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Comments on OPG Variants of LM Tests for Spatial Dependence

The aim of this note is to comment on and clarify the relationship between outer product gradient/martingale difference (OPG) variants of LM tests for spatial dependence with some of the existing literature, including to contributions to LM tests and CLTs for linear quadratic forms by Kelejian and Prucha (2001) and Liu and Prucha (2017), for short KP and LP. The note is written in an intuitive manner.

Consider the simplest case where $y = X\beta + u$ with X exogenous, and we want to test for the absence of correlation in u. Using obvious notation the score is proportional to

$$\widetilde{u}'W\widetilde{u}$$

The approach taken in KP (2001) and LP (2017) is to

1. Show that $n^{-1/2}\widetilde{u}'W\widetilde{u} = n^{-1/2}u'Wu + o_p(1)$
2. Apply the CLT of KP for linear quadratic forms. This CLT was derived by (i) rewriting $u'Wu$ as a sum over martingale differences, i.e., as

$$u'Wu = \sum_{i=2}^{n} u_i\xi_i \text{ with } \xi_i = \sum_{j=1}^{i-1}(w_{ij} + w_{ji})u_j$$

   and (ii) by then applying a CLT for martingale differences. The CLT for linear quadratic form in KP allows for heteroscedasticity. The variance of $u'Wu$ is given by $2tr[\overline{W}\Sigma\overline{W}\Sigma]$ where $\overline{W}$ is the symmetrized weight matrix. The variance of $u'Wu$ can be estimated by $2tr[\overline{W}\widetilde{\Sigma}\overline{W}\widetilde{\Sigma}]$ with $\widetilde{\Sigma} = diag(\widetilde{u}_i^2)$.
3. The above delivers that

$$\begin{aligned} LM &= \frac{(\widetilde{u}'W\widetilde{u})^2}{\widetilde{\Phi}} \xrightarrow{d} \chi(1) \\ &\text{with} \\ \widetilde{\Phi} &= 2tr[\overline{W}\widetilde{\Sigma}\overline{W}\widetilde{\Sigma}] = \sum_{i=2}^{n}\sum_{j=1}^{i-1}(w_{ij} + w_{ji})^2\widetilde{u}_i^2\widetilde{u}_j^2. \end{aligned}$$

Now consider the OPG approach in Born and Breitung (2011), which follows the following steps:

1. Observe that $\widetilde{u}'W\widetilde{u}$ can be rewritten as

$$\widetilde{u}'W\widetilde{u} = \sum_{i=2}^{n} \widetilde{u}_i \widehat{\xi}_i \text{ with } \widehat{\xi}_i = \sum_{j=1}^{i-1} (w_{ij} + w_{ji})\widetilde{u}_j$$

with $\widehat{\xi}_1 = 0$, which leads to the OPG variant of the test statistic

$$\widetilde{LM} = \frac{(\sum_{i=1}^{n} \widetilde{u}_i \widehat{\xi}_i)^2}{\sum_{i=1}^{n} \widetilde{u}_i^2 \widehat{\xi}_i^2} = \frac{(\widetilde{u}'W\widetilde{u})^2}{\widetilde{\Phi}_*}$$

with

$$\widetilde{\Phi}_* = \sum_{i=1}^{n} \widetilde{u}_i^2 \left[\sum_{j=1}^{i-1} (w_{ij} + w_{ji})\widetilde{u}_j \right]^2 = \sum_{i=1}^{n} \widetilde{u}_i^2 \sum_{j=1}^{i-1} \sum_{k=1}^{i-1} (w_{ij} + w_{ji})(w_{ik} + w_{ki})\widetilde{u}_j \widetilde{u}_k$$

Note that $Eu_i^2 u_j u_k = 0$ for $j \neq k$. The difference between $\widetilde{\Phi}$ and $\widetilde{\Phi}_*$ is that the latter is not utilizing this information (and essentially estimates zeros).

2. The paper then shows that $n^{-1/2} \sum_{i=2}^{n} \widetilde{u}_i \widehat{\xi}_i = n^{-1/2} \sum_{i=2}^{n} u_i \xi_i + o_p(1)$ and then applies a CLT for martingales to $n^{-1/2} \sum_{i=2}^{n} u_i \xi_i$ to get the limiting distribution. Born and Breitung then argue that $\widetilde{\Phi}_*$ is a consistent estimator for the asymptotic variance covariance matrix.

3. The above then delivers that

$$\widetilde{LM} = \frac{(\sum_{i=1}^{n} \widetilde{u}_i \widehat{\xi}_i)^2}{\sum_{i=1}^{n} \widetilde{u}_i^2 \widehat{\xi}_i^2} = \frac{(\widetilde{u}'W\widetilde{u})^2}{\widetilde{\Phi}_*} \xrightarrow{d} \chi(1).$$

Comments:

1. From the above it seems that Born and Breitung (2011) in essence and substance have also adopted the approach of KP (2001) in rewriting/transforming the score as a sum over martingale differences. Of course, the idea of a transformation of this kind has its origin in the statistics literature. There is a slight difference in the estimation of the asymptotic variance covariance matrix of the quadratic form.

2. In comparing $\widetilde{\Phi}$ and $\widetilde{\Phi}_*$ we see that the OPG formulation by Born and Breitung (2011) employes an estimator $\widetilde{\Phi}_*$, which does not incorporate all available information, and is likely slightly less efficient than the estimator $\widetilde{\Phi}$, which incorporates all information regarding terms which are zero.

3. Born and Breitung (2011) state that an important advantage of the OPG variant is that it is robust against heteroscedasticity and non-normal disturbances. Of course, the original test of

KP (2001) and the generalizations in LP (2017) have the same advantages. Additionally, the latter approach computes and estimates the limiting variance covariance matrix in a manner that utilizes all information on zero covariances.

4. In establishing that a sequence is a martingale difference sequence it is important to carefully define the sequence of information sets. Of course, central limit theorems for sums of martingale differences postulate various assumptions, which need to be checked carefully. A widely used CLT for martingale differences is Theorem 3.2 of Hall and Heyde (1980). The CLT's for linear quadratic forms in Kelejian and Prucha (2001,2010) utilize Theorem 3.2 of Hall and Heyde (1980). However, as pointed out in Kuersteiner and Prucha (2013), one of the assumptions maintained by Theorem 3.2 of Hall and Heyde (1980) is violated by typical panel data. They introduce as their Theorem 1 an alternative CLT for martingale differences, which covers panel data without the assumption that all regressors are strictly exogenous.

Lit:

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