Dominant-strategy and Bayesian incentive compatibility in multi-object trading environments

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Abstract

When a single-object is to be traded, Bayesian and dominant-strategy incentive compatible mechanisms are interim-utility equivalent in independent, private-values environments; in the same environments, the equivalence breaks down when there are many distinct, indivisible objects to trade. We show that the fixed supply of each type of good imposes strong restrictions on the mechanisms that can be implemented. These restrictions can then be used to determine whether a given Bayesian mechanism has an equivalent dominant strategy mechanism in a multi-unit model.

1 Introduction

Consider the assignment of L distinct and indivisible objects to I agents in a multi-dimensional version of the standard, symmetric, independent private-values model: Each agent observes, as private information, an L-dimensional vector representing the agent's monetary valuations for the various objects. Agents' valuation for any set of objects is the sum of the valuations of each object in the set. Valuation vectors are independent and identically distributed across agents. Money enters utility functions linearly and agents are risk neutral. Within this environment, we investigate when a given Bayesian incentive compatible mechanisms (BIC) has an equivalent dominant strategy incentive compatible (DSIC) mechanism.

We employ a notion of equivalence based on payoffs, not on allocations. Two mechanisms are equivalent if each agent receives the same interim utility in both mechanisms—i.e. the agent receives the same expected payoff given the agent's true valuation and assuming by way of equilibrium analysis that opponents report their valuations truthfully. (We introduced this notion of equivalence to trading mechanisms in Manelli and Vincent (2010)). Thus, two mechanisms that implement different allocations may still be equivalent provided that they both grant the same payoffs to all involved.

When there is a single object ($L = 1$), for *any* BIC mechanism there is a DSIC mechanism that gives each agent the same expected payoff that the agent obtained in the Bayesian mechanism (Manelli and Vincent (2010)). Thus, there is a priori no loss in requiring dominant-strategy incentive compatibility over Bayesian incentive compatibility. An example in Gershkov et al. (2013) shows that equivalence fails when there are multiple objects ($L > 1$). The questions then arise: When are BIC mechanisms also DSIC and what, in general, is the relationship between BIC mechanisms and DSIC mechanisms when there are multiple goods to allocate?

We focus on *finite* Bayesian mechanisms – i.e. mechanisms in which agent types pool into finitely many sets such that each agent within a set receives the same expected probabilities of trade and makes the same expected payment. The interim equilibrium utility of agents in finite BIC mechanisms is a piecewise linear function: the domain of each linear component corresponds to a set of pooling types. Theorem 3 demonstrates that, for any finite BIC mechanism, there is a direct mechanism that generates it which is linear over the same subsets – that is, Bayesian mechanisms which pool agents in terms of *expected* outcomes can always be generated by direct mechanisms that pool agents in terms of *ex post* outcomes. This fact is valuable because in assessing whether a finite BIC mechanism has a DSIC equivalent we will only need to search within finite DSIC mechanisms. Theorem 4 characterizes the interim utilities from finite BIC mechanisms using the feasibility inequalities in Border (1991) and the incentive compatibility property of Rochet (1985). Theorem 5 identifies necessary conditions, implied by the feasibility inequalities, that any finite mechanism must satisfy. A DSIC mechanism that implements a candidate BIC mechanism must satisfy these conditions *and* ex post incentive compatibility. We demonstrate that some candidate BIC mechanisms cannot be implemented in dominant strategies by showing that the two criteria are inconsistent (see the examples in Section 6 and 8). In certain cases the conditions allow us to construct the 'closest' dominant-strategy implementable mechanism. In other cases, the conditions are strong enough to demonstrate when a candidate Bayesian mechanism *can* be implemented as a dominant-strategy incentive compatible mechanism by explicitly constructing the unique mechanism that generates it and then showing that the resulting mechanism is DSIC (Section 7).

Identifying when the BIC-DSIC equivalence holds is valuable. Dominant-strategy mechanisms have advantages over Bayesian mechanisms. For instance, one may be more confident that a rational agent will play a dominant strategy (if one is available) than that the same agent will play a Nash equilibrium strategy.[1]

While the equivalence or lack thereof between BIC and DSIC mechanisms has a long history, equivalence was defined, for much of that history, in terms of allocation: A DSIC mechanism is equivalent to a BIC mechanism if it implements the same allocation. (See, for instance, Mookherjee, D. and S. Reichelstein (1992), and Williams (1999).) In the trading environment that we study, allocative equivalence means that the same probability-of-trade function—that is to say, the probability with which goods are allocated to each agent given the reports made—can be obtained by Bayesian and dominant-strategy mechanisms that only differ on their transfer functions. This is a stronger notion of equivalence than the one we use.

The equivalence between BIC and DSIC mechanisms in single-object environments is robust in various ways and fails to be robust in others. First, it holds for *any mechanism*—not just the efficient mechanism or the revenue maximizing one as it is the case with the first price auction and its equivalent second price auction.[2] Second,

[1]See Mas-Colell, Whinston, and Green (1995), page 870, for a brief discussion of this point.

[2]An extensive literature, including d'Aspremont and Gérard-Varet (1979b), Laffont and Maskin (1979), Makowski and Mezzetti (1994), and Williams (1999), shows in various cases that if an ex post efficient allocation is implemented by a Bayesian incentive-compatible mechanism, then it can also be implemented by a dominant-strategy mechanism. Williams (1999) obtains an equivalence for quasilinear utilities and provides interesting applications, a lucid discussion, and a summary of the

the equivalence holds even with heterogeneous agents and nonsymmetric mechanisms. In particular, it holds when the seller is also privately informed (Manelli and Vincent (2010).) Third, the equivalence has been extended to an independent private values model with finitely competing outcomes (Gershkov et. al. (2013)) and to some instances with non-linear utilities by Kushnir and Liu (2018).

The one-dimensional equivalence fails outside the model described. Gershkov et al. (2013) illustrate this failure in various examples. One of them, with two homogeneous goods and discrete types, shows that the *revenue optimal* BIC mechanism differs from the *revenue optimal* DSIC mechanism. An example in Crémer and McLean (1988, Appendix A) provides an example of equivalence failure with interdependent valuations. We provide examples of the failure of equivalence when $L > 1$ that illustrate the extent of the problem.[3]

Multi-dimensional mechanism design problems are notoriously complex. Our approach focuses on finite mechanisms and combines two separate strands of the implementation literature. For the one good case, Matthews (1984) and Border (1991) characterize the functions that are the expected probability of trade for some mechanism. Maskin and Riley (1984) prove, constructively, a variation of Border's characterization for a particular case. Border (2007) extends his own result to nonsymmetric, one-dimensional environments. These results effectively demonstrate when a candidate mechanism is *feasible* in the sense that it obeys the resource constraint that no more than one object be allocated.

Separately, Rochet (1985) provides necessary and sufficient conditions for incentive compatibility in the case of multi-dimensional buyer types. In brief, the condition requires the convexity of the implied interim utility function for each agent. Combining this result with an adaptation of the above feasibility results allows us to characterize BIC mechanisms in our environment. We then exploit the interaction of convexity constraints with the feasibility constraints to generate our main results concerning the relationship between BIC and DSIC mechanisms.

2 Notation

The vector of all ones in a Euclidean space is denoted by $\mathbf{1}$, the zero vector is $\mathbf{0}$. Given any vector x, x_i denotes its i^{th} component, x_{-i} is obtained by removing x_i from x, and (y, x_{-i}) is constructed by replacing x_i with y in x. If $n \in \mathbb{N}^K$, $\|n\| = \sum_{i=1}^{K} n_i$.

literature.

[3]Other related examples can be found in Jehiel, Moldovanu, and Stacchetti (1998).

The function $\mathbb{1}_C$ is the indicator function that takes the value 1 if condition C is true, otherwise it is 0.

If A is a set, A^c is its complement, A^o is its interior, and $|A|$ is the number of elements in A.

For any real-valued function u on $[0,1]^L$, $\nabla u(x)$ in $\mathbb{R}^L$ denotes its gradient (when it exists) at x. The ℓ^{th} component of $\nabla u(x)$ is denoted by $\nabla_\ell u(x)$. For any real-valued function v on $\times_{i=1}^I [0,1]^L$, $\nabla_{x_i} v(x)$ in $\mathbb{R}^L$ denotes the gradient of v (when it exists) with respect to the vector x_i evaluated at x.

We assume sets and functions are measurable with respect to the corresponding Borel σ-algebras and product spaces are endowed with the product σ-algebras. The integral of a vector-valued function is the vector of the integrals component wise.

3 Model

There are L indivisible objects to allocate to I agents, labelled by $\ell = 1, \dots, L$ and $i = 1, \dots, I$ respectively. Each agent i observes privately the realization of a random vector $x_i \in X = [0,1]^L$. The L-dimensional vector x_i is interpreted as agent i's valuations for the L objects. The restriction of X to be the unit L-cube is a normalization. The random vectors x_i, $i = 1, \dots, I$, are independent and identically distributed, each x_i according to the distribution λ. The distribution λ has full support in X and admits a density function. The product distribution with I factors is denoted by λ^I.

Agent i's preferences over consumption and money transfers are represented by the real-valued function $x_i \cdot q - t$ where q is the L-vector of quantities consumed of each good, and $t \in \mathbb{R}$ is a monetary transfer from the agent to the mechanism designer. All agents are risk neutral.

Definition 1 (Direct mechanism)**.** A direct mechanism is a pair of functions per bidder i, $q_i : X^I \to [0,1]^L$ and $t_i : X^I \to \mathbb{R}$ such that

$$\sum_{i=1}^{I} q_i(x) \leq \mathbf{1}. \tag{1}$$

Bidder 1's expected probabilities of trade when her report is x_1 and all other bidders report truthfully is $Q(x_1) = E_{x_{-1}} q(x_1, x_{-1})$.

We use permutations to define symmetric anonymous mechanisms and to exploit their properties.

Definition 2 (Permutation)**.** A permutation π is a bijection from $\{1, \ldots, I\}$ to itself. Given $y = (y_1, \ldots, y_I)$, $y\pi$ denotes the Ith-tuple $(y_{\pi(1)}, \ldots, y_{\pi(I)})$. The symbol π^i denotes the permutation in which $\pi^i(1) = i$, $\pi^i(i) = 1$ and $\forall j, 1 \neq j \neq i$, $\pi^i(j) = j$.

Definition 3 (Symmetric mechanism)**.** A direct mechanism $\{q_i, t_i\}_{i=1}^I$ is symmetric if $\forall i$, $q_i(x) = q_1(x\pi^i)$ and $t_i = t_1(x\pi^i)$.

Definition 4 (Anonymous mechanism)**.** A mechanism q is anonymous if for every permutation π with $\pi(1) = 1$, $q(x) = q(x\pi)$.

An anonymous symmetric mechanism is described by a single pair of functions, the probability of trade and transfer functions (q_i, t_i) associated with bidder $i = 1$. Henceforth all mechanisms are anonymous and symmetric. We drop the subindex $i = 1$ and denote the mechanism by (q, t).

Definition 5 (Incentive compatible mechanism)**.** The mechanism (q, t) is dominant-strategy incentive compatible (DSIC) if

$$\forall x_1, x_1' \in X, x_{-1} \in X^{I-1}, q(x_1, x_{-1}) \cdot x_1 - t(x_1, x_{-1}) \geq q(x_1', x_{-1}) \cdot x_1 - t(x_1', x_{-1}). \quad (2)$$

It is Bayesian incentive-compatible (BIC) if [4]

$$\forall x_1, x_1' \in X, \int_{X^{I-1}} \left[q(x_1, x_{-1}) \cdot x_1 - t(x_1, x_{-1})\right] \lambda^{I-1}(dx_{-1}) \geq \int_{X^{I-1}} \left[q(x_1', x_{-1}) \cdot x_1 - t(x_1', x_{-1})\right] \lambda^{I-1}(dx_{-1}). \quad (3)$$

Definition 6. Given a mechanism (q, t), define the real-valued functions

$$\begin{aligned} v(x_1, x_{-1}) &= q(x_1, x_{-1}) \cdot x_1 - t(x_1, x_{-1}) \\ u(x_1) &= \int_{X^{I-1}} v(x_1, x_{-1}) \lambda^{I-1}(dx_{-1}) \end{aligned}.$$

If (q, t) is DSIC, then v is the agent's ex post utility function. If (q, t) is BIC then u is the agent's interim utility function.

A typical mechanism problem involves the selection of a function q—satisfying the resource constraint (1), BIC (3) or DSIC (2), and interim individual rationality $(u(x_1) \geq 0)$—to maximize a given objective function.

Essentially, a mechanism is DSIC if and only if its ex post utility $v(x_1, x_{-1})$ is convex in x_1 for any profile x_{-1}. Furthermore, the gradient $\nabla_{x_1} v$ is the probabilities-of-trade

[4] The definition of BIC mechanism is often stated for almost all x_1 and for all x_1'. This distinction is not important in what follows.

function q. See for instance Myerson (1981) for the single-good, many-buyers case; and Rochet (1985), Theorem 1, for the many-goods, single-buyer case. A mechanism is feasible if it allocates no more than one object of each type for any type profile. Anonymous DSIC mechanisms then are characterized by the following theorem.

Theorem 1 (Rochet)**.** *A measurable function $v : X \times X^{I-1} \to \mathbb{R}$ is the ex post utility of a DSIC, symmetric, anonymous mechanism if and only if*

(a) $\forall x_{-1}, v(x_1, x_{-1})$ *is convex in* x_1*. Its gradient,* $\nabla_{x_1} v(x)$*, is defined almost everywhere and is agent 1's probability of trade (DSIC).*

(b) $\sum_{i=1}^{I} \nabla_{x_1} v(x\pi^i) \leq \mathbf{1}$ *(Resource constraint).*

(c) $\nabla_{x_1} v(x) = \nabla_{x_1} v(x\pi)$ *for every permutation* π *with* $\pi(1) = 1$ *(Anonymity).*

The proof follows directly from the characterization of incentive compatibility, see for instance, Theorem 1 and its proof in Rochet (1985).

Similarly, a mechanism, (q, t), is BIC if and only if q satisfies (1), its interim utility u is convex and its gradient ∇u is the expected probabilities-of-trade function, $E_{x_{-1}} q$.

Remark. Let (q, t) be a BIC mechanism with interim utility u. We will use the following notation interchangeably, $Q \equiv E_{x_{-1}} q = \nabla u$ where the equation follows from the characterization of incentive compatibility. If (q, t) is DSIC with ex post utility v, then in addition, $q(x_1, x_{-1}) = \nabla_{x_1} v(x_1, x_{-1})$. In incentive compatible mechanisms, transfers—or expected transfers if only BIC is required – are determined up to a constant by the probability of trade function. See for instance Williams (1999). Thus, we often refer to v or q as a mechanism without mentioning explicitly the transfer function. Similarly, in considering BIC mechanisms, we will variously describe them in terms of the interim equilibrium utilities they generate, u, or their corresponding expected probabilities of trade, Q.

To determine the feasibility of a candidate BIC mechanism, Matthews (1984) and Border (1991) characterize the expected probability of trade functions Q that can be obtained from a mechanism q with I bidders. This amounts to representing the resource constraint (1) in terms of Q. Combining both characterizations, if a function $u(x_1)$ is convex, its gradient ∇u is the only candidate for expected probabilities of trade Q. If $\nabla u = Q$ satisfies the Matthews and Border representation of the resource constraint, then u is the interim utility of a mechanism—that is to say there is q that satisfies (1) and (3) and whose interim utility is u. Theorem 2 makes the argument precise.

Theorem 2 (Matthews-Border-Rochet)**.** *Let* $u : X \to \mathbb{R}_+$*. The function* u *is the interim utility of a BIC mechanism with* I *bidders if and only if* u *is convex,* $\forall x_1 \in$

$X, \nabla u(x_1) \in [0,1]^N$, *a.e. and*

$$\forall A \subset X,\ I \int_A \nabla u(x_1)\, d\lambda \leq [1 - [\lambda(A^c)]^I]\mathbf{1}. \tag{4}$$

Theorems 1 and 2 together yield a program for determining when a given BIC mechanism can be implemented in dominant strategies. For a candidate Bayesian mechanism, represented by its interim utility function, u, Theorem 2 determines whether or not it is Bayesian incentive compatible. The mechanism u is also DSIC if and only if there exists a function $v : X \times X^{I-1} \to \mathbb{R}_+$ satisfying the conditions of Theorem 1 such that for all $x_1 \in X$, $u(x_1) = E_{x_{-1}} v(x_1, x_{-1})$.

4 Finite Mechanisms

Finite mechanisms and their corresponding piecewise linear utility functions play an important role in our results. This is so partly because the interplay of convexity and linearity adds structure to the problem. In addition, the restriction to piecewise linear mechanisms reduces the number of feasibility conditions required by Theorem 2. In classic indirect mechanisms for single object problems such as first and second price auctions, finite mechanisms arise from plausible constraints on the bidding space such as that bidders can only select from a finite number of bids. An equilibrium consequence is that bidders pool into a finite collection of groups in which all members of a group obtain the same expected probability of trade (and make the same expected payment). We incorporate a similar feature in the multi-dimensional case by restricting attention to direct mechanisms where, in equilibrium, bidders separate into a finite number of subsets of their type space and members within the same subset all have the same expected probability of acquiring the available objects.

Definition 7 (Finite mechanism)**.** A mechanism q is finite if there is a finite partition $\mathcal{P}$ of X such that $\forall B \in \mathcal{P}^I$, $\forall x, x' \in B^o$, $q(x) = q(x')$. We say q is finite with partition $\mathcal{P}$. The function $Q = E_{x_{-1}} q$ is finite if $\forall A_k \in \mathcal{P}$, $\forall x_1, x_1' \in A_k^o$, $Q(x_1) = Q(x_1')$.

Throughout, we restrict attention to partitions with elements only of strictly positive λ-measure. Thus, all elements possess non-empty interiors. Given our assumption that λ possesses a density, the behavior of either q or Q on the boundary of any such set is, for the most part, inconsequential. For concision in what follows, we do not make further distinctions between whether types are in the interior or on the boundary of a given subset. When no confusion arises and $\forall x, x' \in B, q(x) = q(x')$, we may write $q(B) = q(x)$; similarly when $\forall x_1, x_1' \in A_k, Q(x_1) = Q(x_1')$ we may write $Q(A_k) = Q(x_1)$ and $\lambda_k = \lambda(A_k)$.

Definition 8 (Counting Function)**.** Fix a finite partition $\{A_k\}_{k=1}^K$ of X. A counting function is a map c, $c : X^I \to \{1, 2, \ldots, I\}^K$, where $c(x) = (c_1(x), c_2(x), \ldots, c_K(x))$ is defined by

$$c_k(x) = \{\#i | x_i \in A_k\}.$$

The definition of the function requires a partition. For concision, we omit this dependence in the notation.

A finite mechanism partitions the set X of consumer types into finitely many groups. Types in each group are treated similarly, in the sense that they face the same probabilities of trade (and pay the same expected transfers). We refer to those groups as market segments. Given a market-segment profile $B = (B_1, \ldots, B_I)$, writing $q(B)$ highlights that the probabilities of trade are the same for any realization of valuations in B. If in addition the mechanism is anonymous, q depends only on the market segment containing bidder 1's valuation, and on the number of bidders with valuations in each market segment, $c(x)$; anonymity renders the names of those bidders immaterial. We record this observation as a lemma.

Lemma 1. *An anonymous mechanism $\bar{q}$ is finite with partition $\mathcal{P} = \{A_k\}_{k=1}^K$ if and only if there exists $p : \{1, \ldots, K\} \times \{n \in \mathbb{N}^K : \|n\| = I\} \to [0, 1]^L$ such that*

$$p(k, n) = \begin{cases} \mathbf{0}, & \text{if } n_k = 0 \\ \bar{q}(x), & \text{if } x_1 \in A_k \text{ and } c(x) = n, \end{cases} \tag{5}$$

$$\forall n, \ \sum_{k=1}^{K} n_k p(k, n) \leq \mathbf{1}. \tag{6}$$

We say $\bar{q}$ is an anonymous, finite mechanism with partition $\mathcal{P}$ and anonymous probability of trade, $p(k, n)$.

Proof. If $\bar{q}$ is a finite mechanism with partition $\mathcal{P}$, then $\bar{q}$ is constant on each element of $\mathcal{P}^I$. Define $p(k, n)$ by (5); since $\bar{q}$ is anonymous, p is well defined. Inequality (1) implies (6). For the converse, suppose p satisfes (6) and $p(k, n) = 0$ for $n_k = 0$. Define $\bar{q}$ as follows: $\forall x \in X^I$, $\bar{q}(x) = p(k, c(x))\mathbb{1}_{x_1 \in A_k}$. Then $\bar{q}$ satisfies (1). By construction, it is anonymous and finite with partition $\mathcal{P}$. □

Note that $p(k, n) \in [0, 1]^L$ is the vector of bidder 1's probabilities of trade for the L goods, when she reports any $x_1 \in A_k$, and each n_j in $n = (n_1, \ldots, n_K)$ is the number of bidders with reported valuations in A_j. The identity of those bidders is irrelevant because of anonymity. Since $\mathcal{P}$ is a partition, every bidder has valuation in some element of the partition, thus $\|n\| = I$. The resource constraint (1) in an anonymous, finite mechanism becomes (6).

Lemma 2 provides a way of approximating arbitrary anonymous mechanisms by finite mechanisms. Taking conditional expectations with respect to any finite partition of the type space yields a finite mechanism that is also anonymous.

Lemma 2. *Let q be an anonymous mechanism and $\mathcal{P} = \{A_k\}_{k=1}^K$ be a partition of X such that $\forall k, \lambda(A_k) > 0$. For each $B \in \mathcal{P}^I$ and $x \in B$, let $n = c(x)$, $k \in \{1, \ldots, K\}$ and define*

$$p(k,n) = \begin{cases} \mathbf{0}, & \text{if } n_k = 0 \\ E[q \mid x \in B], & \text{if } B_1 = A_k \text{ and } \forall 1 \leq j \leq K,\ n_j = |\{i : B_i = A_j\}| \end{cases}$$

and

$$\bar{q}(x) = p(k, c(x)) \mathbb{1}_{x_1 \in A_k}.$$

Then, $\bar{q}$ is an anonymous, finite mechanism with partition $\mathcal{P}$ and anonymous probability of trade p.

Proof. Observe that $p(k, \cdot)$ depends only the number of bidders in each element of the partition $\mathcal{P}$ while the conditioning event on the right side, $x \in B$ describes a specific allocation of bidders to elements of the partition. We verify that $p(k,n)$ is well defined. Let $B, B' \in \mathcal{P}^I$, $B_1 = B'_1 = A_k$, and $\forall j$, $|\{i : B_i = A_j\}| = n_j = |\{i : B'_i = A_j\}|$. Then there is a permutation π with $\pi(1) = 1$ such that $x \in B \iff x\pi \in B'$. By anonymity, $q(x) = q(x\pi)$. Then $E[q(x) \,|\, x \in B] = E[q(x) \,|\, x\pi \in B'] = E[q(x\pi) \,|\, x\pi \in B'] = E[q(x) \,|\, x \in B']$. Thus, $p(k,n)$ satisfies (5).

By Lemma 1, it suffices to verify (6). For any $x \in X^I, \sum_{i=1}^I q(x\pi^i) \leq 1$. Then $\forall B \in \mathcal{P}^I$, $\sum_{i=1}^I E[q(x\pi^i) \,|\, x \in B] \leq 1$. Given $B \in \mathcal{P}^I$, let $I_k = \{i : B_i = A_k\}$, let $n_k = |I_k|$ and $n = (n_1, \ldots, n_K)$. For $i \in I_k$, $E[q(x\pi^i) \,|\, x \in B] = E[q(x) \,|\, x\pi^i \in B] = p(k,n)$. Then, $1 \geq \sum_{i=1}^I E[q(x\pi^i) \,|\, x \in B] = \sum_{k=1}^K \sum_{i \in I_k} E[q(x\pi^i) \,|\, x \in B] = \sum_{k=1}^K n_k p(k,n)$. This proves (6). □

The definition implies $p(k,n)$ is the expectation of the probability of trade $q(x)$ conditional on a particular draw of bidders valuations to sets $\{A_j\}_{j=1}^K$: This explicitly places bidder 1's valuation x_1 in A_k and for every j, n_j bidders must have valuation in A_j. Symmetry, however, implies that $p(k,n)$ is the expectation of the probabilities of trade for *any* bidder i with $x_i \in A_k$ given a realized profile of types, n.

Lemma 3 computes the expected probability of trade by splitting bidders according to the market segment to which they belong.

Lemma 3. *Let $\bar{q}$ be an anonymous, finite mechanism with partition $\mathcal{P} = \{A_k\}_{k=1}^K$ and anonymous probability of trade, $p(k,n)$. Then, for any $j = 1, \ldots, K$, for any good ℓ,*

$$I \int_{A_j} Q_\ell(x_1) \lambda(dx_1) = \sum_{\|n\|=I, n_j>0} \binom{I}{n} \left(\prod_{k=1}^K \lambda^{n_k}(A_k) \right) n_j \, p_\ell(j,n)$$

where $n \in \mathbb{N}^K$.

Proof. Given symmetry and that $\int_{A_j} Q_\ell(x_1)\lambda(dx_1)$ is the expected probability of the event that bidder 1 has a type in A_j and obtains good ℓ, the left side is the expected probability that some bidder with type in the set A_j obtains good ℓ given the mechanism Q. We show that the right side also yields this expected probability.

For any $n \in \mathbb{N}^K$, $\|n\| = I$, $p_\ell(j,n)$ gives the probability that a given bidder with type in A_j (say bidder 1) obtains good ℓ. Since symmetry implies that every bidder with type in A_j has the same probability, given n, the probability that some bidder in A_j obtains the good is $n_j p_\ell(j,n)$. The event that a realization of bidder types generates the profile n occurs with probability

$$\binom{I}{n}\left(\prod_{k=1}^{K}\lambda^{n_k}(A_k)\right).$$

Thus, summing $n_j p_\ell(j,n)$ over all possible realisations of n, $\|n\| = I$ times the probability of each realisation yields the expected probability a trade of good ℓ with a bidder in A_j occurs. The summation need only be over the set of realizations such that $n_j > 0$ because otherwise, $p_\ell(j,n) = 0$. □

5 Results

The results in this section make explicit the relationship between incentive compatible finite anonymous mechanisms and the piecewise linear equilibrium utility functions they correspond to. For certain BIC mechanisms of interest, this relationship imposes strong restrictions on the form that the direct mechanisms can take. In turn, as the following sections demonstrate, these restrictions can in some cases guide the construction of DSIC mechanisms that generate the BIC mechanism, or, in other cases, rule out the existence of DSIC mechanisms that could generate them.

Definition 9. A convex function $u : X \to \mathbb{R}$ is piecewise linear if there is a smallest (by set inclusion) finite family of affine functions $\{f_k\}_{k=1}^K$ defined on X such that $\forall x_1 \in X$, $u(x_1) = \max\{f_k(x_1) : k = 1, \dots, K\}$. The market segment A_k, $k = 1, \dots, K$, is the set

$$A_k = \left\{x_1 \in X \mid f_k(x_1) \geq f_j(x_1) \;\; \forall j \neq k\right\} \setminus \bigcup_{\ell=0}^{k-1} A_\ell$$

where $A_0 = \emptyset$. We denote by M_u the collection of all such market segments. The collection M_u is a partition of X.

Remark. A market segment is a collection of buyer types satisfying finitely many, linear inequalities. Linear pieces that are never a maximum or those that are, at best, a weak maximum, are not considered. A market segment is a convex set of full dimension. The collection M_u of all market segments is a partition of X. The definition of a market segment depends on the labelling of a different linear pieces: for instance, the first market segment is a closed set but any linear piece could be labelled $k = 1$.

If $u : X \to \mathbb{R}$ is a piecewise linear convex function with market segments $\{A_k\}$, then

$$\forall x, x' \in A_k, \nabla u(x) = \nabla u(x').$$

Thus, following our previous abuse of notation, for such functions we write $\nabla u(A_k) = \nabla u(x_1), x_1 \in A_k$. Given any two market segments A_j and A_k, $k \neq j$, then $\nabla u(A_k) \neq \nabla u(A_j)$.

Theorem 3 demonstrates that if an anonymous BIC mechanism u is piecewise linear, then there is a corresponding anonymous finite mechanism $\bar{q}$ defined on the partition, M_u^I that generates u. In this sense, when presented with such a u, in searching for direct mechanisms that generate it, we can restrict attention to direct mechanisms that utilize only u's market segments.

Theorem 3. *Let (q, t) be an anonymous BIC mechanism with interim utility u. Let u be piecewise linear with market segments $M_u = \{A_k\}_{k=1}^K$ and let $\bar{q}$ and p be as in Lemma 2 using the partition M_u. Then $\bar{q}$ is an anonymous finite BIC mechanism with interim utility u. If in addition q is DSIC, then $\bar{q}$ is DSIC.*

Proof. It follows from Lemma 2 that $\bar{q}$ is an anonymous, finite mechanism. Thus $\bar{q}$ satisfies (1) and hence $(\bar{q}, t)$ is a mechanism.

Let $\bar{Q}(x_1)$ be the expected probabilities of trade generated by $\bar{q}$ for a bidder 1 who reports a type x_1 assuming all other bidders report truthfully. Then using the definitions from Lemma 2, for $x_1 \in A_k$,

$$\begin{aligned}
\bar{Q}(x_1) &= E_{x_{-1}}[p(k, c(x_1, x_{-1}))] \\
&= E_{x_{-1}}[E[q \mid x \in B \in \mathcal{P}^I, B_1 = A_k \text{ and } \forall 1 \leq j \leq K,\ c_j(x_1, x_{-1}) = |\{i : B_i = A_j\}|]] \\
&= E_{x_{-1}}[q \mid x_1 \in A_k] \\
&= Q(x_1).
\end{aligned}$$

The first two equalities utilize the definitions from Lemma 2, the third equality applies the law of iterated expectations. Observe that since $Q(x_1) = \nabla u(x_1)$ and u is linear on A_k, $Q(x_1)$ (and $\bar{Q}(x_1)$) are constant on A_k. Since $\bar{q}$ generates the same expected probabilities of trade under truthful reporting as q and (q, t) is BIC, so too is $(\bar{q}, t)$.

We now show that if q is DSIC, then $\bar{q}$ is DSIC. Let v be the ex post utility of q. Then, $\forall x_{-1}$, $v(x_1, x_{-1})$ is convex in x_1, and its gradient $\nabla_{x_1} v = q$, x_1-a.e. (Theorem 1). Then (see for instance Ioffe and Levin (1972), Theorem 1, page 8)

$$\forall B \in M_u^{I-1}, \quad \nabla_{x_1} \int_B v(x_1, x_{-1}) \lambda^{I-1}(dx_{-1}) = \int_B q(x_1, x_{-1}) \lambda^{I-1}(dx_{-1})$$

or equivalently, $\forall B \in M_u^{I-1}, \nabla E_{x_{-1}}[v \,|\, x_{-1} \in B] = E_{x_{-1}}[q \,|\, x_{-1} \in B]$.

The interim utility of q is $u = E_{x_{-1}} v = \sum_{B \in M_u^{I-1}} E_{x_{-1}}[v \,|\, x_{-1} \in B] \lambda^{I-1}(B)$. By hypothesis, the restriction of u to any $A_k \in M_u$ is linear. Each term on the right side is convex and therefore must also be linear when restricted to A_k. Hence for a fixed $B \in M_u^{I-1}$, $\forall x_1 \in A_k$, $\nabla E_{x_{-1}}[v \,|\, x_{-1} \in B] = E_{x_{-1}}[q \,|\, x_{-1} \in B]$ is a constant. This implies that, for any $B \in M_u^{I-1}$, and for all A_k, $x_1 \in A_k$,

$$\begin{aligned} \nabla E_{x_{-1}}[v \,|\, (x_1, x_{-1}) \in (A_k, B)] &= E_{x_{-1}}[q \,|\, (x_1, x_{-1}) \in (A_k, B)] \\ &= p(k, c(x)), (x_1, x_{-1}) \in (A_k, B) \\ &= \bar{q}(x_1, x_{-1}), (x_1, x_{-1}) \in (A_k, B). \end{aligned}$$

The second equality follows by definition of $p(k, n)$ in Lemma 2. The final line follows by definition of $\bar{q}$. Thus, the probabilities of trade generated by $\bar{q}$ for any $x \in (A_k, B)$ equal the gradient vector with respect to x_1 of the expectation of v given $x_{-1} \in B$. Since u is DSIC, $v(x_1, x_{-1})$ is convex in x_1 for each $x_{-1} \in B$ and thus $E_{x_{-1}}[v \,|\, x_{-1} \in B]$ is also convex in x_1. Therefore, $\bar{q}(x_1, x_{-1}), x_{-1} \in B$ is the gradient of a convex function with respect to x_1 for any given B and, thus, $\bar{q}$ is DSIC. □

Definition 10. Let $u : X \to \mathbb{R}$ be piecewise linear and convex with market segments $M_u = \{A_k\}_{k=1}^K$. For any good, ℓ, define A_k^ℓ, $k = 1, \ldots, K^\ell, K^\ell \leq K$, iteratively as

$$A_1^\ell = \cup \{A_k | \nabla_\ell u(A_k) \leq \nabla_\ell u(A_{k'}) \forall A_{k'} \subset M_u\},$$

$$A_k^\ell = \cup \{A_k | \nabla_\ell u(A_k) \leq \nabla_\ell u(A_{k'}) \forall A_{k'} \subset M_u / \cup_{\kappa=1}^{k-1} A_\kappa^\ell\}.$$

The collection of sets $\{A_k^\ell\}_{k=1}^{K^\ell}$ is both a coarsening of M_u and an ordering from lowest to highest values of the gradient of u with respect to good ℓ. While restricting attention to piecewise linear and convex mechanisms reduces the number of inequalities in Theorem 2 to a finite number, it remains large at 2^K. The next result shows that this coarsening reduces the number of inequalities to be tested even further to just K^ℓ for each good, ℓ.

Theorem 4 provides necessary and sufficient conditions for a piecewise linear function, $u : X \to \mathbb{R}_+$ to represent the expected utility of an agent in a Bayesian Nash equilibrium of a finite anonymous mechanism. It is an implication of Proposition 3.2 in Border (1991).

Theorem 4. *Let $u : X \to \mathbb{R}_+$ be piecewise linear and convex. Then u is the interim utility of a BIC mechanism with I bidders if and only if $\nabla u(x_1) \in [0,1]^L$ almost everywhere, and $\forall \ell, \forall k = 1, \ldots, K^\ell$,*

$$I \sum_{\kappa=k}^{K^\ell} \lambda(A_\kappa^\ell) \nabla_\ell u(A_\kappa^\ell) \leq [1 - [\lambda(\bigcup_{\kappa=1}^{k-1} A_\kappa^\ell)]^I]. \tag{7}$$

Proof. From Theorem 1, u is Bayesian incentive compatible if and only if it is convex and has gradients that lie (almost everywhere) in $[0,1]^N$ where the gradients represent the expected probabilities of trade. Fix any good ℓ and let $Q_\ell(x) = \nabla_\ell u(x)$. In turn, $Q_\ell(x)$ can be the expected probability of trade for good ℓ if and only if it satisfies inequality (4) in Theorem 2. We show that for piecewise linear convex functions with gradient in $[0,1]^L$, inequality (4) holds for all $A \subset X$ if and only if inequality (7) holds $\forall \ell, \forall k = 1, \ldots, K^\ell$.

By Proposition 3.2 in Border (1991), inequality (4) holds if and only if

$$\forall \alpha \in [0,1], I \int_{E_\alpha} Q_\ell(x) d\lambda(x) \leq 1 - (\lambda(E_\alpha^c))^I, \tag{8}$$

where

$$E_\alpha = \{x | Q_\ell(x) \geq \alpha\}.$$

For any piecewise linear, convex function, $u : X \to \mathbb{R}_+$ with $\nabla u(x) \in [0,1]^L$ and for any good, ℓ, fix the partition $\{A_k^\ell\}_{k=1}^{K^\ell}$ described in Definition 10. By construction, $\nabla_\ell u(x)$ is constant for all $x \in A_k^\ell$. Denote its value by $\nabla_\ell u(A_k^\ell)$. If $\alpha > \nabla_\ell u(A_{K^\ell}^\ell)$, then E_α is empty and inequality (8) holds trivially. Select any $k = 2, \ldots, K^\ell$. By construction, $0 \leq \nabla_\ell u(A_{k-1}^\ell) < \nabla_\ell u(A_k^\ell) \leq 1$ and for any $\alpha \in (\nabla_\ell u(A_{k-1}^\ell), \nabla_\ell u(A_k^\ell)]$, $E_\alpha = \cup_{\kappa=k}^{K^\ell} A_\kappa^\ell$. Therefore, for α in this range, the inequalities in (8) reduce to the single inequality

$$I \sum_{\kappa=k}^{K^\ell} \lambda(A_\kappa^\ell) \nabla_\ell u(A_\kappa^\ell) \leq [1 - [\lambda(\bigcup_{\kappa=1}^{k-1} A_\kappa^\ell)]^I].$$

If $\nabla_\ell u(A_1^\ell) > 0$ and $\alpha \in [0, \nabla_\ell u(A_1^\ell)]$, then $E_\alpha = X$ and (8) becomes

$$I \sum_{\kappa=1}^{K^\ell} \lambda(A_\kappa^\ell) \nabla_\ell u(A_\kappa^\ell) \leq 1.$$

Thus, the continuum of inequalities in (8) reduce to K^ℓ inequalities, one for each interval $(\nabla_\ell u(A_{k-1}^\ell), \nabla_\ell u(A_k^\ell)]$ for $k > 2$ plus (possibly) the interval $[0, \nabla_\ell u(A_1^\ell)]$. □

Theorem 5 lists three properties of finite mechanisms that are derived from the resource constraint. First, if there are precisely n_k bidders with the same type realization (that is, with types in the same market segment A_k), then a symmetric mechanism cannot assign any object to any of those bidders with probability higher $1/n_k$. Second, if, given a bidder's type realization, her interim probability of trade is zero, then her (ex post) probability of trade must be zero for any realization of her opponent's valuations. Simply, if the expectation of non-negative variables is zero, the realization of those variables must be zero. Third, if given a bidder's type realization, her interim probability of trade is close to its maximum, then her ex post probability of trade must be close its maximum for any realization of her opponent's valuations. This last property is somewhat complex to state because the two maximums must be defined appropriately. The maximum interim probability of trade is the Matthews-Border constraint (4) in Theorem 2. The maximum ex post probability depends on the actual realization of valuations.

Theorem 5. *Let $\bar{q}$ be an anonymous, finite mechanism with partition $\mathcal{P} = \{A_k\}_{k=1}^{K}$ and anonymous probability of trade function p. Then, $\forall \ell = 1, \ldots, L$, $\forall k = 1, \ldots, K$, $\forall n \in \mathbb{N}^K : \|n\| = I$,*

(a) $p_\ell(k, n) \leq \frac{1}{n_k}$,

(b) $\left[Q_\ell(x_1) = 0, x_1 \in A_k\right] \implies p_\ell(k, n) = 0.$

(c) Let $J \subset \{1, 2, \ldots, K\}$, $\{A_j\}_{j \in J} \subseteq \mathcal{P}$, $A = \bigcup_{j \in J} A_j$ and $A^c = X \setminus A$. If $I \sum_{j \in J} Q_\ell(j)\lambda(A_j) = (1 - [\lambda(A^c)]^I)$, then $\forall n$, $\sum_{j \in J} n_j > 0$, $\|n\| = I$,

$$\sum_{j \in J} n_j p_\ell(j, n) = 1$$

and for $j' \notin J$,

$$p_\ell(j', n) = 0.$$

Proof. (a) follows from the resource constraint (1). (b) follows because since $q \geq 0$, if $0 = Q_\ell(x_1) = \sum_{B \in \mathcal{P}^{I-1}} q(A_k, B)\lambda^{I-1}(B)$, then $\forall B$, $q(A_k, B) = 0$.

To prove (c), suppose $\sum_{j\in J} n_j p_\ell(j,n) < 1$. Then we have

$$\begin{aligned}
(1-[\lambda(A^c)]^I) &= I\sum_{j\in J} Q_\ell(j)\lambda(A_j)\\
&= \sum_{j\in J}\left(\sum_{\|n\|=I,\sum_{j\in J} n_j>0}\binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k)n_j p_\ell(j,n)\right)\\
&= \sum_{\|n\|=I,n\neq\tilde{n},\sum_{j\in J} n_j>0}\binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k)\sum_{j\in J} n_j p_\ell(j,n)\\
&< \sum_{\|n\|=I,\sum_{j\in J} n_j>0}\binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k)\\
&= (1-[\lambda(A^c)]^I),
\end{aligned}$$

a contradiction. The second equality applies Lemma 3. The next equality changes the order of summation. The following inequality is by hypothesis. The final line applies the binomial theorem. The result $p_\ell(j',n)=0, j'\notin J$ follows from inequality (6).

□

Theorem 5(c) implies that, there are very strong limitations on the possible values of some parts of the mechanism. These restrictions are made explicit in the following Corollaries. The first corollary demonstrates that, if the Border-Matthews inequality is achieved for some collection of market segments $\{A_j\}_{j\in J}$, then the expected probability of trade for a bidder type in a market segments outside $\{A_j\}_{j\in J}$ must be strictly lower than for bidder types in $\{A_j\}_{j\in J}$.

Corollary 1. *Let $J\subset\{1,2,\ldots,K\}$, $\{A_j\}_{j\in J}\subseteq\mathcal{P}$ and $A=\bigcup_{j\in J} A_j$. If $I\sum_{j\in J} Q_\ell(j)\lambda(A_j) = (1-[\lambda(A^c)]^I)$, then for all $\tilde{j}\notin J$, for all $j\in J$, $Q_\ell(A_{\tilde{j}}) < Q_\ell(A_j)$.*

Proof. Fix a market segment $j\in J$. Define three disjoint subsets of $\{0,1,2,\ldots,I-1\}^K$ as

$$\begin{aligned}
C_{J'} &= \{n|\sum_{j'\in J/j} n_{j'} > 0,,\|n\| = I-1\}\\
C_J &= \{n|n_{j'}=0, j'\in J/j, n_j>0, \|n\|=I-1\}\\
C_J^0 &= \{n|n_j=0, j\in J, \|n\|=I-1\}.
\end{aligned}$$

For a given bidder 1, say, the number of rival bidders in each market segment, n, lies in one of these sets. Let e_j denote the K-dimensional unit vector with 1 in the jth position and zeroes elsewhere. By Theorem 3, we have

$$
\begin{aligned}
Q_\ell(A_j) &= \sum_{n \in C_{J'} \cup C_J \cup C_J^0} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) p_\ell(j, n+e_j) \\
&\geq \sum_{n \in C_J} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) \frac{1}{n_j+1} + \sum_{n \in C_J^c} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right).
\end{aligned}
$$

The first equality follows because, by independence, an allocation of the $I-1$ bidder types to the K market segments, $n \in C_{J'} \cup C_J \cup C_J^0$ occurs with probability $\prod_{k=1}^{K} \lambda^{n_k}(A_k)$ and anonymity allows us to duplicate this allocation $\binom{I-1}{n}$ times. The inequality follows because for $n \in C_{J'}$, $p_\ell(j, n+e_j) \geq 0$ and by Theorem 5(c), for $n \in C_J \cup C_J^0$, $(n_j + 1)p_\ell(j, n+e_j) = 1$. Similarly, for $\tilde{j} \notin J$,

$$
\begin{aligned}
Q_\ell(A_{\tilde{j}}) &= \sum_{n \in C_{J'} \cup C_J \cup C_J^0} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) p_\ell(\tilde{j}, n+e_{\tilde{j}}) \\
&\leq \sum_{n \in C_J^0} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) \frac{1}{n_{\tilde{j}}+1} \\
&< \sum_{n \in C_J^0} \binom{I-1}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) \\
&\leq Q_\ell(A_j).
\end{aligned}
$$

The first inequality follows because, by Theorem 5(c), for $n \in C_{J'} \cup C_J$, $p_\ell(\tilde{j}, n) = 0$ and by Theorem 5(a), for $n \in C_J^0$, $p_\ell(\tilde{j}, n+e_{\tilde{j}}) \leq \frac{1}{n_{\tilde{j}}+1}$. □

Corollary 1 implies that if u is a piece-wise linear BIC mechanism with market segments $\mathcal{P}$ and a collection of market segments J achieves the Border-Matthews bound for good ℓ, then $\cup_{j \in J} A_j$ must equal $\cup_k^{K^\ell} A_k^\ell$ for some $k \in \{1, 2, \ldots, K^\ell\}$ (see Definition 10) – that is, the set $\cup_{j \in J} A_j$ must correspond to an upper contour set of the function $\nabla_\ell u$ and the maximal probabilities of trade can only be achieved via a nested set of market segments .

The next corollary shows that when a nested set of market segments each satisfy the Border inequality exactly, then yet more limits are imposed on the ex post probabilities of trade.

Corollary 2. *Let* $J \subset \tilde{J} \subset \{1, 2, \ldots, K\}$, $\{A_j\}_{j \in J}, \{A_j\}_{j \in \tilde{J}} \subseteq \mathcal{P}$, $A = \bigcup_{j \in J} A_j$, $\tilde{A} = \bigcup_{j \in \tilde{J}} A_j$ *and* $A^c = X \setminus A$, $\tilde{A}^c = X \setminus \tilde{A}$. *If* $I \sum_{j \in J} Q_\ell(j) \lambda(A_j) = (1 - [\lambda(A^c)]^I)$, *and*

$I\sum_{j\in\tilde{J}} Q_\ell(j)\lambda(A_j) = (1-[\lambda(\tilde{A}^c)]^I)$ *then for all* n, $\|n\| = I, n_j = 0, j \in J, \sum_{\tilde{j}\in\tilde{J}} n_{\tilde{j}} > 0$,

$$\sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) = 1,$$

Proof. By hypothesis,

$$\begin{aligned} I\sum_{j\in\tilde{J}} Q_\ell(j)\lambda(A_j) &= I\sum_{j\in J} Q_\ell(j)\lambda(A_j) + I\sum_{j\in\tilde{J}/J} Q_\ell(j)\lambda(A_j) \\ &= (1-[\lambda(A^c)]^I) + I\sum_{j\in\tilde{J}/J} Q_\ell(j)\lambda(A_j) \\ &= (1-[\lambda(\tilde{A}^c)]^I) \end{aligned}$$

Thus,

$$I\sum_{j\in\tilde{J}/J} Q_\ell(j)\lambda(A_j) = (1-[\lambda(\tilde{A}^c)]^I) - (1-[\lambda(A^c)]^I).$$

The left side is the expected probability that some bidder in $\tilde{A}/A$ obtains object ℓ. The right side is the probability of the event that a bidder is in $\tilde{A}/A$ and no bidder is in the set A. Define $N^A = \{n \in \mathbb{N}^K : \|n\| = I, \sum_{j\in J} n_j > 0\}$, $N^{\tilde{A}} = \{n \in \mathbb{N}^K : \|n\| = I, \sum_{j\in\tilde{J}} n_j > 0\}$ and $N^{\tilde{A}/A} = \{n \in \mathbb{N}^K : \|n\| = I, \sum_{j\in J} n_j = 0, \sum_{j\in\tilde{J}} n_j > 0\}$. The right side is the probability that the set $N^{\tilde{A}/A}$ occurs. The definitions then imply

$$(1-[\lambda(\tilde{A}^c)]^I) - (1-[\lambda(A^c)]^I) = \sum_{n\in N^{\tilde{A}/A}} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k). \tag{9}$$

Using the definition of the mechanism $\bar{q}$,

$$\begin{aligned} I\sum_{j\in\tilde{J}/J} Q_\ell(j)\lambda(A_j) &= \sum_{j\in\tilde{J}/J}\sum_{\|n\|=I} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) n_j p_\ell(j,n) \\ &= \sum_{\|n\|=I} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) \sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) \\ &= \sum_{n\in N^{\tilde{A}}} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) \sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) \\ &= \sum_{n\in N^{A}} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) \sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) \\ &\quad + \sum_{n\in N^{\tilde{A}/A}} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) \sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) \\ &= \sum_{n\in N^{\tilde{A}/A}} \binom{I}{n} \prod_{k=1}^K \lambda^{n_k}(A_k) \sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) \end{aligned}$$

The second equality changes the order of summation (feasible because the limits of the two summations are independent of each other). The next equality follows from Theorem 5(c) since if $n \notin N^{\tilde{A}}$ then $p_\ell(j,n) = 0$ for all $j \in \tilde{J}$. The next equality partitions the set $N^{\tilde{A}}$ into two mutually exclusive events, $N^A, N^{\tilde{A}/A}$. The final equality follows from Theorem 5(c) since if $n \notin N^A$ then $p_\ell(j,n) = 0$ for all $j \in \tilde{J}/J$. Equation (9) then implies that for $n \in N^{\tilde{A}/A}$, $\sum_{j\in\tilde{J}/J} n_j p_\ell(j,n) = 1$. $\square$

Remark. Corollaries 1 and 2 demonstrate the power of Theorem 5. To determine if there exists any collection of market segments from a piecewise linear BIC mechanism u that reach the Border-Matthews bound for good ℓ, rather than testing all possible subsets of the K market segments, Corollary 1 implies that only the K^ℓ upper contour sets of $\nabla_\ell u$ need be examined. Suppose the Border-Matthews inequality for good ℓ is satisfied exactly for two collections of market segments, J and J'. Corollary 1 implies that one is a subset of the other, say $J \subset J'$. Corollary 2 implies that for any realization of bidder types with *some* bidders in the market segments J, the direct mechanism that generates u must give good ℓ to one of the bidders in those segments. And, if no bidder type is in those market segments but some bidder type is in J', then the mechanism must give good ℓ to the bidder types in J'/J. If the mechanism is DSIC, then the gradient with respect to x_ℓ of the ex post utility corresponds to the ex post probability of trade and the ex post utility must be convex (Theorem 1). The two facts then impose strong restrictions on the form this mechanism can take. These observations are at the core of the examples that follow.

The Appendix shows that the logic underlying Theorem 5(c) can be used to generate a stronger result, demonstrating that its implications also apply to 'close by' mechanisms:

Theorem 5($\tilde{c}$). *Let $J \subset \{1, 2, \ldots, K\}$, $\{A_j\}_{j\in J} \subseteq \mathcal{P}$, $A = \bigcup_{j\in J} A_j$ and $A^c = X\backslash A$. If $I \sum_{j\in J} Q_\ell(j)\lambda(A_j) = \alpha(1 - [\lambda(A^c)]^I)$, $\alpha \in [0,1]$, then $\forall n$, $\sum_{j\in J} n_j > 0$, $\|n\| = I$,*

$$\sum_{j\in J} n_j p_\ell(j,n) \geq 1 - \frac{1-\alpha}{\binom{I}{n}\prod_{k=1}^K \lambda^{n_k}(A_k)}(1 - [\lambda(A^c)]^I).$$

In the next three sections, we demonstrate implications of Theorems 3, 4 and 5 by analyzing three classes of BIC mechanisms. In the first case, we show how to conclude that a given interim mechanism is BIC by applying Theorem 4. We then apply Theorem 3 and parts (a) and (b) of Theorem 5 to construct the building blocks of all possible ex post mechanisms that could generate the mechanism and show that no DSIC mechanism can be within this set. The second example applies part (c) of Theorem 5 to reduce the set of potential direct mechanism that could generate a particular class of BIC mechanisms and shows that each of them are ex post incentive compatible. Thus all elements of this class of BIC mechanisms are DSIC. The mechanisms within this class are all extreme points of the set of BIC mechanisms. The final example applies a small modification to a mechanism within the previous class to show that there is a set of BIC mechanisms that cannot be implemented in dominant strategies with this modification.

6 A BIC Mechanism That is Not DSIC

Manelli and Vincent (2010) demonstrated that in this environment, with $L = 1$, the interim utility of any BIC mechanism can be generated by a DSIC mechanism. In this section, we provide a simple example that shows the same conclusion does not follow when there are many goods to sell.

We describe a function $u : [0,1]^2 \to \mathbb{R}$ and a prior distribution λ and show that u is the interim utility of a BIC mechanism with two bidders and two goods. To do so we appeal to Theorem 4, the modified Matthews-Border inequality. In Subsection 6.1 we describe the mechanism that generates u and in Subsection 6.2 we prove that u cannot be obtained from a DSIC mechanism.

For $x_1 \in [0,1]^2$, let

$$u(x_1) = \max\Big\{0, \big(3/4, 0\big) \cdot x_1 - 1/2, \big(0, 3/4\big) \cdot x_1 - 1/2, \big(3/4, 3/4\big) \cdot x_1 - 3/4\Big\}.$$

The function u is composed of four linear pieces. Let $M_u = \{A_k\}_{k=1}^{k=4}$ be the domains of the linear pieces; we often call these sets the market segments. The market segments are depicted in Figure 1. The numbers in parenthesis in Figure 1 are $\nabla u(A_k)$, the gradient of u on each A_k. Then u can be written as

$$u(x_1) = \begin{cases} (0,0) \cdot x_1 - 0 & \text{if } x_1 \in A_1 \\ (0, \frac{3}{4}) \cdot x_1 - \frac{1}{2} & \text{if } x_1 \in A_2 \\ (\frac{3}{4}, \frac{3}{4}) \cdot x_1 - \frac{3}{4} & \text{if } x_1 \in A_3 \\ (\frac{3}{4}, 0) \cdot x_1 - \frac{1}{2} & \text{if } x_1 \in A_4. \end{cases}$$

Let the prior λ be any distribution that satisfies $\lambda(A_k) > 0, \forall k, \lambda(A_2) = \lambda(A_4), \lambda(A_3) + \lambda(A_2) = 1/2$.[5]

We verify that u is the interim utility of BIC mechanism with two bidders using the characterization of incentive compatibility in terms of the convexity of u, and the characterization of the resource constraint (1) in terms of u's gradient ∇u (Theorem 4).

[5] Within A_k, the distribution is arbitrary.

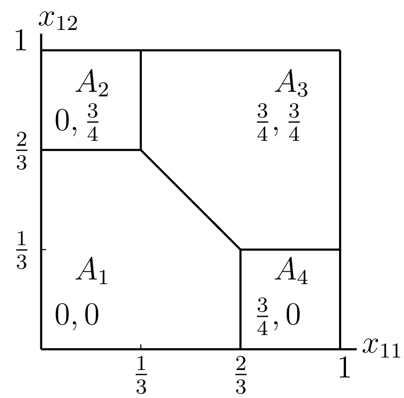


Figure 1: The function u and its gradient $\nabla u(A_k)$

As the pointwise maximum of four linear functions, u is convex. By construction, ∇u is in $[0,1]^2$. It remains to verify that u satisfies inequality (7).

Focus on good 2, the symmetry of the mechanism implies the argument also holds for good 1. To apply Theorem 4, the relevant partition with respect to good 2 is the collection $\{A_1 \cup A_4, A_3 \cup A_2\}$. Inequality (7) requires, (setting $A^2_{K^2} = A_3 \cup A_2$)

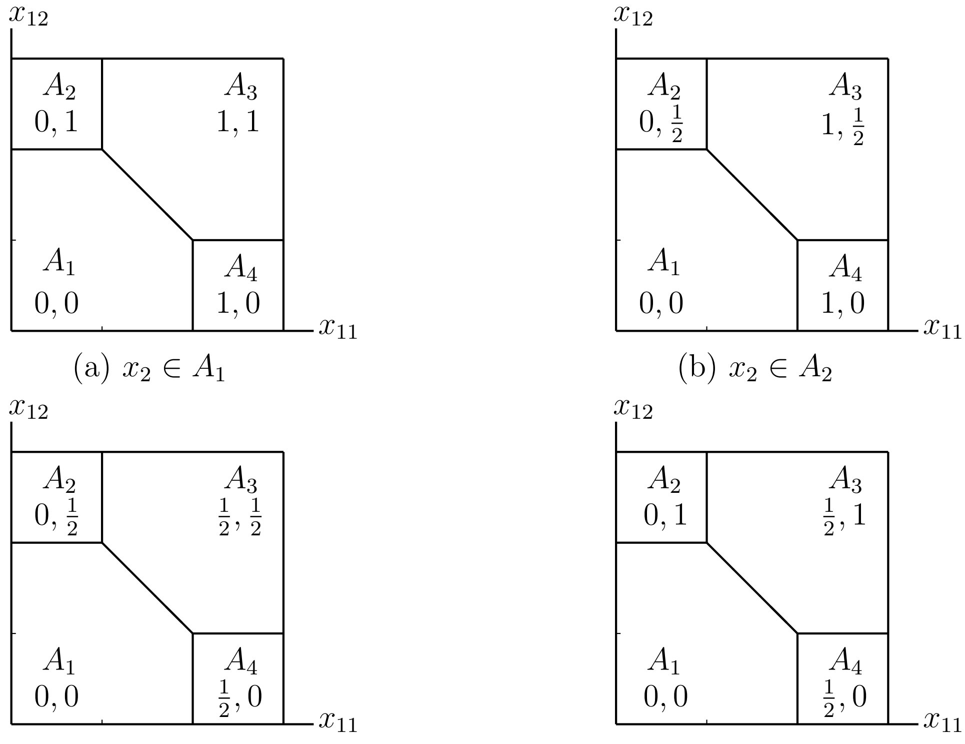
$$I\lambda(A^2_{K^2})\nabla_2 u(A_{K^2}) \leq [1 - [\lambda(A_1 \cup A_4)]^I].$$

Recalling that $\lambda(A_3 \cup A_2) = 1/2$, the left side is $2 \cdot \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{4}$ which equals $1 - (\frac{1}{2})^2$, the right side. So the inequality is (exactly) satisfied. The inequality for the second nested set, $A_1 \cup A_2 \cup A_3 \cup A_4$ holds trivially since the gradient is 0 in $A_1 \cup A_4$. Applying the same argument to good 1 implies u is BIC.

6.1 A BIC mechanism that generates u

Figure 2 depicts a mechanism $q(x_1, x_2)$ that generates u. The mechanism is finite (Definition 7). Thus we write $q(A_k, A_j)$ to indicate bidder 1's probabilities of trade when her valuation is in A_k and bidder 2's valuation is in A_j. Direct calculations determine that $\nabla u(A_k) = \sum_{j=1}^4 \lambda_j q(A_k, A_j)$, that is to say the expected probabilities of trade are the gradient of the interim utility, and the convexity of u implies that q is BIC.

The mechanism q, however, is not DSIC. If it were, its ex post utility $v(x_1, x_2)$ would be convex in x_1 for every x_2. It suffices to inspect the diagram (b) in Figure 2 to note that this is not so. That diagram represents $q(A_k, A_2) = \nabla_{x_1} v(A_k, A_2)$, the probabilities of trade are the gradient (with respect to x_1) of the ex post utility for fixed $x_2 \in A_2$. The gradient does not correspond to the gradient of a convex function since the boundary between A_1 and A_2 has slope -1 which is inconsistent with the



(a) $x_2 \in A_1$ (b) $x_2 \in A_2$

(c) $x_2 \in A_3$ (d) $x_2 \in A_4$

Figure 2: The mechanism $q(x_1, x_2)$

stated gradients. Thus, if it were known that bidder 2's valuation is in A_2, bidder 1 will not report her type truthfully given the probabilities in diagram (b).

6.2 No DSIC mechanism generates u

The previous subsection shows that a given BIC mechanism (that induces u) is not DSIC. We now demonstrate that no DSIC mechanism induces u.

First, since u is piecewise linear, if there is a DSIC mechanism, v, generating u, by Theorem 3 there is a DSIC mechanism that is finite with respect to the partition M_u and the expected probabilities of trade, $\nabla u(A_k)$, are a convex combination of the ex post probabilities of trade, $\nabla_{x_1} v(A_k, A_j)$. More precisely,

$$\forall k, \nabla u(A_k) = \sum_{j=1}^{4} \lambda_j \nabla_{x_1} v(A_k, A_j).$$

Figure 3 illustrates the usefulness of Theorem 3. The top-left diagram shows the gradient of the u on each market segment A_k, and the bottom-left diagram represents the prior distribution λ. The four remaining diagrams, labeled (a) to (d), represent the finite mechanism—or equivalently the gradient of the expost utility $\nabla_{x_1} v(A_k, A_j)$

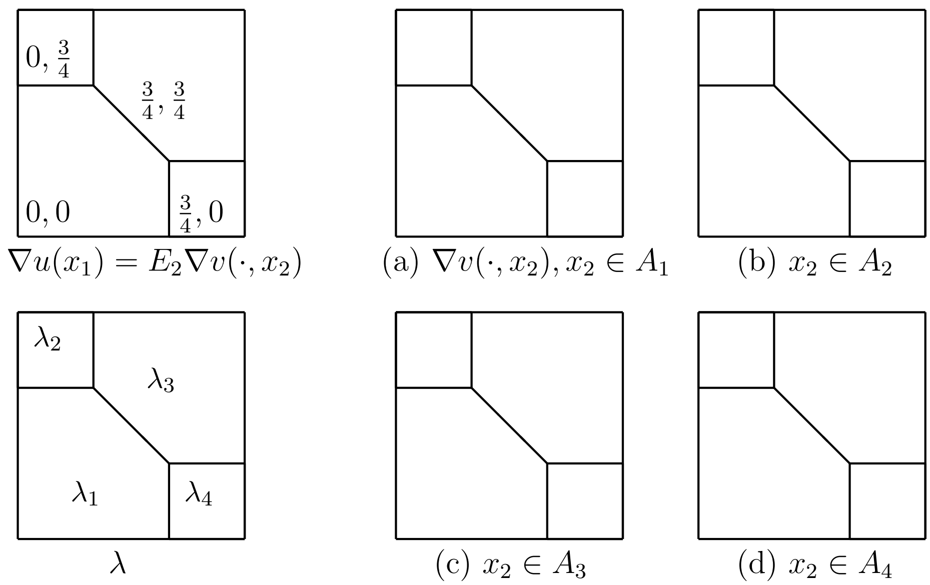


Figure 3: The finite mechanism ∇v

of the finite mechanism. For instance, diagram (b) corresponds to the ex post utility when $x_2 \in A_2$. Within each section of (b), Theorem 3 implies that the gradient vectors of v must be constant. The values of ∇u in the top-left diagram must be obtained as a convex combination of the values to be filled in the four diagrams (a) to (d) using the weights given by λ.

Second, we use a property of finite mechanisms, Theorem 5(b), to obtain the missing gradients in Figure 3. Theorem 5(b) states that if, for some good ℓ, bidder one has valuation in A_k and expects to receive good ℓ with probability zero, then the mechanism must assign good ℓ with probability zero for *any* valuation of bidder 2: To average zero with positive numbers, all numbers must be zero. Therefore we fill with zeros the corresponding missing values in Figure 4.

Third, Theorem 5(a) states that

$$x_1, x_2 \in A_k \implies \nabla_{x_1} v(x_1, x_2) \leq \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

If two bidders have valuation in the same market segment, they cannot be assigned the good with probability higher than $1/2$. This is so because there is at most one unit of each good.

Figure 4 (b) depicts the gradients of the ex post utility v when bidder 2 has valuation in A_2. If bidder 1 also has valuation in A_2, then $\nabla_{x_1} v(A_2, , A_2)$—the ex post probability of assignment—cannot exceed $1/2$. Thus $(0, 1/2)$ is placed in A_2, the top left corner of

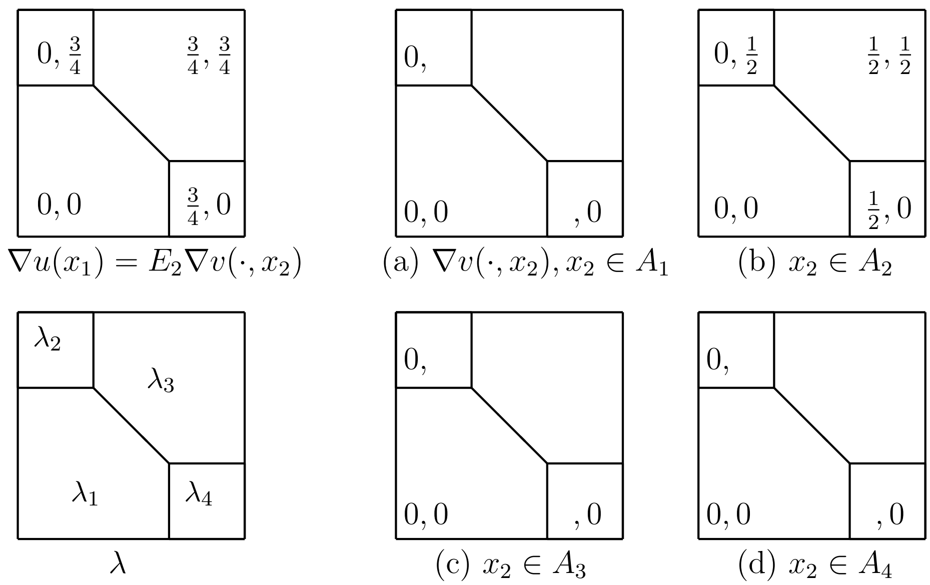


Figure 4: Properties of finite mechanism, Theorem 5

Figure 4 (b).

In turn, this implies that the entry in A_3 in Figure 4 (b) is $\nabla_{x_1} v(A_3, A_2) = (1/2, 1/2)$. This is the only possibility for a finite mechanism given the partition M_u. The boundary between A_2 and A_3 is vertical. Thus the gradient in A_3 must have the form $(\cdot, 1/2)$. (Its first component is not determined.) Since the boundary between A_1 and A_3 has slope -1, the gradient in A_3 must be $(1/2, 1/2)$. Since the boundary between A_3 and A_4 is vertical the gradient in A_4 must be $(1/2, 0)$.

Figure 4 (d) is completed similarly and the results are recorded in Figure 5 (d).

Consider now Figure 4 (c). It depicts the ex post utility when bidder 2's valuation is in A_3. If bidder 1's valuation is also in A_3, then $\nabla_{x_1} v(A_3, A_3) \leq (1/2, 1/2)$. We complete the missing values as we did in Figure 4 (b) and record the results in Figure 5 (c).

It remains to complete Figure 4 (a). In this case bidder 2 has valuation in A_1 and thus receives both goods with zero probability. Therefore both goods can be assigned to bidder 1 with probability one when her valuation is in A_3, the entry in A_3 is $(1, 1)$. Respecting convexity, the missing gradients can be filled in. We record them in Figure 5 (a).

The expectation with respect to λ of the ex post utilities will yield the gradients of the interim utility indicated by Q in the left-bottom diagram in Figure 5. (Symmetry of the ex post mechanisms and the assumption that $\lambda_2 = \lambda_4$ imply that the two gradients

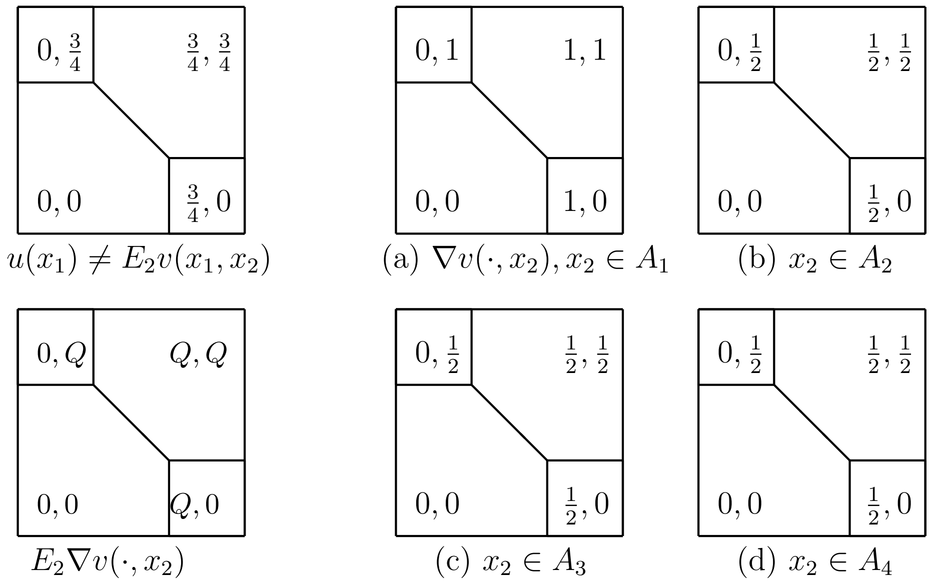


Figure 5: DSIC mechanism

are the same.) Applying the expectation yields (for good 2, for example)

$$
\begin{aligned}
Q &= \lambda_1 1 + \lambda_4 \frac{1}{2} + (\lambda_2 + \lambda_3)\frac{1}{2} \\
&= (\lambda_1 1 + \lambda_4) 1 - \lambda_4 \frac{1}{2} + \frac{1}{4} \\
&= \frac{3}{4} - \lambda_4 \frac{1}{2}.
\end{aligned}
$$

The second line uses the condition $\lambda_2 + \lambda_3 = 1/2$ and the third line uses the condition $\lambda_1 + \lambda_4 = 1/2$. Thus, since $\lambda_4 > 0$ the maximum gradient of an interim utility function generated by a DSIC mechanism must be less than 3/4, which is the gradient of u. This completes the proof that u is not the interim utility of a DSIC mechanism. It also shows that in the left-bottom diagram in Figure 5 is the closest DSIC interim utility (in terms of expected probabilities of trade) to u if the structure M_u is to be preserved.

6.3 Implications of Theorem 5(c)

An alternative and shorter proof that u is not DSIC can be obtained using the conclusion in Theorem 5(c). This alternative approach does not provide the building blocks of all DSIC direct mechanisms that can be used to construct mechanisms with similar market segments, however, it shows there is, in fact, a unique direct mechanism that implements u, the mechanism defined in Subsection 6.1.

Consider good 1 and note that for the collection of sets, $\{A_3, A_4\}$, the equality in Theorem 5(c) holds (and similarly for good 2 using the sets, $\{A_2, A_3\}$). Theorem 5(c) then implies that for $x_2 \in A_1 \cup A_2, x_1 \in A_3 \cup A_4$, $\bar{q}_1(x_1, x_2) = 1$ and similarly for $x_2 \in A_2 \cup A_3, x_1 \in A_1 \cup A_4$, $\bar{q}_2(x_1, x_2) = 1$. This yields the direct mechanism in Figure 2(a) and the gradients with respect to good 1 in Figure 2(b) as well as the gradients with respect to good 2 in Figure 2(d). Additionally, for $x_1, x_2 \in A_3$ or $x_1, x_2 \in A_4$, Theorem 5(c) implies $\bar{q}_1(x_1, x_2) = 1/2$ yielding the gradients in A_3 in Figure 2(c) and A_4 in Figure 2(d). A similar argument yields the gradients with respect to good 2 in A_3 in Figure 2(c) and A_2 in Figure 2(b). From Theorem 5(b), any direct mechanism must have gradients equal to 0 in the same cases as in Figure 2.

It remains to show the gradients for the case $(x_1, x_2) \in (A_3, A_4)$ and $(x_1, x_2) \in (A_4, A_3)$ (and symmetrically for good 2). By Lemma 2 and Theorem 3,

$$\begin{aligned} Q_1(A_4) &= \sum_{k=1}^{4} \lambda(A_k)\bar{q}_1(A_4, A_k) \\ &= \frac{1}{2} + \lambda_4 \frac{1}{2} + \lambda_3 \bar{q}_1(A_4, A_3)). \end{aligned}$$

The second line follows from our conclusions about the gradients in the relevant cases and the condition $\lambda_1 + \lambda_2 = 1/2$. Similarly

$$Q_1(A_3) = \frac{1}{2} + \lambda_3 \frac{1}{2} + \lambda_4 \bar{q}_1(A_3, A_4)).$$

Theorem 5(c) implies $\bar{q}_1(A_3, A_4) + \bar{q}_1(A_4, A_3) = 1$ by setting $J = \{3, 4\}, A = A_3 \cup A_4$. This fact, the above two equations and the fact that $Q_1(A_3) = Q_1(A_4)$ implies $\bar{q}_1(A_3, A_4) = \bar{q}_1(A_4, A_3) = 1/2$. A similar argument yields $\bar{q}_2(A_3, A_2) = \bar{q}_2(A_2, A_3) = 1/2$. This demonstrates that the only ex post mechanism that can yield the BIC mechanism u is the mechanism shown in Figure 2.

7 A Class of DSIC Mechanisms

In this section, we demonstrate how the restrictions imposed on direct mechanisms in Theorem 5 can guide the construction of the unique DSIC mechanism that implements a class of BIC mechanisms. The mechanisms are all extreme points of the set of BIC mechanisms so, in that sense, they are on the boundary of the set of such mechanisms.[6] Furthermore, as extreme points, they are natural candidates for solutions to common

[6]Note that for any BIC mechanism, u, it is straightforward to show that the mechanism $\alpha u, \alpha > 0$ is always DSIC for α small enough.

optimal mechanism design problems where the objective function is linear in u such as revenue maximization. This class is more general than the example in Section 11 in that there are an arbitrary number of market segments and bidders. However, it is more narrow in that the set of good-specific market segments used, for example, in Theorem 4, $\{A_k^\ell\}_{k=1}^{K_\ell}$, is the same for each good.

There are $L = 2$ goods and $I \geq 1$ bidders and assume $\lambda(x) > 0, \forall x$. For any piecewise linear convex mechanism, u, let its market segments, M_u, be $\{A_k\}_{k=1}^{K}$ each with measure $\lambda_k > 0$. A mechanism, u is in the class of examples if

i) M_u is such that, for $k = 2, \dots K$, the set $A_k \cap A_{k-1}$ is a straight line in $[0,1]^2$ with slope $-m_k$, $m_k \in [0,1]$;

ii) $\nabla u(A_1) = (0,0)$;

iii) for all $l = 2, 3, \dots K$,

$$I \sum_{k=l}^{K} \nabla_2 u(A_k)\lambda_k = 1 - \left(\sum_{k=1}^{l-1} \lambda_k \right)^I .$$

Property i) implies that the sets $\{A_k^\ell\}$ used in Theorem 4 are the same for each good. Property iii) implies that the expected probability of obtaining good 2 satisfies the equation in Theorem 5(c). That is, the Border inequality holds exactly for good 2 for all its market segments (except the no-trade segment).

Observe that continuity of u implies that agent types on the boundary manifolds, $A_k \cap A_{k-1}$ must be indifferent between the outcomes offered in the two sets. Define

$$s_k \equiv \nabla_1 u(A_k)/\nabla_2 u(A_k).$$

Convexity implies that $s_2 = m_2$ and (via continuity)

$$s_k = (1 - \frac{\nabla_2 u(A_{k-1})}{\nabla_2 u(A_k)})s_{k-1} + \frac{\nabla_2 u(A_{k-1})}{\nabla_2 u(A_k)} m_k, k = 3, \dots, K. \tag{10}$$

Convexity implies that $\frac{\nabla_2 u(A_{k-1})}{\nabla_2 u(A_k)} < 1$. Since $m_k \in [0,1]$, this implies that $s_k \in [0,1]$ and is fully determined by the $\nabla_2 u(A_k)$s and m_ks. Accordingly, the gradients $\nabla_1 u(A_k)$ are also fully determined for each A_k. Thus, conditions i)-iii) fully determine the expected probabilities of trade associated with u. The expected payments can be derived but are not relevant for the argument. Since u is convex, has gradients in $[0,1]$ and satisfies the Border inequalities, it is BIC. Since, for this class of mechanisms, the Border inequalities bind on good 2, for every market segment except the no-trade segment, A_1, a mechanism in this class is also an extreme point of the set of BIC mechanisms.

Figure 6 provides an example of such a u where $I = 2$ and $\lambda_k = 1/3$ for all k.

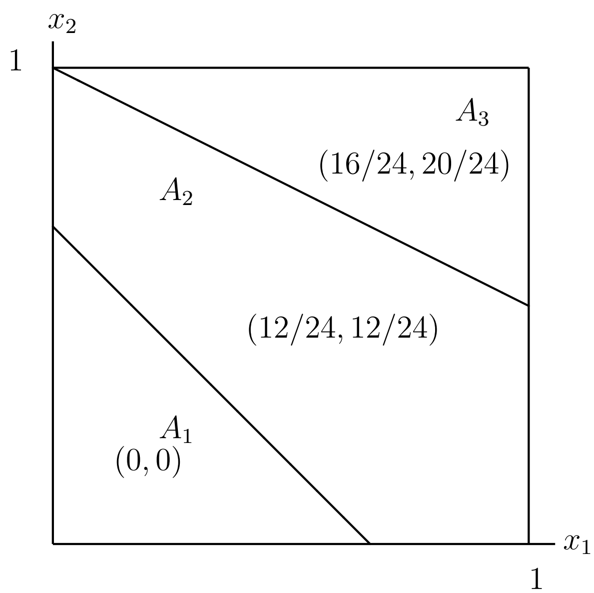


Figure 6: Market Segments for an Example Mechanism

To see that u in this class is DSIC, we show that the properties i)-iii), by applying Theorem 5, yield a unique $p_2(k, n)$ for all k, n. We then assume that the corresponding direct mechanism satisfies $\bar{q}_2(x) = p_2(k, c(x))$ for $x_1 \in A_k$ and assume that $\bar{q}_2(x)$ is the gradient with respect to good 2 (for x_1) of a function $v(x_1, x_{-1})$ that is piecewise linear and convex in x_1. This assumption fully determines the gradient with respect to good 1 of $v(x_1, x_{-1})$. Then, invoking Theorem 1, we assume this gradient corresponds to $\bar{q}_1(x)$. Finally, showing that $E_{x_{-1}}[(\bar{q}_1(x_1, x_{-1}), \bar{q}_2(x_1, x_{-1}))] = (\nabla_1 u(x_1), \nabla_2 u(x_1))$ demonstrates that u is DSIC.

Focus on good 2. For any $x \in X^I$, define $l(x)$ to be the highest market segment containing a bidder type:

$$l(x) = max\{k : c_k(x) > 0\}.$$

Theorem 5(b) implies

$$p_2(1, c(x)) = 0, \forall x, x_1 \in A_1.$$

Corollary 2 and inequality (6) imply that since the Border inequality binds for each nested set A_k^2, if $l(x) > 1$,

$$c_{l(x)}(x) p_2(l(x), c(x)) = 1$$

and

$$p_2(k, c(x)) = 0, k < l(x).$$

Jointly, this implies that for any finite anonymous mechanism that generates u,

$$\begin{aligned}\bar{q}_2(x) &= 0, x_1 \in A_1, \text{ or } x_1 \in A_k, k < l(x) \\ &= \frac{1}{c_{l(x)}(x)}, x_1 \in A_{l(x)}.\end{aligned}$$

Suppose that $\bar{q}_2(x_1, x_{-1})$ equals the second component of $\nabla_{x_1} v(x_1, x_{-1})$, where v is a piece-wise linear convex function with linear pieces defined on a (weakly) coarser partition than M_u. Let $\hat{l}(x_{-1}) = max\{k | x_i \in A_k, i \neq 1\}$. The definition of $\bar{q}_2$ implies that $v(x_1, x_{-1})$ has at most three relevant domains, $\{\cup_{k=0}^{\hat{l}(x_{-1})-1} A_k, A_{\hat{l}(x_{-1})}, \cup_{k=\hat{l}(x_{-1})+1}^{K} A_k\}$ (If $\hat{l}(x_{-1}) = K$, then there are only two market segments, the last one is not present). The probability of trade in the first domain for good 2 is 0 and convexity implies that the probability of trade for good 1 is also 0. The probability of trade for good 2, in the second domain is $1/c_{l(x)}(x)$. Convexity implies that the gradient of v in middle domain is $(m_{\hat{l}(x_{-1})}/c_{l(x)}(x), 1/c_{l(x)}(x))$. The probability of trade for good 2 in the final domain if it is present is 1. Continuity then implies that the gradient of v in the last domain segment is $(s_B, 1)$, where s_B satisfies

$$s_B = (1 - \frac{1}{c_{l(x)}(x)}) m_{\hat{l}(x_{-1})} + \frac{1}{c_{l(x)}(x)} m_{\hat{l}(x_{-1})+1}.$$

(This uses the same argument that was used to construct the s_ks earlier generating equation(10).)

For each x_{-1}, this argument fully characterizes a unique $v(\cdot, x_{-1})$, each of which is convex and satisfies $\nabla v(x_1, x_{-1}) = \bar{q}(x)$ and, so, is a DSIC mechanism. Furthermore, it is a finite anonymous mechanism. Let $\tilde{u} = E_{x_{-1}}[v]$ be the interim utility function that arises from this collection of ex post functions. Since $v(x_1, x_{-1})$ is piece-wise linear and convex for all x_{-1}, $\tilde{u}$ is also piece-wise linear and convex. By construction, the market segments of $\tilde{u}$ are also M_u.

Applying Lemma 3 to the set $A_k^2 = \cup_{\kappa=k}^{K} A_\kappa$,

$$\begin{aligned}I \int_{A_k^2} \nabla_2 \tilde{u}(x_1) \lambda(dx_1) &= \sum_{\|n\|=I, \sum_{j=k}^{K} n_j > 0} \binom{I}{n} \left(\prod_{k=1}^{K} \lambda^{n_k}(A_k) \right) n_j \, p_2(j, n) \\ &= 1 - ((A_k^2)^c)^I \\ &= I \int_{A_k^2} \nabla_2 u(x_1) \lambda(dx_1).\end{aligned}$$

Thus, $\tilde{u}$ generates the same probabilities of trade for good 2 as u.

Since $\tilde{u}$ is convex and has market segments M_u, $\nabla_1 \tilde{u}(A_1) = \nabla_2 \tilde{u}(A_1) = 0$ and

$$\nabla_1 \tilde{u}(A_2)/\nabla_2 \tilde{u}(A_2) = m_1.$$

Set $s_k = \nabla_1 \tilde{u}(A_k)/\nabla_2 \tilde{u}(A_k)$. Convexity now implies that (10) fully defines $\nabla_1 \tilde{u}(A_k) = \nabla_1 u(A_k)$. Thus, $\tilde{u} = u$ and u is DSIC.

8 A Further Example

The class of mechanisms in Section 7 suggested a (restricted) positive result in that a general class of BIC mechanisms were shown to be DSIC. As noted, an important property of this class is that $\{A_k^\ell\}_{k=1}^{K_\ell}$ is the same for both goods. In this section, we show that this property is not sufficient to ensure dominant strategy implementation. Additionally, we describe a 'thick' set of such mechanisms to demonstrate that the failure to achieve DSIC does not merely stem from selecting mechanisms that are extreme points of Bayesian incentive compatible mechanisms.

We first define a piecewise linear function u and show that it is the interim utility of a symmetric BIC mechanism with two bidders. We then show that the interim utility $\alpha u, \alpha \in [0,1]$ cannot be obtained from a DSIC mechanism for α close to 1.

Define $u : [0,1]^2 \to \mathbb{R}$ by

$$u(x_1) = \max\left\{0, \left(\frac{12}{24}, \frac{6}{24}\right) \cdot x_1 - \frac{3}{24}, \left(\frac{19}{24}, \frac{20}{24}\right) \cdot x_1 - \frac{17}{24}\right\}.$$

The function u is composed of three linear pieces. Let $A_k, k \in \{1,2,3\}$ be the domains of each linear piece. These domains are depicted in Figure 7. The pairs of numbers in Figure 7 correspond to the gradients of u, $\nabla u(A_k)$, on each linear piece A_k. The function u can be written as

$$u(x_1) = \begin{cases} (0,0) \cdot x_1 - 0 & \text{if } x_1 \in A_1 \\ (\frac{12}{24}, \frac{6}{24}) \cdot x_1 - \frac{3}{24} & \text{if } x_1 \in A_2 \\ (\frac{19}{24}, \frac{20}{24}) \cdot x_1 - \frac{17}{24} & \text{if } x_1 \in A_3. \end{cases}$$

The prior λ is any distribution that satisfies $\lambda(A_k) = \frac{1}{3}$ for every k.[7]

To determine that u is the interim utility a BIC mechanism, note that u is the pointwise maximum of three linear functions and therefore u is convex. By construction, $\nabla u \in [0,1]^2$. It remains to verify the inequality in Theorem 4. For both goods,

[7] Within each set A_k, the distribution is arbitrary.

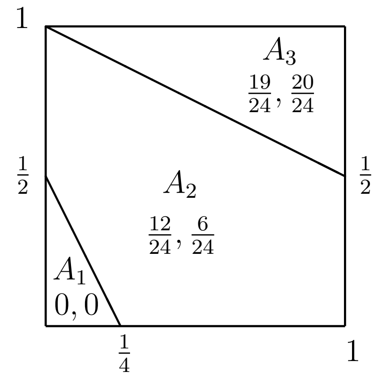


Figure 7: BIC mechanism that has no DSIC equivalent

the collection of nested sets needed to apply the Theorem are

$$\{A_k^\ell\} = \{X, A_3 \cup A_2, A_3\}.$$

Inspection makes it clear that if inequality (7) is satisfied for good 2 using the set A_3, it is also satisfied for good 1 and if it is satisfied for good 1 using the set $A_2 \cup A_3$ it is also satisfied for good 2. As before, the inequality using X will trivially be satisfied as the gradient is zero in the set A_1.

For A_3 and good 2, the left side of (7) evaluates to $2 \cdot \frac{20}{24} \cdot \frac{1}{3} = \frac{40}{72}$ while the right side evaluates to $(1 - (\frac{2}{3})^2) = \frac{5}{9} = \frac{40}{72}$, so the inequality is satisfied exactly. For $A_2 \cup A_3$ and good 1, the left side of (7) evaluates to $2 \cdot (\frac{19}{24} + \frac{12}{24}) \cdot \frac{1}{3} = \frac{62}{72}$ while the right side evaluates to $1 - (\frac{1}{3})^2) = \frac{8}{9} = \frac{64}{72}$ and the inequality is satisfied strictly. Thus u is BIC.

Fix $\alpha \in [0, 1]$. Since u is BIC, so is αu. Also note that

$$2 \int_{A_3} \nabla_2 \alpha u(x_1) d\lambda = \alpha(1 - [\lambda(A_3^c)]^2).$$

We show that there is a $\hat{\alpha} < 1$ such that for all $\alpha \in (\hat{\alpha}, 1]$, αu is not DSIC.

Suppose that αu is DSIC and let $\bar{q}$ be a finite anonymous mechanism that implements it with anonymous probability of trade function $p(k, n)$. Let $n = (n_1, n_2, n_3)$, $\|n\| = 2$ represent the distribution of bidder types across the sets A_1, A_2, A_3. Since αu is a finite mechanism, Theorem 3 implies that, in seeking an ex post $v(x_1, x_2)$ that implements αu, we can restrict attention to piece-wise linear functions such that the domain of the linear segments with respect to x_1 are one of the sets shown in Figure 8 . Furthermore, Theorem 1 implies that $v(x_1, x_2)$ must be convex in x_1 and that the gradient, $\nabla_{x_1} v(x)$ equal $p(j, c(x))$ for $x_1 \in A_j$. Convexity implies that if the market segments for v are as in Figure 8(a), then for all x_1 in $A_2 \cup A_3$, $\nabla_{x_1} v(x)$ is proportionate to $(1/2, 1)$. Similarly, if the market segments for v are as in Figure 8(b), then for all x_1 in A_3, $\nabla_{x_1} v(x)$ is proportionate to $(1, 1/2)$.

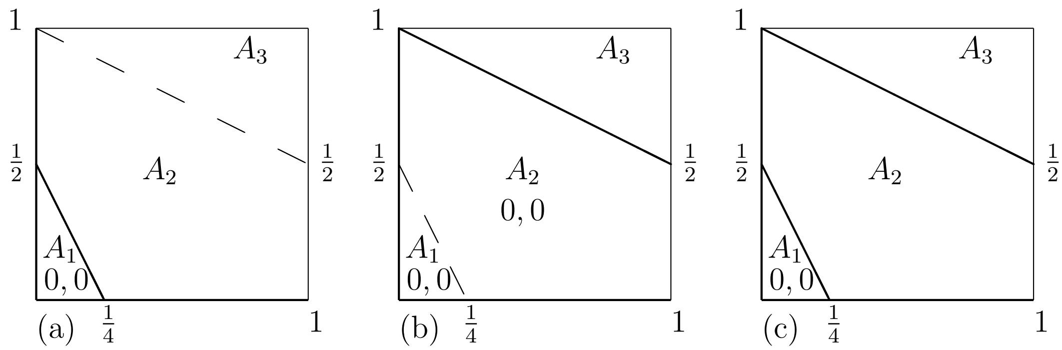


Figure 8: Potential direct mechanisms for αu

Finally, if the market segments for v are as in Figure 8(c), then the gradients of v in A_2 have to be proportionate to $(1/2, 1)$ and the gradient of v in A_2 and A_3 must be consistent with a continuous function. The slope of the manifold between those two sets thus imposes restrictions on these gradients. Specifically, fix x_2 and suppose the market segments of v are as in 8(c). Let $x_1 \in A_2$ and $x_1' \in A_3$. Using the fact that $\nabla_{x_1} v(x) = p(j, c(x))$, this implies

$$-2 = -\frac{p_1(2, c(x_1, x_2))}{p_2(2, c(x_1, x_2))} \tag{11}$$

and

$$-\frac{1}{2} = \frac{p_1(2, c(x_1, x_2)) - p_1(3, c(x_1', x_2))}{p_2(3, c(x_1', x_2)) - p_2(2, c(x_1, x_2))}. \tag{12}$$

Equations (12) and (11) follow by the requirement that the slope of the intersection of the two adjacent linear pieces of v at $A_k \cap A_{k-1}$ correspond to the slope of the manifold at that point. (This is the same logic that generated Equation (10) in Section 7.)

Theorem 5(a) implies that

$$p_1(2, (0, 2, 0)) \leq 1/2. \tag{13}$$

For the cases $n = (0, 1, 1)$ and $n = (1, 0, 1)$, the denominator from Theorem 5($\tilde{c}$) satisfies

$$\binom{I}{n} \prod_{k=1}^{K} \lambda^{n_k}(A_k) = \frac{2}{9}.$$

Applying Theorem 5($\tilde{c}$) to the case $n = (0, 1, 1)$ implies that

$$\begin{aligned} p_2(3, (0, 1, 1)) &\geq 1 - \frac{9(1-\alpha)}{2}(1 - (A_3^c)^2) \\ &= 1 - \frac{9(1-\alpha)}{2}\frac{5}{9} \\ &= 1 - (1-\alpha)\frac{5}{2} \end{aligned}$$

Inequality (7) then implies that $p_2(2,(0,1,1)) \leq (1-\alpha)\frac{5}{2}$ and convexity via (11) implies that

$$p_1(2,(0,1,1)) \leq (1-\alpha)5. \tag{14}$$

Similarly, for the case $n = (1,0,1)$, Theorem 5($\tilde{c}$) implies that $p_2(3,(1,0,1)) \geq 1-(1-\alpha)\frac{5}{2}$. Convexity (via (11) and (12) along with the fact that $p_1(2,(1,0,1)) \leq 1$) implies that

$$\begin{aligned} p_1(2,(1,1,0)) &\leq 4/3 - p_2(2,(1,0,1))) \\ &\leq 4/3 - \frac{2}{3}(1-(1-\alpha)\frac{5}{2}) \\ &= 2/3 + (1-\alpha)\frac{5}{3}. \end{aligned} \tag{15}$$

The expected probability of trade for good 1 given a bidder is in A_2 then must satisfy

$$\begin{aligned} Q_1(A_2) &= \lambda_1 p_1(2,(0,2,0)) + \lambda_2 p_1(2,(0,1,1)) + \lambda_3 p_1(2,(1,1,0)) \\ &\leq \frac{1}{3}(1/2 + (1-\alpha)5 + 2/3 + (1-\alpha)\frac{5}{3}) \\ &= \frac{7}{18} + (1-\alpha)\frac{20}{9}. \end{aligned}$$

The first line uses the definition of a finite anonymous mechanism through Theorem 3 and the second line applies inequalities (15),(13) and (14). This inequality combined with the requirement that $Q_1(A_2) = \alpha/2$ implies that for any $\alpha > \frac{47}{49}$ the mechanism αu cannot be DSIC.

9 Conclusion

The application of Theorem 5 is limited in the sense that restrictions on mechanisms arise only when expected probabilities of trade of a candidate BIC mechanism are close to the Border-Matthews bounds for some subsets of market segments. Nevertheless, such cases are important since many optimization problems will typically require meeting these bounds. For example, classic implementation problems such as maximizing expected seller revenue or maximizing expected social surplus involve maximizing an objective that is linear in the expected probabilities of trade (the Qs) subject to the linear constraints represented by incentive compatibility and the unit good constraint. Solutions to these problems are generically at extreme points of the feasible set and such points require satisfying the Border-Matthews bounds exactly for at least some subsets of the market segments. In these circumstances, Theorem 5 provides a method to rule out the possibility of implementing a potential optimum in dominant strategies (as illustrated in the examples of Sections 6 and 8), or to construct the DSIC mechanism (as done in Section 7) or to provide a roadmap for constructing the DSIC mechanism as close as possible to the optimal mechanism in the sense of retaining the same market segments (as shown in Section 6).

10 References

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11 Appendix: Proof of Theorem 5($\tilde{c}$)

Proof. To prove Theorem 5($\tilde{c}$) consider any $\tilde{n}$, $\|\tilde{n}\| = I, \sum_{j\in J} \tilde{n}_j > 0$. For brevity, define $\sum_{j\in J} \tilde{n}_j p_\ell(j,\tilde{n}) = Z$ and $\binom{I}{\tilde{n}} \prod_{k=1}^{K} \lambda^{\tilde{n}_k}(A_k) = W$. By Lemma 3

$$\begin{aligned}
\alpha(1-[\lambda(A^c)]^I) &= \sum_{j\in J}\left(\sum_{\|n\|=I,\sum_{j\in J} n_j>0} \binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k) n_j p_\ell(j,n)\right) \\
&= \sum_{\|n\|=I,n\neq\tilde{n},\sum_{j\in J} n_j>0} \binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k)\sum_{j\in J} n_j p_\ell(j,n) \\
&+ \binom{I}{\tilde{n}}\prod_{k=1}^{K}\lambda^{\tilde{n}_k}(A_k)\sum_{j\in J}\tilde{n}_j p_\ell(j,\tilde{n}) \\
&= \sum_{\|n\|=I,n\neq\tilde{n},\sum_{j\in J} n_j>0} \binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k)\sum_{j\in J} n_j p_\ell(j,n) \\
&+ \binom{I}{\tilde{n}}\prod_{k=1}^{K}\lambda^{\tilde{n}_k}(A_k)\times 1 \\
&- \binom{I}{\tilde{n}}\prod_{k=1}^{K}\lambda^{\tilde{n}_k}(A_k)(1-Z) \\
&\leq \sum_{\|n\|=I,\sum_{j\in J} n_j>0} \binom{I}{n}\prod_{k=1}^{K}\lambda^{n_k}(A_k) \\
&- \binom{I}{\tilde{n}}\prod_{k=1}^{K}\lambda^{\tilde{n}_k}(A_k)(1-Z) \\
&= (1-[\lambda(A^c)]^I) - \binom{I}{\tilde{n}}\prod_{k=1}^{K}\lambda^{\tilde{n}_k}(A_k)(1-Z) \\
&= (1-[\lambda(A^c)]^I) - W(1-Z)
\end{aligned}$$

The second equality separates out the term in $\tilde{n}$. The next equality adds and subtracts W. The following inequality combines the first two lines of the previous equality and applies (6). The next line rewrites the first line of the previous expression as done in the proof of Lemma 3 and the final line substitutes in W. Rearranging the inequality then yields

$$1-\frac{1-\alpha}{W}(1-[\lambda(A^c)]^I) \leq Z$$

as required. □