\in D_n}\int_0^{\overline{\alpha}_{1,1}(\rho(i,j))} Q_{i,n}(u)Q_{j,n}(u)\mathrm{d}u\\ &\le 2\sum_{i\in D_n}\sum_{j\in D_n}\int_0^{\overline{\alpha}_{1,1}(\rho(i,j))}\left[Q_{i,n}^2(u)+Q_{j,n}^2(u)\right]\mathrm{d}u\\ &\le 4\sum_{i\in D_n}\sum_{m=0}^{\infty}\sum_{j\in D_n:\rho(i,j)\in[m,m+1)}\int_0^{\overline{\alpha}_{1,1}(\rho(i,j))} Q_{i,n}^2(u)\mathrm{d}u\\ &\le 4\sum_{i\in D_n}\sum_{m=0}^{\infty}N_i(1,1,m)\int_0^{\overline{\alpha}_{1,1}(m)} Q_{i,n}^2(u)\mathrm{d}u\\ &\le 4C|D_n|\|X\|_2^2+4C|D_n|\sup_{n,i\in D_n}\sum_{m=1}^{\infty}m^{d-1}\int_0^{\overline{\alpha}_{1,1}(m)} Q_{i,n}^2(u)\mathrm{d}u\\ &\le 4C|D_n|\|X\|_2^2+4C|D_n|\sup_{n,i\in D_n}\int_0^1 \alpha_{inv}^d(u)Q_{i,n}^2(u)\mathrm{d}u \le |D_n|B_2,\end{aligned} \tag{B.12}$$

with $B_2 = 4C(\|X\|_2^2+K_1) < \infty$. The first inequality in the last line follows from Part (a) of Lemma B.1 by setting $\alpha(m) = \overline{\alpha}_{1,1}(m)$ and $f(u) = Q_{i,n}^2(u)$. The last inequality follows from (B.5) and (B.10).

Thus, $\sup_n |D_n|^{-1}\sigma_n^2 < \infty$. By condition (B.6), there exists an N_* and $B_1 > 0$ such that for all $n \ge N_*$, we have $B_1|D_n| \le \sigma_n^2$. Combining this inequality with (B.12) yields for $n \ge N_*$:

$$0 < B_1|D_n| \le \sigma_n^2 \le B_2|D_n|, \tag{B.13}$$

where $0 < B_1 \le B_2 < \infty$.

Since $\widetilde{X}_{i,n}^{(k)}$ is a measurable function of $X_{i,n}$, clearly $\alpha(\sigma(\widetilde{X}_{i,n}^{(k)}), \sigma(\widetilde{X}_{j,n}^{(k)})) \le \alpha(\sigma(X_{i,n}),\sigma(X_{j,n}))$. Using analogous arguments as above, it is readily seen that for each $k > 0$:

$$\widetilde{\sigma}_{n,k}^2 \le 4C|D_n|\sup_{n,i\in D_n}\left\{E\left(\widetilde{X}_{i,n}^{(k)}\right)^2+\int_0^1\alpha_{inv}^d(u)\left(Q_{i,n}^{(k)}(u)\right)^2\mathrm{d}u\right\}. \tag{B.14}$$

In light of the l.h.s. inequality (B.13) and inequality (B.14), we have

$$\begin{aligned}\lim_{k\to\infty}\sup_{n\ge N_*}\frac{\widetilde{\sigma}_{n,k}^2}{\sigma_n^2} &\le 4\frac{C}{B_1}\lim_{k\to\infty}\sup_n\sup_{i\in D_n}E\left(\widetilde{X}_{i,n}^{(k)}\right)^2\\ &\quad+4\frac{C}{B_1}\lim_{k\to\infty}\sup_n\sup_{i\in D_n}\int_0^1\alpha_{inv}^d(u)\left(Q_{i,n}^{(k)}(u)\right)^2\mathrm{d}u.\end{aligned}$$

Observe that $E\left(\widetilde{X}_{i,n}^{(k)}\right)^2 = EX_{i,n}^2\mathbf{1}(|X_{i,n}| > k)$. It now follows from (B.4) and (B.9) that

$$\lim_{k\to\infty}\sup_{n\ge N_*}\frac{\widetilde{\sigma}_{n,k}^2}{\sigma_n^2} = 0, \tag{B.15}$$

and furthermore, utilizing (B.8),

$$\lim_{k\to\infty}\sup_{n\ge N_*}\left|1-\frac{\sigma_{n,k}}{\sigma_n}\right| \le \lim_{k\to\infty}\sup_{n\ge N_*}\frac{\widetilde{\sigma}_{n,k}}{\sigma_n} = 0. \tag{B.16}$$

4. Reduction to bounded variables. We would like to thank Benedikt Pötscher for helpful discussions on this step of the proof. The proof employs a truncation argument in conjunction with Proposition 6.3.9 of Brockwell and Davis (1991). For $k > 0$ consider the decomposition

$$Y_n = \sigma_n^{-1}\sum_{i\in D_n}X_{i,n} = V_{nk} + (Y_n - V_{nk})$$

with

$$V_{nk} = \sigma_n^{-1}\sum_{i\in D_n}(X_{i,n}^{(k)} - EX_{i,n}^{(k)}),$$
$$Y_n - V_{nk} = \sigma_n^{-1}\sum_{i\in D_n}(\widetilde{X}_{i,n}^{(k)} - E\widetilde{X}_{i,n}^{(k)}),$$

and let $V \sim N(0,1)$. We next show that $Y_n \Longrightarrow N(0,1)$ if

$$\sigma_{n,k}^{-1}\sum_{i\in D_n}(X_{i,n}^{(k)} - EX_{i,n}^{(k)}) \Longrightarrow N(0,1) \tag{B.17}$$

for each $k = 1,2,\ldots$. We note that (B.17) will be verified in subsequent steps.

To show that $Y_n \Longrightarrow N(0,1)$ given (B.17) holds, we first verify condition (iii) of Proposition 6.3.9 in Brockwell and Davis (1991). By Markov's inequality

$$P(|Y_n - V_{nk}| > \varepsilon) = P\left(\left|\sigma_n^{-1}\sum_{i\in D_n}(\widetilde{X}_{i,n}^{(k)} - E\widetilde{X}_{i,n}^{(k)})\right| > \varepsilon\right) \le \frac{\widetilde{\sigma}_{n,k}^2}{\varepsilon^2\sigma_n^2}.$$

In light of (B.15)

$$\lim_{k\to\infty}\limsup_{n\to\infty}P(|Y_n - V_{nk}| > \varepsilon) \le \lim_{k\to\infty}\limsup_{n\to\infty}\frac{\widetilde{\sigma}_{n,k}^2}{\varepsilon^2\sigma_n^2} = 0,$$

which verifies the condition.

Next, observe that

$$V_{nk} = \frac{\sigma_{n,k}}{\sigma_n}\left[\sigma_{n,k}^{-1}\sum_{i\in D_n}(X_{i,n}^{(k)} - EX_{i,n}^{(k)})\right].$$

Suppose $r(k) = \lim_{n\to\infty}\sigma_{n,k}/\sigma_n$ exists, then $V_{nk} \Longrightarrow V_k \sim N(0, r^2(k))$ in light of (B.17). If furthermore, $\lim_{k\to\infty} r(k) \to 1$, then $V_k \Longrightarrow V \sim N(0,1)$, and the claim that $Y_n \Longrightarrow N(0,1)$ would follow by Proposition 6.3.9 of Brockwell and Davis (1991). However, in the case of nonstationary variables $\lim_{n\to\infty}\sigma_{n,k}/\sigma_n$ need not exist, and therefore, we have to use a different argument to show that $Y_n \Longrightarrow V \sim N(0,1)$. We shall prove it by contradiction.

Let $\mathcal{M}$ be the set of all probability measures on $(\mathbb{R},\mathfrak{B})$. Observe that we can metrize $\mathcal{M}$ by, e.g., the Prokhorov distance, say $d(.,.)$. Let μ_n and μ be the probability measures corresponding to Y_n and V, respectively, then $\mu_n \Longrightarrow \mu$ iff $d(\mu_n,\mu) \to 0$ as $n \to \infty$. Now suppose that Y_n does not converge to V. Then for some $\varepsilon > 0$ there exists a subsequence $\{n(m)\}$ such that $d(\mu_{n(m)},\mu) > \varepsilon$ for all $n(m)$. Observe that by (B.13) and (B.14) we have $0 \le \sigma_{n,k}/\sigma_n \le B_2/B_1 < \infty$ for all $k > 0$ and all $n \ge N_*$, where N_* does not depend on k. W. l.o.g. assume that with $n(m) \ge N_*$, and hence $0 \le \sigma_{n(m),k}/\sigma_{n(m)} \le B_2/B_1 < \infty$ for all $k > 0$ and all $n(m)$. Consequently, for $k = 1$ there exists a subsubsequence $\{n(m(l_1))\}$ such that $\sigma_{n(m(l_1)),1}/\sigma_{n(m(l_1))} \to r(1)$ as $l_1 \to \infty$. For $k = 2$ there exists a subsubsubsequence $\{n(m(l_1(l_2)))\}$ such that $\sigma_{n(m(l_1(l_2))),2}/\sigma_{n(m(l_1(l_2)))} \to r(2)$ as $l_2 \to \infty$. The argument can be repeated for $k = 3,4,\ldots$. Now construct a subsequence $\{n_l\}$ such that n_1 corresponds to the first element of $\{n(m(l_1))\}$, n_2 corresponds to the second element of $\{n(m(l_1(l_2)))\}$, and so on, then for $k = 1,2,\ldots,$we have:

$$\lim_{l\to\infty}\frac{\sigma_{n_l,k}}{\sigma_{n_l}} = r(k).$$

Moreover, it follows from (B.16) that

$$\begin{aligned}\lim_{k\to\infty}|r(k)-1| &\le \lim_{k\to\infty}\lim_{l\to\infty}\left|r(k)-\frac{\sigma_{n_l,k}}{\sigma_{n_l}}\right|\\ &\quad+\lim_{k\to\infty}\sup_{n\ge N_*}\left|\frac{\sigma_{n,k}}{\sigma_n}-1\right| = 0.\end{aligned}$$

Given (B.17), it follows that $V_{n_l k} \Longrightarrow V_k \sim N(0, r^2(k))$. Then, by Proposition 6.3.9 of Brockwell and Davis (1991), $Y_{n_l} \Longrightarrow V \sim N(0, 1)$ as $l \to \infty$. Since $\{n_l\} \subseteq \{n(m)\}$, this contradicts the hypothesis that $d(\mu_{n(m)}, \mu) > \varepsilon$ for all $n(m)$.

Thus, we have shown that $Y_n \Longrightarrow N(0, 1)$ if (B.17) holds. In light of this, it suffices to prove the CLT for bounded variables. Thus, in the following, we will assume that $|X_{i,n}| \le C_X < \infty$.

5. Renormalization. Since $|D_n| \to \infty$ and $\overline{\alpha}_{1,\infty}(m_n) = O(m^{-d-\varepsilon})$, it is readily seen that we can choose a sequence m_n such that

$$\overline{\alpha}_{1,\infty}(m_n)|D_n|^{1/2} \to 0 \tag{B.18}$$

and

$$m_n^d |D_n|^{-1/2} \to 0 \tag{B.19}$$

as $n \to \infty$. Now, for such m_n define:

$$a_n = \sum_{i,j \in D_n, \rho(i,j) \le m_n} E(X_{i,n} X_{j,n}).$$

Using arguments similar to those employed in derivation of (B.12), it can be easily shown that sufficiently large n, say $n \ge N_{**} \ge N_*$:

$$\sigma_n^2 = a_n + o(|D_n|) = a_n(1 + o(1)). \tag{B.20}$$

For $n \ge N_{**}$, define

$$\bar{S}_n = a_n^{-1/2} S_n = a_n^{-1/2} \sum_{i \in D_n} X_{i,n}.$$

To demonstrate that $\sigma_n^{-1} S_n \Longrightarrow N(0, 1)$, it therefore suffices to show that $\bar{S}_n \Longrightarrow N(0, 1)$.

6. Limiting distribution of $\bar{S}_n$: From the above discussion $\sup_{n \ge N_*} E\bar{S}_n^2 < \infty$. In light of Stein's Lemma (see, e.g., Lemma 2, Bolthausen, 1982) to establish that $\bar{S}_n \Longrightarrow N(0, 1)$, it suffices to show that

$$\lim_{n\to\infty} E[(\mathbf{i}\lambda - \bar{S}_n)\exp(\mathbf{i}\lambda \bar{S}_n)] = 0$$

In the following, we take $n \ge N_{**}$, but will not indicate that explicitly for notational simplicity. Define

$$S_{j,n} = \sum_{i \in D_n, \rho(i,j) \le m_n} X_{i,n} \quad \text{and} \quad \bar{S}_{j,n} = a_n^{-1/2} S_{j,n},$$

then

$$(\mathbf{i}\lambda - \bar{S}_n)\exp(\mathbf{i}\lambda \bar{S}_n) = A_{1,n} - A_{2,n} - A_{3,n},$$

with

$$A_{1,n} = \mathbf{i}\lambda e^{\mathbf{i}\lambda \bar{S}_n}\left(1 - a_n^{-1}\sum_{j \in D_n} X_{j,n} S_{j,n}\right),$$

$$A_{2,n} = a_n^{-1/2} e^{\mathbf{i}\lambda \bar{S}_n} \sum_{j \in D_n} X_{j,n}[1 - \mathbf{i}\lambda \bar{S}_{j,n} - e^{-\mathbf{i}\lambda \bar{S}_{j,n}}],$$

$$A_{3,n} = a_n^{-1/2} \sum_{j \in D_n} X_{j,n} e^{\mathbf{i}\lambda(\bar{S}_n - \bar{S}_{j,n})}.$$

To complete the proof, it suffices to show that $E|A_{k,n}| \to 0$ as $n \to \infty$ for $k = 1, 2, 3$. The latter can be verified using Lemma A.1 and arguments analogous to those in Guyon (1995), pp. 112-113. A detailed proof of these statements is available on the authors' webpages. This completes the proof of the theorem. ■

Proof of Corollary 1. As in Theorem 1, let $\overline{Q}_{i,n}^{(k)} := Q_{|Z_{i,n}/c_{i,n}|} \mathbf{1}_{(|Z_{i,n}/c_{i,n}|>k)}$ and let $\alpha_{inv}(u)$ be the inverse of $\overline{\alpha}_{1,1}(m)$ as given in Definition 2. By Hölder's inequality,

$$\lim_{k\to\infty} \sup_n \sup_{i \in D_n} \int_0^1 \alpha_{inv}^d(u) \left(\overline{Q}_{i,n}^{(k)}(u)\right)^2 du \le \left[\int_0^1 \alpha_{inv}^{d(2+\delta)/\delta} du\right]^{\delta/(2+\delta)}$$
$$\times \left[\lim_{k\to\infty} \sup_n \sup_{i \in D_n} \int_0^1 \left(\overline{Q}_{i,n}^{(k)}(u)\right)^{2+\delta} du\right]^{2/(2+\delta)}. \tag{B.21}$$

In light of Remark 1 and condition (3) maintained by the lemma, we have

$$\lim_{k\to\infty} \sup_n \sup_{i \in D_n} \int_0^1 \left(\overline{Q}_{i,n}^{(k)}(u)\right)^{2+\delta} du = \lim_{k\to\infty} \sup_n \sup_{i \in D_n} E\left[\left|Z_{i,n}/c_{i,n}\right|^{2+\delta} \times \mathbf{1}\left(\left|Z_{i,n}/c_{i,n}\right| > k\right)\right] = 0.$$

Hence to complete the proof, it suffices to show that the first term on the r.h.s. of (B.21) is finite. To see this, observe that by Part (b) of Lemma B.1 with $\alpha(m) = \overline{\alpha}_{1,1}(m)$ and $q = d(2+\delta)/\delta$ we have

$$\int_0^1 \alpha_{inv}^{d(2+\delta)/\delta}(u) du \le \frac{d(2+\delta)}{\delta} \sum_{m=1}^{\infty} \overline{\alpha}_{1,1}(m) m^{[d(2+\delta)/\delta]-1} < \infty,$$

where the r.h.s. is finite by condition (4) maintained by the lemma.

Finally, we verify the claim made in the discussion of Corollary 1 that the mixing condition

$$\sum_{m=1}^{\infty} \overline{\alpha}_{1,1}(m) m^{[d(2+\delta)/\delta]-1} < \infty \tag{B.22}$$

is weaker than the condition

$$\sum_{m=1}^{\infty} m^{d-1} \overline{\alpha}_{1,1}(m)^{\delta/(2+\delta)} < \infty \tag{B.23}$$

used in the previous version of the CLT. To see this, observe that condition (B.23) implies that

$$m^d \overline{\alpha}_{1,1}(m)^{\delta/(2+\delta)} \to 0. \tag{B.24}$$

Next, note that the ratio of the summands in (B.22) and (B.23) equals $\left[m^d \overline{\alpha}_{1,1}(m)^{\delta/(2+\delta)}\right]^{2/\delta}$ and therefore, tends to zero as $m \to \infty$. Hence, condition (B.23) indeed implies (B.22). ■

Proof of Claim before Corollary 1. By Lemma B.2, $Q_{|Z_i|}^{(k)}(u) \le Q_{|Z_i|}(u)$ for each $k > 0$ and all $u \in (0, 1)$. Consequently, for all $k > 0$:

$$\int_0^1 \alpha_{inv}^d(u) \left(Q_{|Z_i|}^{(k)}(u)\right)^2 du \le \int_0^1 \alpha_{inv}^d(u) Q_{|Z_i|}^2(u) du < \infty.$$

Furthermore, by Lemma B.2, we have $\lim_{k\to\infty} Q_{|Z_i|}^{(k)}(u) = 0$. Therefore, by the Dominated Convergence Theorem:

$$\lim_{k\to\infty} \int_0^1 \alpha_{inv}^d(u) \left(Q_{|Z_i|}^{(k)}(u)\right)^2 du = 0,$$

as required. ■

Appendix C. Proofs of ULLN and LLN

Proof of Theorem 2. In the following we use the abbreviations ACL_0UEC $[ACL_pUEC]$ $[[a.s.ACUEC]]$ for L_0 $[L_p]$, $[[a.s.]]$ stochastic equicontinuity as defined in Definition 3. We first show that ACL_0UEC and the Domination Assumption 6 for $g_{i,n}(Z_{i,n}, \theta) = q_{i,n}(Z_{i,n}, \theta)/c_{i,n}$ jointly imply that the $g_{i,n}(Z_{i,n}, \theta)$ is ACL_pUEC, $p \ge 1$.

Given $\varepsilon > 0$, it follows from Assumption 6 that we can choose some $k = k(\varepsilon) < \infty$ such that

$$\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} E(d_{i,n}^p \mathbf{1}(d_{i,n} > k)) < \frac{\varepsilon}{3 \cdot 2^p}. \tag{C.1}$$

Let

$$Y_{i,n}(\delta) = \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} \left|g_{i,n}(Z_{i,n}, \theta) - g_{i,n}(Z_{i,n}, \theta')\right|^p,$$

and observe that $Y_{i,n}(\delta) \leq 2^p d_{i,n}^p$, then

$$\begin{aligned} E\left[Y_{i,n}(\delta)\right] &\leq \varepsilon/3 + EY_{i,n}(\delta)\mathbf{1}(Y_{i,n}(\delta) > \varepsilon/3, d_{i,n} > k) \\ &\quad + EY_{i,n}(\delta)\mathbf{1}(Y_{i,n}(\delta) > \varepsilon/3, d_{i,n} \leq k) \\ &\leq \varepsilon/3 + 2^p E d_{i,n}^p \mathbf{1}(d_{i,n} > k) + 2^p k^p P(Y_{i,n}(\delta) > \varepsilon/3). \end{aligned} \tag{C.2}$$

From the assumption that the $g_{i,n}(Z_{i,n}, \theta)$ is ACL_0UEC, it follows that we can find some $\delta = \delta(\varepsilon)$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} P(Y_{i,n}(\delta) > \varepsilon) = \limsup_{n\to\infty} \frac{1}{|D_n|} \\ &\times \sum_{i\in D_n} P\left(\sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| g_{i,n}(Z_{i,n}, \theta) - g_{i,n}(Z_{i,n}, \theta') \right| > \varepsilon^{\frac{1}{p}} \right) \\ &\leq \frac{\varepsilon}{3\,(2k)^p}. \end{aligned} \tag{C.3}$$

It now follows from (C.1)–(C.3) that for $\delta = \delta(\varepsilon)$,

$$\begin{aligned} &\limsup \frac{1}{|D_n|} \sum_{i\in D_n} EY_{i,n}(\delta) \\ &\leq \varepsilon/3 + 2^p \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} E d_{i,n}^p \mathbf{1}(d_{i,n} > k) \\ &+ 2^p k^p \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i\in D_n} P(Y_{i,n}(\delta) > \varepsilon/3) \leq \varepsilon, \end{aligned}$$

which implies that $g_{i,n}(Z_{i,n}, \theta)$ is ACL_pUEC, $p \geq 1$.

We next show that this in turn implies that $Q_n(\theta)$ is AL_pUEC, $p \geq 1$, as defined in Pötscher and Prucha (1994a), i.e., we show that

$$\limsup_{n\to\infty} E\left\{ \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| Q_n(\theta) - Q_n(\theta') \right|^p \right\} \to 0 \quad \text{as } \delta \to 0.$$

To see this, observe that

$$\begin{aligned} &E \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| Q_n(\theta) - Q_n(\theta') \right|^p \\ &\leq \frac{1}{|D_n|} \sum_{i\in D_n} E \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| q_{i,n}(Z_{i,n}, \theta) - q_{i,n}(Z_{i,n}, \theta') \right|^p / c_{i,n}^p \\ &= \frac{1}{|D_n|} \sum_{i\in D_n} EY_{i,n}(\delta) \end{aligned}$$

where we have used inequality (1.4.3) in Bierens (1994). The claim now follows since the lim sup of the last term goes to zero as $\delta \to 0$, as demonstrated above. Moreover, by Theorem 2.1 in Pötscher and Prucha (1994a), $Q_n(\theta)$ is also AL_0UEC, i.e., for every $\varepsilon > 0$

$$\limsup_{n\to\infty} P\left\{ \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| Q_n(\theta) - Q_n(\theta') \right| > \varepsilon \right\} \to 0 \quad \text{as } \delta \to 0.$$

Given the assumed weak pointwise LLN for $Q_n(\theta)$, the i.p. portion of part (a) of the theorem now follows directly from Theorem 3.1(a) of Pötscher and Prucha (1994a).

For the a.s. portion of the theorem, note that by the triangle inequality

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| Q_n(\theta) - Q_n(\theta') \right| \leq \limsup_{n\to\infty} \frac{1}{|D_n|} \\ &\times \sum_{i\in D_n} \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| g_{i,n}(Z_{i,n}, \theta) - g_{i,n}(Z_{i,n}, \theta') \right|. \end{aligned}$$

The r.h.s. of the last inequality goes to zero as $\delta \to 0$, since $g_{i,n}$ is $a.s.ACUEC$ by assumption. Therefore,

$$\limsup_{n\to\infty} \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} \left| Q_n(\theta) - Q_n(\theta') \right| \to 0 \quad \text{as } \delta \to 0 \text{ a.s.}$$

i.e., Q_n is $a.s.AUEC$, as defined in Pötscher and Prucha (1994a). Given the assumed strong pointwise LLN for $Q_n(\theta)$ the a.s. portion of part (a) of the theorem now follows from Theorem 3.1(a) of Pötscher and Prucha (1994a).

Next observe that since $a.s.ACUEC \implies ACL_0UEC$ we have that $Q_n(\theta)$ is AL_pUEC, $p \geq 1$, both under the i.p. and a.s. assumptions of the theorem. This in turn implies that $\overline{Q}_n(\theta) = EQ_n(\theta)$ is $AUEC$, by Theorem 3.3 in Pötscher and Prucha (1994a), which proves part (b) of the theorem. ■

Proof of Theorem 3. Define $X_{i,n} = Z_{i,n}/M_n$, and observe that

$$[|D_n|\, M_n]^{-1} \sum_{i\in D_n} \left(Z_{i,n} - EZ_{i,n}\right) = |D_n|^{-1} \sum_{i\in D_n} \left(X_{i,n} - EX_{i,n}\right).$$

Hence, it suffices to prove the LLN for $X_{i,n}$.

We first establish mixing and moment conditions for $X_{i,n}$ from those for $Z_{i,n}$. Clearly, if $Z_{i,n}$ is α-mixing [ϕ-mixing], then $X_{i,n}$ is also α-mixing [ϕ-mixing] with the same coefficients. Thus, $X_{i,n}$ satisfies Assumption 3(b) with $k = l = 1$ [Assumption 4(b) with $k = l = 1$]. Furthermore, since $Z_{i,n}/c_{i,n}$ is uniformly L_1 integrable, $X_{i,n}$ is also uniformly L_1 integrable, i.e.,

$$\lim_{k\to\infty} \sup_n \sup_{i\in D_n} E\left[\left|X_{i,n}\right| \mathbf{1}\left(\left|X_{i,n}\right| > k\right)\right] = 0. \tag{C.4}$$

In proving the LLN we consider truncated versions of $X_{i,n}$. For $0 < k < \infty$ let

$$X_{i,n}^{(k)} = X_{i,n}\mathbf{1}\left(\left|X_{i,n}\right| \leq k\right), \qquad \widetilde{X}_{i,n}^{(k)} = X_{i,n} - X_{i,n}^{(k)} = X_{i,n}\mathbf{1}\left(\left|X_{i,n}\right| > k\right).$$

In light of (C.4)

$$\lim_{k\to\infty} \sup_n \sup_{i\in D_n} E\left|\widetilde{X}_{i,n}^{(k)}\right| = 0. \tag{C.5}$$

Clearly, $X_{i,n}^{(k)}$ is a measurable function of $X_{i,n}$, and thus $X_{i,n}^{(k)}$ is also α-mixing [ϕ-mixing] with mixing coefficients not exceeding those of $X_{i,n}$.

By Minkowski's inequality

$$\begin{aligned} &E\left|\sum_{i\in D_n} \left(X_{i,n} - EX_{i,n}\right)\right| \\ &\leq 2E \sum_{i\in D_n} \left|\widetilde{X}_{i,n}^{(k)}\right| + E\left|\sum_{i\in D_n} \left(X_{i,n}^{(k)} - EX_{i,n}^{(k)}\right)\right| \end{aligned} \tag{C.6}$$

and thus

$$\begin{aligned} &\lim_{n\to\infty} \left\| |D_n|^{-1} \sum_{i\in D_n} \left(X_{i,n} - EX_{i,n}\right) \right\|_1 \leq 2 \lim_{k\to\infty} \sup_n \sup_{i\in D_n} E\left|\widetilde{X}_{i,n}^{(k)}\right| \\ &+ \lim_{k\to\infty} \lim_{n\to\infty} \left\| |D_n|^{-1} \sum_{i\in D_n} \left(X_{i,n}^{(k)} - EX_{i,n}^{(k)}\right) \right\|_1 \end{aligned} \tag{C.7}$$

where $\|.\|_1$ denotes the L_1-norm. The first term on the r.h.s. of (C.7) goes to zero in light of (C.5). To complete the proof, we now demonstrate that also the second term converges to zero. To that effect, it suffices to show that $X_{i,n}^{(k)}$ satisfies an L_1-norm LLN for fixed k.

Let $\sigma_{n,k}^2 = \mathrm{Var}\left[\sum_{i\in D_n} X_{i,n}^{(k)}\right]$, then by Lyapunov's inequality

$$\left\| |D_n|^{-1} \sum_{i\in D_n} \left(X_{i,n}^{(k)} - EX_{i,n}^{(k)}\right) \right\|_1 \leq |D_n|^{-1} \sigma_{n,k}. \tag{C.8}$$

Using Lemma A.1(iii), the mixing inequality of Thereom A.5 of Hall and Heyde (1980) and arguments as in Step 2 of the proof of the CLT, we have in the α-mixing case:

$$\sigma_{n,k}^2 \le 4|D_n| \left(k^2 + CKk^2\right)$$

with $C < \infty$, and $K = \sum_{m=1}^{\infty} m^{d-1}\overline{\alpha}_{1,1}(m) < \infty$ by Assumption 3(b). Consequently, the r.h.s. of (C.8) is seen to go to zero as $n \to \infty$, which establishes that the $X_{i,n}^{(k)}$ satisfies an L_1-norm LLN for fixed k. The proof for the ϕ-mixing case is analogous. This completes the proof. ■

Proof of Proposition 1. Define the modulus of continuity of $f_{i,n}(Z_{i,n}, \theta)$ as

$$w(f_{i,n}, Z_{i,n}, \delta) = \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} \left| f_{i,n}(Z_{i,n}, \theta) - f_{i,n}(Z_{i,n}, \theta') \right|.$$

Further observe that $\left\{\omega : w(f_{i,n}, Z_{i,n}, \delta) > \varepsilon\right\} \subseteq \left\{\omega : B_{i,n}h(\delta) > \varepsilon\right\}$. By Markov's inequality and the i.p. part of Condition 1, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} P\left[w(f_{i,n}, Z_{i,n}, \delta) > \varepsilon\right] \\
&\le \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} P\left[B_{i,n} > \frac{\varepsilon}{h(\delta)}\right] \\
&\le \left[\frac{h(\delta)}{\varepsilon}\right]^p \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} EB_{i,n}^p \le C_1 \left[\frac{h(\delta)}{\varepsilon}\right]^p \to 0 \\
&\quad \text{as } \delta \to 0
\end{aligned}$$

for some $C_1 < \infty$, which establishes the i.p. part of the theorem. For the a.s. part, observe that by the a.s. part of Condition 1 we have a.s.

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} w(f_{i,n}, Z_{i,n}, \delta) \\
&\le h(\delta) \limsup_{n\to\infty} \frac{1}{|D_n|} \sum_{i \in D_n} B_{i,n} \le C_2 h(\delta) \to 0 \quad \text{as } \delta \to 0
\end{aligned}$$

for some $C_2 < \infty$, which establishes the a.s. part of the theorem. ■

Proof of Proposition 2. The proof is analogous to the first part of the proof of Theorem 4.5 in Pötscher and Prucha (1994a), and is therefore omitted. ■

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