Optimal Dynamic Hotel Pricing[*]

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Abstract

We analyze a confidential reservation database provided by a luxury hotel, "hotel 0", based in a major US city that enables us to observe individual reservations and cancellations at a daily frequency over a 37 month period. We show how the hotel sets prices for various classes of customers and how its prices vary over time. Hotel pricing is a challenging high-dimensional problem since hotels must not only set prices for each current date, but they must also quote prices for a range of future dates, room types and customer types. We formulate and estimate a structural model of optimal dynamic hotel pricing using the Method of Simulated Moments (MSM). The estimated model provides accurate predictions of the actual prices set by this firm and resulting paths of bookings and cancellations. Prices quoted for bookings generally decline as the arrival date approaches on non-busy days, but can increase dramatically in the final days before arrival on busy days when there is a high probability of sell-out. Hotel 0's prices co-move strongly with its competitors' prices and we show that a simple *price-following* strategy where hotel 0 undercuts its competitors' average price by a fixed percentage provides a good first approximation to its pricing behavior. However we show that simple price-following is suboptimal: when hotel 0 expects to sell out, it is optimal to depart from price-following and increase its price significantly above its competitors. Though price-following has the superficial appearance of collusive behavior mediated by the use of a commercial revenue management system (RMS), our results suggest that hotel 0's pricing is competitive and is best described as a rational best response to its beliefs about demand and the prices set by its competitors.

Keywords price discrimination, dynamic pricing, price-following strategies, Bertrand price competition, dynamic programming, method of simulated moments, revealed beliefs, revenue optimization, revenue management systems, algorithmic collusion

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1 Introduction

We analyze a unique new micro panel dataset of daily observations of reservations, prices, and occupancy of a luxury hotel based in a major US city. Due to the confidential nature of the data we are unable to reveal the name of the hotel or the city where it is located. Hereafter we refer to it as "hotel 0" since it is one of 7 competing luxury hotels (with its competitors labeled 1 to 6) that constitute a local market in a small but highly desirable location of this city. We formulate and estimate a dynamic model of optimal pricing by hotel 0: it sets its prices to maximize its expected profits (revenue less cost of cleaning/servicing rooms) as a best response to its beliefs about the arrival of customers and the dynamics of its competitors' prices. Our main finding is that our model provides surprisingly accurate predictions of the prices set by hotel 0. This suggests that hotel 0 is setting prices in an approximately optimal fashion and is consistent with the hypothesis that there is a dynamic Bertrand price equilibrium in this particular luxury hotel market.

Hotel pricing is a challenging problem since beside setting different prices for various room categories (standard rooms, deluxe rooms, penthouse suites, etc) and customer categories (tourist versus business guests, group discounts for corporations, governments, etc.) a hotel manager must be able to continuously update and quote a large array of *future prices* since most of its customers book rooms well in advance of their planned arrival date. Optimal pricing depends critically on accurate knowledge of customer demand, and there are two key aspects to this: 1) recognizing the stochastic nature of demand and bookings and being able to use pricing to accommodate large day-to-day swings in the number of customers wishing to stay in one of the hotels in this market, and 2) understanding customers' evaluation of the relative desirability of the competing hotels and their degree of price sensitivity, and being able to exploit differences along these dimensions among its various types of customers.

We introduce a dynamic model of hotel demand that captures these two key aspects of demand. Customers arrive stochastically and reserve a room at hotel 0 or one of its competitors at randomly distributed lead times prior to arrival. Our model allows for stochastic cancellations but not overbooking: the dynamic allocation of capacity subject to "hard" capacity constraints is central to our explanation of hotel 0's price setting behavior. Though we have daily observations of the *best available rate* (BAR) of comparable rooms quoted by the six competing hotels, we only observe the number of new reservations (and cancellations) at hotel 0, but not at its competitors. Thus, we face a problem of *censoring* that makes it challenging to estimate customer demand, and without knowing demand, it is hard to set good prices.

Via a matched dataset provided by STR, we observe the total occupancy and average daily rate (ADR) for all seven hotels on a daily basis. The ADR is an average of different prices paid by different customers who reserved at different times and may have been eligible for various group or corporate/government discounts. If we use ADR in place of the price customers were actually charged, at a minimum we have a problem of errors in variables. But there is a more serious problem of *endogeneity* in hotel prices due to the strong co-movement of prices of the seven hotels who independently raise or lower prices in response to shocks to the aggregate demand for luxury hotel rooms in this part of the city. Prices peak to ration the available supply of rooms on days where demand is high and occupancy is close to 100%, but prices can fall precipitously on days when demand is low and there is significant excess capacity. Regressions of hotel occupancy on hotel prices therefore produce spurious positively sloped demand functions due to the effect of demand shocks on endogenously determined prices. There are few relevant instrumental variables that can successfully deal with the endogeneity problem. Also, hotel demand is not given by a simple linear demand equation but by a conditional probability distribution that is generally nonlinear in prices, derived from micro aggregation of individual discrete choices of hotel by a random number of customers who book rooms at various future arrival dates. It is not obvious how to control for endogeneity in our stochastic nonlinear dynamic model even if we did have good instruments.

We show how the censoring, errors-in-variables, and endogeneity problems can be solved using structural econometric methods. We provide credible structural estimates of the stochastic arrival process of customers and their preferences for the competing hotels using the method of simulated moments of McFadden (1998) as extended to dynamic structural models with continuous decisions and endogenous censoring by Merlo, Ortalo-Magne, and Rust (2015) and Hall and Rust (2018).[1] Our key identifying assumption, besides parametric restrictions on consumer arrival and demand, is the maintained assumption that hotel 0 is an expected profit maximizer. In essence, our structural estimation can be regarded as process for inferring the hotel manager's beliefs about customer demand that are implicit in the array of prices the hotel sets on a daily basis. As such, our structural estimation method can be regarded as a procedure for inferring the hotel manager's *revealed beliefs* about customer demand from observations of the prices they set, similar to the way that structural estimation is used to infer the revealed preferences of consumers from observations of their choices, see e.g. McFadden (1976).

However just because we *assume* that hotel 0 maximizes profits does not imply that our relatively simple and parsimoniously parameterized model will be able to provide reasonable estimates of demand or good predictions of the prices the hotel actually charges. We show, via simulations of counterfactual pricing strategies, that our model and optimal pricing algorithm provides intuitively reasonable counterfactual predictions of occupancy and revenues. We showed the predictions to the manager of hotel 0, who agrees that they are plausible. We can use the model to simulate a wide range of counterfactual pricing

[1] See also MacKay and Miller (Harvard Business School, typescript, 2018) who provide a related method of moments based method for "instrument-free demand estimation."

strategies and quantify the forgone profits relative to a dynamically optimal strategy.

Our model generates optimal prices for hotel 0 in virtually any scenario. The optimal strategy entails both *price following* and *price undercutting* under typical conditions, but it is optimal for hotel to raise its prices unilaterally to values significantly above its competitors when it expects to sell out. However it is not optimal for hotel 0 to decrease its prices unilaterally in the face of expected excess capacity unless its competitors also decrease their prices. Thus, the optimal strategy takes the form of a *conditional price following rule:* undercut competitors' price by a roughly fixed percentage unless hotel 0 expects to sell out. In the latter case optimal prices rise in a way that resembles an auction for scarce room capacity.

Our paper contributes to the academically understudied area of *applied revenue management.* A key reference to this literature is Phillips (2005) who notes that despite the fact that pricing decisions "are usually critical determinants of profitability" "pricing decisions are often badly managed (or even unmanaged)." (p. 38). He documents the growth of commercial revenue management systems that originated in the 1980s when American Airlines was threatened by the entry of the low-cost carrier PeopleExpress.

> "In response, American developed a management program based on differentiating prices between leisure and business travelers. A key element of this program was a "yield management" system that used optimization algorithms to determine the right number of seats to protect for later-booking full-fare passengers on each flight while still accepting early-booking low-fare passengers. This approach was a resounding success for American, resulting in the ultimate demise of PeopleExpress." (p. 78).

Commercial RMSs are now widely used both by the airlines and in the hospitality industry due to the similar nature of the problem of advance booking and optimally allocating a finite and perishable "inventory" to stochastically arriving customers with differing willingness to pay. Examples include IDeaS (a SAS subsidiary), JDA, PROS, and Revenue Analytics. According to Anderson and Kimes (2011) "At its most basic level, RM is about a hotel's ability to segment its consumers and price and control room inventory differently across these segments — in essence practicing some form of price discrimination. In many instances RM used in the hotel industry has been shown to increase revenue by 2 to 5 percent." (p. 192).

Revenue management systems are proprietary so we do not know what sort of optimization principles they use and what types of data and econometric methods they employ. McAfee and te Veld (2008) note that "At this point, the mechanism determining airline prices is mysterious and merits continuing investigation because airlines engage in the most computationally intensive pricing of any industry." (p. 437). Phillips (2005) notes that "The tools that pricers use day to day are far more likely to be drawn from the fields of statistics or operations research than from economics." (p. 68) and he credits marketing (which he regards as a subfield of operations research and management science) noting that "marketing science has brought some science to what was previously viewed as a 'black art"' (p. 70). Yet "there remains a gap between marketing science models and their use in practice. The reasons for this gap are numerous. Many marketing models have been build on unrealistically stylized views of consumer behavior. Other models have been build to 'determine if what we see in practice can happen in theory.' Other models seem limited by unrealistically simplistic assumptions." (p . 70).

Phillips' book and the related literature on revenue management systems contain many important practical insights and offer many *heuristic principles* for revenue management such as the advice of Anderson and Kimes (2011) to "Be careful with rate reductions because you could lower your rates (and dilute your ADR) without improving occupancy." (p. 195). However these studies make no mention of a key tool for calculating optimal dynamic prices — *dynamic programming* (DP). In fact, there is a substantial literature in operations research/management science that uses DP to characterize optimal dynamic pricing strategies for perishable inventories over a finite horizon, see for example Gallego and van Ryzin (1994) and McAfee and te Veld (2008) and references in these papers to literature dating back to the 1960s. Recent work has focused on numerical calculation of optimal dynamic pricing strategies specifically for hotel revenue management, see Ivanov (2014), Anderson and Xie (2012), Zhang and Lu (2013), and Zhang and Weatherford (2016). Still, most of the OR/management science literature is highly theoretical and to our knowledge only Zhang and Weatherford (2016) provide any empirical evidence of how well the DP algorithms perform in practice, and they conclude that though the "relative magnitude of the revenue improvement is small" (approximately 0.19%) "this truly can be a significant improvement, especially given that DP decomposition is the state-of-the-art in the industry in terms of implemented algorithms."

In fact, *dynamic pricing* has not been widely adopted by most RMSs. Instead they practice *yield management* an orientation inherited from their origin in the airline industry. This involves controlling *quantities* using a predefined set *time-invariant prices* "many revenue management (RM) applications are based on product availability control, in which product prices are fixed and product availability is adjusted dynamically over time. Static pricing, whereby the price for each product is fixed, is also frequently observed in practice." (p. 102). As Gallego and van Ryzin (1994) note, "Airlines and hotels must, for a variety of operational and customer-relations reasons, offer a limited number of fares that remain relatively static, at least in the sense of spanning several problem instances." However they argue that dynamic pricing can be approximated using yield management strategies "a static set of fare classes together with a dynamic allocation scheme can be used to synthesize different prices for each instance. This interpretation better explains both the magnitude of revenue increases and the disparity in fare prices found in yield management practice." (pp 1000-1001).[2]

[2] See also Board and Skrzypacz (2016) who characterize optimal selling strategies using mechanism design when buyers are forward looking. They find that "in the continuous-time limit, the optimal mechanism can be implemented by posting anony-

There have been a number of claims that yield management and RMS have lead to significant improvements in profitability. Gallego and van Ryzin (1994) claim that "The benefits of yield management are often staggering; American Airlines reports a five-percent increase in revenue worth approximately \$1.4 billion dollars over a three-year period, attributable to effective yield management." (p. 1000). However, we are unaware of studies that provide scientific validation (say via controlled experiments or other means) of the claims that commercial RMS have resulted in significant increases in hotel revenues and profits. The only study we found was Ortega (2016) who used a database of chain hotels with 3 star ratings and ANOVA methods to analyse whether hotels that use a RMS outperform non-RMS-users in a context of decreasing demand. This study concludes that "RMSs have been more effective in improving occupancy than in achieving higher rates." (p. 656). We are not aware of any other studies that analyze the algorithms that RMS use to allocate rooms or set recommended prices, or any comparisons of the profitability of commercial RMS relative to expert human revenue managers.

Since our econometric model calculates optimal recommended prices in real time for any possible scenario, it can be regarded as a prototype RMS. We can subject our model, and in principle any commercial RMS, to scientific validation and testing such as using holdout samples to validate its performance. The ultimate validation is via *controlled field experiments* that compare the profitability of a "treatment location" where prices are set by a RMS with the profitability of a "control location" where prices are set by an expert human revenue manager. This type of field experiment was conducted in Cho and Rust (2010) to demonstrate that DP can improve the profitability of rental car rate-setting and replacement decisions. Unfortunately, there is only a single hotel 0 so it is not possible to compare differences in profitability between a treatment and control location: at best we could evaluate our model using a "before-after" comparison similar to the one done in Misra and Nair (2011). The design of effective field experiments to validate our model (or commercial RMS) is beyond the scope of this paper, but we show that our model enables us to conduct simulated field experiments that provide considerable insight into how an effective experiment must be designed to make valid inferences about whether one decision procedure (or RMS) is better than another given the inherent variability in outcomes driven by stochastic shocks to the demand for hotel rooms.

The closest available study to our's methodologically is the recent econometric study by Williams (2018) who uses a dynamic structural estimation approach that is very similar to the one we use in this study, but using data from a particular airline.[3] To our knowledge Williams is the first to use an empirically

mous prices." (p. 1046).

[3] Other relevant papes include Lazarev (2013) and Sweeting (2012). Lazarev also studies airline pricing on monopoly routes, and Sweeting studies dynamic pricing of major league baseball games using secondary market data from eBay and StubHub. To the extent that particular baseball games are one-time events, they are essentially dynamic auctions by a monopolist that differ in

estimated dynamic programming model to show how dynamic pricing in the face of stochastic demand complements intertemporal price discrimination in airline markets. He concludes that "By having fares respond to demand shocks, airlines are able to secure seats for late-arriving consumers. These consumers are then charged high prices. While airlines utilize sophisticated pricing systems that result in significant price discrimination, these systems also more efficiently ration seats." (p. 47). The airline that Williams studied used a commercial RMS to help set its prices, and his results suggest that commercial RMSs are capable of recommending nearly optimal prices.

We find very similar conclusions for the hotel we study, except that the *human revenue manager* frequently overrides the recommended prices from the RMS. Thus, we conclude that the hybrid human-supervised/RMS system results in a nearly optimal dynamic pricing strategy. Similar to Williams we show that dynamic pricing results in significantly higher expected profits than fixed price strategies, calling into question the empirical revelance of a major conclusion of Gallego and van Ryzin (1994) "that policies that have no price changes are asymptotically (as the expected volume of sales increases) optimal over the class of policies that allow an unlimited number of price changes at no cost." (p. 1001). The major difference between our study and Williams' (besides the difference in application area, hotels versus airlines) is that due to lack of data on airfares of competing airlines Williams focused on *monopoly routes.* However it is obvious that the value of dynamic pricing is even greater in a competitive context where the hotel needs to adjust its own prices in response to changes in the prices of its competitors.

With only a 13% revenue/occupancy share in the local market where it operates, hotel 0 is far from having monopoly power, so it is not surprising that the prices of its competitors is a key state variable in hotel 0's pricing strategy. As we noted above, hotel 0's pricing strategy can be well approximated by a combined price-following and price-undercutting strategy where it discounts its price relative to its competitors' by a fixed percentage. In fact a simple regression of hotel 0's prices on the average price of its competitors and seasonal and weekday dummy variables has an R^2 of .86.

The fact that most hotels use commercial RMS systems that provide recommended prices and have real time access to the prices charged by their competitors has raised concerns about the potential for *algorithmic collusion* that may not be technically illegal given current US Anti-trust law (see Harrington (2017) and Ezrachi and Stucke (2016)). Yet Harrington admits that "there is currently no evidence of

key respects from repeated competitive pricing game that occurs in the hotel market we analyze. There are also methodological differences: Sweeting uses a two-step approach that differs from the fully structural estimation approach that our study and the Williams study employs. In the first step Sweeting estimates the demand for tickets using instrumental variable methods, and in the second step he tests whether a first order Euler equation condition for optimal dynamic pricing holds given the estimated demand curve. Sweeting finds that "the simplest dynamic pricing models describe very accurately both the pricing problem faced by sellers and how they behave, explaining why sellers cut prices dramatically, by 40 percent or more, as an event approaches. The estimates also imply that dynamic pricing is valuable, raising the average sellers expected payoff by around 16 percent." (p. 1133).

collusion by autonomous price-setting agents in actual markets, and research has yet to be conducted to investigate whether such collusion can occur in a reasonably sophisticated simulated market." (p. 71).

If hotel 0's price setting can be described as a price-following strategy, is this evidence of algorithmic collusion fostered by the hotels' real time access to each others' prices and their use of a commercial RMS that might be recommending collusive prices? A time series plot of the hotels' prices shows price cycles with high price periods interspersed with briefer periods of deep price cuts. Is this evidence of tacit collusion by these hotels with periodic "price wars" that punish hotels that deviate from the collusive price recommended by their RMS? Kimes (2009) analyzes an international survey of hotel revenue managers who cite "price wars" as one of their chief concerns. One respondent wrote "Price wars! Keep your cool and be a price leader also in rough times. Your comp. set will follow (eventually)." (p. 9).

However in the market we study, we see little evidence of tacit collusion and price wars. The price-following behavior we observe can be explained by stochastic demand shocks that result in highly correlated movements in occupancy and ADRs of the hotels in this market. The hotels raise their prices sharply to effectively "auction off" scarce collective capacity on particularly busy days when all the hotels are nearly sold out, but they cut prices in a manner predicted by a model of Bertrand price competition on non-busy days where the hotels have significant excess capacity.

We find that price following with proportional price undercutting is a best response by hotel 0 to its competitors except in situations where hotel 0 expects to sell out. Though price following has the superficial appearance of collusive behavior mediated by the use of a commercial revenue management systems (RMS), our results suggest that a dynamic competitive Bertrand equilibrium provides a better description of the outcomes in this market. Further, the fact that hotel 0 frequently *disregards* the recommended prices of its RMS also casts doubt on the hypothesis of RMS-mediated collusion. In any event, we think it is unlikely that the RMS used by hotel 0 recommends collusive prices, and effective collusion would seem to require all of the hotels in this market to subscribe to the same RMS, which is also unlikely.

Section 2 describes the data set and documents the price following behavior by hotel 0 relative to its competitive set. Section 3 introduces our dynamic model of demand for hotel rooms and the dynamic programming problem we solve to provide our own version of "recommended prices." Section 4 describes the method of simulated moments estimator we use to uncover the hotel manager's beliefs about stochastic demand for hotel 0 and presents our estimation results and main empirical findings. We show that the optimal prices from our dynamic programming model are close to the prices hotel 0 actually sets. Section 5 illustrates the predictions of the model by considering several counterfactual pricing strategies. Section 6 summarizes our conclusions and discusses topics we plan to explore in future work.

Table 1: Hotels in the local market in our study

Property	Avg. BAR	Star	Class	Chained Brand	Rate	Relative Capacity	Distance to mass transit	Cancel Policy
hotel 0	$ 293.26	4	Luxury	No	4.4	79%	3 min	1 day before
hotel 1	$ 282.64	4.5	Upper Up	No	4.4	81%	5 min	3 day before
hotel 2	$ 285.16	4	Upper Up	No	4.4	63%	3 min	1 day before
hotel 3	$ 338.29	4	Upper Up	Yes	4.2	99%	8 min	2 day before
hotel 4	$ 397.09	4	Luxury	No	4.6	100%	10 min	Strict
hotel 5	$ 253.51	4	Upper Up	No	4.2	47%	8 min	3 day before
hotel 6	$ 454.30	5	Luxury	Yes	4.7	52%	10 min	1 day before

2 Data

As we noted in the introduction, due to a non-disclosure agreement with the hotel that provided the data for our study, we are unable to provide too much detail about the local market in which hotel 0 operates to guarantee the anonymity of the hotel and the owner. We can say that it is a luxury hotel located in a highly desirable downtown location of a major US city. The company that owns this hotel operates a small chain of "boutique hotels" in leading cities worldwide.

Hotel 0 is one of seven luxury hotels operating in a tightly defined local area that is recognized by OTAs and other travel agents. Though customers can book at other luxury hotels in other parts of this city, the locations of these other luxury hotels are sufficiently far from this particular desirable area that they are not regarded as relevant substitutes for customers who wish to stay in this specific area of the city. Table 1 lists some summary information about the seven hotels: all are 4-star or higher rated hotels that are classified as upscale class or luxury class. To avoid identifying the hotels we show only the relative capacity, where we normalize the capacity of the largest hotel to 1. However our model uses all relevant information including the actual capacity, which we will show is quite important to the optimal pricing rule we derive.

2.1 Data sources

The customers of the hotel are both business/government customers who mainly stay in the hotel on weekdays and tourists who typically stay on weekends. Since business customers and government customers are reimbursed for their travel expenses, we can expect them to be more price inelastic than tourists. On the other hand, many government agencies and large corporations that do frequent business in this city have negotiated government and corporate discounted rates with this hotel. These discounted rates are typically a fixed percentage, often 15 to 20%, off the currently quoted price that is called the *best available rate* (BAR). The revenue manager of hotel 0 is in charge of updating an array of BARs for different room classes and different future arrival dates and posting these prices to the web via the GDS and via its own website. As we noted above, the revenue manager uses a *uniform price strategy* and does not sell blocks of rooms to wholesalers under contracts that give wholesalers discretion to set their own prices for the blocks of rooms they purchase. Thus, there is no ability to "arbitrage" prices of rooms for hotel 0 by searching different OTAs. However hotel 0 does pay a significant commission, ranging from 15 to 25%, for reservations that are made via OTAs such as Expedia. The GDS that hotel 0 uses allows the revenue manager to change prices as frequently as she desires, though there is a short lag before the prices are propagated everywhere on the Internet including the leading OTAs. However for hotel 0's own website and reservation system, price changes take place instantaneously, and hotel 0 has its own loyalty program that provides discounts to customers who are members of the program. There are other groups that include weddings that involve a larger group of guests that are typically individually negotiated with the hotel revenue manager, but the discounts to these groups are typically quoted as a percentage discount off the BAR similar to corporate and government contract rates.

As we noted above, hotel 0 subscribes to a RMS that provides recommended prices. The hotel revenue manager uses her own discretion to select a relatively small number of different possible BARs (effectively, she discretizes the pricing space) which are treated as a predefined choice set that is entered into the RMS. Based on a proprietary algorithm that considers remaining availability, seasonal effects, cancellation rates and competitors' prices, the RMS communicates a recommended BAR to the revenue manager at the start of each business day. Even though the revenue manager has some control over the prices the RMS can recommend via her choice of a predefined finite set of possible BARs, she typically ignores the recommended price from the RMS and instead sets her own BARs based on her own experience, judgement and intuition.

We do not know to what extent the RMS is able to observe and adapt to the knowledge that the revenue manager is disregarding their recommended prices. This would seem to be important information that any RMS would want to collect, including the revenue manager's feedback about the overall quality of the recommended prices from the system. We can imagine that manual "price overrides" are common for newly launched hotels where the RMS may initially not have enough data to form good predictions about demand, or when there are unexpected changes to demand or entry/exit of other hotels in the local market. In these cases we might expect that the recommended prices from the RMS would be less trustworthy until sufficient data are accumulated to enable the RMS to provide an updated model of customer demand that

Table 2: Data sources used in this study

Data	The first day of occupancy	The last day of occupancy	Observations	Description
market vision	2010-09-21	2014-08-13	609,181	competitors' price
reservation raw	2009-09-01	2013-10-31	201,176	reservations detail information
cancellation raw	2009-09-01	2013-10-31	29,241	cancel detail information
daily pick-up report	2010-09-16	2014-05-21	475,187	daily revenue report
STR market data	2010-01-01	2014-12-31	1,731	competitors' occupancy
Data range	**2010-10-01**	**2013-10-31**		**37 months**

provides accurate predictions for the local market in question. But price overrides are the norm for hotel 0, even though it has been using the RMS for many years and market conditions are reasonably stable (e.g. no major entry or exit of competing luxury hotels or expansions of its competitors' capacity, etc). It is puzzling that the RMS does not appear to adapt and respond to the fact that hotel 0's revenue manager repeatedly ignores its recommendations.

Hotel 0 provided us information from its reservation database that enabled us to track all bookings, cancellations, and prices for a 37 month period between September 2010 and October 2013. In addition, we were provided aggregate daily reports and their competitive daily rates of hotel 0's six competitors from a service called *Market Vision* provides quotes from hotel 0's six competitors for several room rate categories several times per day. While Market Vision provides excellent data on prices, it provides no information on the reservations at hotel 0's competitors. This information does not seem to be readily available, but we were able to obtain data on the occupancy of hotel 0's competitors on a daily basis thanks to data provided by STR. Table 2 summarizes the data sources we used for our study.

Market Vision's provides prices sampled from all channels such as GDS, OTAs/Meta sites, and hotel websites. Although it collects only the lowest priced rooms for each hotel, it also collects prices relevant to different customer segments such as groups like AAA, Advance purchase, Any Non-qualified rates, Government, Unrestricted/No Merchant, and Unrestricted. Advance purchase rates can provide customers discounts of 10-15% off the current BAR if booked more than 7 days prior to arrival and often include a deposit or pre-payment to guarantee the reservation. Any Non-qualified rates are the lowest of the Unrestricted, Advance Purchase/No Merchant Model but exclude qualified rates that require membership, association or identification and contract customers such as government. Unrestricted/no merchant rates are prices available to all customers without qualification or advance purchase requirements except that other merchants and wholesalers are excluded. Unrestricted prices are the residual set that are offered

Table 3: Room Types

Code	Description	% of rooms (before renovation)	% of rooms (after renovation)	Rack Rate
B1K	Superior, 1 King	57	43	$203.15
B2D	Superior, 2 double beds	33	19	$ 203.15
A1K	Deluxe, 1 King	4	14	$ 253.15
A2D	Deluxe, 2 double beds	1	14	$ 253.15
GD1K	Grand Deluxe, 1 King	0	3	$ 303.15
GD2D	Grand Deluxe, 2 double beds	0	1.5	$ 303.15
others	Suites, etc	5	5.5	>$600 or negotiated

to any customer including merchants and room wholesalers and typically have a 24 hour cancellation window, i.e. there is no penalty for cancellation provided it is done more than 24 hours prior to the standard check-in time on the date of arrival. Market vision separately collects special price offers that come with non-standard cancellation penalties.

Our data are unique in the level of detail we have on reservations and cancellations. Our reservation database contains the full history of each individual booking, including the channel through which the booking was made. Each booking is identified with a unique reservation identification number that is created when the reservation is initiated and becomes the permanent identifier for each reservation along with time stamps and dates of arrival and departure and amounts actually paid including incidental charges.

Among these 11 room types, Hotel 0 essentially has two basic categories: regular rooms and luxury suites but 95% of the rooms in the hotel are regular rooms. Typically, BAR is the rate for room categories B1K and B2D in Table 3. We rarely observe the hotel overbooking the rooms in these categories, though on the few occasions where this happens the overflow customers are automatically upgraded to the next highest tier of rooms such as A1K or A2D.

There are around 200 rate codes which can be broken into 14 categories summarized in Table 4. To simplify the analysis, we divided the codes into two; transient and group bookings. Transients are individual travelers who pay the BAR or discounted BAR. Although the net of commission price that hotel 0 receives differs depending on which channel was used to do the booking (i.e. an OTA versus hotel 0's own website), transient customers themselves pay the same price regardless of channel, namely the BAR in effect at the time they booked. Group bookings are also generally based on the BAR in effect when they booked, however it will vary by pre-negotiated contract discount rate that differs from different groups (rate codes).

A field in the reservation database, "share amount," records how much the guest has paid for the room

Table 4: Hotel Reservation type

Category	Market Segment	Title	Description	Booking Share
Transient	BAR	Best Available Rate	Best available rates that have hotel house cancellation policy, rate codes BAR only applicable in this segment	68.4%
	CON	Consortia/TMC	Consortia, Travel Management Companies bookings	
	RESW	Restricted-Web	Advance purchase and/or any promotional offers available in Hotel 0 collection web site with restrictions such as pre-paid/non-refundable i.e. 10% off 7 day advance purchase, 2mlos at 20% off, or limited time offer	
	CORL	Corporate LRA	Corporate/local negotiated rates with last room availability	
	CORN	Corporate NLRA	Corporate/local negotiated rates with Non-last room availability	
	GOV	Government	Federal or state government per diem and/or accounts with per diem equivalent rates	
	PAK	Package	Room package	
	FIT	Wholesale	Locally negotiated wholesale accounts and Third party vacation package	
	DIS	Qualified Discount	AAA, AARP, Employee rate or any qualified discounted rates	
	RESO	Restricted-OTAs	Same rates as restricted segment available in OTA merchant sites	
	OPQ	Opaque	Hotwire/ Priceline	
Group	CGP	Corporate	corporate group	31.6%
	CGV	Government	government group	
	ASS	Association	convention group	
	TOT	Tour & Travel	tour group	
	group	group	uncategorized group	

per night excluding tax. The share amount is generally the gross is revenue that hotel 0 earns from that customer on the given date. However when a guest books through a traditional travel agency, the hotel must subsequently pay a commission to the travel agent, typically 10% of share amount. However if the booking is made via an OTA, the share amount is net of the OTAs commission, which is typically 22% for hotel 0. The reservation database also allows us to observe cancellations. Due to the 24 hour standard cancellation policy at hotel 0, we observe the highest rate of cancellations a day before the scheduled arrival date. As we show below, it is critical to account for cancellations, and the way cancellations are modeled (including whether or not cancellations are "strategic" and respond to changes in the BAR that Hotel sets after the reservation was initially made) has a significant impact on its pricing strategy.

Note that while the hotel reservation database tracks each separate reservation and cancellation, we had to use these data to reconstruct the occupancy and revenues earned by the hotel on a day by day basis over our sample period. hotel 0 has an information system that provides a daily summary of its bookings and revenue called the "Daily Pace Report". We used this information to provide a check on the occupancy and revenues that we constructed directly from hotel 0's reservation database. On most days we can exactly replicate the summary numbers in the Daily Pace Reports from our own constructed totals using the reservations and cancellation information in the reservation database. When there were discrepancies, the differences only amounted to a few rooms, or about only 2-3% of total occupancy. However since the time interval for the Daily Pace Reports is a subset of the time interval of reservations in the reservation database, we restricted our analysis to the subinterval from September 16, 2010 to October 31, 2013 where it was possible for us to cross-check our constructed occupancy and revenue totals.

Although the data we have on hotel 0 provides an incredible level of detail, as we show in the next section, our model requires more data about the reservation/cancellation quantity dynamics of hotel 0's competitors that are not provided in the Market Vision data, which provide only competitors' prices. The information on the total number of consumers who "arrive" and book rooms at one of the seven hotels in this local market is critical for our inferences about customer demand, and especially how customers respond to daily fluctuations in the relative BARs of the seven competing hotels. Unfortunately we do not have access to the reservation databases of hotel 0's competitors, so we are unable to observe the total number of new reservations that are made in at all the hotels and at which prices (including group, corporate discounts, etc) besides hotel 0. However as we show in the next section, it is possible to make inferences on the booking and reservation/cancellation dynamics of hotel 0's competitors given their prices if we can at least observe the total final occupancy rates of its competitors. Fortunately we were able to obtain this information from STR via an academic research contract it has with Georgetown University. In addition to total occupancy at each competing hotel on a daily basis, the STR data provide information on the competitors' ADRs and total revenue. The STR data turn out to be crucial for our ability to estimate a credible demand model.

2.2 Data summary

Figure 1 illustrates the cyclicality of reservations and prices, both over a given week and over the year, reflecting seasonal variations in the demand for hotels. The bars in the left hand panel of figure 1 show a typical weekly cycle of occupancy for hotel 0 where the lowest occupancy is on Sunday, but a peak occupancy on Saturday, and a midweek peak occupancy on Tuesdays and Wednesdays. The ADR peaks on Tuesday, and the higher rates during the weekdays reflects price discrimination for less price elastic business guests, whereas the lower rates on Fridays and Saturdays are designed to attract more price elastic tourists. Occupancy is lowest on Sundays when tourists are checking out to return home for work on Monday, whereas a typical business guest checks in during the middle of the week and departs before the weekend. The right hand panel of figure 1 shows the price and occupancy dynamics over the year. Occupancy rates are the highest in the spring and early fall, and are lowest around holidays such as Thanksgiving, Christmas and New Year's. The black line in the figure plots hotel 0's ADR and total revenues, and we seek that both of these move in sync with the ups and downs in occupancy rates. This suggests that prices and revenues at hotel 0 are highly "demand driven".

Figure 1: Booking and price dynamics over the week and year

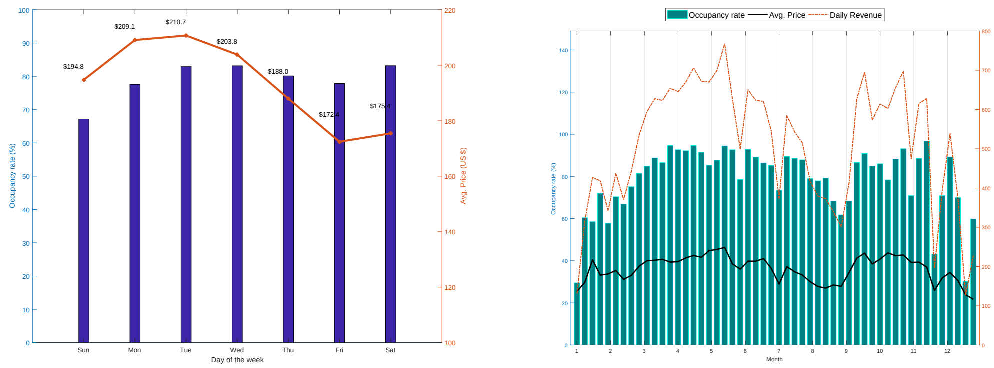


Figure 2: Annual price dynamics for all seven hotels

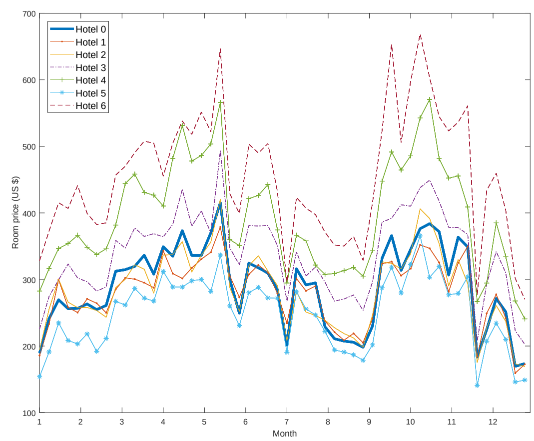


Figure 2 compares the price dynamics for hotel 0 to those of its six competitors over the year. It plots the weekly average BAR from October 2010 to October 2013 for same-day reservations using the Market vision data, though we would obtain similar results if we plot a time series of ADRs using the STR data. The bold line plots the average BAR of hotel 0 while the other lines indicate BAR of six competitor hotels. We see strong co-movement in the prices of the seven hotels, and that they follow similar cyclical fluctuations, though hotel 0 tends to underprice its competitors with the exception of hotel 5. Similar the prices in figure 1 we find that prices are highest in the spring and the fall with peaks in early May and mid-September and October. Prices are lowest at the key holidays: Thanksgiving, Christmas, New Year's, as well as early July and August. During peak periods the average BAR of hotel 0 can be over \$350 per night, whereas in the lowest periods it averages about \$200.

The pattern of co-movement in the prices in this market might be described as "price following" and given the fact that most hotels use RMS and have extensive knowledge of their competitors' prices from services such as Market Vision, it could raise concerns about the possibility that the RMS enable these hotels to engage in algorithmic collusion. The price troughs following price peaks might be interpreted as "price wars" that are designed to punish hotels that deviate from the recommended prices that are highest when prices are peaking. However we do not think this is the correct interpretation or conclusion to draw from these price patterns.

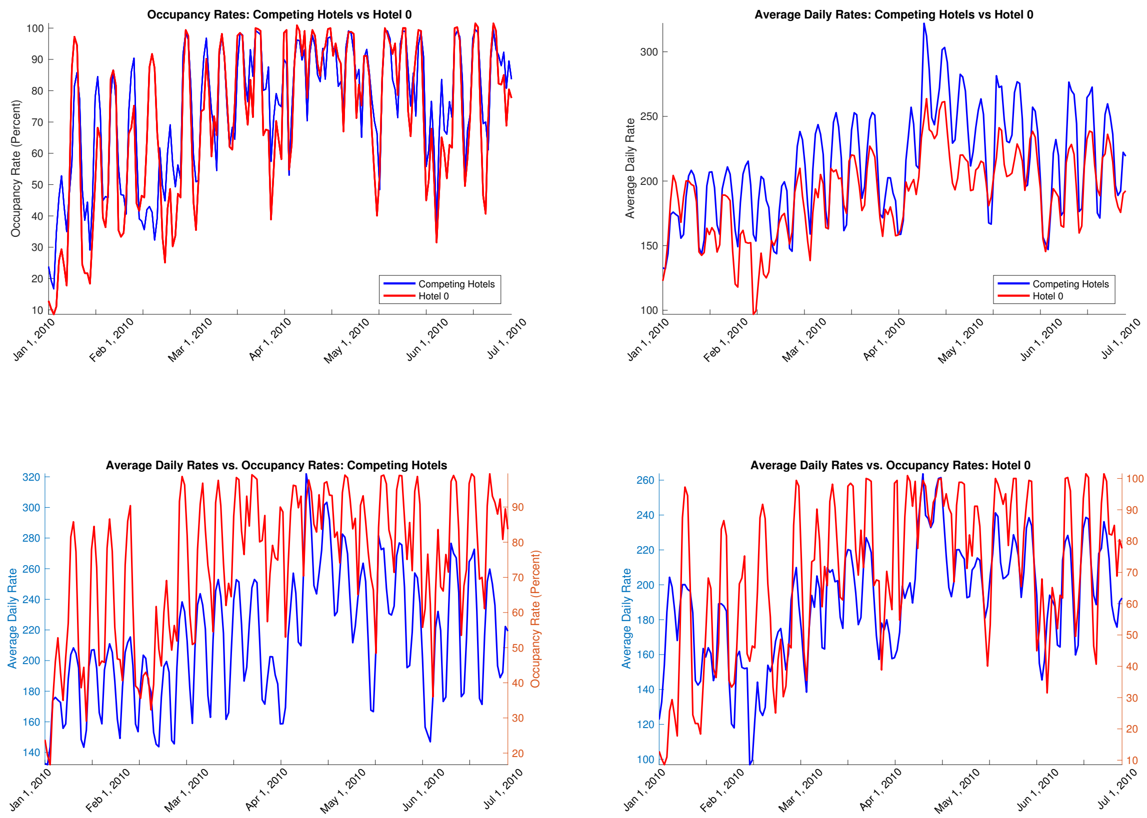
Figure 3 plots the time series of ADRs and occupancy rates for all seven hotels in this market for the first half of 2010 using the STR data. The top left panel plots the occupancy rate for hotel 0 versus the occupancy rate of its competitors, where the competitor occupancy rate is defined as the total occupancy at the six competing hotels divided by the total room capacity of those hotels. With few exceptions, we see that occupancy follows the same weekly cycle at all of the hotels that we illustrated in the left panel of figure 1 for hotel 0, as well as the seasonal fluctuations (i.e. higher in the spring but lower at end of June) that we observed in the right panel of figure 1. The top right panel of figure 3 shows that all seven hotels also have strong weekly cycles in their ADRs and the reasons are likely to be much the same as we conjecture for hotel 0: higher mid-week prices to discriminate against less price elastic business guests and lower weekend rates to try to attract the more price elastic tourists.

The lower two panels of Figure 3 plot the cycles in occupancy rates (red lines) and ADRs (blue lines) for hotel 0 (right hand panel) versus its competitors (left hand panel). The data suggests that the weekly price cycles are driven not only by different compositions of guests (business versus tourists) but also to ration scarce capacity, since these hotels tend to be fully booked midweek but not on weekends. Both hotel 0 and its competitors follow similar weekly occupancy and price cycles, as well as similar seasonal price/occupancy cycles. For example we see that ADRs for both hotel 0 and its competitors peaked in mid April 2010, during a period where occupancy was close to 100% both mid-week and on the weekends.

It is natural to ask the question: which motive is more important for hotel 0? That is, does the revenue manager increase prices mainly to ration scarce capacity, or to try to exploit the more inelastic demand of business travelers who are more likely to be staying in the hotel midweek? Or, is hotel 0 simply following the prices of its competitors? If so, is this price following behavior a sign that all of the hotels are following the recommended prices from their RMS, and could this be evidence of tacit collusion mediated by the RMS?

Table 5 provides some insight into this question by presenting the results of a simple OLS regression of the logarithm of hotel 0's ADR on the average ADR of its six competitors and on its own and competitors' occupancy rates. This simple model results in an R^2 of 86% when we also add dummies for different days

Figure 3: Co-movement in ADR and cccupancy rates for all seven hotels



of the week and months of the year to capture the weekly and seasonal price cycles.

Note that the occupancy also affects hotel 0's pricing but in a counterintuitive fashion: hotel 0's occupancy rate has a negative coefficient, but the occupancy rate of its competitors has a much larger positive coefficient. We may suspect that the co-movement in occupancy rates leads to a collinearity issue but hotel 0's own occupancy has a negative coefficient even after we move the occupancy of the competing hotels from the regression. The coefficient estimate for Hotel 0's own occupancy rate only turns positive when we remove the ADR of the competing hotels, but then the fit of the model drops precipitously, to an R^2 of 0.17.

The regression findings suggest that the effect of occupancy on hotel 0's pricing decisions are second order relative to the dominant effect of the prices set by its competitors. To a first approximation, hotel 0 sets its prices at 70% of the average of its competitors' prices. The R^2 drops to 0.69 when we remove ADR_c from the regression but retain occupancy variables and daily and seasonal dummies. Overall, the regression results suggest that the revenue manager is setting prices in accordance with a "price following"

Table 5: Ordinary least squares regression with dependent variable ADR_0

Variable	Estimate	Standard Error
constant	27.93	2.24
ADR_c	0.73	0.01
OCC_0	-0.09	0.027
OCC_c	0.273	0.044
$N = 1277$, $R^2 = 0.83$		

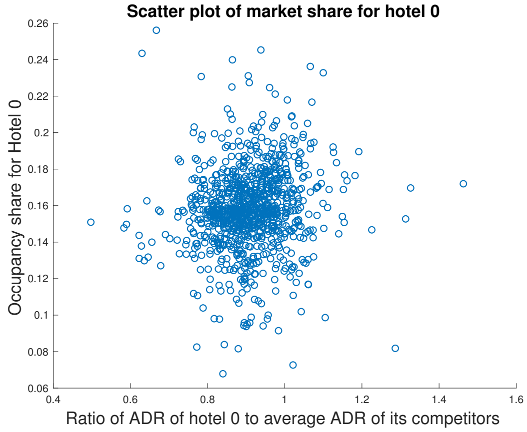
strategy, and that knowledge of her competitors' prices is the most important piece of information besides the day of the week and the season of the year that she uses to set her own prices. The fact that hotel 0's own occupancy appears to have only a second order effect on its price setting once we condition on the prices of competitors suggests that raising prices to ration scarce capacity is not an important motive for hotel 0.

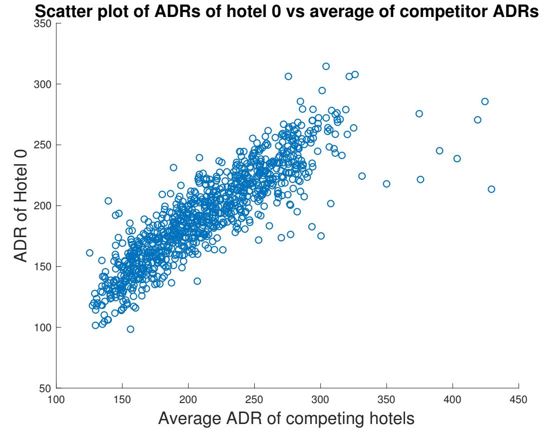
On the other hand it is not clear whether the fact that hotel 0's behavior is well approximated by "price following strategy" is evidence in favor of "algorithmic collusion" that Ezrachi and Stucke (2016) and Harrington (2017) discuss. Even if demand for rooms cycles in a systematic way during the week versus weekends, it is not clear that collusive prices would necessarily follow the same cyclical pattern that we observe in this market. In particular, we would expect that if the hotels in this market operated as a cartel, their prices would rise sufficiently high that there would be excess capacity even during the peak weekday periods, and the excess capacity would serve in part as a credible threat to engage in a price war that would deter any of the hotels that contemplated deviating from the collusive recommended prices, see Benoit and Krishna (1987) and Davidson and Deneckere (1990).

An alternative hypothesis is that this market is best approximated by a dynamic competitive equilibrium in a market characterized by strong Bertrand price competition subject to fixed capacity constraints. Stochastic shocks to demand lead to the price cycles we observe, with prices peaking to ration the available capacity in periods where demand exceeds available supply, but prices falling significantly as predicted by Bertrand price competition in periods of low demand where there is excess capacity. In this paper we will argue that the latter explanation is more likely to be closer to the truth, especially given what we have already reported about hotel 0's disinclination to follow the recommended prices of its RMS, combined with the fact that the revenue manager believes that the recommended prices are too low.

Regardless of the interpretation, the strong co-movement of hotel 0's prices with the prices of its competitors creates real difficulties for demand estimation. We can see the problem in figure 4. The right hand panel lots the ADR of hotel 0 against the average ADR of its competitors. The co-movement in

Figure 4: Price scatterplots for hotel 0 and its competitors





prices is evident it the positive correlation in prices we see in the figure, something already captured in the regression results in table 5. However we might expect that on days where the *relative price* of hotel 0 is higher that there should be fewer guests booking its rooms. But the left hand panel of figure 4 shows that there is little evidence in favor of this hypothesis. The scatterplot of hotel 0's share of total occupancy on the ratio of hotel 0's ADR to the average ADR of its competitors is roughly a circle of dots, which explains why we do not obtain a negative coefficient on hotel 0's price in an OLS regression of its market share on the ADR of hotel 0 relative to its competitors.

These results strongly suggest a problem of *endogeneity* in the prices we observe in this market. If the market is well approximated as a dynamic Bertrand equilibrium but subject to large stochastic demand shocks, then we would expect to see high prices set to ratio demand when demand is high but low prices as the hotels compete for the available demand in periods where there is excess supply of rooms. This type of competition will generate a positively sloped scatterplot of prices similar to what we observe in figure 4 and generally a positively sloped relationship between ADRs and occupancy for hotels individually. Thus, simple OLS regressions will infer *positively sloped demand curves* in this market.

There are no obvious instrumental variables that can solve this endogeneity problem. One possible instrumental variable is a decrease in capacity of the hotel. If we regard the hotel as setting prices to ration demand, then in periods where there is a reduction in available rooms for exogenous reasons (such as a bursted pipe or other problems that remove rooms from service, or planned upgrades to rooms that take rooms out of services for a period of time, similar to what we showed in table 3 when the hotel converted 23 of its standard rooms to deluxe rooms), then the decrease in supply of rooms may serve as an instrumental variable that may allow us to estimate a negatively sloped demand curve.

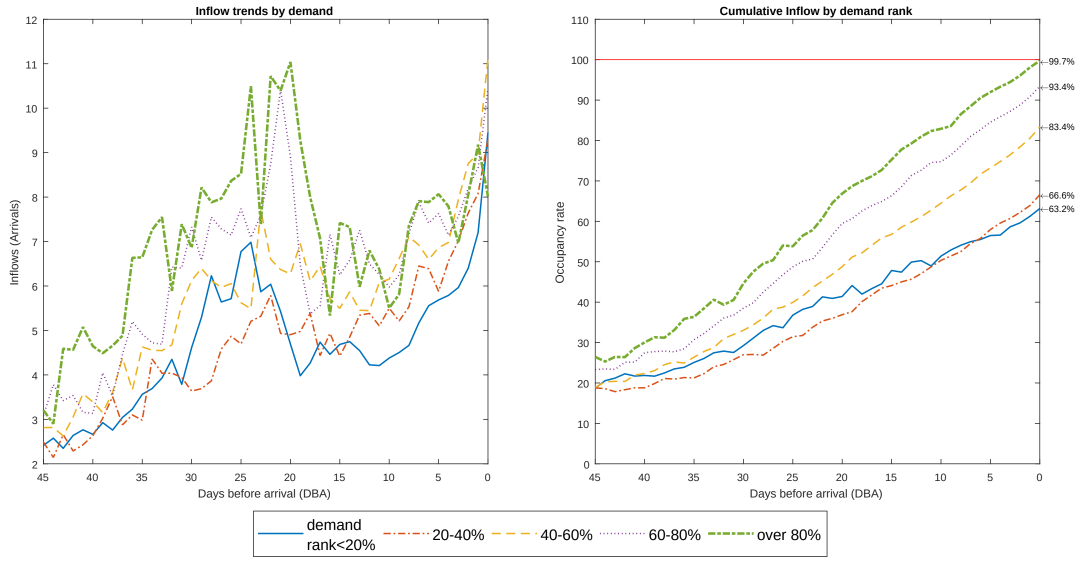
Unfortunately when we tried to use available capacity as an instrument we find highly unreliable and generally insignificant results. Depending on the subsample we use, that estimated coefficient for 2SLS of the log of the ratio of the ADR of hotel 0 to the average ADR of its competitors ranges from -4.02 to 7.76 but the maximum t-statistic for any of these subsamples is 1.2. Most likely the capacity instruments are weak instruments since the F-statistic in the first stage regressions ranges from 0.03 to 5.43. There is not enough exogenous variation in hotel 0's available capacity to make this a good instrument for estimating the effect of hotel 0's price on demand. An additional complication is that the model of demand we specify in section 3 is not a simple linear demand model but a stochastic nonlinear demand function that results from a micro-aggregation of the individual discrete choices of consumers who are arriving at random times prior to occupancy to book a room at one of these hotels and are choosing the best option given the BARs quoted by these hotels at that time.

2.3 Booking and pricing dynamics

Figure 5 plots the inflows and occupancy distribution by days before arrival (DBA), which were drawn from actual reservation records, i.e. reservation raw data. We classify our data into quintiles based on total occupancy. The highest demand quintile results in a sellout and near 100% capacity on the date of occupancy. The right hand panel of figure 5 plots the average occupancy trajectory leading up to the sellout and we see from the top green line, even 10 days away from arrival the hotel has still only sold 80% of its total capacity. Thus, there are peaks in reservations that occur at 20 days before arrival and in the last day before arrival (i.e. the date the reservation starts when customers occupy the room). Overall, while the hotel may be able to predict well *ex ante* which days will be busy, the pattern of bookings suggests that the hotel will typically not know if it will be sold out until the arrival date.

Figure 6 plots the reservation trajectories on particular busy days to provide further understanding of the co-evolution of bookings and price setting dynamics. The top left panel of figure 6 shows bookings and the path of BARs for April 18, 2013. The revenue manager knew in advance that this would be an extremely busy day due to spring festivals in the city. At 45 days out, she sets the BAR to be nearly twice as high as she would set for less busy days. The red line in the figure also plots the average BAR of her competitors, and we can see this is a relatively rare example where hotel 0 sets its price higher than the average price of its competitors. Despite the high price, hotel 0 has sold out 20 days prior to the arrival date, April 18, 2013. We can see that the hotel has overbooked itself (selling 106% of its capacity) 13 days out, and in response to continued interest the hotel raised its BAR up to $1050 10 days out. However cancellations in the last 10 days enabled the hotel to book a few more customers, but evidently the fear of more last minute cancellations prompted the hotel to reduce the BAR to $399, which is even lower than

Figure 5: Inflow and reservation dynamics prior to occupancy



the price it had initially set 45 days out. There was a strong response to this price drop and evidently this raised hotel 0's expectation of a sellout, so in the last several days prior to arrival, the manager increased the the BAR to $559, matching the average BAR of her competitors.

The upper right panel of figure 6 shows the path of reservations and BARs prior to the arrival date July 7, 2014. This is also a busy day but not quite as busy as April 18, 2013. The revenue manager appears to realize that there is a high chance that this will not be a sellout date, so she sets the BAR 45 days ahead of arrival at a more typical level of $209, just slightly undercutting the average BAR of her competitors. Although the occupancy of hotel 0 is already reasonably high 45 days before the arrival date, the manager decreases the BAR to $169 40 days out. There is a strong demand response to this price cut and 15 days out the hotel is nearly 80% booked. The manager raises the price back to $209 about 5 days out, but then appears to reconsider and drops the price back to $169. A strong response to this final price cut enables the hotel to reach 100% capacity and then two days before arrival the manager raises the price to $295.75, significantly higher than the average BAR of her competitors.

The bottom left panel of figure 6 shows the path for November 18, 2010, which comes at the end of peak period but just before Thanksgiving which is a slack holiday period for hotel 0. However November 18, 2010 was expected to be a busy day as evident from the unusually high BAR, $319, that the manager sets 45 days out. This price is below the average BAR quoted by her competitors and we see a jump in bookings about 42 days out. In response the manager increased the BAR to $339 37 days out, and a further

Figure 6: Occupancy and BAR dynamics on busy days

increase to $379 15 days out. This latter increase results in her BAR being higher than the average BAR of her competitors. Perhaps as a result of this, the growth of occupancy in the next several days slows down and 9 days out she drops the price back to $319. This price drop results in a surge in new bookings that results in a sellout, so the manager raises the price up to $379 for the last few days prior to arrival. Although there are several cancellations in the last few days, the hotel is nearly 100% booked.

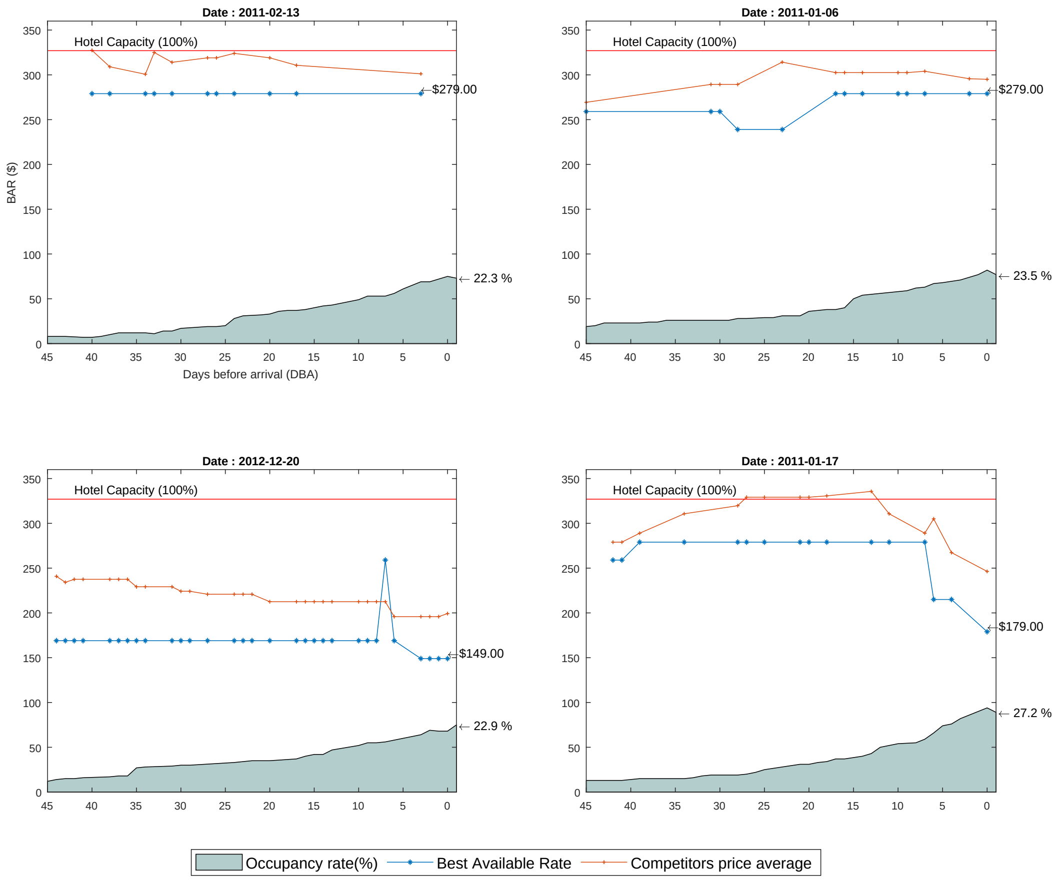
The bottom right panel of figure 6 shows the path for March 22, 2013. There was high occupancy on the same day in 2012 and both days are weekdays. The revenue manager expects relatively high demand by business travelers and sets a relatively high BAR of $239 but perhaps due to uncertainty about how much demand will materialize, this price is not as high as the prices she set 45 days out on days where she has more optimistic expectations of demand. Indeed, due to a relatively lackluster rate of new bookings, the manager reduces the BAR to $185 40 days out and maintains this price until 20 days out when the number of rooms booked appears to be large enough to make her optimistic about the changes of selling out. She raises the price on a succession of days, but by 11 days out she appears to conclude that she has raised the price too high and is not on a trajectory to sell out. She lowers the BAR again and keeps it at a lower level until about 3 days out when it seems clear that the hotel will not sell out. She drops the BAR again and this appears to result in enough additional bookings in the last few days that she again raises the price, then making a small final price cut to $219 on the arrival date.

Note that our narrative of these trajectories represents our *ex post* interpretation of factors motivating the revenue manager to change her prices, but we may indeed be reading too much into the data to ascribe specific motives based on specific information that caused her to make various price changes. Similarly, though we do see changes in bookings in response to various price changes that seem reasonable predictable, we cannot be sure that the hotel's price changes "caused" these changes in bookings. In particular, our narrative above may make the reader imagine that the revenue manager's goal in setting BAR was to try to sell out on these particular days, and this may or may not be her actual objective. Given our discussion in the introduction, the revenue manager would probably strongly disavow that her goal is to maximize occupancy, and indeed one reason she routinely ignores the recommended prices from hotel 0's RMS is because she believes its prices are set too low with the objective of trying to maximize occupancy rather than expected profits.

Thus, it is helpful to show the hotel's occupancy slack days where there is no hope that the hotel can sell out. On these days we observe both lower BARs and fewer changes in the BAR. We also see that on non-busy days, hotel 0 systematically undercuts the average BAR of its competitors at every booking date prior to arrival.. Figure 7 illustrates four specific slack days where the occupancy rate ends up below 30%. The upper left hand panel shows the trajectory for February 13, 2011. For this day, the revenue manager sets a single BAR of $279 and makes no further changes. The top right panel illustrates the trajectory leading up to January 6, 2011 and in this case she does make several changes to the BAR, lowering it between 30 and 28 days out, but then appearing to respond to an increase in her competitors' BAR about 22 days out, she increases her BAR to $279 where it remains unchanged until the arrival date. In this case we do not see any evident demand response to the temporary "experiment" of the drop in BAR between 30 and 22 days out.

The lower left panel of figure 7 shows the occupancy and price trajectories leading up to arrival on December 20, 2012. In this case the manager underprices her competitors except for a single upward blip in her BAR about 9 days out. She then reduces the price back to the initial value and then makes a further price cut to $149 4 days out. Again, there appears to be no obvious demand reaction to the temporary increase in prices, though the final price drop may have brought in a few extra bookings. The lower right panel illustrates the booking history prior to arrival on January 17, 2011. In this case we see mixed

Figure 7: Occupancy and BAR dynamics on non busy days



evidence for the "price following" strategy, since though the revenue manager systematically undercuts the BAR of her competitors, she appears to raise her BAR in response to an increase of her competitors' BARs 40 days, but she does not continue to increase her BAR as her competitors continue to raise their BARs up about 12 days prior to the arrival date. However the competitors start cutting their BARs at this point and though hotel 0 does not immediately respond to the price cuts, at 7 days out she does drop hotel 0's BAR and does another price cut, down to $179 in the final night prior to arrival.

Overall, our analysis of individual trajectories (observations) is consistent with our earlier conclusion that hotel 0's pricing responds strongly to the prices set by its competitors, but not so much to capacity, except on days where the manager expects Hotel to sell out, when we do observe increases in BAR as the arrival date approaches and examples where hotel 0 can set its BAR higher than the average BAR of its competitors. But on non busy days we rarely observe hotel 0 setting a higher BAR than its competitors.

2.4 New reservation arrival dynamics

Central to our model of hotel 0's price dynamics is the stochastic arrival of customers wishing to book rooms in this market. Let r_t be the number of new transient and g_t be the number of group reservations booked t days before arrival. We observe the exact values of $\{r_t, g_t\}$, the number of bookings in advance of arrival at hotel 0 for all t and all possible arrival days (via its reservation database) but not at its competitors, where we only observe total occupancy on the arrival day (i.e. the day the reservation starts). As we have seen above, there is substantial day to day variability in the number of reservations made on any given day which we denote by the random variable $\tilde{d}$. We modeled reservation inflow to this market using both a Poisson and a Negative binomial distribution, but found the latter distribution provided a better fit to the data due to the well known restriction inherent in the Poisson distribution that its mean and variance are equal. A negative binomial distribution has two parameters (ϕ, q) and its probability distribution for the $\pi(r|\phi, q)$ is given by

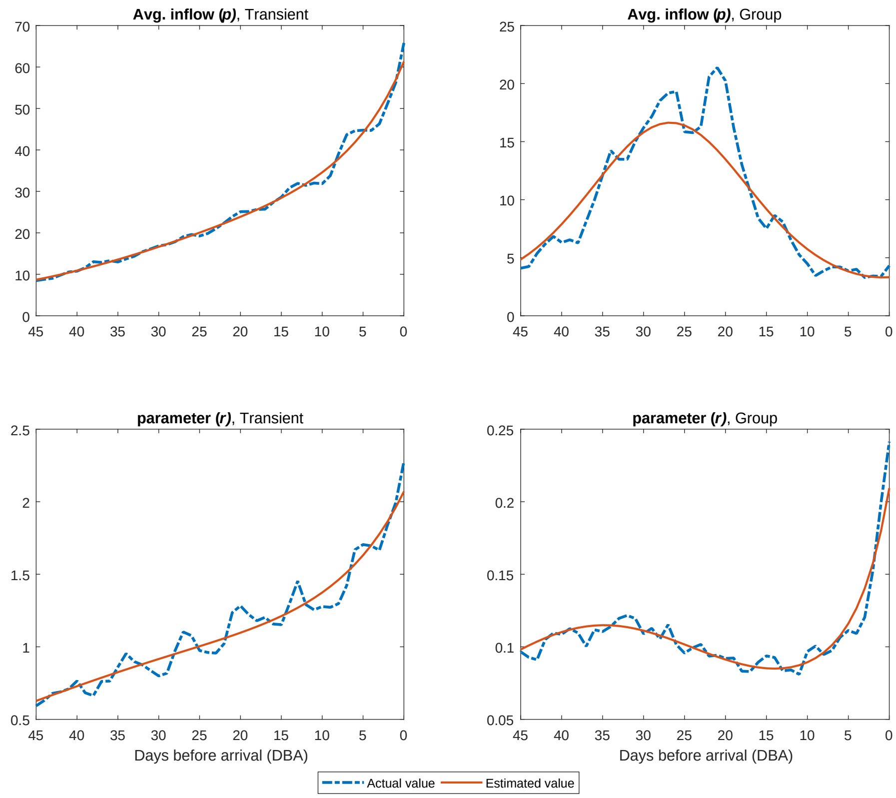
$$\pi(r|\phi,\mu) = \binom{r+\phi-1}{r} q^{\phi}(1-q)^r \tag{1}$$

with mean and variance given by $\mu = E\{\tilde{r}\} = (1-q)\phi/q$ and $\text{var}(\tilde{r}) = (1-q)\phi/q^2$ where $q \in (0,1)$ and ϕ is a positive real number. The main advantage of the negative binomial over the Poisson distribution is that the negative binomial allows for "overdispersion" i.e. the possibility that the variance of arrivals exceeds the mean number of arrivals whereas the Poisson restricts the mean and variance of the number of arrivals to be the same. When modeling, we find it convenient to re-parameterize negative binomial distribution with (ϕ, μ) instead of (ϕ, q).

Using the reservation data for hotel 0, we estimated the parameters (ϕ_t, μ_t) for negative binomial distributions of the number of bookings by transient and group customers separately. We estimated different (ϕ_t, μ_t) parameters for each group for up to 45 days in advance of arrival by maximum likelihood, where we assumed that the number of reservations $\tilde{r}_t$ for each booking t prior to arrival are independent but non-identically distributed negative binomial random variables.

Figure 8 plots the estimated (ϕ_t, μ_t) parameters and we see distinct dynamics for the two different types of customers. For transient customers (ϕ_t, μ_t) are decreasing in t which implies that the mean and variance of the number of transient bookings increase as the arrival date approaches, suggesting a greater proclivity towards "last minute" bookings. Group customers are much more likely to book in advance, with the mean number of bookings peaking about 25 days in advance of arrival. This makes sense since group bookings may often be for conferences or business meetings that require more advance planning than for tourists who may have more flexibility to come on the spur of the moment. Simulations of the estimated negative binomial model of bookings produce simulated booking dynamics that are very similar

Figure 8: Estimated negative binomial parameters and polynomial approximations

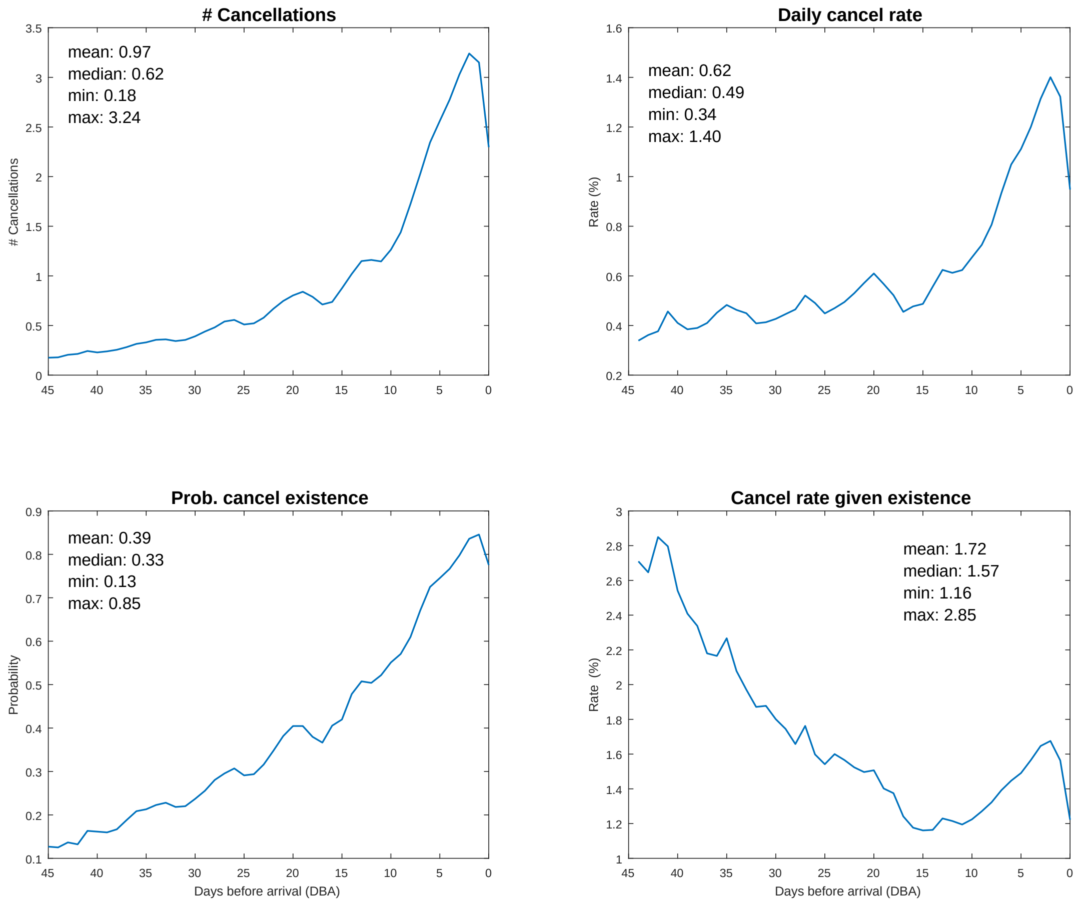


to the ones we illustrated in figure 8, including the peak in reservations about 20 to 30 days prior to arrival, the dip in reservations about 10 days prior to arrival and a rapid increase in last minute reservations in the last few days prior to arrival. The solid red lines in figure 8 show that the dynamics in the estimated (ϕ_t, μ_t) parameters can be well approximated by low order polynomial functions of t. Based on these findings, we used 3rd degree polynomial approximations to capture systematic trends in booking dynamics prior to arrival date in our structural estimation results in section 5.

2.5 Cancellation dynamics

Hotel 0's standard policy is to allow free cancellations (i.e. with no penalty) up to 24 hours before check in time on the date the reservation starts. A customer who cancels within 24 hours of checkin forfeits the price of the room for the first night of the reservation, but can be refunded the amount paid for additional nights beyond the first. No shows are customers who book but never arrive for their stay at the hotel: these customers are also charged though there are far fewer of them: only 1.7% of all reservations recorded in our database. We are more concerned about cancellations than no shows since a cancellation gives the

Figure 9: Cancellation dynamics



hotel an opportunity to rebook the room if the cancellation happens early enough. A no show is actually a good thing for the hotel: the customer pays for the room but the hotel incurs no room cleaning or other charges, though if the hotel were able to predict no shows accurately enough it could potentially factor them into its booking strategy and be a bit more aggressive in how it books rooms on days it expects to sell out.

Figure 9 plots statistics relevant to cancellations at hotel 0. The upper left panel plots the expected number of cancellations on a daily basis based on the number of days before arrival. Cancellations start to increase rapidly about 10 days prior to arrival and peaks at over 3 cancellations per day before it starts to decrease sharply two or three days before arrival. Customers who forget to cancel prior to the 24 hour window prior to check in may end up as no-shows, however if they have a multiple day reservation, it makes sense to cancel and be refunded at least part of the cost for their stay rather than incur the full charge as a no show.

The upper right panel of figure 9 plots the cancellation rate, i.e the fraction of booked customers who cancel as a function of the number of days prior to arrival. The cancellation rate has a similar shape to the expected number of cancellations, increasing from under 0.5% 40 days prior to arrival to a peak of 1.4% a few days prior to arrival. We can decompose the cancellation rate into the product of the probability at least one customer cancels on any given day times the expected number of cancellations given at least one cancellation occurs. The lower left panel of figure 9 plots the probability that at least one cancellation occurs. This is essentially a monotonically decreasing function of the number of days before arrival except for a small downturn on the arrival date which is likely a reflection of the cancellation penalty. The expected number of cancellations given that at least one cancellation occurs is an increasing function of the number of days before arrival at least up to about 15 days out when it reaches a minimum of 1.16 cancellations. The cancellation rate increases as we approach the check in date between 15 days out until about 3 days out, and then it drops sharply, also likely reflecting the penalty for cancelling within 24 hours of check in.

Williams (2018) found that there are no significant gains to the strategic timing of purchasing tickets for the airline flights he analyzes, and he used this fact to simplify his dynamic programming analysis of optimal airline pricing. We would like to follow a similar approach for hotels but there are much stiffer penalties for cancelling an airline reservation than a hotel reservation. Most airlines have a significant cancellation penalty (typically $200 or more) for cancellations or changes in reservations outside a 24 hour window when a flight is booked. Hotels have typically been laxer in their cancellation policies, and most have no penalty as long as the reservation is cancelled more than 24 hours prior to check in. hotel 0 has this cancellation policy.[4]

These lax cancellation policies could encourage consumers to engage in dynamic price shopping that can impede a hotel's ability to engage in dynamic pricing. For example a consumer may book far in advance of their intended arrival date to lock in a base price and then continue to monitor the hotel's website and other OTAs to search for an even better deal. If the consumer finds one, then they can costlessly cancel the initial reservation and rebook at a lower price. If sufficiently many consumers follow this type of strategy, it limits a hotel's ability to cut its BAR as the arrival date approaches to try to attract additional guests with lower willingness to pay. It sufficiently many of the hotels already booked guests are motivated to engage in strategic cancellations then any these price cuts would come at the cost of partially cannibalizing revenue from customers the hotel may have already booked at higher BARs.

[4]Jet (2017) notes that hotels are experimenting with cancellation policies, becoming more like the airlines in penalizing customers who cancel. For example, he notes that some hotels are considering tiered cancellation policies where "you might be able to cancel for free a week or more in advance and the hotel will slowly ratchet up the fee as the check-in date approaches. Ultimately costing one or two nights." He also notes that "Hotel chains are also experimenting with nonrefundable booking (also known as an advanced booking/purchase)."

We do not find strong evidence that strategic cancellations are an issue for hotel 0. We estimated a simple binary logit model of the decision to cancel a reservation, where the probability of cancelling a reservation depends on the difference in the BAR that the customer paid when they made their reservation and subsequent path of BARs posted by the hotel up to the arrival date. The estimated coefficient on the price differential is negative (indicating that customers are more likely to cancel if a subsequent BAR is lower than the BAR that they made their reservation at), but the estimated coefficient is small and barely significant. Thus, even though there are negligible penalties for cancelling and rebooking a hotel reservation, our findings suggest that relatively few consumers engage in sophisticated dynamic strategies that involve an initial booking relatively far in advance of arrival combined with periodic monitoring of prices to find a better deal up to 24 hours before arrival (when the cancellation penalty kicks in). If a significant fraction of consumers were following this type of strategy we would expect to find strong evidence of strategic cancellations but we don't.

2.6 Price dynamics of competing hotels

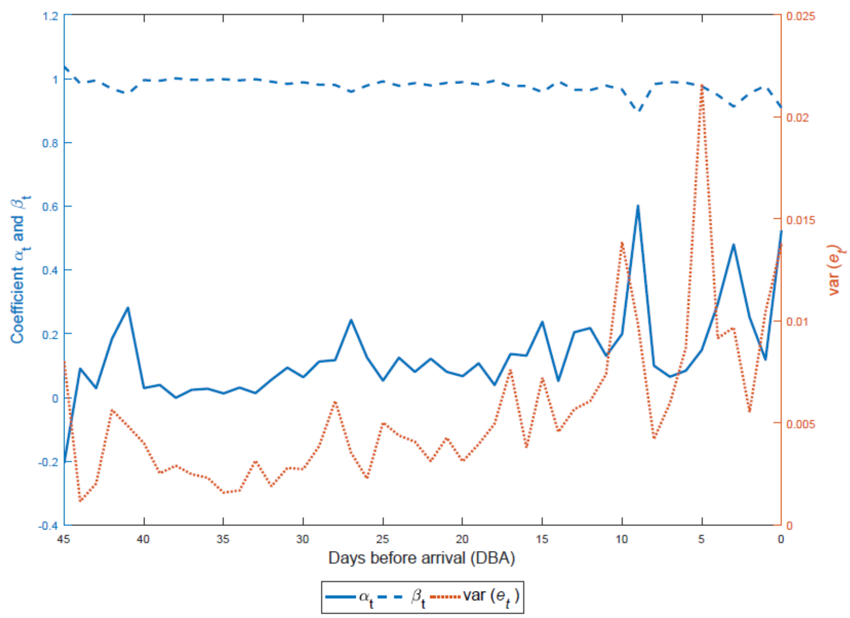
The model we introduce in the next section requires hotel 0 to not only have knowledge of the BARs set by their competitors, but they also need to have expectations about how their competitors' BARs will evolve in the future for each different arrival date. We do not develop a full dynamic equilibrium model of the hotel market in this paper, and assume that the local market we study is approximately in equilibrium and stationary in the sense that the price dynamics of the hotels may differ between busy and less busy days but the price dynamics are not shifting with calendar time (such as due to entry or exit of additional hotels, which has not happened during our sample period). Thus, we econometrically estimate a transition probability for the average BAR of hotel 0's competitors that take the form of an AR(1) process in logs of the average BAR charged by hotel 0's competitors which we denote by ρ_t:

$$\log \rho_{t-1} = \alpha_t + \beta_t \log \rho_t + e_t. \tag{2}$$

We assume that the error term in equation (2) is a normally distributed *IID* error process with mean 0 and variance σ_t^2 where t indexes the number of days prior to arrival. Thus, hotel 0 treats the prices of its competitors as "exogenous" and while we do allow the coefficients $(\alpha_t, \beta_t, \sigma_t^2)$ to vary based on the number of days t prior to arrival, the $\{\rho_t\}$ is otherwise stochastically stationary in the sense that these coefficients are not shifting in calendar time (i.e. the coefficients $(\alpha_t, \beta_t, \sigma_t^2)$ that are valid t days before arrival are the same for an arrival date in 2010, 2011 or 2012, etc).

Figure 10 plots the estimated $(\alpha_t, \beta_t, \sigma_t^2)$ coefficients as a function of t from $t = 45$ days prior to arrival to $t = 0$, the day of arrival. To a first approximation we find that competitor prices evolves as random walks

Figure 10: Estimated competitor BAR transition parameters



without drift: the β_t coefficents are very close to 1 and the α_t coefficient estimates are very close to 0. The pattern of estimated σ_t^2 parameters in figure 10 indicate that there is particularly high price volatility 10 and 5 days ahead of arrival, as well as on the day of arrival itself. The pattern of the estimated intercept coefficients α_t also indicates that hotel 0 expects its competitors to raise their BAR at 9 and 4 days prior to arrival, respectively. Given the low estimated values of σ_t^2, a plot of the conditional lognormal distribution for ρ_{t-1} given ρ_t is nearly symmetrically distributed about its mean. For example when $\alpha_t = 0$, $\beta_t = 1$, $\sigma_t^2 = 0.005$ and $\rho_t = 350$, the distribution of ρ_{t-1} is highly concentrated near its current value, $\rho_t = 350$ with an expected value of $E\{\rho_{t-1}|\rho_t = 350\} = 350.876$ and a conditional standard derivation of 24.8 and a probability of nearly 98% that ρ_{t-1} will fall in the interval $[346, 352]$.

3 Dynamic programming model of optimal hotel pricing

We assume hotel 0 chooses a dynamic pricing strategy to maximize its expected profits and focus on the BAR as its key decision variable. As we noted in the introduction, some RMS set fixed price tiers for different types of rooms and use a dynamic quantity allocation strategy instead of a dynamic pricing strategy. We have already documented that both hotel 0 and its RMS use dynamic pricing, and we believe this is a superior strategy when implemented correctly, since it provides the hotel more flexibility in how it tailors prices to different types of customers and how it responds to unexpected demand shocks.[5]

As we noted in the introduction, hotel 0 does not sell blocks of rooms at wholesale rates and follows a uniform pricing strategy so its prices are the same regardless of whether a customer books via an OTA or hotel 0's own website. However hotel 0 does pay a commission for reservations that are made via OTAs so for the roughly 20% of all of its bookings that are done via OTAs, hotel 0's revenue is its BAR less the OTA's commission. Though hotel 0 can choose not to accept reservations via OTAs if it believes the commission is too large, in practice it does pay the commission and accept bookings via all major OTAs. In addition hotel 0 has negotiated corporate and government discounts that are typically a fixed percentage reduction off the BAR prevailing at the time of the booking. We treat hotel 0's decision to adopt a uniform pricing strategy, to accept reservations from all OTAs, and its negotiated discounts with corporate and government customers, as given and do not include these as additional decisions in our DP model. The only decision we focus on is hotel 0's choice of its BAR, which we assume is updated at the start of each day.

3.1 State and control variables

Our analysis in the previous section suggests that there are three key pieces of information that the revenue manager needs to consider when setting the BAR and predicting revenue and occupancy in the hotel: $(n, \overline{p}, \rho)$ where n is the number of rooms reserved for occupancy on a specific date, $\overline{p}$ is the ADR (average price of rooms booked so far), and ρ is the average BAR of competing hotels for a comparable room.[6] Our optimal dynamic pricing strategy uses these three variables as the state variables of the dynamic programming (DP) problem. There are other implicit non-time-varying variables such as the capacity of

[5]Technically, we solve for the optimum within the class of uniform pricing strategies. However hotels could use other selling mechanisms that might potentially raise higher revenue. Maskin, Riley, and Hahn (1989) show that in a static, one-shot environment when n units are to be sold to n_b buyers who each demand one unit and have *IID* valuations their units, the optimal mechanism can be implemented using a reserve price and bids, where bidders with the $\min(n, n_b)$ highest bids are allocated a unit at a price equal to their bid. However they note if their are different types of bidders, or their valuations are correlated, the optimal selling mechanism may have a very different form. Board and Skrzypacz (2016) consider the optimal selling mechanism in a dynamic context with forward looking buyers. "A seller wishes to sell multiple goods by a deadline, for example, the end of a season. Potential buyers enter over time and can strategically time their purchases. Each period, the profit-maximizing mechanism awards units to the buyers with the highest valuations exceeding a sequence of cutoffs. We show that these cutoffs are deterministic, depending only on the inventory and time remaining" (p. 1046). However their analysis does not consider the effect of competition, and behavioral-based realities that customers may prefer the immediacy, convenience and familiarity of posted prices relative to auctions, which involve uncertainties and a delays while bidders are informed whether their bids were winning bids. However auctions seem preferable to random rationing of excess demand when a hotel is overbooked, yet it is not clear why auctions are rarely used in the hotel industry even as a priority queuing device when they are oversold.

[6]Though the demand for rooms could potentially depend on the full vector of competitors' prices, this would increase the dimensionality of the pricing problem, since the manager would have to keep track of the six different BARs set by the six competing hotels. We have found that the average of these six BARs represents a "sufficient statistic" for the competitors' BARs that enables us to model the demand for hotel 0's room with sufficient accuracy. We believe the gains in accuracy in predicting demand from incorporating all six prices is outweighed by the curse of dimensionality in solving the dynamic programming problem using six individual BARs as state variables. However a formal evaluation and validation of this conjecture would require solving the model using all six BARs, something we have not attempted to do yet.

the hotel $\overline{n}$ and its attributes as well as the capacity and attributes of its competitors. The role of these factors be clear when we introduce a model of stochastic demand for hotel rooms in this market.

In section 3 we showed that the number of days prior to occupancy t is also an important state variable that affects the hotel 0's pricing. There is one continuous decision variable in our model, the revenue manager's choice of the BAR, which we denote by p_t. We assume that the revenue manager updates the BAR at the start of each day and this updating is instantaneous, so all consumers who wish to book a room on day t will observe hotel 0's BAR p_t as well as the BARs of its competitors, ρ_t, and will choose one of them based on a simple static utility maximization. Allowing for unobserved heterogeneity in consumer choices, our model of hotel demand (to be described in more detail below) implies that a consumer of type τ has a probability $P_t(p_t, \rho_t, \tau)$ of making a reservation at hotel 0 t days prior to arrival.

We start the backward induction calculation to solve the DP problem by defining the value function $V_{-1}(n, \overline{p}, \rho)$ for realized profits on the morning *after* the arrival date, $t = 0$. On this date there are no further decisions by the hotel: the value function simply summarizes the realized profits earned by the hotel that become known on the arrival date when n of its rooms are occupied at ADR of $\overline{p}$. $V_1(n, \overline{p}, \rho)$ equals the total revenues received from the guests net of any discounts they were given, and net of the hotel's costs which includes room cleaning or other costs involved in serving the guests less any commissions to OTAs or other travel agencies.[7] Note that on $t = -1$ or the night of the arrival date, the prices set by hotel 0's competitors ρ does not affect hotel 0's profits, and hence does not enter V_0. However ρ_t does affect how many customers book at hotel 0 prior to that night so it is a critical state variable that affects hotel 0's pricing decisions and expected profits for all $t \geq 0$.

Hotel 0's pricing problem starts some fixed number of days T prior to any given arrival date, where T is the maximum number of days in advance of arrival that the hotel will book a room. We assume that $T = 45$, though in practice hotel 0 does book rooms even further in advance than this. In principle the hotel is solving many DP in "parallel" and is setting prices not only for the current date (e.g. Dec 3, 2012 for customers who arrive on the same day as "walk-in" clients) but it must be able to set future prices for advance bookings up to T days in the future. We now describe further assumptions to make these DP problems tractable, enabling us to solve the relevant DPs in parallel to obtain BAR decision rules that enable hotel 0 to set prices and book rooms at all future dates in real time.

Our DP solution distinguishes several types or categories of days that have similar patterns of occupancy and customers, such as weekdays versus weekends and busy versus non busy days. As we noted,

[7] The hotel may also earn additional revenues from guests from in-house restaurants, bars, video/internet services, room service charges and so forth. We account for the profits from these add-on services as a reduction in the costs of serving its guests.

business customers are more likely to stay in the hotel during a weekday whereas tourists are more likely to stay during weekends. Our stochastic demand model will take account of differential arrival rates of different types of customers on different types of arrival dates, as well as their differential willingness to pay for rooms. We will assume that the hotel revenue manager knows in advance that certain days are likely to be particularly busy, and hence the arrival rate for reservations will be higher than for less busy days. For example, the revenue manager will know when there are large conferences or events occurring in the city or due to other seasonal reasons (i.e. graduation dates, sports events, and so forth). We will solve separate DPs for the different occupancy categories that we can identify from our data. The model we estimate will have 8 different dynamic programs for 4 occupancy quartiles that index how busy the hotel is likely to be as well as differentiating weekdays versus weekends.

We assume that these categories are essentially generic: i.e. we assume that the stochastic process governing the pattern of arrivals and cancellations, the types of customers who make reservations, and their willingness to pay for hotel 0 relative to other hotels are the same for all days of a given type. This allows us to pool all days of a given type for purposes of econometric analysis and for solving the optimal pricing strategy for the firm. Thus, if there are K types of occupancy days, we will need to estimate K separate stochastic demand/arrival processes and solve K corresponding dynamic programs. Let $p^*_{t,k}(n_t, \overline{p}_t, \rho_t),\ k \in K$ be the optimal price hotel 0 will charge as a BAR reservation at non-contract rates for an occupancy day of type k t days before arrival. If we have solved all K dynamic programs, then will have K corresponding optimal pricing rules $\{p^*_1, \dots, p^*_K\}$ and we can then regard these as "dynamic price schedules" that the hotel can supply to its GDS and quote to its customers via its own website.

Note how our formulation of the DP problem has used the principle of *decomposition* to significantly simplify the overall decision problem the hotel faces. Without an appeal to decomposition, the hotel must potentially solve separate DPs for each possible arrival date. Since there are 365 possible days per year, in principle we might expect to have to solve 365 separate DPs to determine the hotel's pricing problem for every day in the year. However by grouping arrival dates into a smaller number of K similar types of days with similar occupancy dynamics, we only need to solve $K = 8 < 365$ DPs to be able to generate a dynamic pricing strategy that enables hotel 0 to set its prices in any given day in the year. We also apply the principle of decomposition to the way we analyze reservations that involve multiple day stays at the hotel. To do this, we need to ignore the possibility of length of stay based discounts, i.e. a lower daily rate for customers who book a room for a longer period of time. Initial empirical analysis using our data seems to indicate that length of stay based discounts are not easy to see in the data, and this suggests that they do not play a major role in attracting customers to stay in this hotel. This assumption could be wrong, however, and we note that it is an assumption we might want to relax in future work if we find from an analysis of consumer demand that length of stay based discounts could be effective in attracting more customers and generating more revenue. But initially we will treat a single customer who wishes to reserve a room for S successive days as equivalent to S individual customers making separate, independent 1 day reservations.

Finally, we ignore substitution across the 9 different room classes in our data set, i.e. between standard rooms and the luxury suites (much larger rooms on higher floors of the hotel, with balconies, kitchens and additional living areas, and so forth). Most of hotel 0's available space are allocated to the standard rooms at the lower price tiers. We ignore the possibility that customers who are looking to book a luxury suite would choose a standard room instead because of the significantly lower price of a standard room. Similar to our exogenous assumption about occupancy dates and length of stay, we assume that the customers make a choice of room class in advance and may substitute between hotels in an attempt to, say, find a luxury suite in one of the other luxury hotels in this market on a given date based on price, but these customers represent a separate market segment and are not likely to substitute and book a standard room based on a simultaneous price comparison between prices of luxury suites and standard rooms in all seven hotels in this market. This assumption is another application of the decomposition principle that allows us to analyze the room reservation and pricing decisions for standard rooms and luxury suites separately. Thus, we need to solve 2 separate DPs instead of a single DP involving a choice of up to 11 different BARs for the 11 different room types that hotel 0 has.[8]

3.2 The Bellman Equation

We now introduce a bit of further notation and we are ready to write the Bellman equation which defines the objective and solution to the optimal dynamic hotel pricing problem. We need to distinguish between reservations that are booked at the current BAR or discounted BAR by individual (which we referred to as transient reservations) versus any party of customers who are eligible for a pre-negotiated discount off the BAR, which are typically group reservations for corporations and governments. There are two different types of transient customers, business and leisure, which we can distinguish in the reservation database. Our demand model will allow separate utility parameters for business and leisure customers, including the possibility that business customers are less price elastic. Though business customers are typically

[8] If our analysis, or discussion with the revenue manager, leads us to conclude that there is significant substitution between narrower room type categories, then we can relax these assumptions in future work. For example if there is significant substitution between the lowest tier of rooms, B1K/B2D, and the next highest tier, A1K/A2D, then we could solve the DP as a 6-dimensional problem with state variables $(n_1, \overline{p}_1, \rho_1, n_2, \overline{p}_2, \rho_2)$ where n_1 is the number of reservations for B1K/B2D rooms and n_2 is the number of reservations for A1K/A2D rooms, and so forth.

corporate or government employees, if they are classified as transients, they will generally pay the BAR in effect at the time they make their reservation. If the business or leisure traveler is part of a group, then they are eligible for a discount off the BAR that can differ depending on the contract negotiated between hotel 0 and the group that they are a part of. Our DP model assumes that group reservations are exogenous random events that do not depend on the BAR.

Hotel 0 has a variety of corporate and government contracts that allow their employees to reserve certain rooms in the hotel at pre-negotiated discounts subject to room availability. Though there are different government and corporate discounts for group reservations and different restrictions associated with each, we will let p_g denote these group rates. Most of the group discounts are provided as a fixed discount relative to the prevailing BAR, $p_g = \delta_g p$ where p is the BAR and $\delta_g \in (0,1)$ is the negotiated discount rate for the group. Actual corporate and government contracts are more complicated and include "block out dates" such as holidays where the pre-negotiated rate is not applicable, and there may be different rates for weekend vs weekday, or the rates can vary over the season of the year. We can take some of these details into account in subsequent work but initially our objective is to keep our DP model as simple as possible, so we treat any group customer as having the right to book on any date at the discounted price $p_g = \delta_g p$ subject to room availability.

Let c_t be the total number of cancellations by existing customers (both group and transients) t days prior to arrival, and let $e_t(c_t|n,p,\rho,\overline{p})$ be the corresponding conditional probability density for c_t. Note that while we allow the conditional density for cancellations e_t to depend on (p,ρ) and potentially allow for the possibility of strategic cancellations, our empirical results in section 3.4 indicate that none of the price variables $(p,\rho,\overline{p})$ are significant predictors of the number of cancellations, so the model we actually solve excludes prices from e_t and thus we assume exogenous cancellations for both group and transient customers.

Let g_t^d denote the number of *new* group reservation requests received by Hotel 0 t days before arrival, and let $g_t(g_t^d|p,\rho)$ be the corresponding probability distribution for this random variable. Similarly, let r_t^d be the number of *new* transient reservation requests from non-group customers who are generally ineligible for a discount off the BAR. However some transient customers may be eligible for discounts based on standard, non-negotiated discounts that the hotel provides to some classes of customers such as senior discounts, discounts to military, and so forth. Let $f_t(r_t^d|p,\rho)$ denote the conditional probability distribution of the number of new requests by transient customers which depends on state variables (p,ρ). To enforcing the capacity constraint on the number of hotel rooms, $\overline{n}$, we distinguish the actual number of bookings of group and transient customers, (g_t,r_t) from the total number of *desired bookings* (g_t^d,r_t^d).

The actual number of bookings result from censoring the desired number of bookings to ensure that the hotel capacity constrain $\overline{n}$ is satisfied with probabiliy 1 for all t.

We assume that hotel 0 strictly enforces its capacity constraint $\overline{n}$ at every date t prior to arrival, and thus never overbooks. Though we do see temporary periods where the number of rooms booked $n_t > \overline{n}$, we rarely observe overbooking on the final arrival date, i.e. $n_{-1} > \overline{n}$. Though hotel 0 can feasibly allow overbooking sufficiently far in advance of the arrival date and gamble that cancellations will occur in the intervening period as we see in the left panels of figure 6, in our DP model we assume that the occupancy constraint is enforced with probability 1 every day t prior to arrival. Thus on any date t where demand for rooms exceeds remaining available capacity, the hotel accepts new bookings according to some predetermined censoring rule $\eta(.)$ until there are no rooms left. This "no overbooking assumption" has important implications for the optimal pricing strategy.

We let the function η encode the demand censoring rule that enables us to satisfy the hotel capacity constraint with probability 1 for each t,

$$(r_t, g_t) = \eta(r_t^d, g_t^d, c_t, n_t, \overline{n}). \tag{3}$$

The function η satisfies following conditions:

1 if capacity constraint is not binding and the number of group and transient guests who desire to stay at hotel do not exceed its capacity, i.e. if $\overline{n} > n_t - c_t + r_t^d + g_t^d$, then $(r_t, g_t) = (r_t^d, g_t^d)$,

2 otherwise, the total desired number of bookings (r_t^d, g_t^d) are reduced according to some random allocation rule approximating "first-come, first served", so $n_{t-1} = \overline{n}$ with probability 1.

Hotel 0's group contracts have 'subject to availability" clauses that imply that in any situation where $n_t + r_t^d + g_t^d - c_t > \overline{n}$ the group reservation requests will be denied first. If total remaining demand $n_t + r_t^d - c_t$ still exceeds capacity after the denying the group reservation request, the hotel takes as many transient reservations on a first-come, first served basis until it sells out. Since we do not model the order of arrival of bookings with the day, this amounts to a probabilistic censoring of r_t^d and g_t^d, starting with group bookings first and continuing to the transient bookings to ensure that $n_{t-1} \leq \overline{n}$ with probability 1. Given this, the law of motion for n_t, the number of rooms booked t days prior to arrival is given by

$$n_{t-1} = n_t - c_t + r_t + g_t, \tag{4}$$

where (r_t, g_t) are given by the censoring rule (3).

Let $\overline{p}_t$ be the ADR at hotel 0 for all bookings that been made t days prior to arrival. We provide an accounting identity below that serves as a "law of motion" for the ADR in our DP model that enables us to keep track of revenues from rooms already booked using this $\overline{p}_t$ in conjunction with the number of rooms already booked n_t. We define

$$\overline{p}_{t-1} = \frac{(n_t - c_t)\overline{p}_t + \delta_r p_t r_t + \delta_g p_t g_t}{n_{t-1}}, \tag{5}$$

where δ_r is the average discount provided to transient customers, δ_g is the average discount for group customers. Thus, equation (5) simply specifies $\overline{p}_{t-1}$ to be the total revenues booked $t-1$ days prior to arrival divided by the total number of customers booked, i.e. the ADR. In equation (5) we assume that when cancellations occur, on average the hotel must refund cancelled reservations at the existing ADR $\overline{p}_t$. Let $\overline{p}_{t-1} = \lambda(n_t, r_t, g_t, c_t, \overline{p}_t, p_t)$ denote the law of motion for the ADR given in equation (5) and let $h_t(\rho'|\rho)$ represent the (exogenous) transition probability for the average BAR of hotel 0's competitors. Now we have the notation we need to write down the Bellman equation. Let $V_{-1}(n, \overline{p}, p)$ be the hotel's realized profits on the morning after the occupancy date, $t = 0$, which we denote by $t = -1$ and is given by

$$V_{-1}(n, \overline{p}, \rho) = n \cdot (\overline{p} - \omega), \tag{6}$$

where $\overline{n}$ is hotel 0's total room capacity and ω is the marginal cost of servicing a room, net of per customer profits on incidental spending and services at the hotel such as in its bar and restaurant. Equation (6) implicitly enforces a hard constraint on room capacity through $\eta(.)$.[9] We also allow the ω parameter to capture commissions that the hotel must pay to OTAs for some of its bookings, so ω reflects the average or expected net marginal cost to the hotel for each room it books. We use the term "marginal cost" since we do not attempt to allocate fixed costs such as depreciation/amortization of the hotel building or other "front office" costs including the salary of hotel management, including the revenue manager.

Given the terminal value V_{-1} in (6) the Bellman equation recursively defines the expected profit functions $\{V_0, V_1, \ldots, V_T\}$ via the recursion relation

$$\begin{aligned} V_t(n, \overline{p}, \rho) = & \max_p \left[\int_{\rho'} \sum_{r^d} \sum_{g^d} \sum_c V_{t-1}(n', \bar{p}', \rho') e_t(c|n) f_t(r^d|p, \rho) g_t(r^d|p, \rho) h_t(\rho'|\rho) \right] \\ & s.t. \quad n' = n - c + r + g \\ & \qquad \overline{p}' = \lambda(n, r, g, c, \overline{p}, p) \\ & \qquad (r, g) = \eta(r^d, g^d, c, n, \overline{n}). \end{aligned} \tag{7}$$

[9]Hotel 0 rarely overbooks, but when it does, it accommodates any unexpected additional guests in one of its higher class rooms. There may be a higher marginal cost of servicing a guest in a higher class room and for reputational reasons, the hotel seeks to avoid a situation where it effectively significantly underprices its available higher class rooms by letting overbooked customers stay in them for the price of the lower class tier that they reserved at. In future work we will provide a deeper analysis of overbooking, but given that it rarely occurs we have decided to use the simpler specification where the capacity of standard class rooms $\overline{n}$ is treated by the revenue manager as a hard constraint.

The value of p_t that maximizes the right hand side in (7) defines the optimal dynamic pricing strategy $\{p_t^*(n,\overline{p},\rho)\}$ that specifies the BAR that the hotel should charge, $p_t = p_t^*(n,\overline{p},\rho)$, in any state and for each day t in advance of arrival. Though realized profits on the arrival date, $V_{-1}(n,\overline{p},\rho)$, do not depend on ρ, the BAR of competing hotels, ρ_t is a critical state variable since it affects hotel 0's pricing and the number of customers who book at hotel 0 for any day $t \geq 0$ which includes the morning of the arrival date $t = 0$. Intuitively, hotel 0 needs to pay attention to ρ_t when it sets its own BAR p_t, since if it sets p_t too high it increases the probability that new customers who are booking rooms will book at one of its competitors.

Note that for notational simplicity we have omitted the index $k \in \{1,\ldots,K\}$ of the type of occupancy date and room type (regular vs luxury suite). In principle each of the stochastic laws of motion in the Bellman equation (7), g_t, f_t and h_t, also depend on k and thus there are implicitly k different value functions V_t^k and corresponding optimal pricing strategies that we compute by solving K separate DP problems. In our empirical work we ignore luxury suites since they constitute such a small fraction of hotel 0's total rooms and we do not have enough data to provide reliable parameter estimates for these rooms. Instead we focus on regular rooms, which are 95% of the total rooms at hotel 0 and solve $K = 8$ DP problems for the 4 quartiles of occupancy and weekend vs weekday arrivals. Each of these DPs implies a corresponding optimal pricing strategy $p_{t,k} = p_{t,k}^*(n,\overline{p},\rho)$ that provides a complete operating plan for the hotel in all days of the year and under any eventuality. Our interest is to see how well the optimal pricing rule predicts the actual prices charged by hotel 0. For notational simplicity we will drop the k index, though we will make it clear in the following section how the results depend on the type of occupancy day k.

3.3 Properties of an optimal dynamic pricing strategy

The optimal dynamic pricing strategy depends potentially on four key variables: 1) number of days prior to arrival, t, 2) number of rooms already booked, n_t, 3) the average BAR of competing hotels, ρ_t, and 4) hotel 0's own ADR $\overline{p}_t$. The dependence of pricing decisions on the first three items seems completely intuitive. In particular, knowing how many rooms are already booked, n_t along with knowledge of the capacity of the hotel (which is an implicit but non-time-varying state variable) determines the number of rooms left to be sold, and it may be optimal for the hotel to raise its prices when it expects to sell out. However the potential dependence of BAR on ADR variable seems unintuitive, since one might expect that pricing is a *forward looking decision* whereas the ADR summarizes the average price at which past bookings were made. While knowledge of the ADR is important for forecasting the ultimate revenue and profit the hotel will earn, we now discuss conditions under which ADR will have no effect on the optimal BAR. That is, we now ask under what conditions will $p_t^*(n,\overline{p},\rho)$ depend on t, n and ρ but not $\overline{p}$? We will show there are conditions under which p_t^* will depend on ADR ρ_t even though it is true that pricing is a primarily a forward looking decision. However if cancellation decisions are endogenous, then they will generally depend on both the current BAR p and the ADR $\overline{p}$ and tend to increase when p falls below $\overline{p}$. In this case hotel 0 has to trade-off the gain from cutting the BAR in the last few days before arrival against the loss in revenue from existing bookings if enough already booked customers decide to cancel and rebook at the lower BAR.

First we show that the value function V_t can be decomposed into the sum of two components: 1) a "backward looking component" V_t^b that provides the expected profits from customers who are already booked, and 2) a "forward looking component" V_t^f that provides the expected profits from customers who will arrive and book rooms in the future. This representation holds regardless of whether cancellations are endogenous or exogenous. If cancellations are exogenous the optimal dynamic pricing rule p_t^* will indeed be independent of ADR $\overline{p}$, but if cancellations are endogenous then p_t^* depends on $\overline{p}$.

Theorem 1 *For each $t \in \{1,\dots,T\}$ the value function V_t has the representation*

$$V_t(n,\overline{p},\rho) = V_t^f(n,\overline{p},\rho) + V_t^b(n,\overline{p},\rho) \tag{8}$$

where V_t^f is the "forward looking component" that equals the expected profits from rooms that are not yet booked, whereas V_t^b is the "backward looking component" that equals expected profits from rooms that are already booked.

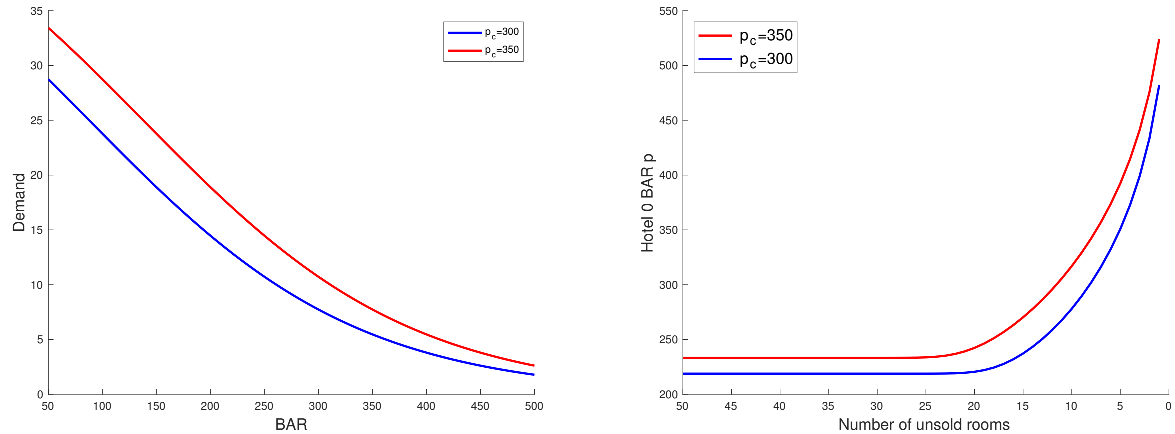
We now introduce an assumption about the stochastic demand function for hotel rooms that is satisfied by the static discrete choice model that we introduce in the next section.

Assumption 1 *The conditional probability distributions for the number of new transient and group reservation requests, r_t^d and g_t^d are independent of the hotel's ADR $\overline{p}$.*

Note that we can see that Assumption 1 is satisfied by virtue of the form of the conditional probability distributions $f_t(r^d|p,\rho)$ and $g_t(g^d|p,\rho)$ entering the Bellman equation (7): neither of these conditional densities depend on the ADR, $\overline{p}$. Intuitively if future customers do not care about the ADR in making their decisions, then it is optimal for the hotel revenue manager to disregard ADR when it comes to setting BAR.

While it is tempting to conjecture that Assumption 1 implies that the optimal decision rule for BAR should also be independent of ADR (since the revenue manager sets BAR to attract future customers to book some or all of the hotels available rooms), the result depends on the additional assumption of exogenous cancellations, given below.

Figure 11: Example demand functions and optimal prices $p_1^*(n_1, \rho)$ for $\rho = 300$ and $\rho = 350$



Assumption 2 (Exogenous cancellations) *The conditional probability distributions for the number of cancellations, c_t, by existing customers does not depend on the hotel 0's BAR p or ADR $\overline{p}$.*

Assumption 2 holds if the conditional probability density $e_t(c|n, p, \rho, \overline{p})$ in the Bellman equation (7) does not depend on $(p, \overline{p})$. As we observed in section 3.4 we find only weak evidence that cancellation decisions depend on hotel 0's BAR and ADR.

Theorem 2 *If Assumption 1 and 2 hold, then for each $t \in \{1, \dots, T\}$ the forward looking component of the value function V_t^f is independent of $\overline{p}$, i.e. it can be written as $V_t^f(n, \rho)$ and depends on (n, ρ) but not $\overline{p}$.*

Theorem 3 *If Assumptions 1 and 2 hold then for each $t \in \{1, \dots, T\}$ the optimal decision rule for BAR p_t^* is independent of $\overline{p}$, i.e. it can be written as $p_t^*(n, \rho)$ and depends on (n, ρ) but not $\overline{p}$.*

When the exogenous cancellation condition in Assumption 2 holds, it provides additional special structure that we can exploit to substantially speed up the solution to the DP problem, since we only have to compute optimal prices over a two-dimensional grid of points (n, ρ) instead of a three dimensional grid $(n, \rho, \overline{p})$, via a backward induction recursion to calculate the forward-looking component of the value function, $V_t^f(n, \rho)$. The gain in speed from exploiting this additional structure can be important for structural estimation of the model since as we see in the next section, the MSM estimator of the model's unknown parameters requires repeated trial solutions of the hotel's DP problem as we search for structural parameters that enable the model to best fit a vector of moments characterizing the hotel's actual pricing behavior and the occupancy and cancellation decisions of its customers.

The left panel of figure 11 illustrates example demand curves for business guests using parameter estimates from the model that we present in the next section. For this example we assume that the revenue manager is certain that a total of $k_0 = 50$ business guests are making reservations on the arrival date. However the revenue manager is uncertain about how many of these guests will choose to book a room at hotel 0. Under our assumption that the guests make independent decisions, the distribution of demand for hotel 0 will have a binomial distribution with parameters $(50, P_0(p,\rho))$ where $P_0(p,\rho)$ is the probability any of these consumers will book a room at hotel 0 given that its BAR is p and the average BAR of its competitors is ρ. In the next section we will describe a more general stochastic demand model that allows for a random number of customers to arrive on any given day to book rooms. This model will imply that r_t^d, the number of new transient reservation requests at hotel 0, is a mixture of binomials. We will also derive the functional form of $P_0(p,\rho)$ from a simple static binary choice model.

But to simplify the illustration, assume that the number of arrivals k_0 is known to be some fixed value such as $k_0 = 50$. Then the expected demand (requests) curve for hotel 0 is particularly simple: it can be written as $D_0(p,\rho) = k_0 P_0(p,\rho)$. In figure 11 we illustrate how expected demand depends on the average price ρ of hotel 0's competitors. The blue line plots the expected demand for hotel 0's rooms 1 day in advance of arrival when $\rho = 300$ and the red line plots the expected demand when $\rho = 350$. Thus, via a straightforward substitution effect in customers' choices, the increase in ρ results in an upward shift in the demand for hotel 0. As we show below, this simple demand substitution effect is the key reason why Hotel 0's optimal price strategy can be described as "price following" — the rise in the price of competing hotels increases the demand for hotel 0 and this makes it optimal for hotel 0 to raise its prices in response.

The right hand panel of figure 11 plots the optimal pricing rule $p_0^*(n,\rho)$ for the demand model above under the assumption that the marginal cost of servicing a room is $\omega = 50$. We have assumed exogenous cancellations so by Theorem 3, p_0^* does not depend on ADR $\overline{p}$. However we can see that it does clearly depend on competitors' BAR ρ as well as the number of remaining unsold rooms $\overline{n} - n_0$. We see that BAR is essentially flat as a function of n_0 for values of n_0 sufficiently below the hotel's capacity $\overline{n}$. However it starts to rise steeply, and well above the prices ρ set by its competitors, as n_1 gets close to $\overline{n}$ and hotel 0 expects to sell out. Thus, we see a clear price asymmetry: it is optimal to increase prices to ration scarce capacity, but when the hotel has too much excess capacity it is not optimal to cut its BAR to try to increase its occupancy. Instead it is better to keep its prices high (though undercutting its competitors) and accept the fact that there will be many unsold rooms.

It is tempting to frame the hotel's optimal pricing problem as a simple "Econ 101 problem" where the hotel has a supply function with a constant marginal cost of ω until its capacity $\overline{n}$ is reached, at which point its marginal cost curve becomes a vertical line. Would it be valid to calculate the optimal price as the value of p that equates expected marginal revenue to marginal cost? Unfortunately this simplistic approach does not provide the correct solution.

The actual calculation of optimal prices for hotel 0 is more complicated due to the stochastic nature of demand and the need to enforce the capacity constraint with probability 1. To see why, let's illustrate the breakdown of V_0^f with only one segment of customers. Suppose that at $t=0$, the morning of arrival, hotel 0 has already booked a total of n_0 rooms, so it has a remaining capacity of $\overline{n}-n_0$ rooms left to sell. Let $\tilde{r}_0(p,\rho)$ be a binomially distributed random variable with parameters $(50, P_0(p,\rho))$ that represents the stochastic demand (number of new bookings) at hotel 0 by the 50 customers who are booking rooms in this market on the arrival day and face prices (p,ρ) for hotel 0 and its competitors, respectively. Then we have

$$V_0^f(n,\rho) = \max_p E\left\{\min[\tilde{r}_0(p,\rho), \overline{n}-n_0](p-\omega)\right\}, \tag{9}$$

and $p_0^*(n,\rho)$ is the value of p that maximizes the forward looking expected profit in equation (9).

Let $b(r|k_1, P_1(p,\rho))$ and $B(r|k_1, P_1(p,\rho))$ be the probability density and cumulative distribution functions for the binomial random variable $\tilde{r}_1(p,\rho)$ with parameters $k_1 = 50$ and $P_1(p,\rho)$. Then we can write the expectation on the right hand side of (9) as

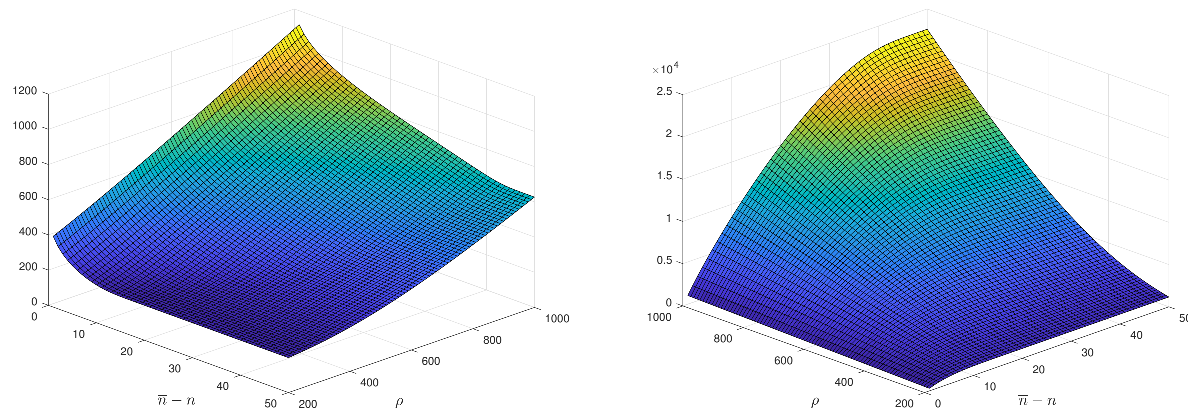
$$\begin{aligned} E\left\{\min[\tilde{r}_0(p,\rho), \overline{n}-n_0](p-\omega)\right\} \quad = \quad & (\overline{n}-n_0)[1-B(\overline{n}-n_0-1|k_0, P_0(p,\rho))](p-\omega) + \\ & \sum_{r=0}^{\overline{n}-n_0-1} rb(r|k_0, P_0(p,\rho))(p-\omega). \end{aligned} \tag{10}$$

The left panel of figure 12 plots the optimal price function $p_0^*(n,\rho)$ and the forward looking profit function $V_0^f(n,\rho)$ for this example by numerical maximization of expected profits in equation (10) over a grid of points over the number of unsold rooms $\overline{n}-n_1$ ranging from 1 to 50 and over a uniformly spaced grid of points over ρ from $\rho = 200$ to $\rho = 1000$. The right panel plots the corresponding value of maximized expected profits, $V_1^f(n,\rho)$. Profits are monotonically increasing in ρ but are decreasing in n since there are fewer remaining unsold rooms $\overline{n}-n$ left to sell to new customers. We also see that $V_0^f(n,\rho)$ is neither convex nor concave. While it is generally concave in available unsold capacity $\overline{n}-n$ for all values of ρ, its shape as a function of ρ depends on n. $V_0^f(n,\rho)$ is convex in ρ when available capacity $\overline{n}-n$ is sufficiently large, but is concave in ρ when the hotel is close to selling out.

We see that $p_1^*(n,\rho)$ is a monotonically increasing function of ρ, so the optimal pricing rule displays the "price following" behavior we found in the regression results for hotel 0's actual prices in table 5. We also have $p_0^*(n,\rho) \leq \rho$ when hotel 0's occupancy n is sufficiently below its capacity $\overline{n}$, so the optimal pricing rule generally results in hotel 0 undercutting the prices of its competitors.

The optimal pricing rule appears to differ from the empirical pricing rule we uncovered for hotel 0 from the regression results in table 5: the regression estimates showed that hotel 0's occupancy had a

Figure 12: Example of optimal pricing rule $p_0^*(n,\rho,\overline{p})$ and $V_0^f(n,\rho)$



negative but insignificant effect on its ADR. In figure 12 we see it is optimal for hotel 0 to depart from price following and raise its prices significantly above the average BAR of its competitors as n approaches $\overline{n}$. The optimal price function is a convex function of $(\overline{n}-n,\rho)$ and the optimal price schedule increases particularly rapidly when there are fewer than 10 rooms left to be sold.

However when there is significant excess capacity, it is not optimal for hotel 0 to cut its BAR to try to attract more customers: the optimal price schedule $p_1^*(n,\rho)$ flattens out when remaining unsold capacity $\overline{n}-n$ becomes sufficiently large, especially when the average BAR of competitors is low. Thus, the optimal pricing strategy displays a *conditional* version of "price following" and "price undercutting" — it is only optimal to do this when hotel 0's occupancy rate is sufficiently low. When hotel 0 is close to selling out, it is optimal to raise its prices sharply, even if this means that its BAR will be higher than its competitors' BAR. We observed this type of behavior for hotel 0 in the top left panel of figure 6.

We conclude that an optimal pricing strategy does not imply price aggressive unilateral price cutting when occupancy rates are low and there is little chance that the hotel will be sold out on the arrival date. If the hotels are pricing optimally, then on non-busy days we should not expect to see hotel 0 makes any unilateral price cuts to try to sell more rooms. Instead, it should only cut its BAR in response to price cuts by its competitors. But on busy days, it is optimal to increase prices both unilaterally to try to ration scarce remaining capacity, and also in response to price rises by the hotel's competitors. These reinforcing effects of price increases in response to shocks to market demand that lead all of the hotels in this market to be close to selling out at the same time can generate the sharp pricing peaking behavior we observed for this hotel market at both seasonal and weekly frequencies that we observed in figures 2 and 3.

3.4 A Stochastic Model of Hotel Demand

Our model of stochastic demand for hotel rooms is based on an assumption of inelastic but stochastic arrival of customers who wish to book a room in one of the seven hotels in this market. At each day t prior to an intended arrival to stay at one of the hotels in the particular neighborhood of the city, a random number of customers k_t "arrive" and consider the attributes and BARs of the seven hotels in this neighborhood and choose to book at one of them. When we use the term "inelastic" we mean that the stochastic process governing the number of consumers who arrive k_t is independent of the prices of the hotels in this market. However we can allow the choice of an "outside good" which can be interpreted as a choice not to book a reservation at any of the seven luxury hotels in this market if all of their prices are too high. When we allow for an outside option, then the demand to stay in one of the seven hotels is not really inelastic, since a sufficiently high price for all of the hotels will cause an increasing fraction of consumers who arrive to try to book a room to choose the outside good, which can be interpreted as either the decision to cancel or reschedule their visit for another date (such as if they are a tourist) or to stay a some nearby hotel that is outside the immediate neighborhood where the seven hotels are located.

Our consumer demand model ignores the more complicated possibility that customers solve dynamic programs to calculate optimal dynamic search strategies for when to book a room at a particular hotel in this market. As we noted in section 3.4, if customers were using these strategies we would expect to observe *endogenous cancellations* i.e. consumers would tend to book early and monitor the prices in the market and cancel their reservations and rebook if the BAR falls sufficiently prior to their intended arrival date. The fact that we do not find any statistically significant effect of reductions in BAR on cancellations suggests that a model of *exogenous cancellations* (where current BAR does not affect the cancellation rate) is a reasonable approximation to consumer behavior in this market.

We will shortly discuss our assumptions about the stochastic process from which the realization for the number of arriving customers k_t is drawn from, but conditional on k_t, the k_t individual reservations involve independent trinomial choices of whether to reserve a room at hotel 0 at price p_t, or to make the reservation at one of the competing hotels at price ρ_t, or to choose the "outside good" to either stay outside this neighborhood or cancel or reschedule their trip to some other less busy date.

Let the consumer's "type" be indexed by τ and assume that a consumer of type τ chooses to reserve the room at the hotel that provides the highest utility, taking into account Type 1 extreme value distributed shocks that represent other idiosyncratic factors affecting their choice of which hotel to reserve at. With seven hotels in this local market and the outside good, the choice model is a multinomial choice model with 8 alternatives (the seven hotels plus the outside good as the 8th choice). However we make an approximation to simplify the model by assuming that the probability of choosing to reserve at hotel 0 can be well approximated with a 3 choice model consisting of 0) hotel 0, 1) the outside good, or 2) booking at one of the other 6 hotels. We normalize the net utility of the outside good to be 0 and the net utility of the hotel we are studying to be a_τ, and let $b_\tau > 0$ denote consumer τ's degree of price-sensitivity. We also normalize the intercept representing the average utility of choosing one of other competing hotels to be zero. Then a consumer of type τ chooses to book at hotel 0, which we denote by the choice $d = 0$ if

$$a_\tau + b_\tau \delta_\tau p_t + \varepsilon_0 \geq \max[\varepsilon_1, b_\tau \delta'_\tau \rho_t + \varepsilon_2], \tag{11}$$

where δ_τ and δ'_τ denotes any discount off the BAR that a customer of type τ might be entitled to (such as if the customer is part of a group) at different hotels [10], and $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ are independent Type 1 extreme value distributed random variables with mean 0 and scale parameters normalized to 1. This implies that the probability the consumer chooses to reserve at hotel 0, which we denote by the choice $d = 0$, is given by

$$Pr\{d = 0|\tau, p_t, \rho_t\} = \frac{\exp\{a_\tau + b_\tau \delta_\tau p_t\}}{1 + \exp\{a_\tau + b_\tau \delta_\tau p_t\} + \exp\{b_\tau \delta'_\tau \rho_t\}}. \tag{12}$$

Assume that $m_t(\tau)$ is the probability at t that an individual consumer is of type τ and assume there are a finite number L of types of consumers, then we have

$$P_t(p_t, \rho_t) \equiv Pr\{d = 0|p_t, \rho_t\} = \sum_{l=1}^{L} Pr\{d = 0|\tau_l, p_t, \rho_t\} m_t(\tau_l) \tag{13}$$

It follows that conditional on the number of arrivals k_t, the number of new reservation requests at hotel 0 will be $\tilde{r}_t^d \sim \text{bin}(k_t, P_t(p_t, \rho_t))$, i.e. a binomial distribution with parameters k_t and $P_t(p_t, \rho_t)$. We allow the distribution of types, $m_t(\tau)$, to differ by the number of days prior to arrival and on the type of day, i.e. weekend vs. weekday, busy vs. non busy, etc.

Unfortunately allowing an outside good in the choice model, while natural and appealing for several reasons, creates identification problems. How do we distinguish empirically between a case where there is a high rate of arrival of customers wishing to book a room in this market but a high fraction of these customers end up choosing the outside good from an alternative case where there is a lower rate of arrivals but fewer of these consumers choose the outside good? In both cases, the number of reservations being made at hotel 0 and its competitors could be approximately the same. Unless we can somehow observe the total number of people visiting OTA websites who are considering booking at one of the seven hotels in this particular hotel market, it seems dubious that we can identify the parameters of a choice model that allows for the possibility of an outside good.

[10] In estimation, we impose $\delta'_\tau = \delta_\tau$.

Thus it is not clear that we can identify the parameters of the stochastic process governing the arrival of customers $\{k_t\}$ and the parameters of choice model that allows for the choice of an outside good. Our "solution" to the identification problem is to rule out the possibility of the outside good in the consumer choice model. Thus, our analysis presumes that every consumer wishing to book a room in this market chooses to book at hotel 0 or one of its competitors. Given this restriction, the probability that a consumer of type τ chooses hotel 0 is given by the following binomial logit model

$$Pr\{d=0|\tau,p_t,\rho_t\} = \frac{\exp\{a_\tau + b_\tau(\delta_\tau p_t - \delta'_\tau \rho_t)\}}{1+\exp\{a_\tau + b_\tau(\delta_\tau p_t - \delta'_\tau \rho_t)\}}. \tag{14}$$

We assume that the total number of customers who book new reservations at one of the hotels in this market t days before their intended arrival date, k_t, is a realization of a negative binomial distribution.

Now we derive the conditional probability of the number of new transient reservation requests r^d, $f_t(r^d|p_t,\rho_t)$, from our assumptions on the stochastic arrival of customers and their individual discrete choices of which hotel to book at. While we can observe r_t^d, which is the same as r_t in hotel 0's reservation database, is capacity constraint is not binding, we (and also hotel 0) will generally not observe k_t, the total number of consumers wishing to book a room in this market of the city t days prior to any given arrival date. However if there are r_t^d new transient requests received by hotel 0, we can conclude that $k_t \geq r_t^d$. Thus, we have

$$f_t(r|p_t,\rho_t) = \sum_{k\geq r} \binom{k}{r} P_t(p_t,\rho_t)^r [1-P_t(p_t,\rho_t)]^{(k-r)} \pi(k|\phi_t,\mu_t), \tag{15}$$

where the choice probability $P(p_t,\rho_t)$ of booking at hotel 0 is given in equation (13) and $\pi(k|\phi_t,\mu_t)$ is the negative binomial density (1).

In principle the parameters of our stochastic demand model could be estimated by the method of maximum likelihood, by pooling observations of new reservations r_t at the various different dates t prior to occupancy at dates of the same type (i.e. weekdays vs weekends, etc). However identification is problematic if we do not observe k_t, for reasons similar to the one that motivated us to exclude the possibility of an outside good in the choice model. Even with a restriction to a binary choice model, it is not obvious that it is possible to identify the demand parameters, especially for different observed and unobserved types of customers. For example if we observe, say, a total of 10 new reservations made on a given day, was this because hotel 0 managed to attract 50% of a total of the $k_t = 20$ new reservations that were made that day in this market, or only 10% of $k_t = 100$ new reservations?

While we do not observe k_t for any t, the data we obtained from STR enables us to observe occupancy at $t = 0$ at hotel 0's competitors. This provides identifying restrictions, since knowing the capacity and arrival day occupancy at the other hotels helps us to infer the overall number of arrivals leading up to the occupancy date. The cumulative number of arrivals cannot be too high or too low, otherwise occupancy rates at both hotel 0 and its competitors could not match the values we observe. In the next section we propose a method of simulated moments (MSM) estimator and show that the STR data, combined with the reservation and cancellation data we have from hotel 0 enables us to identify the parameters of our stochastic demand model and control for the endogeneity problem we discussed in section 3, even in the absence of having any relevant instrumental variables to control for endogeneity.

4 Results

In the previous section we showed that the dynamic programming model can generate solutions that are qualitatively consistent with the "price following" behavior by hotel 0 that we described in section 3, though not in all respects (e.g. the regression results suggest that hotel 0's prices do not depend on its occupancy). In this section we show how to estimate the model so we can make a more rigorous quantitative assessment of how well the model can approximate the actual pricing, occupancy, and cancellation data from hotel 0.

4.1 Estimation method: MSM

Let θ be a $M \times 1$ vector of the unknown parameters of the model, such as the parameters of the stochastic demand and cancellation model. The preferred method of estimating θ is maximum likelihood, since it allows us to best fit the realized values of the data we observe and results in an asymptotically efficient estimator for θ. However there are several reasons why direct maximum likelihood estimation is not feasible in this case. The key problem is that the model we formulated in section 4 is "statistically degenerate" in the sense that the likelihood of the data will be zero regardless of value of θ. The reason is that the dynamic programming model results in a *deterministic* optimal decision rule for the BAR, $p_t = p_t^*(n_t, \rho_t, \overline{p}_t, \theta)$, and so there will be observed values of $(p_t, n_t, \rho_t, \overline{p}_t)$ that will not lie on the graph of this function regardless of what value of θ we choose. In other words the dynamic programming model is incapable of predicting any such observation.

One way to avoid the zero likelihood problem is to include an additional state variable η_t that can be regarded as information observed by hotel 0 that affects its choice of BAR that we do not observe as the econometrician. Under a sufficiently flexible specification of the hotel 0's dynamic programming problem it might be possible that an augmented decision rule that incorporates η_t as an *unobserved state variable* could result in a non-degenerate decision rule for the BAR of the form $p_t = p_t^*(\eta_t, n_t, \rho_t, \overline{p}_t, \theta)$. That is, for any given θ we can find at least one value of η_t such that $(p_t, \eta_t, n_t, \rho_t, \overline{p}_t)$ lies on the graph of p_t^*. If this were possible, then we could write a likelihood for p_t given $(n_t, \rho_t, \overline{p}_t)$ that will be non-zero for any θ, and in this case we could estimate θ by maximum likelihood.

However we are not aware of any specification of a dynamic programming model where we can do this, at least if p_t is treated as a continuous random variable. However if we are willing to discretize the set of possible BARs and assume that hotel 0 chooses its BAR from a pre-defined *finite set* it would be possible to model hotel 0's choice of prices as a dynamic discrete choice problem as in Rust (1987). Unfortunately the set of BARs chosen by hotel 0 is too large to be well approximated by a pre-defined finite set of discretized prices and trying to impose a coarse discretization on the choice of BARs could lead to an artificially suboptimal pricing policy for hotel 0.

Even if we used a finer discretization on the set of possible BARs (which is computationally burdensome), there are additional econometric issues of censoring and endogeneity that make it challenging to infer the parameters of our stochastic model of demand by maximum likelihood. The ideal data set would allow us to observe the total number of customers k_t who arrive to book a room in this market t days prior to their intended arrival date, and enable us to observe which hotels they chose given the prices available at date t. However as we discussed in section 4.4, we do not observe k_t or the number of people reserving rooms at hotel 0's competitors prior to any given arrival date. Instead we only observe r_t, and g_t the number of new reservations made by transient and group customers of hotel 0, respectively. We also observe total occupancy at hotel 0's customers on a daily basis, but not the trajectory of bookings and cancellations leading up to the final occupancy each day at hotel 0's competitors. We would need to "integrate out" a full likelihood for the data to match the subset of information we actually observe, and this lead to a high dimensional numerical integration problem that is intractable, at least using deterministic quadrature rules.

An even more serious econometric problem is the endogeneity in the hotel prices that we noted in the introduction. Stochastic shocks to the total number of customers wishing to book rooms in this local hotel market cause the prices of all of the hotels in this market to be strongly positively autocorrelated: when there is high demand for rooms hotels are likely both raise their prices substantially and sell out, whereas on days where demand is low we will see excess capacity and lower prices. The result is a strong positive correlation in prices and occupancy that we observed in the right hand panel of figure 4, and an ambiguous relationship between the occupancy share for hotel 0 versus the ratio of its ADR to its competitors' average ADR as shown in the left hand panel of the figure. We are not aware of any instrumental variables that would be effective for solving this endogeneity problem and enable us to estimate a plausible negatively sloped demand curve for hotel 0.

Our solution to these problems is to estimate the unknown parameters of our model using the method of simulated moments (MSM). The basic idea is to simulate the set of prices, bookings and cancellations for all seven hotels in this market and then censor the simulated data in the same way the data we observe is censored, namely, we exclude observations on the paths of bookings and cancellations leading up to each arrival date at hotel 0's competitors and use only the realized occupancy on the arrival dates to compare to the comparable data that we have from STR. MSM requires us to specify a vector of $J \geq M$ moments based on the observed, censored data, which we denote by the $J \times 1$ vector $\overline{m}_T$ where T denotes the total number of arrival day observations we have in our data set. Using our dynamic programming model we simulate a corresponding set of moments based on the simulated, censored data, which we denote by $\overline{m}_{S,T}(\theta)$, where S denotes the number of independent simulations of the dynamic programming model that were averaged to help reduce simulation noise, which we will specify more explicitly below. The MSM estimator is then defined by a value $\hat{\theta}_T$ that satisfies

$$\hat{\theta}_T = \underset{\theta}{argmin}[\overline{m}_T - \overline{m}_{S,T}(\theta)]' W_T [\overline{m}_T - \overline{m}_{S,T}] \tag{16}$$

where W_T is a $J \times J$ positive definite weighting matrix to be specified below.

Let $\vec{x}_t$ be a vector of variables such as the paths for BARs, bookings, cancellations and final occupancy at a given calendar date t for hotel and its six competitors. Due to the censoring we noted above, while we observe final occupancy on date t and the full path of BARs leading up to the arrival date t for both hotel 0 and its competitors, we only observe the paths of bookings and cancellations for hotel 0 but not for its competitors. Let $h(\vec{x}_t)$ be a function that maps $\vec{x}_t$ into R^J. Thus, $h(\vec{x}_t)$ constitutes a $J \times 1$ vector of statistics summarizing the path of BARs, bookings, cancellations and final occupancy at hotel 0 and its competitors on date t. Let $\overline{m}_T$ denote the time average of these statistics over T different arrival dates,

$$\overline{m}_T = \frac{1}{T} \sum_{t=1}^{T} h(\vec{x}_t). \tag{17}$$

We assume that the paths $\vec{x}_s$ and $\vec{x}_t$ leading up to distinct arrival dates $s \neq t$ are independently distributed, which implies that $h(\vec{x}_s)$ and $h(\vec{x}_t)$ are independently distributed $J \times 1$ vectors. Then under suitable regularity conditions, the Law of Large Numbers implies that with probability 1 we have $\overline{m}_T \to E\{h(\vec{x}_t)\}$. Similarly, the Central Limit Theorem implies that

$$\sqrt{T}[\overline{m}_T - E\{h(\vec{x}_t)\}] \Longrightarrow N(0, \Omega), \tag{18}$$

where Ω is the $J \times J$ variance-covariance matrix given by

$$\Omega = E\left\{[h(\vec{x}_t) - E\{h(\vec{x}_t)\}][h(\vec{x}_t) - E\{h(\vec{x}_t)\}]'\right\}. \tag{19}$$

Table 6: Definition of estimation subsamples

Sample (Weekday)	Number of Observations	MSM Criterion	Sample (Weekend)	Number of Observations	MSM Criterion	Demand Description
00	132	10,658.7	01	103	9,248.0	Lowest quartile
10	132	12,482.4	11	103	11,605.6	2nd quartile
20	132	20,670.8	21	103	14,111.5	3rd quartile
30	132	45,978.7	31	103	20,950.7	Highest quartile

We assume that the stochastic processes generating the observed paths of bookings, cancellations and prices are independently distributed for different arrival days. So the $J \times 1$ vector of moments $\overline{m}_T$ are formed as sample averages of various functionals of the realizations of these independent stochastic processes, and a Law of Large Numbers for *IID* observations can be used to establish that $\overline{m}_T$ converges with probability 1 to $J \times 1$ vector m^* equal to the expectation of the individual random vectors entering the average $\overline{m}_T$.

4.2 Model specification

In previous discussion of demand model, we assumes that there are two segments of customers: 1) transients and 2) groups, and there are multiple unobserved types of customers in each segment. In our estimation, we adopt three segments of customers 1) leisure 2) business and 3) groups, and there're no unobserved types within each segment. Basically, we further divide "transient" into "leisure" and "business" according to observed market code in the database. This specification wouldn't change our theorems concerning value function and optimal policies [11].

We estimated a total of $K = 8$ separate DP models by MSM on $K = 8$ corresponding subsamples based on classifying the 1731 days over our sample period into groups based on whether each day was a weekend versus a weekend and based on four quartiles for total occupancy on the arrival date, $t = 0$. The various samples and the MSM estimation criterion is listed in table 6. We define a "weekend" as consisting of the three days Thursday, Friday and Saturday and the other days of the week as "weekdays". Table 7 shows the fractions of business, leisure and group customers as well as the average occupancy rate in the 8 subsamples. As we would expect, there are more business and group customers staying in the hotel on weekdays, and relatively more leisure travelers on weekends. Occupancy rates are generally higher on the weekends, so the lower ADRs on weekends are presumably caused by the higher fraction of more price elastic leisure travelers who stay at hotel 0 on the weekends.

Before turning to the parameter estimates, we provide further information on our choice of functional

[11] In appendix, we provide proof for arbitrary number of segments and unobserved types.

Table 7: Customer distribution by subsample

Sample (Weekday)	Customer share business	leisure	group	Occupancy rate	Sample (Weekend)	Customer share business	leisure	group	Occupancy rate
00	0.18	0.61	0.21	51.6 %	01	0.12	0.67	0.21	58.7 %
10	0.20	0.51	0.29	73.9 %	11	0.13	0.65	0.22	81.5 %
20	0.25	0.39	0.36	88.5 %	21	0.14	0.60	0.26	91.0 %
30	0.26	0.30	0.44	99.2 %	31	0.15	0.56	0.30	95.3 %

forms for the probability densities of cancellations, each segment's arrival process (the random k's) respectively, t days before arrival when a total of n rooms are booked. Starting first with cancellations, though it is tempting to predict cancellations using a simple average per day cancellation rate, there is a lot of day to day variability in cancellations that is not evident in our non-parametric plots of cancellation probabilities and rates in figure 9. In particular, cancellations tend to be a very "spikey" process with zero cancellations on most days and clusters of cancellations on others. This is also true for group reservations and to a lesser extent transient reservations.

Instead of a negative binomial, we used zero-inflated negative binomial (ZINB) distributions to capture the spikey aspects of reservations. And a zero-inflated binomial (ZIB) for c_t. A ZINB random variable equals 0 with probability $\gamma_t \in (0,1)$ and with probability $1-\gamma_t$ it is a draw from a negative binomial distribution with parameters (ϕ_t, μ_t). Similarly, a ZIB random variable c_t equals 0 with probability γ_t and is a draw from a binomial distribution with probability $1-\gamma_t$. We allow the probabilities of zero outcomes, γ_t to differ for each segment and to depend on the number of days t prior to arrival. This implies that cancellations, and new reservation arrivals are independent but non-identically distributed events and contemporaneously independent of each other. We also assume that all existing reservations have the same cancellation probability regardless of which segment they are in.

As a dimensionality reduction device, instead of estimating separate values for the parameter $(\gamma_t, \phi_t, \mu_t)$ governing the zero-inflated distributions for each segment for each of the $T = 45$ days prior to arrival, we specified these parameters to be 3rd order polynomials of t and estimated four polynomial coefficients to capture the trajectory for these parameters as a function of t rather than estimate $T = 45$ different values for each parameter in each of the $K = 8$ market segments. As we illustrated in figure 8 the 3rd order polynomials are able to accurately capture how these parameters vary with t but much more parsimoniously.

4.3 Estimation procedure

We adopted a two-step structural estimation approach. In the first step we used the reservation data from hotel 0 to estimate the average discount rates δ_τ applicable to various types of customers, i.e. transients versus groups and business versus leisure travelers. Using the data from Market Vision, we estimated the parameters of the lognormal $AR(1)$ specification for the competing hotels' average BAR, $\{\rho_t\}$, given in equation (2). Treating these estimates as given, we estimated the remaining parameters of the model such as the binomial logit demand parameters (a_τ, b_τ) for the different types of customers and the parameters for the stochastic arrival of transient customers, group reservations and cancellations using the 3rd order polynomial approximations to capture the time variation in these parameters as a function of t, the number of days prior to arrival by MSM using nested numerical solution and simulation of the DP problem of optimal dynamic pricing developed in section 4.[12]

We formed a total of $J = 481$ moments to estimate the $M = 46$ parameters in each model (or a total of 368 parameters in total for all of the $K = 8$ models). Table 8 summarizes the moments we used to estimate the model: in general terms we used observations of the mean and variance of BAR and occupancy trajectories, the number of new reservations by group and transients for each day t prior to arrival, and the number of cancellations at hotel 0, the distribution of ADRs and occupancy for hotel 0 on each occupancy date, as well as means, variances and covariances between hotel 0's ADRs and occupancy rates and the ADRs and occupancy rates at hotel 0's competitors over the 1731 day period in the data set we obtained from STR.[13]

4.4 Estimation results

The fit of the estimated model is very good and the difference between the simulated moments from our model and the actual moments is illustrated graphically in figure 13. The round-solid line indicates the moments created by the actual data, while the star-solid line indicates the moments of simulation. We used moments for estimation and almost all moments look very close to each other for sample20. The fit of the model in the other 7 subsamples is equally good in terms of the graphical deviation between actual and simulated moments. The values of the minimized SMD criterion differ across subsamples in table 6

[12]Prior to estimating the model on the actual data, we verified that the MSM could accurately estimate the model parameters via a small scale monte carlo study where we generated artificial data from the DP model and verified that the MSM parameter estimates were close to the true values and that the asymptotic approximation to the sampling distribution was approximately normally distributed with a covariance matrix formed from a misspecification-consistent version of the covariance matrix for the MSM estimator derived in Hall and Rust (2018).

[13]We pre-estimated the AR(1) processes for the average BAR of hotel 0's competitors using the data from Market Vision. There are an additional 135 parameters from these estimation results that were taken as given in our MSM of the remaining parameters of the DP models. Note that we did not attempt to estimate the marginal cost parameter ω which we set to zero. We are in the process of re-estimating the model where we add ω as part of the overall vector θ.

Table 8: List of Moments

Hotel	Description of Moment	Number of Moments
Hotel 0	avg. occupancy rate, by t	47
	distribution of occupancy on t=0	28
	avg. Transient reservations (Leisure+Business), by t	47
	variance of Transient reservations, by t	47
	prob. of no Group reservations, by t	47
	avg. Group reservations, by t	47
	prob. of non-zero cancellations, by t	47
	avg. cancellation rate, by t	46
	avg. BAR, by t	47
	avg. ADR on t=0	1
	distribution of ADR on t=0	28
All Hotels	avg. occupancy rate on t=0	1
	distribution of occupancy rate on t=0	48
Total		481

mainly due to different sample sizes.

As we noted above, our stochastic demand model has a total of 46 unknown parameters after we do the dimensionality reduction of using 3rd order polynomials to capture systematic changes in the densities governing the number of reservation arrivals and cancellations, respectively, t days prior to arrival on an occupancy day of type k. We estimated $K = 8$ separate DP models by MSM given the classification of occupancy dates provided in table 6.

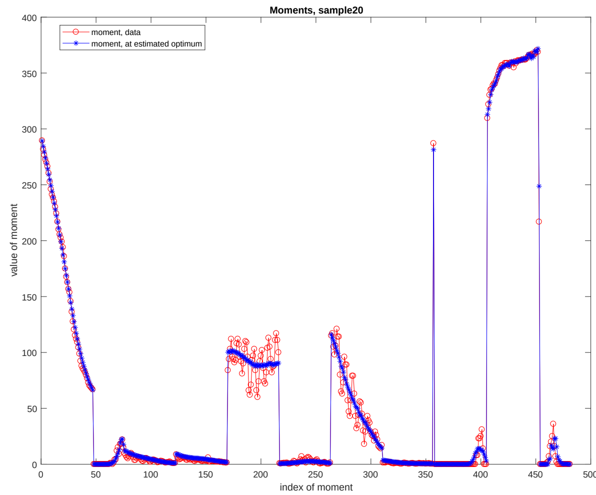
We start by presenting the estimated choice probability parameters (a_τ, b_τ) for the three different classes of customers, leisure, business and group, in table 9. Note that the binomial logit specification (14) implies the following demand elasticity

$$\eta_p = \frac{dQ}{dP} \cdot \frac{P}{Q} = b_\tau \cdot p_t. \tag{20}$$

From table 9 we see that all of the estimated price coefficient estimates b_τ have the *a priori* correct sign (i.e. they imply a downward sloping demand curves) with reasonable implied demand elasticities. For example at an average BAR of \$180 the estimated value of $\hat{b}_\tau = -.006$ for a business traveler on a non-busy weekday implies a price elasticity of $\eta_p = -1.06$. The estimated b_τ coefficients have larger estimated values but also larger variances in the highest demand sample, and so we hesitate to speculate whether b_τ is really higher during the busiest periods, which seems counterintuitive.

However we do find the intuitively plausible result that $\hat{b}_\tau$ is higher for leisure travelers than business travelers except for the case of weekdays in the busiest occupancy subsample, which we already noted are

Figure 13: Difference between simulated and actual moments, sample 20



estimated imprecisely. The minimum occupancy rate of seven hotels in the highest demand subsample was greater than 92.0%. Considering the fact that 95.4% rooms of hotel 0 are regular rooms, almost all of the regular rooms of the luxury hotels are sold out by arrival date in this subsample. For these reasons we do not think the estimates for the highest demand weekday subsample are necessarily fully reliable whereas the $\hat{b}_\tau$ coefficients in the other subsamples show less variability and are more consistent with each other. Overall, we believe structural MSM estimator provides plausible estimates of demand, which is remarkable considering the severe econometric problems of censoring and endogeneity that we noted in the introduction. All of the traditional econometric methods we tried such as instrumental variables failed to produce price and reasonable estimates and frequently implied upward sloping estimated curves.

The remaining parameter estimates are for the parameters of the stochastic reservation arrival and cancellation processes and are presented in tables 10, 11, 12 and 13 in the Appendix. Tables 10 and 11 present the estimated coefficients of the 3rd degree polynomial for the μ_t parameters of the negative binomial probability for leisure customers for weekdays and weekends, respectively. The coefficients of the cubic terms (t^3) are near zero and the coefficients of the quadratic terms (t^2) are concentrated around 0.002. Thus, the trends in the μ_t parameter are dominated by the linear terms. The estimated coefficients for the 3rd degree polynomial specification governing the ϕ_t parameters of the negative binomial model that govern the arrival of transient customers to book rooms on one of the hotels in this market are similar: the linear terms are the largest. However, the main difference is in the estimated coefficients of the third and fourth powers of t which are rather different than the corresponding estimated coefficients for the μ_t parameters.

Table 9: Estimates of Choice Parameters (a_τ, b_τ)

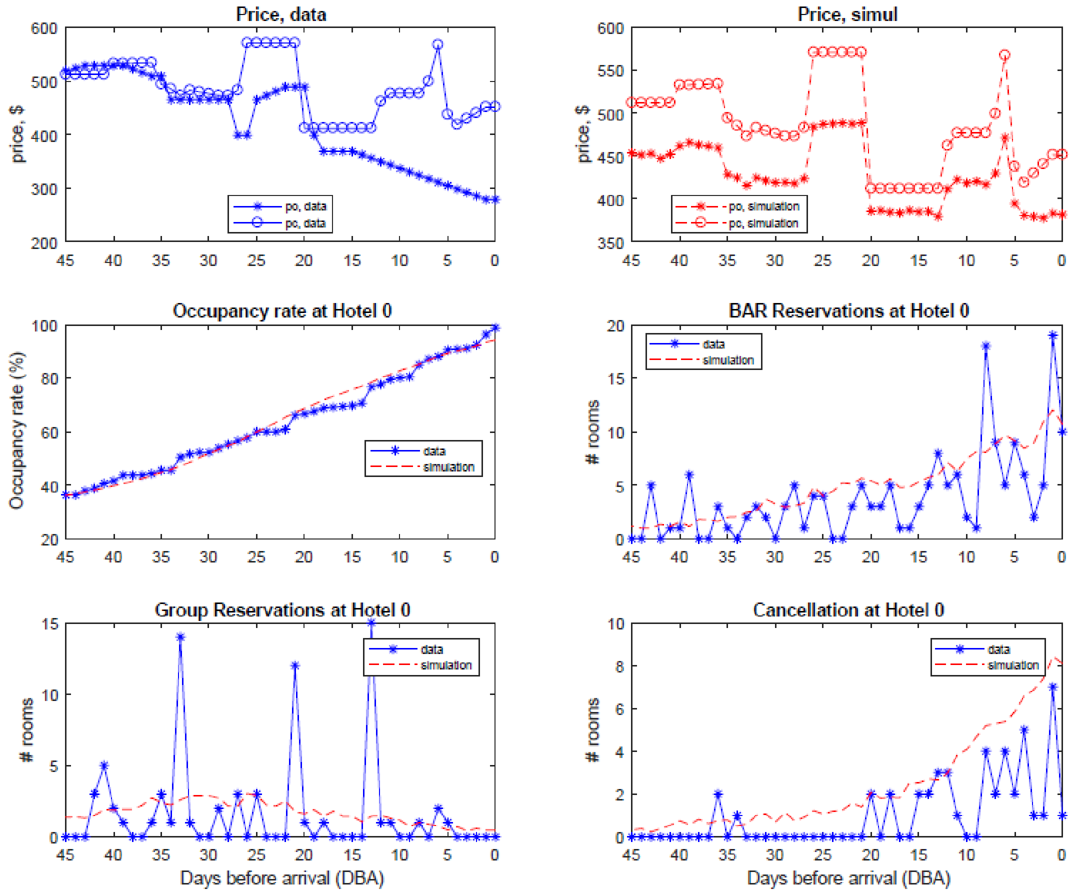
	Segment	Parameter	Lowest Demand (0-25%)	Medium-Low Demand (25-50%)	Medium-high Demand (50-75%)	Highest Demand (75-100%)
Weekday	Leisure	a_τ	-1.698 (0.384)	-1.546 (0.338)	-1.329 (0.174)	-2.300 (2.798)
		b_τ	-0.008 (0.001)	-0.007 (0.001)	-0.010 (0.001)	-0.074 (0.036)
	Business	a_τ	-1.618 (1.151)	-1.904 (0.150)	-1.047 (0.134)	-2.564 (0.742)
		b_τ	-0.006 (0.002)	-5.8E-3 (2.7E-4)	-0.006 (0.001)	-0.091 (0.091)
	Group	a_τ	-0.539 (1.115)	-0.935 (0.152)	-1.167 (0.360)	-1.370 (1.362)
		b_τ	-0.012 (0.005)	-0.011 (0.002)	-0.012 (0.002)	-0.094 (0.055)
Weekend	Leisure	a_τ	-1.580 (0.091)	-1.803 (0.328)	-0.296 (0.515)	-3.821 (15.325)
		b_τ	-0.008 (0.001)	-0.009 (0.002)	-0.035 (0.046)	-0.128 (0.980)
	Business	a_τ	-1.358 (0.149)	-1.262 (0.314)	-2.203 (2.480)	-3.874 (5.172)
		b_τ	-0.007 (0.001)	-0.007 (0.003)	-0.007 (0.010)	-0.076 (0.269)
	Group	a_τ	-0.813 (0.076)	-0.913 (0.217)	-0.002 (0.003)	-2.537 (4.421)
		b_τ	-0.012 (0.003)	-0.017 (0.002)	-0.015 (0.010)	-0.134 (0.194)

Note: standard errors in parentheses.

Tables 12 and 13 present the estimated coefficients of a 3rd degree polynomials for the coefficients of the cancellation probability $e_t(c|n)$ for weekdays and weekends, respectively. The parameters are fairly stable across subsamples and predict cancellation rates that are very close to the non-parametrically estimated values shown in figure 9. The fact that our estimated cancellation rates are not highly sensitive to demand conditions is consistent with our previous analysis where we showed that cancellation rates do not depend on the current BAR, so the hypothesis of exogenous cancellation rates seems to be a good approximation for this market.

We now turn graphical comparisons that illustrate how the estimated model is able to capture many of the features we see in the data. Figure 14 compares the predictions of the model and the actual outcomes for a *specific day* to show that the model does a good job of capturing dynamics on a day by day basis and not just on average. The specific day comes from sample 20 which is a weekday and the second highest demand quartile. The top left panel of Figure 14 plots the BAR of our hotel and the average BAR of competing hotels, while the top right panel shows the simulated optimal price of hotel 0 and its competitors. Both panels use the same average BAR of hotel 0's competitors. We see that the predicted optimal BAR for hotel 0 closely follows the BAR of the competitors, consistent with our "price following" finding in section 3. The middle left panel compares the simulated and actual occupancy trajectories on

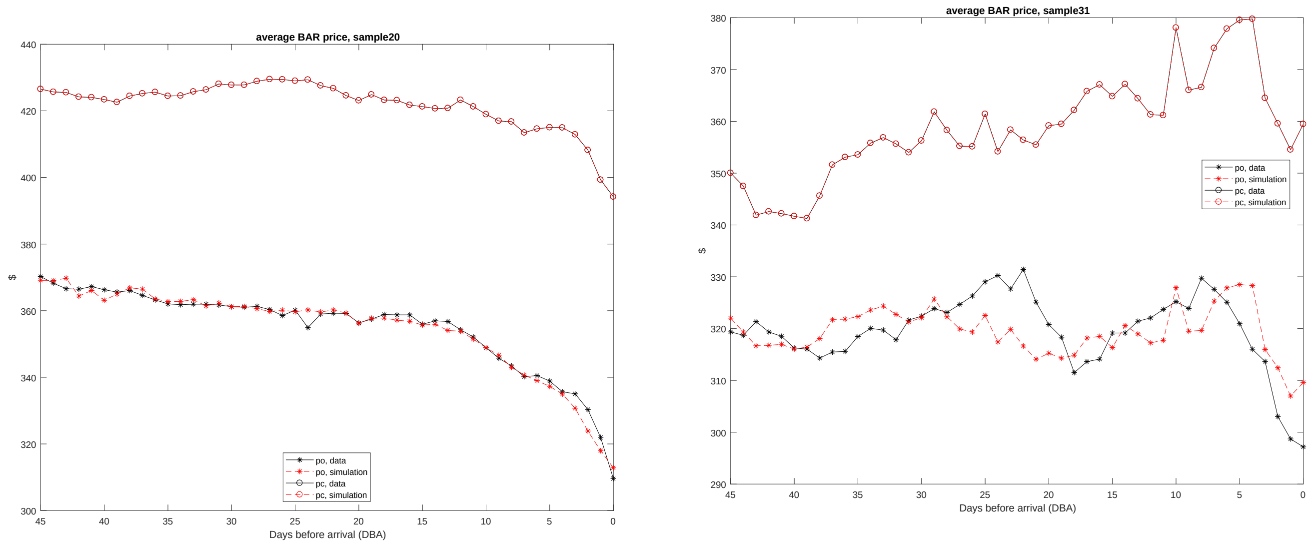
Figure 14: Simulated versus actual outcomes, day 41 of sample20



this particular day. The blue dashed line plots the simulated occupancy for hotel 0 while the star-solid line plots the actual occupancy trajectory on the same day. At 10 days before arrival, simulated occupancy (dashed-line) is slightly higher than actual. But as the arrival date approaches, actual occupancy increases slightly faster than simulated occupancy, so the model slightly underpredicts total occupancy on the arrival date. The higher actual occupancy is likely due to the fact that the DP model sets a higher BAR in the last few days prior to arrival than what hotel 0 actually set as seen in the top panels of figure 14.

The remaining panels of figure 14 compare simulated and actual daily bookings and cancellations. The middle right panel shows the trajectory of transient reservations from both leisure and business customers. The actual inflow marked with the star-dash line fluctuates quite a bit, so it is hard for the model to exactly track its daily movements. Despite this, the model is able to track actual realizations on specific days remarkably well. The bottom two panels of figure 14 plot the trajectories for group reservations and cancellations. Unlike the previous panels, we see some large discrepancies between the data and

Figure 15: BAR trajectories for busy weekday and busiest weekend



the simulation. Given the spikey and random nature of group reservations, we cannot expect simulations from our model to track actual outcomes as well. However the simulation errors in group and transient reservations and cancellations tend to average out in a way that leads the overall simulated path of bookings (top left panel) to track the actual path remarkably closely.

Figure 15 plots average simulated BAR trajectories for hotel 0 and its competitors in two separate samples: 20 (busy weekdays) and 31 (busiest weekends). Starting with the left hand panel, we see that the optimal BARs from the DP model track hotel 0's actual BARs very closely, including the overall downward trend in the BAR and an acceleration in price cutting that occurs in the last few days prior to arrival. Given our characterization of the optimal pricing rule in section 4, the price cutting by hotel 0 is a result of price following in response to price cuts by its competitors rather than unilateral price cuts.

The right hand panel of figure 15 plots the average BAR trajectories for the most busy weekends in the sample. In this case the average BAR trajectory is essentially flat as the arrival date approaches, except for price cutting in the last 5 days prior to arrival. For this sample the simulated BARs do not track actual BARs quite as closely but they still follow by overall pattern very well, including a downturn in prices in the week before arrival. Notice that in comparison to figure 15 prices on most busy weekend are lower on average than the prices on a busy (but not most busy) weekday. This is due to the greater price elasticity and lower arrival rate of leisure customers for weekends compared to business travelers on weekdays. The same pattern occurs in the competitors prices ρ_t which are marked as pc in the figures.

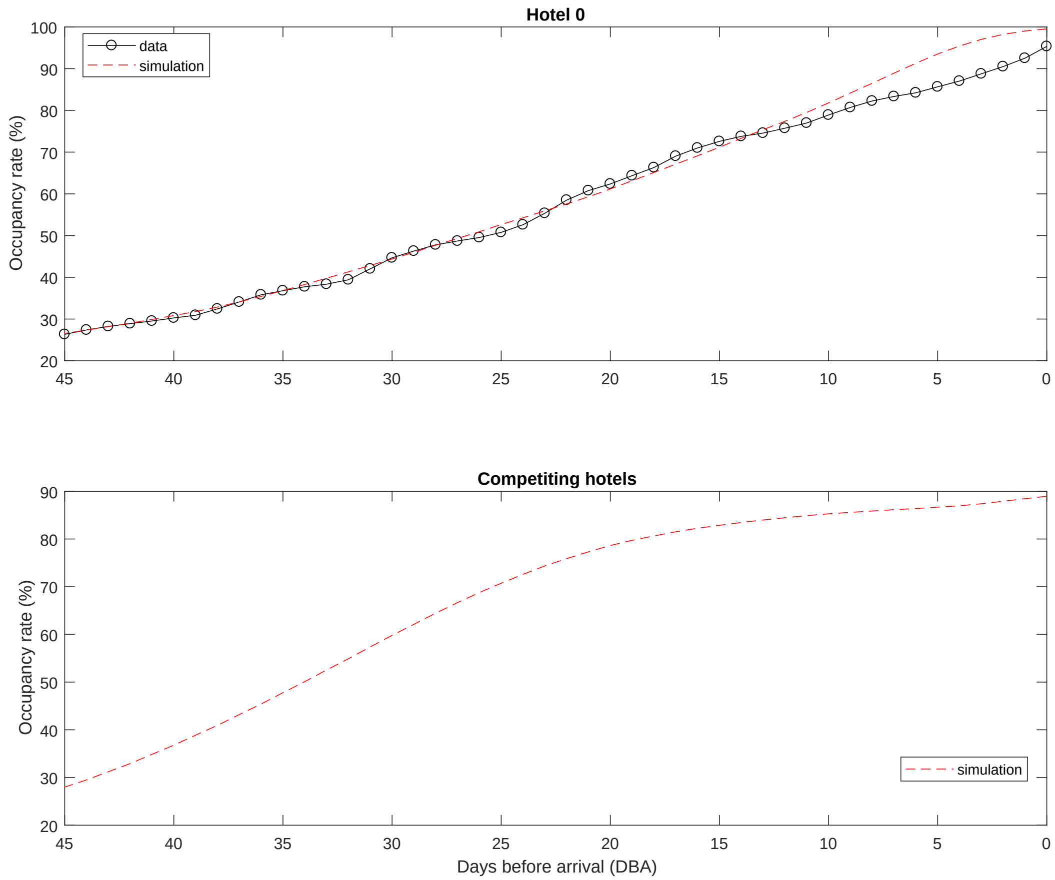
Figure 16 compares the average simulated booking trajectory from the estimated DP model to the actual average trajectory for sample 31, busiest weekends. Though we do not observe the booking trajectories of hotel 0's competitors, we are able to simulate them and we present the average simulated trajectory for the competitors as well. Note that our simulations indicate that the competitor hotels are not completely sold out by the arrival date, and since we do observe the occupancy rates at the competing hotels on each arrival date, we can confirm that our model does predict the final occupancy at the competing hotels accurately. Note that our model simulations predict that the competing hotels tend to book up faster compared to hotel 0. For example, 25 days away from arrival the model predicts that competing hotels are 70% booked whereas hotel 0 is only 50% booked. In figure 16 we see that the estimated DP model slightly overestimates occupancy rates at hotel 0 in the last few days prior to arrival, but generally provides a very close prediction of the actual booking trajectory.

Lastly, we show how the DP model is able to track ADRs and occupancy rates. Figure 17 compares the simulated ADRs for hotel 0 and its competitors to the actual realized ADRs over the interval from February 16, 2012 to May 26, 2012. The top panel compares simulated and actual ADRs for hotel 0 and the bottom panel does the same for the average ADRs of hotel 0's competitors. The simulated ADRs for hotel 0's competitors were generated from our estimated random walk model for $\{\rho_t\}$ given in equation (2). Generally the DP model tracks the weekly cycles in the ADRs at hotel 0 quite well, though it tends to overpredict the ADRs during weekdays, in some cases by significant amounts. This same pattern of prediction errors is also true of the more reduced-form AR(1) for the average ADRs of hotel 0's competitors.

Note that the ADR is generally different from the BAR due to various discounts provided to different types of customers and the stochastic variability in how many customers book at different BARs prior to the arrival date. Even if two customers book reservations on the same day for occupancy on the same date in the future, they may be eligible for different discounts. Also some hotels do not enforce a uniform price strategy that hotel 0 uses, and when hotels sell blocks of rooms to wholesalers, we can see quite a bit of dispersion in the BAR across different OTAs even for the same hotel on the same occupancy date. This high price dispersion is precisely why we see entry by meta sites such as Kayak.com and Trivago.

The additional price dispersion due to different discounts offered to different customers complicates our simulations: we need to draw from the distribution of possible discount rates applicable when simulating each booking. Since we cannot observe this distribution for hotel 0's competitors, we used average discount rates using ADRs provided by STR. We can observe the full distribution of discount rates using hotel 0's reservation database, since we see every customer and contract category and the discount rate

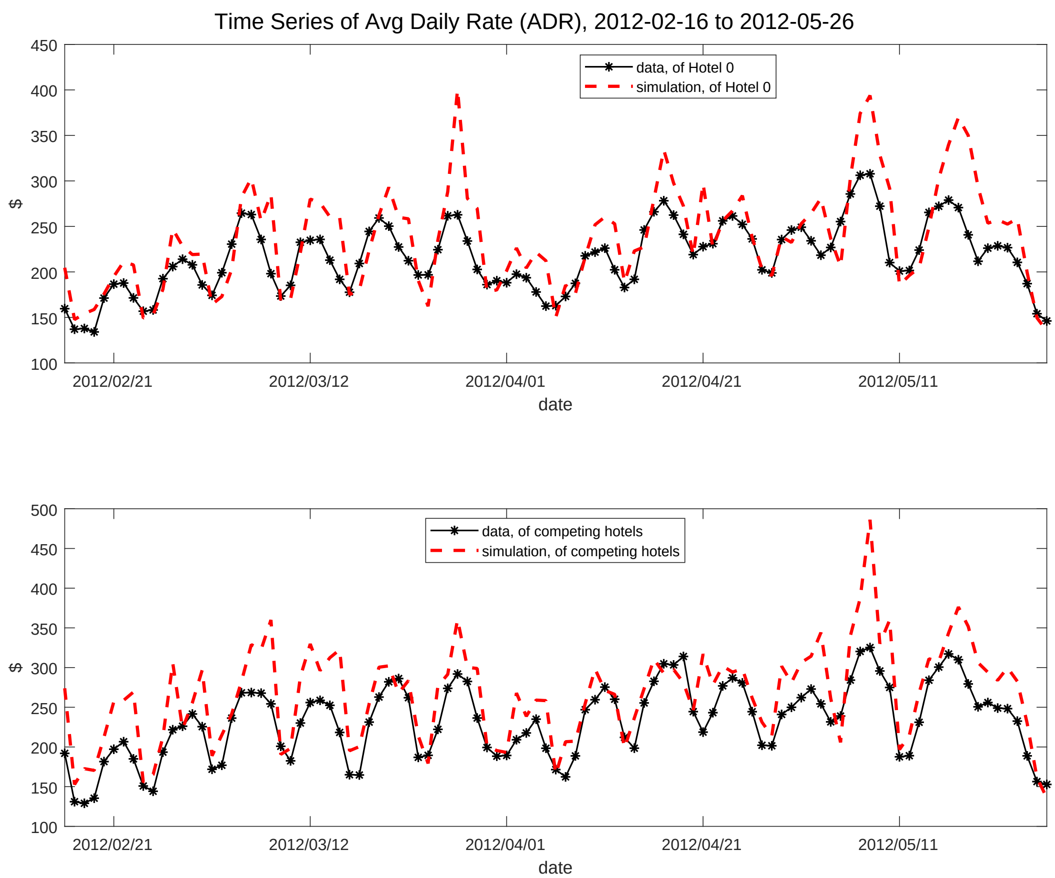
Figure 16: Occupancy trajectory for sample 31, busiest weekend



applicable to each. However in this version of the model we used the average discount rate for hotel 0's simulated bookings as well. Thus, some of the prediction errors for ADRs in figure 17 are likely due to our use of average discount rates to calculate estimated ADRs for specific days, especially weekdays. For example we do observe large group reservations at especially low discounted rates for events such as conferences that occur on weekdays, and when such events occur, the model's predicted ADR using an average discount rate will naturally overpredict the actual ADR.

Figure 18 plots the arrival day occupancy rates for hotel 0 and its competitors over the same period. For each arrival day, we used the reservation database information from hotel 0 to calculate its actual occupancy rate, whereas for hotel 0's competitors we obtained occupancy rates on each arrival day from the STR data. The top panel compares actual and simulated occupancy rates for hotel 0 and the bottom

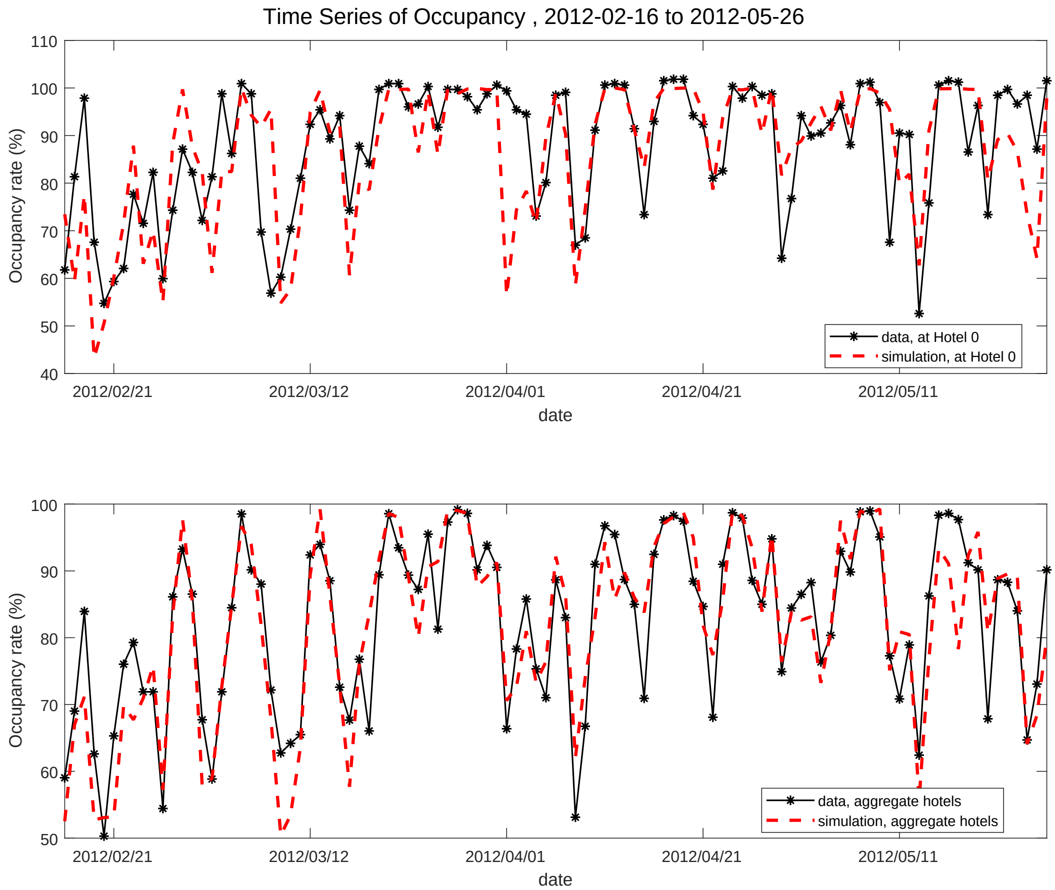
Figure 17: Time Series of Average Daily Rate (ADR)



panel does the same for hotel 0's competitors. We see that in general, the model is able to track both prices and occupancy rates in this market quite well.

We also observe occasional large differences between simulated and actual rates in figure 18. The prediction errors are even smaller for the competing hotels as we see in the bottom panel of figure 18. Overall the two time series appear to match each other rather closely with the exception of a few dates such as April 1, 2012 where the simulation produces an anomalously low final occupancy rate. The unusually high actual occupancy rate on this date may be due to a special group reservation at a deeply discounted rate. Indeed, we see an unusually high share of reservations of April 1, 2012 that were booked under codes 'RESO' and 'OPQ' which are special discounted group codes for travel agency bookings.

Figure 18: Time Series of Occupancy



4.5 Counter-factual experiments

We conducted several counterfactual price experiments to help judge whether the estimated stochastic demand model provides a good approximation to reality. We had an opportunity to present our findings to the revenue manager of hotel 0, and by in large, the revenue manager confirmed that the counterfactual predictions seem reasonable and are broadly in line with her own intuitive prediction of how these various changes to pricing would affect bookings, cancellations, occupancy, and overall profits. We presented three different experiments using the estimated model. In experiment 1 we fixed BAR at a specific, time-invariant value instead of adopting the optimal dynamic pricing strategy. In experiment 2 we instituted a 20% discount off the optimal BAR calculated by our DP model starting 15 days prior to arrival. In experiment 3 we increased the BAR by 20% over the optimal BAR starting 15 days ahead of arrival. Though the results below show a single simulated path, it is quite easy to simulate many paths and construct a distribution of outcomes given different realizations for customer arrivals and decisions. We will show the entire distributions of simulated outcomes below, but it is helpful to start by showing a single typical realized path for each of the three experiments described above.

Figures 19 and 20 compare simulated versus actual paths of occupancy under experiment 1, where we turn off dynamic pricing and exogenously fix the BAR to a single constant value for all booking days t prior to arrival. Both of these experiments were done for busy Thursdays in April, 2012. In each case, we compare the actual occupancy path to the mean and 95% confidence interval of paths simulated by our model of stochastic demand but conditioning on the actual path of BARs for hotel 0's competitors.

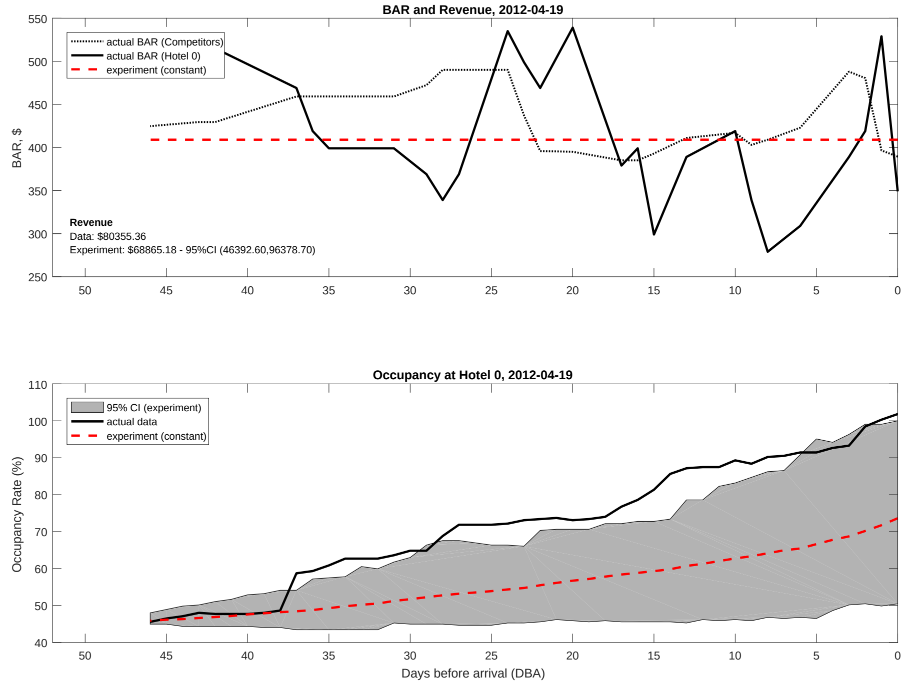
The top panel of figure 19 compares the counterfactual constant BAR of $p = 410$ (marked with a dashed line) with the actual BAR (solid line). The actual BAR trajectory ranges between \$280 and \$550. We see the effect of price cuts by hotel 0 around $t = 40$ days prior to arrival which appear to have resulted in a sharp increase in bookings about $t = 35$ days prior to arrival. Another price decrease between 30 and 25 days prior to arrival appear to have caused a jump in bookings over this same interval. However for the fixed price policy, bookings and overall occupancy increase at a much more steady pace, which seems intuitive given that prices are fixed in the counterfactual scenario.

Despite the fact that the fixed price of $p = 410$ is initially lower than the dynamically varying BAR that hotel 0 actually charged, the simulated mean occupancy rate is close to the actual occupancy rate until 38 days before arrival. But at that point the price cuts in the actual BAR appear to have stimulated a significant increase in bookings and subsequently the actual path of bookings lies above the upper 95% confidence interval of simulated booking paths under a fixed price policy. Actual revenues were \$80,355 on April 19, 2012 which is significantly higher than the mean revenues of \$68,865 that our model predicts hotel 0 would have earned had it maintained a constant price of $p = 410$.

Figure 20 provides another illustration of experiment 1, but for a different day: April 26, 2012, where we fixed the BAR at a lower price of $p = 360$ for all days t prior to arrival. Here again we see that the dynamically changing price path that hotel 0 actually chose resulted in higher final occupancy and revenues than the model predicts under a counterfactual suboptimal time-invariant BAR. We see that the lower actual price charged by hotel 0 at $t = 46$ days before arrival leads to initially higher bookings, but a price increase between $t = 40$ and $t = 35$ days before arrival dampened the growth in bookings, and actual prices are closer to the counterfactual flat price of $p = 360$ between $t = 35$ and $t = 25$ days before arrival, so the actual booking trajectory moves closer to the mean trajectory in our counterfactual simulations.

However at $t = 17$ days before arrival, the hotel cut its BAR to $p = 260$ and this lead to a large jump

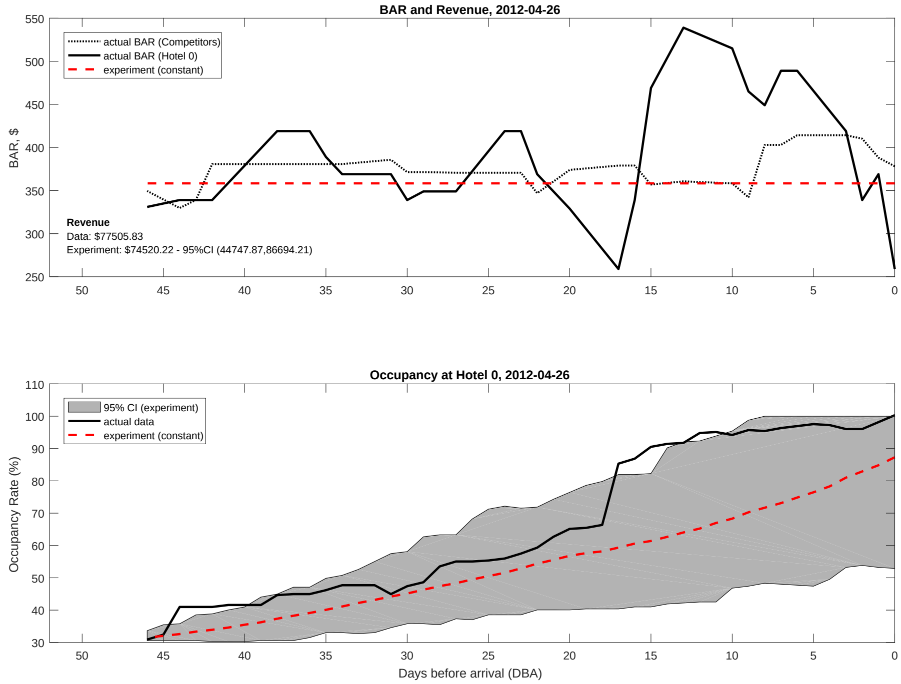
Figure 19: Experiment 1: constant BAR on April 19, 2012



in occupancy. After this large jump in its occupancy, hotel 0's actual bookings remains near the top of the 95% confidence interval of bookings under the flat price counterfactual. The jump in bookings due to its apparently strategically timed price cut at $t = 17$ days before arrival seems to have made hotel 0 more confident of a sell out and at $t = 14$ days before arrival it raised its BAR to over \$500, far above the average BAR of its competitors. Then in the remaining two weeks it more or less steadily cuts its BAR, to $p = 260$ on the final day, well below the average BAR of its competitors. The actual revenues earned by hotel 0 on this day were \$77,505, which is higher than the mean simulated revenues of \$74,520, though given the relatively high variability in simulated outcomes, actual revenues are not statistically significantly higher than mean simulated revenues in this case.

Though we caution not to read too much into individual simulations, they provide helpful illustrations of the dynamic process of price adjustment and how hotel 0 responds to occupancy and the prices set by its competitors, and illustrate the potential gains from the use of a dynamic pricing strategy. The remaining counterfactual experiments focus on evaluating the overall level of prices and are based on comparing simulations using the optimal pricing strategy from the estimated DP model and two counterfactual paths: one that is 20% higher than the optimal BAR and the other which is 20% lower. Though we simulated prices for hotel 0, we used the actual average BARs quoted by hotel 0's competitors leading up to the

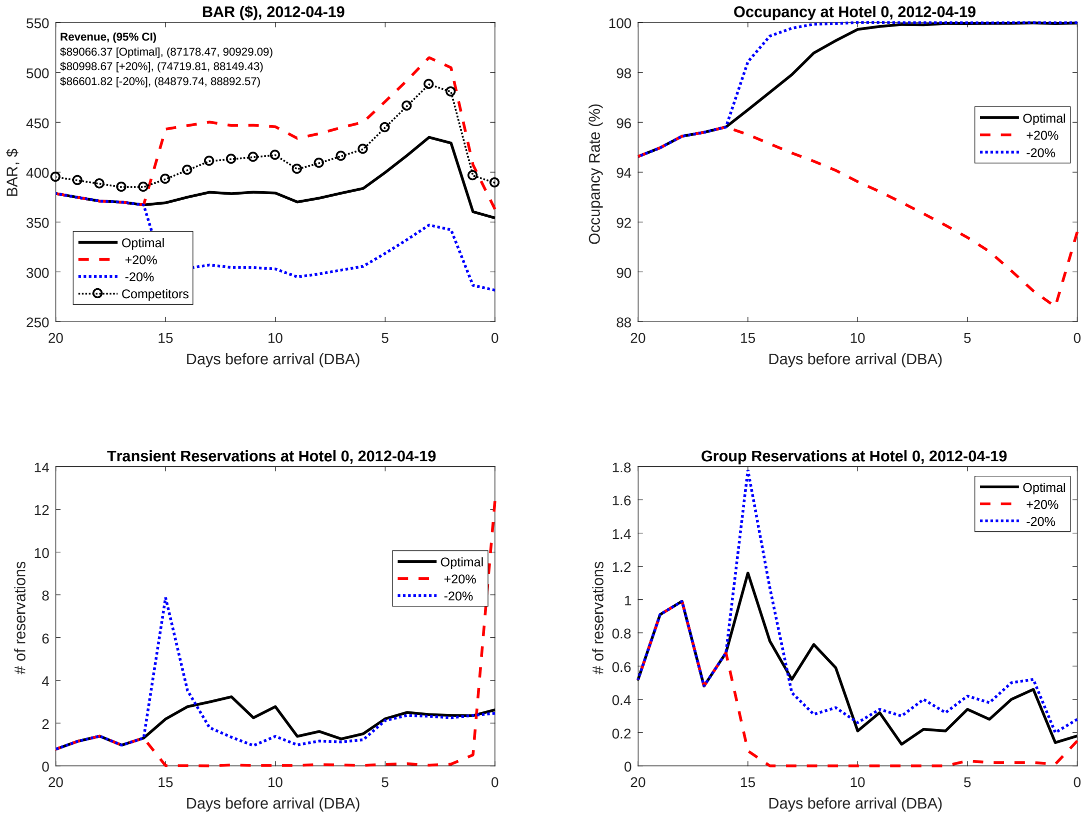
Figure 20: Experiment 1: constant BAR on April 26, 2012



same arrival date, April 19, 2012, that we illustrated in figure 19. We instituted the counterfactual price changes (relative to the optimal prices) starting at $t = 15$ days prior to arrival.

Figure 21 plots the results of experiments 2 and 3. We show the simulation starting at $t = 20$ days prior to arrival and the black curve in the top left hand panel of the figure plots the optimal BAR calculated by the DP model. Notice that it displays the "price following" property much more strongly than the actual BAR trajectory that hotel 0 chose that is illustrated in the top panel of figure 19. In particular, the dotted line in the latter figure plots the average BAR trajectory $\{\rho_t\}$ of hotel 0's competitors and as we see in figure 21 the optimal BAR trajectory from the DP model tracks the shape of $\{\rho_t\}$ relatively closely, and the optimal pricing rule does prescribe that hotel 0 price undercuts its competitors. Also note that the revenue earned under the optimal pricing rule is \$89,066, which is 11% higher than the actual revenue earned on April 19, 2012 of \$80,355. Though there is "simulation noise" in counterfactual simulation of optimal prices from the DP model (due to stochastic simulation of arrival rates of transient and group customers), we have conditioned on the same path of average BARs for hotel 0's competitors and this conditioning considerably reduces the variability in the counterfactual simulations compared to a scenario where we also simulate competitor BARs as well as stochastic arrival of customers. Later, we will show the entire counterfactual distribution of occupancy and revenues that factors in the effect of stochastic

Figure 21: Experiment examples of BAR addition/deduction



arrival of customers so you can judge the variability created by the stochastic arrival of customers, which does turn out to be substantial as we show below. However we think it is helpful for the understanding of the model to show a single simulation where we condition on the same BAR trajectory for hotel 0's competitors and overall number of arrivals of customers at each day t before arrival in order to "control" for these other factors and focus purely on the impact of the changes in prices on the outcomes.[14]

In figure 21 the counterfactual experiments start at $t = 15$ days before arrival. The dashed red line plots the counterfactual BAR for experiment 2, where we increase the BAR by 20% in the remaining two weeks prior to arrival. The blue dotted line plots the counterfactual BARs for experiment 3, where we decrease prices by 20% relative to the optimal values until arrival day. The remaining panels of figure 21 use the estimated demand model to simulate the implications of these counterfactual price trajectories on bookings, cancellations, occupancy and total revenues, controlling for prices of competitors' BARs and the stochastic arrival of customers as described above, so we can isolate the pure effect of the counterfactual

[14]We also conditioned on the same set of uniform random "seeds" that we used to simulate the choices of each customer who arrives each day and chooses to stay either at hotel 0 or one of its competitors. Thus, in some sense our computer experiment provides a type of controlled experiment that could never actually be done in reality.

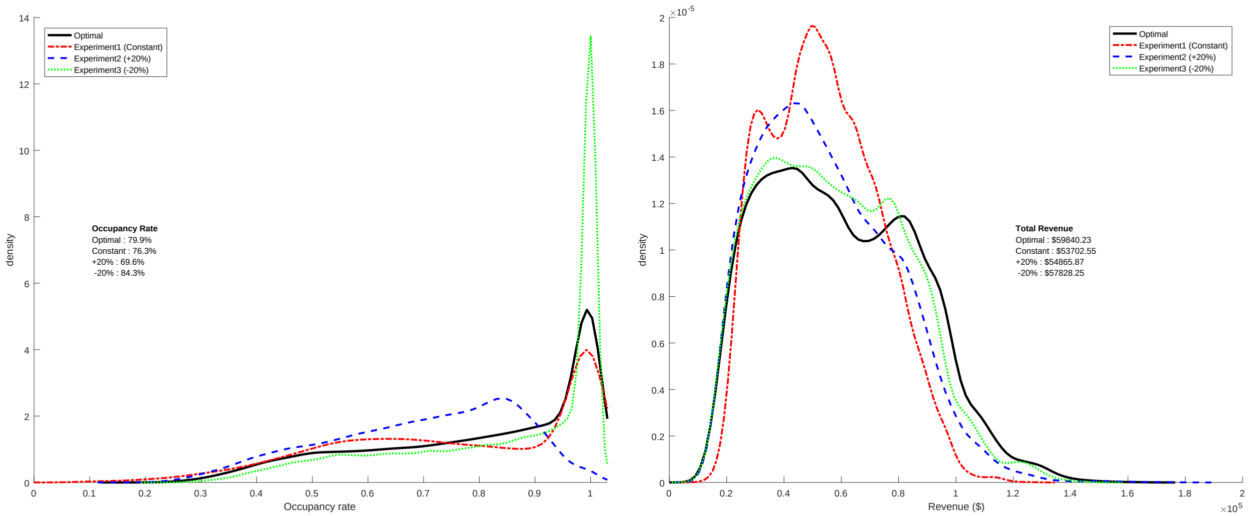
price changes on the outcomes. The top right hand panel of the figure shows the impact on occupancy rates. As we would expect, the 20% decrease in BAR leads to an immediate jump in both transient and group bookings shown by the blue dotted lines in the bottom two panels of the figure. The jump in bookings results in a rapid increase in occupancy rates, which reach 100% at $t = 12$ days before arrival, 7 days earlier compared to what happens under the optimal pricing strategy. Conversely the 20% price increase causes transient and group bookings to drop to nearly zero until just a few days prior before arrival, so occupancy rates decrease under this experiment until experiencing a slight rebound just a few days before arrival.

In sum, final occupancy rates on the arrival date, April 19, 2012, are predicted to be 100% under both the optimal pricing strategy and the 20% price decrease counterfactual, but are below 92% under the 20% price increase counterfactual. The price increase results in revenues of $80,998, or 9% lower than the revenues under the optimal pricing strategy. Revenues under the price decrease counterfactual are $86,601, or 3% lower than the revenues hotel 0 earns under the optimal pricing rule. Both of these counterfactual revenues are higher than revenues that the model predicts under a non-dynamic fixed BAR scenario, $68,865 as in figure 19.

Thus, this is a result of the property of an optimal dynamic pricing rule that we discussed in section 4.3, namely, that it is not optimal for hotel 0 to unilaterally cut its prices when it is close to selling out. However the counterfactual does show there is a danger in "overpricing" which drives a significant number of potential customers to hotel 0's competitors and results in significant loss of revenue compared to the optimal pricing rule. In summary, the change in the optimal price affects the new reservations and occupancy rate and the results of experiments seem quite reasonable and are consistent with our expectations. When we showed these predictions to the revenue manager at hotel 0, she also agreed that the counterfactual predictions seem very reasonable given her own experience and beliefs about how the hotel customers react to price changes.

Finally, figure 22 plots the distributions of occupancy and revenues for different realizations of stochastic shocks, but conditioning on the actual realized path of BARs for hotel 0's competitors as shown in dotted black line in the top left panel of figure 21. The left hand panel plots the distribution of occupancy rates on the arrival day implied by the optimal pricing strategy and the three counterfactual scenarios. We see huge variability in occupancy rates for different simulated arrivals of customers under each pricing scenario. For example under the optimal pricing strategy, the support of the distribution of occupancy rates is approximately $[.2, 1]$ with an expected value of .8. The mode of the distribution is at 100% occupancy but there is a long left tail corresponding to low occupancy rates on days when insufficiently many

Figure 22: Distribution of occupancy and revenue in the experiments



customers arrive to book rooms.

Occupancy rates are highest on average under the 20% price decrease (84%), and lowest under the 20% price increase scenario which is not surprising. However it is interesting to note that while ADRs are higher under the optimal pricing strategy compared to a constant price, occupancy under the optimal dynamic pricing strategy is actually *higher* — 80% versus 76%. This is an important indication of the efficiency gain to dynamic pricing.

Of course the hotel is interested in maximizing profit, not occupancy. The right hand panel of figure 22 plots the distribution of revenues under the optimal pricing rule and the three counterfactual pricing scenarios. The variability in arrival rates of customers generates considerable variance in realized revenues. However not surprisingly, expected revenues are highest under the optimal pricing rule and lower under the other three suboptimal pricing rules. Expected revenues are lowest under experiment 1, the scenario where the hotel does not adopt dynamic pricing. The failure to price according to an optimal dynamic pricing rule would result in a loss of 10% in expected revenues, a significant amount. The losses are lowest in experiment 3, i.e. the scenario where the hotel cuts its BAR 20% relative to the optimal values starting $t = 15$ before arrival. Expected revenues are nearly 4% below the optimal expected value under this scenario. However we see that there is a significant cost to “overpricing” — under experiment 2, the 20% price increase scenario, expected revenues would fall by nearly 9%.

5 Conclusion

This paper has introduced a dynamic programming (DP) model of optimal dynamic hotel pricing. We have shown how to infer the parameters of a stochastic model of hotel demand using the method of simulated moments (MSM), and shown that the estimator produces reasonable, downward sloping demand curves despite severe econometric problems such as censoring (i.e. inability to observe the total number of customers booking rooms on any given date) and endogeneity (which results in a strong positive correlation between hotel prices and occupancy). Thus, our structural MSM estimator enables us to make accurate and valid inferences about demand, in a situation where traditional instrumental variables methods do not apply. Instead, our estimation approach imposes a behavioral assumption — expected profit maximization — which imposes strong restrictions that link the hotel's beliefs about the stochastic process generating demand for hotel rooms to the prices it sets which we can observe. In essence, our structural estimation method can be viewed as a method of inverting observed pricing decisions to uncover the "revealed beliefs" of the hotel about the rate of arrival of customers wishing to book rooms and their preferences for the competing hotels in this market.

We estimated the model using reservation data from an actual hotel in major US city and showed it provides a remarkably good approximation to the hotel's actual pricing behavior. In particular, our model provides considerable insight into the apparent "price following" and "price undercutting" strategy that hotel 0 uses. The strong co-movement of prices of the seven hotels in this local market has the superficial appearance of tacit collusion that could be sustained by commercial price shopping services that inform the hotels of each others' prices in real time, combined with the hotels' use of revenue management systems (RMS) that provide "recommended prices." However our analysis leads us to a much more benign conclusion. We find that hotel 0's price setting behavior is competitive and is well approximated as a best response to the dynamic pricing of its competitors. The strong co-movement in prices in this market is entirely consistent Bertrand price competition among all of the hotels, which are subject to aggregate demand shocks that cause their occupancy rates and prices to move together. We showed that when a hotel expects to sell out, it is optimal for it to increase its price to ration scarce capacity. This can lead a hotel to increase its price far above the price of its competitors, even though under more typical scenarios where the hotel is far from selling out, it is optimal for the hotel to price below its competitors in a manner that closely approximates the "price following" behavior that we showed provides a very good approximation to the way hotel 0 actually sets its prices.

Though there may be legitimate concern about the possibility that machine learning algorithms could learn to collude via repeated interactions in real world markets as suggested by Harrington (2017) and

Ezrachi and Stucke (2016), these concerns seem a long way off for the particular hotel market we studied. Our own DP model can itself be regarded as a prototype RMS, and our MSM estimation method can be regarded as a type of machine learning. However any model or algorithm can only be as good as the information it is based on. The key information for an effective RMS is having an accurate model on customer demand. It seems quite doubtful to us that machine learning algorithms would be capable of learning about the nature of customer demand on their own in a vacuum, much less learn how to collude especially if different hotels are using different RMSs. Instead, we think the more interesting and relevant question is: how do commercial RMSs learn about customer demand given that they are serving so many thousands of distinct hotel markets simultaneously? How do these systems solve the highly challenging econometric problems we encountered that prevented us from using traditional demand estimation methods (such as instrumental variables) for estimating consumer demand in these markets?

We have been able to solve these problems, but only because we have both highly detailed data and the ability to observe the actual hotel pricing decisions set by the hotel's existing RMS (as occasionally overridden by the human revenue manager) which we have used to "train" our RMS. The downside is that in doing this, we needed to assume that the hotel's RMS sets prices optimally. But if this is truly the case, then the hotel has no need for our RMS! We are currently investigating whether it is possible to relax the assumption of expected profit maximization. We hope to demonstrate that it is possible to infer customer demand even if the hotel is not setting optimal prices. This estimator relies on non-parametric estimation of the hotel's price-setting rule which would then be used in a semi-parametric two-step version of our MSM estimator, where we use the non-parametrically estimated pricing rule in place of the optimal pricing rule from the solution to the firm's DP problem. If it is possible to infer demand via this approach, we can use the price setting behavior of humans or by machines (e.g. RMS) to learn about demand without relying on the assumption that either is behaving optimally. Given good estimates of consumer demand, we can then employ our DP algorithm to provide optimal pricing and use field experiments to demonstrate and validate (via scientific methods rather than marketing hype) that such an approach can help firms improve their profitability, and thus outperform existing commercial RMS, as complemented by the oversight of expert human revenue managers. But for the foreseeable future our advice to hotel 0 is: keep your RMS and your human revenue manager. They appear to work very well together and are probably worth every penny it pays for them.

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6 Appendix

6.1 Appendix 1: Estimated parameters and standard errors

Table 10: Estimates of ZINB Parameters (γ,ϕ,μ), Weekdays

	Parameter	Polynomial Coefficient	Lowest Demand (0-25%)	Medium-Low Demand (25-50%)	Medium-high Demand (50-75%)	Highest Demand (75-100%)
	μ_t,	t^3	-2.6E-5 (1.5E-5)	1.3E-5 (7E-6)	-6E-5 (2.9E-5)	-2.5E-5 (5.4E-6)
	Negative	t^2	0.002 (0.002)	0.001 (0.001)	0.002 (0.001)	0.002 (0.001)
	Binomial	t	-0.111 (0.050)	-0.110 (0.033)	-0.093 (0.031)	-0.088 (0.033)
	(Leisure)	1	3.677 (2.931)	3.639 (1.269)	3.397 (0.563)	3.372 (0.579)
	ϕ_t,	t^3	-2.8E-5 (2.5E-5)	-4.4E-5 (3.3E-5)	-3.4E-5 (1.5E-4)	-2.6E-5 (2.2E-5)
	Negative	t^2	0.001 (5.5E-5)	0.002 (0.001)	0.003 (0.002)	0.002 (0.001)
	Binomial	t	-0.060 (0.099)	-0.108 (0.025)	-0.013 (0.014)	-0.038 (0.019)
	(Leisure)	1	0.548 (1.181)	0.291 (0.204)	0.420 (0.540)	-0.093 (0.081)
	μ_t,	t^3	3.9E-5 (4.8E-6)	2.6E-5 (3.3E-5)	7.7E-6 (1.3E-6)	1.82E-5 (1.6E-5)
	Negative	t^2	-0.003 (0.001)	-0.002 (0.001)	-0.002 (3E-4)	-0.003 (0.003)
	Binomial	t	0.048 (0.015)	0.029 (0.022)	0.034 (0.004)	0.036 (0.063)
Weekday	(Business)	1	0.933 (1.310)	2.749 (0.554)	2.246 (0.639)	3.003 (2.184)
	ϕ_t,	t^3	3.7E-5 (7.5E-6)	5.8E-6 (1.4E-5)	2.2E-5 (5.8E-6)	-2.7E-5 (6.6E-6)
	Negative	t^2	-0.003 (0.001)	-8.4E-4 (2.2E-4)	-1.8E-3 (4.3E-4)	1.2E-3 (4.6E-4)
	Binomial	t	0.012 (0.012)	-0.008 (0.010)	0.009 (0.001)	-0.022 (0.011)
	(Business)	1	-1.096 (1.630)	-0.677 (0.541)	-0.378 (0.049)	-0.648 (0.338)
	γ_t,	t^3	1.4E-18 (5.5E-18)	9.8E-19 (2.1E-18)	3.58E-18 (1.06E-17)	1.3E-18 (6.29E-18)
	Zero	t^2	-1.08E-16 (2.2E-16)	-2.11E-17 (7.41E-17)	-1.49E-17 (2.59E-17)	-4.32E-17 (1.41E-16)
	Inflation	t	1.32E-15 (6.4E-15)	8.97E-16 (2.33E-15)	8.97E-16 (9.50E-16)	7.42E-16 (7.75E-16)
	(Leisure)	1	18.796 (55.835)	26.195 (79.966)	32.996 (76.012)	19.045 (22.840)
	γ_t,	t^3	1.11E-18 (2.2E-18)	1.19E-18 (1.28E-18)	1.55E-18 (4.09E-18)	7.56E-19 (3.48E-18)
	Zero	t^2	-8.03E-17 (8.17E-17)	-4.67E-17 (1.79E-16)	-6.71E-17 (1.23E-16)	-4.75E-17 (1.83E-16)
	Inflation	t	2.5E-15 (5.81E-15)	1.07E-15 (3.57E-15)	7.72E-16 (1.75E-15)	7.13E-16 (1.99E-15)
	(Business)	1	21.427 (126.721)	25.461 (33.661)	15.283 (30.505)	20.292 (13.631)

Note : t denotes number of days before occupancy. Standard errors in parentheses.

6.2 Appendix 2: Proof of Theorems

6.2.1 Review of Model Setup

We use " $\sim$ " sign to indicate random variable. There are S segments in total, indexed by s, $s \in \mathcal{S} = \{1,2,...,S\}$. t days before arrival, state variables are $(n_t, \bar{p}_t, \rho_t)$. A total of $\tilde{k}_{st}$ customers of segment s enter the market. $\tilde{k}_{st}$'s are exogenous and independent processes across s and t, $\tilde{k}_{st} \sim f_{st}(k)$. Observing spot BAR prices (p_t, ρ_t) and relevant discount rate δ_s, customers decide which hotel to stay and send out their reservation requests. The number of reservation requests received by Hotel 0 from segment s is $\tilde{n}^d_{st}$. It follows a distribution which is a mix of $f_{st}(k)$ and binomial distribution,

$$\tilde{n}^d_{st} \sim l_{st}(n|p_t, \rho_t) \tag{21}$$

Table 11: Estimates of ZINB Parameters(γ, ϕ, μ), Weekends

	Parameter	Polynomial Coefficient	Lowest Demand (0-25%)	Medium-Low Demand (25-50%)	Medium-high Demand (50-75%)	Highest Demand (75-100%)
Weekend	μ_t, Negative Binomial (Leisure)	t^3	-3.15E-5 (1.56E-5)	-2.13E-5 (1.71E-5)	-2.39E-5(3.35E-5)	-4.34E-5(3.5E-4)
		t^2	2.5E-3 (2.7E-4)	0.002 (5.3E-4)	0.002 (0.003)	2.2E-3 (3.6E-3)
		t	-0.090 (0.022)	-0.107 (0.044)	-0.110 (0.098)	-0.146 (0.479)
		1	3.144 (0.547)	4.094 (0.926)	2.364 (2.501)	3.832 (6.629)
	ϕ_t, Negative Binomial (Leisure)	t^3	1.02E-5 (7.6E-6)	-3.41E-7 (1.26E-6)	-2.99E-5 (5.83E-5)	2.39E-5(2.59E-4)
		t^2	-4.4E-4 (2.9E-4)	3.63E-4 (1.83E-4)	2.8E-3 (3.1E-3)	-1.7E-3 (7.5E-3)
		t	-0.032 (0.029)	-0.010 (0.022)	-0.108 (0.086)	0.006 (0.020)
		1	0.686 (0.352)	0.098 (0.119)	0.733 (1.374)	0.191 (0.937)
	μ_t, Negative Binomial (Business)	t^3	2.65E-5 (1.3E-5)	3.81E-5 (7.61E-6)	2.79E-5 (3.59E-5)	8.74E-6 (6.68E-5)
		t^2	-0.002 (3.4E-4)	-0.003 (0.001)	-0.002 (0.003)	-1.2E-3 (0.010)
		t	0.022 (0.003)	0.071 (0.003)	0.046 (0.020)	0.010 (0.190)
		1	2.021 (0.830)	1.561 (0.484)	2.313 (3.803)	1.661 (9.124)
	ϕ_t, Negative Binomial (Business)	t^3	2.19E-5 (9.23E-6)	2.22E-5(6.02E-6)	-1.74E-6 (4.22E-6)	-6.24E-6 (7.78E-5)
		t^2	-1.3E-3 (3.8E-4)	-1.4E-3 (1.1E-4)	2E-4 (4E-4)	2.8E-4 (1.8E-3)
		t	-0.009 (0.003)	-0.008 (0.004)	-0.037 (0.038)	-0.026 (0.116)
		1	-1.487 (0.408)	-0.871 (0.460)	-1.002 (2.426)	-0.744 (3.426)
	γ_t, Zero Inflation (Leisure)	t^3	7.07E-19 (1.13E-18)	6.91E-19 (2.66E-18)	9.11E-19 (6.61E-18)	8.16E-19 (1.53E-17)
		t^2	-4.16E-17 (1.54E-16)	-4.33E-17 (6.37E-17)	-1.79E-17 (1.12E-16)	-5.11E-17 (8.56E-15)
		t	1.15E-15 (3.92E-15)	1.03E-15 (1.49E-15)	9.74E-16 (1.16E-15)	7.84E-16 (1.74E-13)
		1	19.701 (25.148)	28.028 (57.937)	17.821 (101.211)	21.158 (153.839)
	γ_t, Zero Inflation (Business)	t^3	8.90E-19 (2.17E-18)	1.2E-18 (3.3E-18)	6.97E-19 (2.41E-18)	8.04E-19 (1.36E-16)
		t^2	-5.35E-17 (1.20E-16)	-5.20E-17 (1.21E-16)	-3.35E-17 (8.07E-17)	-5.71E-17 (1.26E-14)
		t	7.52E-16 (8.58E-16)	9.18E-16 (6.99E-16)	7.5E-16 (1.99E-15)	7.96E-16 (7.71E-14)
		1	23.215 (89.462)	16.588 (26.847)	26.186 (44.460)	21.073 (403.918)

Note : t denotes number of days before occupancy. Standard errors in parentheses.

(p_t, ρ_t) enter into (21) since they governs the underlying binomial distribution. δ_s is omitted here since it is not a choice variable or a state variable, but is predetermined. [15]

The number of requests approved by the manager, in another word, the number of new reservations is $\tilde{n}_{st}$. We use a deterministic mapping $\eta(.)$ to capture the manager's approval procedure, i.e.

$$(\tilde{n}_{1t}, \tilde{n}_{2t}, ..., \tilde{n}_{St}) = \eta(\tilde{n}^d_{1t}, \tilde{n}^d_{2t}, ..., \tilde{n}^d_{St}, n_t, \tilde{c}_t) \tag{22}$$

In particular, $\eta(.)$ can accommodate "no overbooking" assumption, where approvals depend on available capacity.

Let $\tilde{c}_t$ be the number of cancellations (all segments combined) taking place in Hotel 0 (we assume it is realized before the manager apply $\eta(.)$). It follows a binomial distribution characterized by state variable n_t and a uniform cancellation rate α_t. The cancellation rate could potentially depend on $(p_t, \bar{p}_t, \rho_t)$, when customers make strategic cancellations. Thus in general,

$$\tilde{c}_t \sim e_t(c|n_t, \bar{p}_t, \rho_t, p_t) \tag{23}$$

$$\mathbb{E}_t(\tilde{c}_t|n_t, \bar{p}_t, \rho_t, p_t) = \alpha_t(p_t, \bar{p}_t, \rho_t) \cdot n_t \tag{24}$$

[15] We may allow for multiple unobserved types within segment. Unobserved types differ in price elasticity. So the static choice probability in binomial distribution is the average across types weighted by type fractions. As long as type fractions are exogenous, process of $\tilde{n}^d_{st}$ only depends on (p_t, ρ_t).

Table 12: Estimates of Other Parameters, Weekdays

	Parameter	Polynomial Coefficient	Lowest Demand (0-25%)	Medium-Low Demand (25-50%)	Medium-high Demand (50-75%)	Highest Demand (75-100%)
Weekday	γ_t, Zero Inflation (Group)	t^3	-6.36E-5 (5.52E-5)	-1.15E-4 (5.56E-5)	-1.1E-4 (2.51E-4)	-9.37E-5 (4.07E-5)
		t^2	3.06E-3 (2.41E-3)	0.006 (0.002)	0.008 (0.003)	1.15E-2 (1.06E-2)
		t	-0.019 (0.010)	-0.054 (0.018)	-0.027 (0.037)	-0.048 (0.054)
		1	-0.971 (0.960)	0.402 (0.401)	0.636 (0.333)	-0.714 (0.632)
	Mean Arrivals (if arrival >0) (Group)	t^3	-8.57E-5 (2.35E-5)	4.71E-6 (6.86E-6)	-2.03E-5 (5.53E-5)	2.91E-5 (2.04E-5)
		t^2	2.82E-3 (3.18E-4)	-3.60E-3 (9.7E-4)	-5.04E-4 (1.42E-3)	-5.78E-3 (1.27E-3)
		t	0.058 (0.037)	0.151 (0.042)	0.076 (0.004)	0.201 (0.067)
		1	0.457 (2.140)	0.304 (1.511)	0.792 (0.166)	0.423 (0.513)
	Probability of cancel >0	t^3	2.41E-5 (2.09E-5)	3.83E-5 (4.68E-6)	4.67E-5 (1.4E-5)	8.98E-5 (2.54E-5)
		t^2	-0.003 (0.001)	-3.57E-3 (8.44E-4)	-4.02E-3 (0.92E-3)	-7.19E-3 (2.13E-3)
		t	0.147 (0.031)	0.157 (0.019)	0.175 (0.021)	0.240 (0.055)
		1	-1.028 (0.073)	-1.472 (0.153)	-1.989 (0.201)	-2.522 (0.418)
	Cancellation Rate (if cancel>0)	t^3	3.83E-5 (1.47E-5)	1.65E-5 (6.07E-6)	-2.08E-5 (3.96E-6)	1.92E-5 (5.25E-6)
		t^2	-0.003 (0.001)	-1.47E-3 (1.48E-4)	4.11E-4 (1.41E-4)	-2.15E-3 (6.79E-4)
		t	0.060 (0.025)	0.061 (0.012)	0.025 (0.011)	0.047 (0.024)
		1	3.884 (3.923)	3.206 (1.025)	3.432 (1.042)	4.043 (0.729)

Note : t denotes number of days before occupancy. Standard errors in parentheses.

Combining (21), (22) and (23), we have joint distribution L_t

$$(\tilde{n}_{1t}, \tilde{n}_{2t}, ..., \tilde{n}_{St}) \sim L_t(n_1, n_2, ..., n_S | n_t, \bar{p}_t, \rho_t, p_t) \tag{25}$$

Variable dependency is important for later discussion. Note that $(\tilde{n}^d_{1t}, ..., \tilde{n}^d_{2t})$ doesn't depend on $(n_t, \bar{p}_t)$ according to (21). The intuition is that new requests are static decisions by randomly arriving new customers wishing to book a room, and make their choices based on the curent prices but not historical prices nor how many rooms are already occupied. However, the distribution of $(\tilde{n}_{1t}, ..., \tilde{n}_{2t})$, L_t, does depend on $(n_t, \bar{p}_t)$. This is because residual capacity, $\bar{N} + \tilde{c}_t - n_t$, matters when the realization of $(\tilde{n}^d_{1t}, ..., \tilde{n}^d_{2t})$ is high and capacity constraint is binding. This binding case has positive probability, since $\tilde{n}^d_{st}$ has negative binomial component and its outcomes are not bounded. So with positive probability, $\tilde{c}_t$ and n_t kick in. Consequently, $\bar{p}_t$ enters into L_t through (and only through) the cancellation channel as in (23). The law of motion for n_t and $\bar{p}_t$ is

$$n_{t-1} = n_t - c_t + \sum_s n_{st} \equiv n'(n_t, c_t, n_{1t}, ..., n_{St}) \tag{26}$$

$$\bar{p}_{t-1} = \frac{(n_t - c_t)\bar{p}_t + p_t \sum_s \delta_s n_{st}}{n_{t-1}} \equiv \lambda(n_t, \bar{p}_t, p_t, c_t, n_{1t}, ..., n_{St}) \tag{27}$$

Next period revenue-on-the-book and profit-on-the-book

$$n_{t-1}\bar{p}_{t-1} = (n_t - c_t)\bar{p}_t + p_t \sum_s \delta_s n_{st} \tag{28}$$

$$n_{t-1}(\bar{p}_{t-1} - \omega) = (n_t - c_t)(\bar{p}_t - \omega) + \sum_s (\delta_s p_t - \omega) n_{st} \tag{29}$$

With all notations above, the Bellman equation is

Table 13: Estimates of Other Parameters, Weekends

	Parameter	Polynomial Coefficient	Lowest Demand (0-25%)	Medium-Low Demand (25-50%)	Medium-high Demand (50-75%)	Highest Demand (75-100%)
Weekend	γ_t, Zero Inflation (Group)	t^3	-9.29E-5 (5.5E-5)	-1.31E-4 (4.16E-5)	1.12E-4 (5.87E-5)	-1.06E-4 (1.98E-4)
		t^2	5.28E-3 (2.74E-3)	0.007 (0.001)	0.007 (0.004)	0.008 (0.002)
		t	-0.069 (0.032)	-0.057 (0.063)	-0.080 (0.070)	-0.106 (0.172)
		1	0.158 (0.296)	-0.661 (1.238)	-0.281 (0.261)	-0.991 (3.938)
	Mean Arrivals (if arrival >0) (Group)	t^3	-8.34E-5 (4.98E-5)	-5.22E-5 (1.42E-5)	-1.79E-5 (1.59E-5)	3.87E-5 (3.89E-5)
		t^2	3.09E-3 (1.87E-3)	8.92E-4 (1.34E-4)	-0.75E-3 (1.21E-3)	-0.006 (0.010)
		t	4.12E-3 (5.11E-3)	0.058 (0.065)	0.099 (0.028)	0.228 (0.427)
		1	0.707 (3.132)	0.549 (0.427)	0.444 (0.428)	0.035 (0.031)
	Probability of cancel >0	t^3	8.78E-5 (1.49E-5)	4.16E-5 (1.89E-5)	6.58E-5 (1.43E-4)	7.96E-5 (1.71E-4)
		t^2	-7.42E-3 (1.27E-3)	-5.46E-3 (0.9E-3)	-6.62E-3 (0.011)	-0.008 (0.012)
		t	0.227 (0.018)	0.239 (0.016)	0.253 (0.243)	0.276 (0.204)
		1	-1.322 (0.116)	-2.162 (0.103)	-2.465 (1.506)	-2.757 (1.293)
	Cancellation Rate (if cancel >0)	t^3	2.48E-5 (2.03E-5)	1.79E-5 (9.17E-6)	1.1E-5 (1.8E-5)	3.44E-5 (2.04E-4)
		t^2	-2.78E-3 (0.77E-3)	-1.69E-3 (0.69E-3)	-0.003 (0.006)	-0.003 (0.014)
		t	0.060 (0.008)	0.050 (0.011)	0.065 (0.045)	0.077 (0.768)
		1	4.397 (0.299)	3.840 (1.125)	4.674 (4.595)	4.548 (2.403)

Note : t denotes number of days before occupancy. Standard errors in parentheses.

$$\begin{aligned} V_t(n,\bar{p},\rho) = &\max_p \left\{ \int_{\rho'}\int_{n_1}\cdots\int_{n_S}\int_{c} \Big[V_{t-1}(n',\bar{p}',\rho')\cdot \right. \\ &\left. e_t(c|n,\bar{p},\rho,p)\cdot L_t(n_1,\ldots,n_S|n,\bar{p},\rho,p)\cdot h_t(\rho'|\rho)\Big]\right\} \\ \equiv &\max_p \left\{ \mathbb{E}_t\left[V_{t-1}(n',\bar{p}',\rho') \middle| n,\bar{p},\rho,p\right]\right\}, \\ &\forall\, 0\le t\le T \end{aligned} \tag{30}$$

At $t=-1$,

$$V_{-1}(n,\bar{p},\rho) = (\bar{p}-\omega)\cdot n \tag{31}$$

6.2.2 Theorem and Proof

Theorem 1. *For each $t\in\{0,\ldots,T\}$ the value function V_t has the representation*

$$V_t(n,\overline{p},\rho) = V_t^f(n,\overline{p},\rho) + V_t^b(n,\overline{p},\rho) \tag{32}$$

where V_t^f is the "forward looking component" that equals the expected profits from rooms that are not yet booked, whereas V_t^b is the "backward looking component" that equals expected profits from rooms that are already booked.

Proof. We will first prove by induction that V_t has the additive decomposition. Then we will show the components V_t^f and V_t^b have the desired interpretation. Before the main argument, let's denote $I_t \equiv \{n_t, \bar{p}_t, \rho_t, p_t\}$, $I_t^* \equiv \{n_t, \bar{p}_t, \rho_t, p_t^*(n_t, \bar{p}_t, \rho_t)\}$. I_t gathers all the information on the morning of day t after the manager sets arbitrary BAR price p_t, but before any random process (e.g. reservation requests, cancellations, etc) realizes. I_t^* is a similar information set, except for that the manager sets the optimal price p_t^* with respect to the state $(n_t, \bar{p}_t, \rho_t)$. We will drop the time subscript wherever the context is clear.

Step 1: Prove the theorem holds for $t = 0$.
At $t = 0$,

$$
\begin{aligned}
&\mathbb{E}_0\left[V_{-1}(n', \bar{p}', \rho')\middle| n, \bar{p}, \rho, p\right] \\
=&\mathbb{E}_0\left[(n - c \cdot (\bar{p} - \omega)\middle| I_0\right] + \mathbb{E}_0\left[\sum_s (\delta_s p - \omega) \cdot n_s \middle| I_0\right] \\
=&(\bar{p} - \omega) \cdot n \cdot \mathbb{E}_0\left[1 - c/n\middle| I_0\right] + \mathbb{E}_0\left[\sum_s (\delta_s p - \omega) \cdot n_s \middle| I_0\right] \\
=&(\bar{p} - \omega) \cdot n \cdot [1 - \alpha_0(p, \bar{p}, \rho)] + \mathbb{E}_0\left[\sum_s (\delta_s p - \omega) \cdot n_s \middle| I_0\right]
\end{aligned}
$$

The first equality is due to (29). The last equality is due to (24). Define

$$EV_0^b(n, \bar{p}, \rho, p) \equiv (\bar{p} - \omega) \cdot n \cdot [1 - \alpha_0(p, \bar{p}, \rho)] \tag{33}$$

$$EV_0^f(n, \bar{p}, \rho, p) \equiv \mathbb{E}_0\left[\sum_s (\delta_s p - \omega) \cdot n_s \middle| I_0\right] \tag{34}$$

Thus,

$$\mathbb{E}_0\left[V_{-1}(n', \bar{p}', \rho')\middle| n, \bar{p}, \rho, p\right] = EV_0^b(n, \bar{p}, \rho, p) + EV_0^f(n, \bar{p}, \rho, p) \tag{35}$$

(33) is the expected "backward-looking" value. It is the product of existing reservations n, their average profit margin $(\bar{p} - \omega)$, and the expected fraction $(1 - \alpha_0)$ of them that will stay until next period (i.e. will show up and check in). Its dependency on $(n, \bar{p}, \rho, p)$ is obvious. (34) is the expected "forward-looking" value. It is the expected product of new reservations n_s and profit margins of new reservations $(\delta_s p - \omega)$. The dependency of EV_0^f on $(n, \bar{p}, \rho, p)$ is due to (25), (26), (23). It is worth restating that $\bar{p}$ enters into EV_0^f in a non-trivial way, through (and only through) cancellation channel (specifically, the $\bar{p}$-dependent cancellation rate α_t) embedded in the $\eta(.)$ mapping. In another word, it is the capacity constraint mechanism that entangles future reservations, which stem from "non-backward-looking" static consumer choices, with $\bar{p}$ — the information of past reservations. We will explore this finding further in theorem 2 and 3. According to (30) and (35),

$$
\begin{aligned}
V_0(n, \bar{p}, \rho) &= \max_p \left\{ \mathbb{E}_0\left[V_{-1}(n', \bar{p}', \rho')\middle| n, \bar{p}, \rho, p\right] \right\} \\
&= \max_p \left\{ EV_0^b(n, \bar{p}, \rho, p) + EV_0^f(n, \bar{p}, \rho, p) \right\}
\end{aligned}
$$

It is obvious that optimal policy p_0^* depends on $(n, \bar{p}, \rho)$

$$p_0^*(n, \bar{p}, \rho) = arg\max_p \left\{ EV_0^b(n, \bar{p}, \rho, p) + EV_0^f(n, \bar{p}, \rho, p) \right\} \tag{36}$$

Define

$$
\begin{aligned}
V_0^b(n,\bar{p},\rho) &\equiv EV_0^b(\ n\ ,\ \bar{p}\ ,\ \rho\ ,\ p_0^*(n,\bar{p},\rho)\ ) \\
&= n\cdot(\bar{p}-\omega)\cdot[\ 1-\alpha_0(p_0^*,\bar{p},\rho)\ ] \\
&= n\cdot(\bar{p}-\omega)\cdot A_0(n,\bar{p},\rho,p_0^*) \\
\textit{where}\ \ A_0(n,\bar{p},\rho,p_0^*) &\equiv 1-\alpha_0(p_0^*,\bar{p},\rho)
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
V_0^f(n,\bar{p},\rho) &\equiv EV_0^f(\ n\ ,\ \bar{p}\ ,\ \rho\ ,\ p_0^*(n,\bar{p},\rho)\ ) \\
&= \mathbb{E}_0\left[\sum_s[\ \delta_s\cdot\ p_0^*(n,\bar{p},\rho)-\omega\ ]\cdot n_s\ \middle|\ I_0^*\right]
\end{aligned}
\tag{38}
$$

As a result,

$$
V_0(n,\bar{p},\rho)=V_0^b(n,\bar{p},\rho)+V_0^f(n,\bar{p},\rho)
\tag{39}
$$

Therefore, theorem holds for $t=0$.

Step 2: Suppose for $t-1$,

$$
\begin{aligned}
V_{t-1}(n,\bar{p},\rho) &= V_{t-1}^b(n,\bar{p},\rho)+V_{t-1}^f(n,\bar{p},\rho) \\
\textit{where}\ \ V_{t-1}^b(n,\bar{p},\rho) &= n\cdot(\bar{p}-\omega)\cdot A_{t-1}(n,\bar{p},\rho,p_{t-1}^*)
\end{aligned}
\tag{40}
$$

we want to prove similar result holds for t, i.e.

$$
\begin{aligned}
V_t(n,\bar{p},\rho) &= V_t^b(n,\bar{p},\rho)+V_t^f(n,\bar{p},\rho) \\
\textit{where}\ \ V_t^b(n,\bar{p},\rho) &= n\cdot(\bar{p}-\omega)\cdot A_t(n,\bar{p},\rho,p_t^*)
\end{aligned}
\tag{41}
$$

We now prove (41). Denote $A_t^*\equiv A_t(n_t,\bar{p}_t,\rho_t,p_t^*)$

$$
\begin{aligned}
&\mathbb{E}_t\left[V_{t-1}(n',\bar{p}',\rho')\ \middle|\ n,\bar{p},\rho,p\right] \\
=\ &\mathbb{E}_t\left[V_{t-1}^b(n',\bar{p}',\rho')+V_{t-1}^f(n',\bar{p}',\rho')\ \middle|\ I_t\right] \\
=\ &\mathbb{E}_t\left[n'\cdot(\bar{p}'-\omega)\cdot A_{t-1}^*+V_{t-1}^f(n',\bar{p}',\rho')\ \middle|\ I_t\right] \\
=\ &\mathbb{E}_t\left[A_{t-1}^*\cdot(n-c)\cdot(\bar{p}-\omega)\ \middle|\ I_t\right]+ \\
&\mathbb{E}_t\left[A_{t-1}^*\cdot\sum_s(p\cdot\delta_s-\omega)\cdot n_s+V_{t-1}^f(n',\bar{p}',\rho')\ \middle|\ I_t\right] \\
=\ &n\cdot(\bar{p}-\omega)\cdot\mathbb{E}_t\left[A_{t-1}^*\cdot(1-c/n)\ \middle|\ I_t\right]+ \\
&\mathbb{E}_t\left[A_{t-1}^*\cdot\sum_s(p\cdot\delta_s-\omega)\cdot n_s+V_{t-1}^f(n',\bar{p}',\rho')\ \middle|\ I_t\right]
\end{aligned}
$$

Define

$$A_t(n,\bar{p},\rho,p) \equiv \mathbb{E}_t\left[\, A^*_{t-1}\cdot(1-c/n)\,\middle|\, I_t\,\right] \tag{42}$$

$$EV_t^b(n,\bar{p},\rho,p) \equiv n\cdot(\bar{p}-\omega)\cdot A_t(n,\bar{p},\rho,p) \tag{43}$$

$$EV_t^f(n,\bar{p},\rho,p) \equiv \mathbb{E}_t\left[\, A^*_{t-1}\cdot\sum_s(p\cdot\delta_s-\omega)\cdot n_s + V^f_{t-1}(n',\bar{p}',\rho')\,\middle|\, I_t\,\right] \tag{44}$$

Similar to $t=0$ case, we have

$$\begin{aligned} V_t(n,\bar{p},\rho) &= \max_p\left\{\mathbb{E}_t\left[V_{t-1}(n',\bar{p}',\rho')\middle| n,\bar{p},\rho,p\right]\right\} \\ &= \max_p\left\{EV_t^b(n,\bar{p},\rho,p)+EV_t^f(n,\bar{p},\rho,p)\right\} \end{aligned}$$

Likewise, optimal policy p_t^* depends on $(n,\bar{p},\rho)$

$$p_t^*(n,\bar{p},\rho) = arg\max_p\left\{EV_t^b(n,\bar{p},\rho,p)+EV_t^f(n,\bar{p},\rho,p)\right\} \tag{45}$$

Define

$$\begin{aligned} V_t^b(n,\bar{p},\rho) &\equiv EV_t^b(n,\bar{p},\rho,p_t^*(n,\bar{p},\rho)) \\ &= n\cdot(\bar{p}-\omega)\cdot A_t(n,\bar{p},\rho,p_t^*) \end{aligned} \tag{46}$$

$$V_t^f(n,\bar{p},\rho) \equiv EV_t^f(n,\bar{p},\rho,p_t^*(n,\bar{p},\rho)). \tag{47}$$

As a result,

$$\begin{aligned} V_t(n,\bar{p},\rho) &= V_t^b(n,\bar{p},\rho)+V_t^f(n,\bar{p},\rho) \\ where\ \ V_t^b(n,\bar{p},\rho) &= n\cdot(\bar{p}-\omega)\cdot A_t(n,\bar{p},\rho,p_t^*) \end{aligned} \tag{48}$$

Therefore, (41) is proved.

Combining results from step 1 and step 2, we have proved by induction that the additive decomposition holds, i.e.

$$V_t(n,\overline{p},\rho) = V_t^f(n,\overline{p},\rho)+V_t^b(n,\overline{p},\rho)\ \ ,0\le t\le T \tag{49}$$

, with recursive relation

$$V_t^b(n,\bar{p},\rho) = n\cdot(\bar{p}-\omega)\cdot A_t^* \tag{50}$$

$$V_t^f(n,\bar{p},\rho) = \mathbb{E}_t\left[A_{t-1}^*\cdot\sum_s(p_t^*\cdot\delta_s-\omega)\cdot n_s + V_{t-1}^f(n',\bar{p}',\rho')\,\middle|\, I_t^*\right] \tag{51}$$

$$A_t(n,\bar{p},\rho,p) = \mathbb{E}_t\left[A_{t-1}^*\cdot(1-c/n)\,\middle|\, I_t\right] \tag{52}$$

$$A_t^* = A_t(n,\bar{p},\rho,p_t^*) = \mathbb{E}_t\left[A_{t-1}^*\cdot(1-c/n)\,\middle|\, I_t^*\right] \tag{53}$$

$$V_0^f(n,\rho) = \mathbb{E}_0\left[\sum_s(p_0^*\cdot\delta_s-\omega)\cdot n_s\,\middle|\, I_0^*\right] \tag{54}$$

$$A_{-1}^* = 1 \tag{55}$$

Next, we will explain why V_t^f and V_t^b can be interpreted as claimed in the theorem.
With above recursion, A_t^* can be written as,

$$A_t^* = \mathbb{E}_t\left[(1-\frac{c_t}{n_t})\cdot\mathbb{E}_{t-1}\left[(1-\frac{c_{t-1}}{n_{t-1}})\cdot\mathbb{E}_{t-2}\left[(1-\frac{c_{t-2}}{n_{t-2}})\cdot\cdots\,\middle|\, I_{t-2}^*\right]\,\middle|\, I_{t-1}^*\right]\,\middle|\, I_t^*\right] \tag{56}$$

Note that $\mathbb{E}_t[1-c_t/n_t|I_t^*]$ measures the "expected" probability at the beginning of day t that an existing reservation will not be canceled on day t, given optimal BAR p_t^*. Applying this interpretation recursively, A_t^* measures the "expected" t-day-ahead probability that an existing reservation will not be canceled thereafter, if the manager follows optimal policies $\{p_t^*, p_{t-1}^*, ..., p_0^*\}$. In another word, A_t^* is the *t-day-ahead survival rate* of existing reservations along optimal policy path. V_t^b is the product of existing reservations n, their average profit margin $(\bar{p}-\omega)$ and the corresponding survival rate A_t^*. Obviously, V_t^b is the expected profit from existing reservations — a "backward-looking" component.

To see why V_t^f is "forward-looking", denote $A_{t-1}^*\cdot[\sum_s(p_t^*\cdot\delta_s-\omega)\cdot n_{st}]$ as π_t^*. π_t^* is the $(t-1)$-day-ahead expected profit from a particular realization of new reservations n_s made on day t, given optimal price path thereafter. It is a product of three factors: number of new reservations n_{st}, profit margin of new reservations $(p_t^*\cdot\delta_s-\omega)$, and survival rate A_{t-1}^*. [16] Apply (51) recursively. We have,

$$V_t^f(n_t,\bar{p}_t,\rho_t) = \mathbb{E}_t\left[\pi_t^* + \mathbb{E}_{t-1}\left[\pi_{t-1}^* + \mathbb{E}_{t-2}[\pi_{t-2}^* + \ \ldots\ |I_{t-2}^*]\middle|I_{t-1}^*\right]\middle|I_t^*\right] \tag{57}$$

V_t^f is evidently the expected profit from all future reservations (made on day t, day $t-1$, ..., day 0), thus the "forward-looking" value at the beginning of day t.

□

Assumption 2 (Exogenous cancellations) *The conditional probability distributions for the number of cancellations, c_t, by existing customers does not depend on the hotel 0's BAR p or ADR $\overline{p}$.*

[16] Cancellation process (23) or (58) implicitly assumes that cancellation happens with existing reservations, but not with new reservations. New reservation never gets canceled on the same day it is made. So the relevant survival rate for day t reservation is A_{t-1}^*.

Theorem 2. *If Assumption 1 and 2 hold, then for each $t \in \{1,\ldots,T\}$ the forward looking component of the value function V_t^f is independent of $\overline{p}$, i.e. it can be written as $V_t^f(n,\rho)$ and depends on (n,ρ) but not $\overline{p}$.*

Theorem 3. *If Assumptions 1 and 2 hold, then for each $t \in \{1,\ldots,T\}$ the optimal decision rule for BAR p_t^* is independent of $\overline{p}$, i.e. it can be written as $p_t^*(n,\rho)$ and depends on (n,ρ) but not $\overline{p}$.*

Proof of theorem 2 and theorem 3. Before the main argument, let's revisit variable dependency given the exogenous cancellation assumption that α_t does not depend on $(p_t,\bar{p}_t)$. A direct implication is that (23) and (24) now become

$$\tilde{c}_t \sim e_t(c|n_t,\rho_t) \tag{58}$$

$$\mathbb{E}_t(\tilde{c}_t|n_t,\bar{p}_t,\rho_t) = \mathbb{E}_t(\tilde{c}_t|n_t,\rho_t) = \alpha_t(\rho_t)\cdot n_t \tag{59}$$

As discussed in model setup, $\bar{p}$ enters into L_t , the distribution of $(n_1,...,n_S)$, only through the distribution of cancellation. Now that the distribution of cancellation doesn't depend on $\bar{p}$, neither does L_t. (25) reduces to

$$(\tilde{n}_{1t},\tilde{n}_{2t},...,\tilde{n}_{St}) \sim L_t(n_{1t},n_{2t},...,n_{St}|n_t,\rho_t,p_t) \tag{60}$$

L_t's dependence on n_t is not trivial because of the censoring procedure $\eta(.)$. Next day state variable n_{t-1} is random on the morning of day t after the manager setting p_t. $\tilde{n}_{t-1}$ is a function of $(n_t,\tilde{c}_t,\tilde{n}_{1t},...,\tilde{n}_{St})$ through accounting identity (26). By (60) and (58), distribution of $\tilde{n}_{t-1}$ only depends on (n_t,ρ_t,p_t)

$$\tilde{n}_{t-1} \sim \zeta_t(n'|n_t,\rho_t,p_t) \tag{61}$$

(60) and (61) are keys to the proof. Now let's move on to the main argument. t subscript is dropped wherever the context is clear.

Step 1: At t=0, equation (33) now has a straightforward dimension reduction as follows,

$$EV_0^b(n,\bar{p},\rho,p) \equiv [1-\alpha_0(\rho)]n(\bar{p}-\omega) \equiv EV_0^b(n,\rho,\bar{p}) \tag{62}$$

Equation (34) has dimension reduction as follows,

$$EV_0^f(n,\bar{p},\rho,p) \equiv \sum_s(\delta_s p-\omega)\,\mathbb{E}_0\left[n_s\middle|n,\bar{p},\rho,p\right] = \mathbb{E}_0\left[\sum_s(\delta_s p-\omega)n_s\middle|n,\rho,p\right]. \equiv EV_0^f(n,\rho,p) \tag{63}$$

where the second equality follows from (60). Thus, the Bellman equation becomes,

$$\begin{aligned} V_0(n,\bar{p},\rho) &= \max_p\left\{\mathbb{E}_0\left[V_{-1}(n',\bar{p}',\rho')\middle|n,\bar{p},\rho,p\right]\right\} \\ &= \max_p\left\{EV_0^b(n,\bar{p},\rho)+EV_0^f(n,\rho,p)\right\} \\ &= EV_0^b(n,\bar{p},\rho)+\max_p\left\{EV_0^f(n,\rho,p)\right\} \end{aligned}$$

By the decomposition above, it should be obvious p_0^* depends on (n,ρ) but not $\bar{p}$. We have $p_0^*(n,\rho)$. Thus Theorem 3 holds for $\mathbf{t}=\mathbf{0}$. Furthermore, equations(37) and equation (38) can be written as,

$$
\begin{aligned}
&V_0^b(n,\bar{p},\rho) = A_0(\rho)\cdot n\cdot(\bar{p}-\omega)\\
&where\ \ A_0(\rho) \equiv 1-\alpha_0(\rho)
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
V_0^f(n,\bar{p},\rho) &= EV_0^f(\ n\ ,\ \rho\ ,\ p_0^*(n,\rho)\ )\\
&= \mathbb{E}_0\left[\sum_s(\ \delta_s\ \cdot\ p_0^*(n,\rho)\ -\ \omega\ )\cdot n_s\middle| n,\rho,p_0^*(n,\rho)\right]\\
&\equiv V_0^f(n,\rho)
\end{aligned}
\tag{65}
$$

Thus Theorem 2 holds for $\mathbf{t=0}$ and the decomposition of the value function is given by

$$
\begin{aligned}
V_0(n,\bar{p},\rho) &= V_0^b(n,\bar{p},\rho)+V_0^f(n,\rho)\\
&= A_0(\rho)\cdot n\cdot(\bar{p}-\omega)+V_0^f(n,\rho)
\end{aligned}
\tag{66}
$$

Step 2: (induction step) Suppose for $t-1$, $V_{t-1}(n,\bar{p},\rho)$ has following representation,

$$
V_{t-1}(n,\bar{p},\rho) = A_{t-1}(\rho)\cdot n\cdot(\bar{p}-\omega)+V_{t-1}^f(n,\rho)
\tag{67}
$$

We can prove that for t, policy function p_t^* only depends on (n,ρ) and $V_t(n,\bar{p},\rho)$ has similar representation,

$$
V_t(n,\bar{p},\rho) = A_t(\rho)\cdot n\cdot(\bar{p}-\omega)+V_t^f(n,\rho)
\tag{68}
$$

Combining this result with step 1, theorem 2 and theorem 3 can be proved by induction. Following is how to prove (68):

$$
\begin{aligned}
&\mathbb{E}_t\left[\ V_{t-1}(n',\bar{p}',\rho')\ \middle|\ n,\bar{p},\rho,p\ \right]\\
=\ &\mathbb{E}_t\left[\ A_{t-1}(\rho')\cdot n'\cdot(\bar{p}'-\omega)+V_{t-1}^f(n',\rho')\ \middle|\ n,\bar{p},\rho,p\ \right]\\
=\ &\mathbb{E}_t\left[\ A_{t-1}(\rho')\cdot(n-c)\cdot(\bar{p}-\omega)\ \middle|\ n,\bar{p},\rho,p\ \right]+\\
&\mathbb{E}_t\left[\ A_{t-1}(\rho')\cdot\sum_s(p\cdot\delta_s-\omega)n_s+V_{t-1}^f(n',\rho')\ \middle|\ n,\bar{p},\rho,p\ \right]
\end{aligned}
$$

The second term is expected "forward-looking" value EV_t^f. The expectation is over random variables $(n',\rho',n_1,n_2,...,n_S)$, distributions of which collectively depend on (n,ρ,p) due to (60), (58) and (61). So the "forward-looking" part does not depend on $\bar{p}$. We have

$$
\begin{aligned}
&\mathbb{E}_t\left[\ A_{t-1}(\rho')\cdot\sum_s(p\cdot\delta_s-\omega)n_s+V_{t-1}^f(n',\rho')\ \middle|\ n,\bar{p},\rho,p\ \right]\\
=\ &\mathbb{E}_t\left[\ A_{t-1}(\rho')\cdot\sum_s(p\cdot\delta_s-\omega)n_s+V_{t-1}^f(n',\rho')\ \middle|\ n,\rho,p\ \right]\\
\equiv\ &EV_t^f(\ n\ ,\ \rho\ ,\ p\ )
\end{aligned}
\tag{69}
$$

The first term is expected "backward-looking" value EV_t^b, which can be further simplified to illustrate its independence from p,

$$\begin{aligned}
&\mathbb{E}_t\left[A_{t-1}(\rho')\cdot(n-c)\cdot(\bar{p}-\omega) \,\middle|\, n,\bar{p},\rho,p \right] \\
&= (\bar{p}-\omega)\cdot\left\{ n\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, n,\bar{p},\rho,p \right] - \mathbb{E}_t\left[A_{t-1}(\rho')\cdot c \,\middle|\, n,\bar{p},\rho,p \right]\right\} \\
&= (\bar{p}-\omega)\cdot\left\{ n\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right] - \mathbb{E}_t\left[A_{t-1}(\rho')\cdot c \,\middle|\, n,\rho \right]\right\} \\
&= (\bar{p}-\omega)\cdot\left\{ n\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right] - n\cdot\alpha_t(\rho)\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right]\right\} \\
&= n\cdot(\bar{p}-\omega)\cdot\left\{ \mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right]\cdot[1-\alpha_t(\rho)]\right\} \\
&= n\cdot(\bar{p}-\omega)\cdot A_t(\rho) \\
&\equiv EV_t^b(n,\bar{p},\rho), \text{where} \\
&A_t(\rho) = \mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right]\cdot[1-\alpha_t(\rho)]
\end{aligned} \tag{70}$$

The second equality is due to the fact that distribution of ρ' only depends on ρ and distribution of c only depends on (n,ρ). The third equality uses the Law of Iterated Expectations,

$$\begin{aligned}
&\mathbb{E}_t\left[A_{t-1}(\rho')\cdot c \,\middle|\, n,\rho \right] \\
=&\mathbb{E}_t\left[\mathbb{E}_t[\, A_{t-1}(\rho')\cdot c \mid n,\rho,\rho' \,] \,\middle|\, n,\rho \right] \\
=&\mathbb{E}_t\left[A_{t-1}(\rho')\cdot\mathbb{E}_t[c|n,\rho,\rho'] \,\middle|\, n,\rho \right] \\
=&\mathbb{E}_t\left[A_{t-1}(\rho')\cdot\mathbb{E}_t[c|n,\rho] \,\middle|\, n,\rho \right] \\
=&\mathbb{E}_t\left[A_{t-1}(\rho')\cdot[\, n\cdot\alpha_t(\rho)\,] \,\middle|\, n,\rho \right] \\
=&n\cdot\alpha_t(\rho)\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, n,\rho \right] \\
=&n\cdot\alpha_t(\rho)\cdot\mathbb{E}_t\left[A_{t-1}(\rho') \,\middle|\, \rho \right]
\end{aligned}$$

Now the Bellman equation is

$$
\begin{aligned}
V_t(n,\bar{p},\rho) &= \max_p \left\{ \mathbb{E}_t \left[V_{t-1}(n',\bar{p}',\rho') \middle| n,\bar{p},\rho,p \right] \right\} \\
&= \max_p \left\{ EV_t^b(n,\bar{p},\rho) + EV_t^f(n,\rho,p) \right\} \\
&= EV_t^b(n,\bar{p},\rho) + \max_p \left\{ EV_t^f(n,\rho,p) \right\}
\end{aligned}
$$

p only enters EV_t^f. So optimal policy p_t^* only depends on (n,ρ). Thus, V_t has the additive decomposition into forward and backward looking components as claimed,

$$
\begin{aligned}
V_t(n,\bar{p},\rho) &= EV_t^b(n,\bar{p},\rho) + EV_t^f(n,\rho,p_t^*(n,\rho)) \\
&= V_t^b(n,\bar{p},\rho) + V_t^f(n,\rho),
\end{aligned}
$$

where V_t^b and V_t^f are given by

$$V_t^b(n,\bar{p},\rho) = n \cdot (\bar{p} - \omega) \cdot A_t(\rho) \tag{71}$$

$$V_t^f(n,\rho) = \mathbb{E}_t \left[A_{t-1}(\rho') \cdot \left(\sum_s [\, p_t^*(n,\rho) \cdot \delta_s - \omega \,] \cdot n_s \right) + V_{t-1}^f(n',\rho') \;\middle|\; n,\rho,p_t^*(n,\rho) \right] \tag{72}$$

$$A_t(\rho) = \mathbb{E}_t \left[A_{t-1}(\rho') \;\middle|\; \rho \right] \cdot [1 - \alpha_t(\rho)] \tag{73}$$

$$V_0^f(n,\rho) = \mathbb{E}_0 \left[\sum_s (\, \delta_s \cdot p_0^*(n,\rho) - \omega \,) \cdot n_s \;\middle|\; n,\rho,p_0^*(n,\rho) \right] \tag{74}$$

$$A_{-1}(\rho) = 1 \tag{75}$$

Therefore we have proved theorem 2 and theorem 3. Now equation (57) becomes

$$A_t(\rho_t) = [1-\alpha_t(\rho_t)] \cdot \mathbb{E}_t \left[[1-\alpha_{t-1}(\rho_{t-1})] \cdot \mathbb{E}_{t-1} \left[[1-\alpha_{t-2}(\rho_{t-2})] \cdot \ldots \middle| \rho_{t-1} \right] \middle| \rho_t \right] \tag{76}$$

For the special case where α_t does not depend on ρ_t we have

$$
\begin{aligned}
V_t^b(n,\bar{p},\rho) = V_t^b(n,\bar{p}) &= n \cdot (\bar{p} - \omega) \cdot A_t \\
A_t &= \prod_{\tau=0}^{t} [1-\alpha_\tau].
\end{aligned} \tag{77}
$$

$\square$