

A NOTE ON THE ESTIMATION OF NONSYMMETRIC DYNAMIC FACTOR DEMAND MODELS*

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In a recent article Epstein and Yatchew (1985) introduced a simplified procedure for the estimation of symmetric dynamic factor demand models. This procedure hinges on a reparametrization of the model, results in closed form analytic expressions for the firm's factor demand, and can be carried out by standard econometric packages. The purpose of this note is to extend the procedure to the case of nonsymmetric dynamic factor demand models.

1. Introduction

In a recent article Epstein and Yatchew (1985) introduced a simplified procedure for the estimation of a class of linear dynamic factor demand systems. While the procedure is presented in terms of a dynamic factor demand model the procedure can also be applied towards the estimation of other linear rational expectations models which have a similar structure.

The Epstein and Yatchew procedure is similar to that suggested by Hansen and Sargent (1980, 1981) in that the solution to the firm's (stochastic closed loop) optimal control problem is obtained by solving the stable roots backwards and the unstable roots forwards, and it incorporates the transversality condition. The class of models considered by Epstein and Yatchew is less general than that considered by Hansen and Sargent in that adjustment costs only depend on first-order changes in the factor inputs and are specified as separable from the rest of the technology.

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Epstein and Yatchew assume that all new investment becomes immediately productive. This results in a symmetric second-order system of difference equations for the optimal factor inputs. The method suggested by Epstein and Yatchew hinges on a reparametrization of the model. Due to this reparametrization it is possible to obtain closed form analytic expressions for the firm's factor demand. As a consequence, this procedure can be carried out by standard econometric packages and avoids during estimation the need for the repeated solution of the firm's control problem by numerical methods.

Prucha and Nadiri (1986) adopt (and trivially modify) the Epstein and Yatchew procedure to the case where all new investment becomes productive with a one-period lag. Linear rational expectations models that correspond to a symmetric second-order difference equation system have recently also been discussed by Kollintzas (1985).

Many models of interest correspond to a set of nonsymmetric difference equations. For example, symmetry is lost if we allow inputs to become productive at different points in time.¹ In general symmetry is also lost if we allow for nonseparability between the adjustment costs and the levels of the inputs unless further restrictions [as, e.g., in Kollintzas (1985)] are imposed.

The purpose of this note is to extend the Epstein and Yatchew procedure to the nonsymmetric case. This is achieved in two steps. In the first step we demonstrate that also in the nonsymmetric case both the backward- and the forward-looking portion of the optimal control solution can be expressed solely in terms of two matrices (apart from the forcing functions and the discount factor), where one of these matrices is the accelerator matrix. This demonstration is essential for the second step, in which we then extend the reparametrization method and derive closed form analytic expressions for the firm's factor demand.

In the symmetric case the demonstration that the optimal control solution can be expressed solely as a function of two matrices is readily achieved by diagonalizing the second-order difference equations, i.e., Euler equations, that characterize the optimal control solution; compare, e.g., Lucas (1967) and Kollintzas (1985). In the nonsymmetric case this approach is not applicable. Kollintzas (1986) derives an expression for the optimal control solution in the nonsymmetric case by rewriting the Euler equations as a first-order system and by employing the Jordan decomposition for this system. However, Kollintzas (1986) does not provide an expression for the optimal control solution in the above described form. In this note we develop an alternative solution method for the nonsymmetric case that results in an expression for the optimal control solution in the form needed for the reparametrization step. The solution

¹We note that in empirical and theoretical studies capital is often assumed to only become productive with a lag, while labor is typically modeled as immediately productive; see, e.g., Berndt and Morrison (1981), Kydland and Prescott (1982).

method is quite simple. As a byproduct we also obtain certain properties for the product of the two matrices that appear in the expression for the optimal control solution. These symmetry properties can be exploited during estimation.

2. A generalization of a closed form estimation procedure

Consider a firm that produces output y_t from the $k \times 1$ input vector $x_t = [n'_t, s'_t]'$, where n_t is the $k_1 \times 1$ vector of immediately productive inputs and s_t is the $k_2 \times 1$ vector of inputs that only become productive with a one-period lag. The firm's production set is defined by the following production function:

$$\begin{aligned}
 y_t &= F(x_t, x_{t-1}, \Delta x_t) \\
 &= a_0 + [n'_t, s'_t]a + \frac{1}{2}[n'_t, s'_t]A[n'_t, s'_t]' \\
 &\quad + \frac{1}{2}[\Delta n'_t, \Delta s'_t]B[\Delta n'_t, \Delta s'_t]' + [n'_t, s'_t]C[\Delta n'_t, \Delta s'_t]',
 \end{aligned}$$

with

$$a = \begin{bmatrix} a_n \\ a_s \end{bmatrix}, \quad A = \begin{bmatrix} A_{nn} & A_{ns} \\ A_{sn} & A_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_{nn} & B_{ns} \\ B_{sn} & B_{ss} \end{bmatrix}, \quad C = \begin{bmatrix} C_{nn} & C_{ns} \\ C_{sn} & C_{ss} \end{bmatrix},$$

and where $\Delta x_t = x_t - x_{t-1}$. We assume that the production function satisfies the following concavity properties: The matrix composed of A and B as diagonal blocks and C and C' as off-diagonal blocks is symmetric and negative semidefinite and B is negative definite. Adjustment costs in terms of foregone output are reflected by $\Delta x_t = [\Delta n'_t, \Delta s'_t]'$ as an argument in the production function.

The firm is assumed to choose its inputs according to a stochastic closed loop feed back control policy in order to maximize the expected present value of future profits.² More specifically, the firm sets the current input vector and chooses a contingency plan for setting its inputs in future periods according to the following optimization problem:

$$\begin{aligned}
 \max E_t \sum_{\tau=t}^{\infty} \gamma^{\tau-t} & \left[F(x_\tau, x_{\tau-1}, \Delta x_\tau) - (Q_w w_\tau)' x_\tau \right. \\
 & \left. - (Q_q q_\tau)' (x_\tau - (I - \delta)x_{\tau-1}) \right], \tag{2}
 \end{aligned}$$

²Since B is negative definite all inputs are taken to be quasi-fixed. Variable factors can be readily incorporated by specifying the firm's technology in terms of the restricted profit function. We note that our discussion in terms of a profit maximization problem is only illustrative. The discussion also applies, with trivial modifications, to a cost minimization problem.

for given initial stocks x_{t-1} and $\gamma = (1+r)^{-1}$, $r > 0$. Here E_t denotes the expectations operator conditional on information available at time t . With w_t and q_t we denote $m_1 \times 1$ and $m_2 \times 1$ vectors of nonzero factor prices normalized by the output price where $m_i \leq k$ ($i = 1, 2$). The elements of w_t may be thought of as representing short-run costs such as wages, the elements of the q_t may be thought of as representing after tax acquisition prices. Since the dimension of w_t and q_t may be less than that of x_t , we have introduced selector matrices Q_w and Q_q . Those matrices are of dimension $k \times m_1$ and $k \times m_2$, respectively, with $\text{rank}[Q_q, Q_w] = k$; they select the appropriate elements from the x_t vector. The diagonal matrix of depreciation rates (some of which may be zero) is denoted by δ , the real discount rate is denoted by r .

The prices q_t and w_t are known at time t and are exogenous to the firm. As in Epstein and Yatchew (1985) we assume that they are determined by an autoregressive process:

$$[q'_t, w'_t]' = v + \sum_{i=1}^p \Theta_i [q'_{t-i}, w'_{t-i}]' + \xi_t, \quad (3)$$

where the ξ_t 's are distributed i.i.d.. We assume that the price process is of mean exponential order less than $(1+r)^{1/2}$.³

Furthermore we restrict the solution space to be the class of processes x_t that are of mean exponential order less than $(1+r)^{1/2}$. This ensures, in particular, the finiteness of the objective function for all processes in the solution space.

The objective function in the above specified control problem is linear-quadratic. Certainty equivalence then implies that the optimal inputs in period t corresponding to the stochastic control problem (2) are identical to those obtained by solving the following nonstochastic control problem:

$$\max_{\tau=t}^{\infty} \sum \gamma^{\tau-t} \left[F(x_\tau, x_{\tau-1}, \Delta x_\tau) - (E_t Q_w w_\tau)' x_\tau - (E_t Q_q q_\tau)' (x_\tau - (I - \delta)x_{\tau-1}) \right], \quad (4)$$

for given initial stocks x_{t-1} . The input sequence optimizing (4) must satisfy the following set of deterministic Euler equation ($\tau = t, \dots, \infty$):

$$-\underline{B}x_{\tau+1} + \underline{G}x_\tau - (1+r)\underline{B}'x_{\tau-1} = -(1+r)\underline{a} + (1+r)\underline{a}_\tau, \quad (5)$$

³A vector process, say η_t , is said to be of mean exponential order less than κ if there exist constants K and $0 < \rho < \kappa$ such that $E_t \|\eta_{t+j}\| \leq K\rho^{t+j}$ for all t and $j > 0$.

where

$$\underline{G} = \begin{bmatrix} (1+r)(A_{nn} + C_{nn} + C'_{nn}) + (2+r)B_{nn} & (1+r)C_{ns} - C'_{sn} + (2+r)B_{ns} \\ (1+r)C'_{ns} - C_{sn} + (2+r)B_{sn} & A_{ss} - C_{ss} - C'_{ss} + (2+r)B_{ss} \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} B_{nn} + C'_{nn} & B_{ns} \\ B_{sn} + C'_{ns} - C_{sn} - A_{sn} & B_{ss} - C_{ss} \end{bmatrix},$$

$$\underline{a} = [a'_n, a'_s/(1+r)]',$$

$$\underline{a}_\tau = E_t Q_w w_\tau + E_t Q_q q_\tau - [(I - \delta)/(1+r)] E_t Q_q q_{\tau+1}.$$

Furthermore, this optimizing input sequence must belong to the restricted solution space described above. We assume that \underline{B} is nonsingular. Eq. (5) differs from the corresponding equation in Epstein and Yatchew (1985) in that the matrix \underline{B} is (possibly) nonsymmetric. We henceforth refer to situations with $\underline{B} = \underline{B}'$ and $\underline{G} = \underline{G}'$ as the symmetric case and to situations with $\underline{B} \neq \underline{B}'$ and $\underline{G} \neq \underline{G}'$ as the nonsymmetric case.

It is well-known and simple to show that the characteristic roots of the homogeneous difference equation system corresponding to eq. (5) come in pairs multiplying to $(1+r)$. We assume that these roots are distinct. It then follows that exactly k of the roots are less than $(1+r)^{1/2}$ in modulus. Let Λ be the $k \times k$ diagonal matrix of these roots and let V be the $k \times k$ matrix of solution vectors corresponding to these roots. We assume, as in Kollintzas (1986), that V is nonsingular, and define $M = I - V\Lambda V^{-1}$. In the appendix we prove the following theorem:

Theorem.⁴ The optimal factor inputs at time t corresponding to control problem (3) and (because of the certainty equivalence principle) to control problem (2), are uniquely given by the following accelerator model:

$$x_t = M\bar{x}_t + (I - M)x_{t-1}, \quad \bar{x}_t = \underline{A}^{-1}(J_t - \underline{a}), \tag{6}$$

$$J_t = D \sum_{\tau=t}^{\infty} (I + D)^{-(\tau-t+1)} \underline{a}_\tau, \quad D = (1+r)(I - M')^{-1} - I,$$

$$\underline{A} = (I - M')^{-1}(rI + M')\underline{B}M/(1+r) = D\underline{B}M/(1+r).$$

⁴As discussed in the introduction Kollintzas (1986) provides an alternative expression for the optimal control solution involving V and an analogous matrix associated with the roots exceeding $(1+r)^{1/2}$ in modulus. The theorem here establishes also the symmetry of S .

The accelerator matrix M satisfies

$$-\underline{B}(I - M)^2 + \underline{G}(I - M) - (1 + r)\underline{B}' = 0. \quad (7)$$

Furthermore, $S = \underline{B}(I - M)$ is symmetric.

The structure of the above solution for x_t resembles closely that given in Epstein and Yatchew (1985) for the symmetric case.⁵ However, there are subtle differences. Firstly, note that in the symmetric case $\underline{B}M = [\underline{B}M]'$; in the nonsymmetric case $\underline{B}(I - M) = [\underline{B}(I - M)]'$ but in general $\underline{B}M \neq [\underline{B}M]'$. Secondly, note that in the symmetric case $A = A'$; however, in the nonsymmetric case we find that generally $\underline{A} \neq \underline{A}' = \underline{B}M(M + rI)(I - M)^{-1}/(1 + r)$.⁶ Consequently, it is generally incorrect to use the latter formula for \underline{A} in the nonsymmetric case. The latter formula corresponds to formula (9) in Epstein and Yatchew (1985). Hence, while this formula is appropriate in the symmetric case only its transpose is appropriate in the nonsymmetric case.

Given the structure of the above solution for x_t , we can now extend the methodology introduced by Epstein and Yatchew to the nonsymmetric case. A basic difficulty in estimating the dynamic factor demand model (6) stems from the fact that in general (7) cannot be solved explicitly for M in terms of the original model parameters. However, upon making use of the expressions for \underline{B} and \underline{G} given in (5), an inspection of (7) reveals that it is possible to solve this equation for A in terms of B , C , and M , or alternatively, in terms of B , C , and S :

$$\begin{aligned} A_{nn} &= \{ S_{nn} - (1 + r)C_{nn} - (1 + r)C'_{nn} - (2 + r)B_{nn} \\ &\quad + (1 + r)[(B'_{nn} + C_{nn})S^{nn}(B_{nn} + C'_{nn}) + \bar{B}'_{sn}S^{sn}(B_{nn} + C'_{nn}) \\ &\quad + (B'_{nn} + C_{nn})S^{ns}\bar{B}_{sn} + \bar{B}'_{sn}S^{ss}\bar{B}_{sn}] \} / (1 + r), \\ A_{ss} &= S_{ss} + C_{ss} + C'_{ss} - (2 + r)B_{ss} + (1 + r)[(B'_{ss} - C'_{ss})S^{ss}(B_{ss} - C_{ss}) \\ &\quad + B'_{ns}S^{ns}(B_{ss} - C_{ss}) + (B'_{ss} - C'_{ss})S^{sn}B_{ns} + B'_{ns}S^{nn}B_{ns}], \\ A_{sn} &= A'_{ns} = B_{sn} - \bar{B}_{sn} + C'_{ns} - C_{sn}, \end{aligned} \quad (8)$$

⁵In somewhat more detail, the case considered by Epstein and Yatchew corresponds to the following special case of the model considered here: $x_t = n_t = [\bar{n}'_t, n'_t]'$, $a = a_n$, $A = A_{nn}$, $B = B_{nn}$ diagonal, $C = 0$, $Q_q = [I_h, 0_{k-h \times h}]'$, $Q_w = [0_{k \times k-h}, I_{k-h}]'$, where k , h , and $k - h$ correspond to the dimensions of the vectors n_t , \bar{n}_t , and n_t . Their analysis generalizes trivially to the general symmetric case.

⁶That in the nonsymmetric case it is generally no longer true that $\underline{A} = \underline{A}'$ was checked in terms of a specific counterexample.

with

$$\begin{aligned} \bar{B}_{sn} = & - [B'_{ns}S^{ns} + (B'_{ss} - C'_{ss})S^{ss}]^{-1} \\ & \times \{S_{sn} - (1+r)C'_{ns} + C_{sn} - (2+r)B_{sn} + (1+r) \\ & \times [B'_{ns}S^{nn}(B_{nn} + C'_{nn}) + (B'_{ss} - C'_{ss})S^{sn}(B_{nn} + C'_{nn})]\} \\ & / (1+r), \end{aligned}$$

and where S_{ij} and S^{ij} denote the (i, j) th block of, respectively, S and S^{-1} ($i, j = n, s$).

We expect that in most empirical applications the dimensions of the matrices in (8) will be small. Consequently, for typical applications, explicit expressions for the elements of A can be readily obtained from the above formulas. We note further that the above formulas simplify considerably if adjustment costs are taken to be separable, i.e., $C = 0$, and the adjustment cost matrix B is diagonal.

In constructing J_t we assume that expectations on q_t and w_t are formed rationally from the autoregressive process (3). Consequently,

$$J_t = \alpha + \sum_{i=0}^{p-1} \beta_i [q'_{t-i}, w'_{t-i}]'. \tag{9}$$

Rather than express the vector α and the matrices β_i in terms of the vector ν and the matrices Θ_i , it is easier to express the latter in terms of the former. By, e.g., analogous argumentation as in Epstein and Yatchew (1985) it follows that

$$\begin{aligned} \nu = RD\alpha, \quad R = & \{ \beta_0 - D[(I - \delta)Q_q/(1+r), 0] \}^{-1} \\ \Theta_1 = R \{ & (I + D)\beta_0 - D[Q_q, Q_w] - \beta_1 \}, \\ \Theta_i = R \{ & (I + D)\beta_{i-1} - \beta_i \}, \quad i = 2, \dots, p, \end{aligned} \tag{10}$$

where $\beta_p = 0$, $D = B'S^{-1}/(1+r) - I$ and the null matrix in the expression for R is of dimension $\bar{k} \times m_2$. The above equations closely resemble analogous equations given in Epstein and Yatchew (1985).⁷

Based on (7)–(10) we can now reparametrize the production function (1), the system of factor demand equations (6), and the price process (3) in terms

⁷The model considered in Epstein and Yatchew corresponds to a model where $Q_q = [I, 0]'$ and $Q_w = [0, I]'$; compare footnote 5. We note that completely analogous to Epstein and Yatchew it is readily possible to also incorporate deterministic time trends into the above analysis.

of a , B , C , S (or M), α , and $\beta_0, \dots, \beta_{p-1}$.⁸ Since the reparametrized model is described by closed form analytic expressions, it can be readily estimated by standard econometric packages such as TSP. Also, estimation of the model in its reparametrized form seems computationally advantageous: Firstly, it avoids repeated numerical solutions of the firm's optimization problem for different sets of trial parameter values; secondly, derivatives of the statistical objective function can be taken analytically rather than numerically.

Clearly, the estimation approach considered in this note is not restricted to the above specific dynamic factor demand model. The approach is, however, restricted to rational expectations models that result in a second-order difference equation.⁹

Appendix: Proof of the theorem

It follows directly from the definition on Λ and V that

$$-BV\Lambda^2 + \underline{G}V\Lambda - (1+r)\underline{B}'V = 0. \quad (\text{A.1})$$

Observe that by definition $M = I - V\Lambda V^{-1}$. Eq. (7) then follows from (A.1) upon postmultiplication with V^{-1} . Next we demonstrate that S is symmetric. Define $\Omega = V'\underline{B}V$, and let $\omega_{i,j}$ and λ_i denote, respectively, the (i, j) th element of Ω and the i th diagonal element of Λ . Premultiplication of (A.1) with V' and post-multiplication with Λ^{-1} yields

$$\Omega\Lambda + (1+r)\Omega'\Lambda^{-1} = V'\underline{G}V. \quad (\text{A.2})$$

Since \underline{G} is symmetric it follows that also the matrix on the RHS of the above equation is symmetric. The (i, j) th and (j, i) th elements of that matrix are given by, respectively, $\omega_{i,j}\lambda_j + (1+r)\omega_{j,i}/\lambda_j$ and $\omega_{j,i}\lambda_i + (1+r)\omega_{i,j}/\lambda_i$. Equating the two elements yields $\omega_{i,j}\lambda_j = \omega_{j,i}\lambda_i$. Hence $\Omega\Lambda$ and consequently $S = \underline{B}(I - M) = V'^{-1}\Omega\Lambda V^{-1}$ is symmetric.

It remains to be shown that (6) represents the optimal control solution. We define g_τ implicitly from the following equation ($\tau = t, \dots, \infty$):

$$x_\tau = (I - M)x_{\tau-1} + g_\tau. \quad (\text{A.3})$$

⁸For econometric estimation we need to add stochastic disturbance terms to (1) and (6). Following Epstein and Yatchew (1985), we may interpret those disturbance terms as measurement and random optimization errors. The latter may also be interpreted as random shocks to the technology that are observed by the firm but not by the researcher; compare Hansen and Sargent (1981).

⁹For example, the approach still applies if the production function (1) is generalized by adding the following adjustment cost term: $\frac{1}{2}\sum_i[\Delta n'_{t-i}, \Delta s'_{t-i}]B_i[\Delta n'_{t-i}, \Delta s'_{t-i}]'$. Note that the resulting Euler equation is still of second order since all cross-products in the objective function involve at most a time difference of one period.

Clearly then $x_{\tau+1} = (I - M)^2 x_{\tau-1} + (I - M)g_{\tau} + g_{\tau+1}$. Substitution of those expressions for x_{τ} and $x_{\tau+1}$ into (5) and making use of (7) yields

$$\underline{B}g_{\tau+1} - (1 + r)\underline{B}'(I - M)^{-1}g_{\tau} = -(1 + r)h_{\tau}, \tag{A.4}$$

$$h_{\tau} = -\underline{a} + \underline{a}_{\tau}.$$

We note that the roots of this first-order difference equation are given by $(1 + r)\Lambda^{-1}$ and are hence explosive. The backward solution for g_{τ} that satisfies the condition that x_{τ} is of mean exponential order less than $(1 + r)^{1/2}$ is unique and is easily seen to be given by

$$g_t = (1 + r)\underline{B}^{-1} \sum_{\tau=t}^{\infty} [\underline{B}(I - M)\underline{B}'^{-1}/(1 + r)]^{(\tau-t+1)} h_{\tau}. \tag{A.5}$$

Observe that $(I + D)^{-1} = \underline{B}(I - M)\underline{B}'^{-1}/(1 + r) = (I - M')/(1 + r)$. The solution given in (6) is then readily obtained upon substitution of (A.5) into (A.3).

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