Introduction to Matching Problems: Set-up and Definitions

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Outline

- Examples of matching problems.
- Two-sided matching: one to one, many to one.
- One-sided matching markets.
- Concepts and definitions.
- One-to-one matching – the Deferred Acceptance Algorithm.
- One-sided matching markets – The Top Trading Cycles Algorithm.

Examples

- Marriage markets, dating services.
- Labor markets.
- National Residency Matching Program, law clerkships.
- College assignment problems, Greek Rush(?).
- School choice issues.
- Office or room allocations. Kidney exchange.

One-to-one Matching

- A one-to-one matching problem is a list of three items, $(M, W, \succ)$.
- M, W is the list of agents to be matched with one another, $\succ$ is a description for each agent of their preferences over being matched with specific agents on the other side (or not being matched at all.)
- A ‘matching’ is a pairing of each agent in one group with one agent in the other, or a leaving the agent alone.
- A matching can leave some agents unmatched.

Many-to-one Matching

- A Many-to-one matching problems extends the idea of one-to-one matching so that one side might gain many agents from the other side.
- In this case, we also need to specify the maximum number of matches it can have.
- Applicable for Greek rush, matching new doctors to residency programs.

One-sided Matching Markets

- A one-sided matching market is a list of four items, $(I, H, \succ, \mu)$.
- I is the list of agents to be matched with one of the H distinct objects, $\succ$ is a description for each agent (say i) of his (strict) preferences over the objects to be matched with.
- The additional component μ is an initial allocation reflecting the fact that agents already are allocated objects and might wish to trade.
- The most interesting application of these problems are kidney exchanges.

One-sided Matching Allocations

- A one-sided matching matching allocation differs from matching markets in that there is no initial assignment of objects to agents.
- It is very much like many-to-one two-sided matching problem.
- The main difference is that, instead of preferences for objects, there are criteria that objects have over agents. (Known and agreed upon? Where do they come from?)
- A typical application is school choice problems. We will not study these problems.

Efficiency, Individual Rationality

- Relevant for one and two-sided matching? Role of $\succeq_S$.
  - A matching is (Pareto) *efficient* if there is no other matching such that no agent is worse off in the new matching and at least some agent is strictly better off.
  - Observe the different role of preferences in this assessment with one-sided vs. two-sided markets.
  - In particular, contrast preferences in college matching problems with priorities in school choice problems.
  - A matching is *individually rational* if all agents prefer their match to being unmatched.

Blocking Pairs and Stability

- Relevant only for two-sided matching.
  - For any given matching, (w, m) form a *blocking pair* if agent m prefers w to its match and w prefers m to its match, (Temptation to divorce.)
  - A matching is *stable* if there does not exist a blocking pair.
- Result: If a matching is stable, then it is efficient.

Strategy Proofness

- Consider the question of how to implement a matching without knowledge of agents' preferences.
- If we tried to implement a matching outcome, would agents behave strategically to frustrate our attempts?
- A matching is *strategy proof* if for all possible preferences and all agents, it is a weakly dominant strategy to behave according to your true preferences.

One-to-one Two-sided Matching – The Deferred Acceptance Algorithm (DAA)

- Fix a one-to-one matching problem and operate as follows.
  - Step 1. Every man 'proposes' to his most preferred woman. Each woman 'holds' the most acceptable offer she receives and rejects the rest.
  - Step k: Every man rejected in step $k-1$ proposes to the most preferred woman who has not yet rejected him (or no proposal). Every woman holds her most preferred offer to date and rejects the rest.
  - The process stops when there are no more rejections.
- There is a similar Woman Optimal algorithm with the roles reversed.
- For each algorithm, the process stops after a finite number of steps.

Demonstration

- Link(add#)

Stability

- The outcome from the deferred acceptance algorithm is stable.
- The M-optimal version yields the most preferred for M of all stable outcomes.
- The W-optimal version yields the least preferred for M of all stable outcomes.
- Behaving according to the true preferences is a dominant strategy for M in the M-optimal DAA.

Housing Markets and Top Trading Cycles

- Step 1: Each agent points to the owner of its favorite house.
- A *cycle* is a subset of agents such that each agent in the set is pointed to by a single agent.
- Note, at least one cycle has to occur (prove it).
- For each cycle, the trade occurs and the agents in the subset are removed.
- Step *k*: Repeat step 1 with the remaining agents and houses.
- The process ends when there are no more houses.

Results

- The TTC ends in a final number of steps and is the unique matching in the core (define the core for this problem).
- Implementing this process as a direct mechanism (having agents report their types and running the TTC on the basis of reports) is strategy proof.
- The TTC outcome is the only efficient, individually rational and strategy proof outcome.

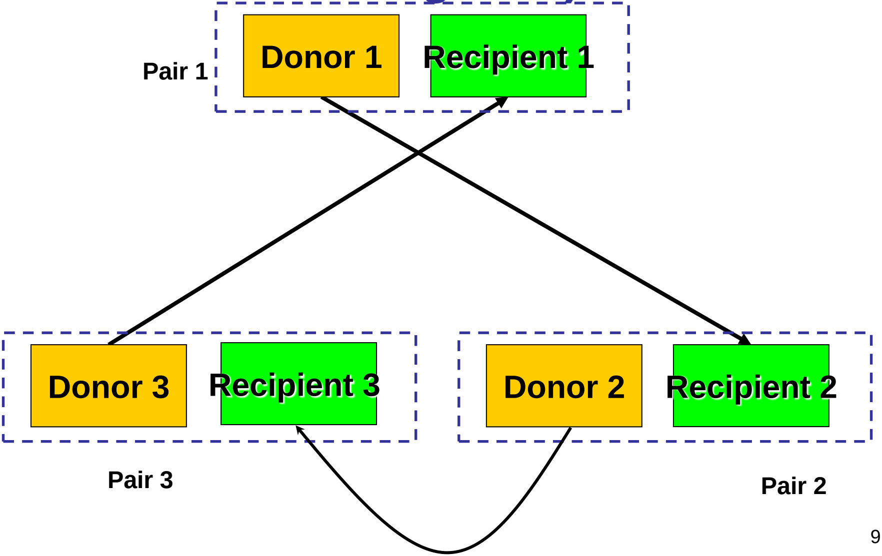
Kidney Exchange

- Each year, over 100,000 patients are listed on a registry requiring kidney transplants.
- Every year, about 11,000 kidneys are donated through fatalities, another 6,000 are voluntary donations.
- The consequence is a severe shortage of available kidneys.
- Market response?? Iran.
- Some recipients acquire kidneys from family members or friends but success requires a compatible match which is often unavailable.

Cycles and Chains

- Many unachieved matches can be resolved through simultaneous exchanges of compatible kidneys across pairs.
- Suppose John is willing to donate a kidney to his partner, Jane but is incompatible.
- Mary is willing to donate to her partner, Melissa but is incompatible.
- If John's kidney is compatible with Melissa's and Mary's is compatible with Jane, a 'cycle' arises and the two pairs can simultaneous donate across couples.
- A process that emerged from a branch of matching theory known as 'Top Trading Cycles.' (See the diagram from Al Roth's lecture notes.)

3-pair exchange (6 simultaneous surgeries)



Cycles and Chains

- The application of matching theory increased the number of feasible transplants but....
- Simultaneous ycles are relatively rare.
- For strategic reasons, cycles need to appear almost simultaneously.
- If non-directed donations are included, and non-simultaneous transplants allowed, the number of feasible transplants can increase greatly.
- The possibility of cycles emerging also increases with the size of the pool Internationalize the pool?

Non-directed donors: cycles plus <u>chains</u>