Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly

Alejandro M. Manelli*
Department of Economics
W.P. Carey School of Business
Arizona State University
Tempe, Az 85287
and

Daniel R. Vincent *
Department of Economics
University of Maryland
College Park, MD 20742

This Draft: December 2005

Abstract

The seller of N distinct objects is uncertain about the buyer's valuation for those objects. The seller's problem, to maximize expected revenue, consists of maximizing a linear functional over a convex set of mechanisms. A solution to the seller's problem can always be found in an extreme point of the feasible set. We identify the relevant extreme points and faces of the feasible set. We provide a simple algebraic procedure to determine whether a mechanism is an extreme point. We characterize the mechanisms that maximize revenue for some well-behaved distribution of buyer's valuations.

Keywords: extreme point, faces, non-linear pricing, monopoly pricing, multi-dimensional, screening, incentive compatibility, adverse selection, mechanism design.

*We are very grateful to Nicolas Zalduendo, Eric Balder, Kim Border, Thomas Kittsteiner, and Nicholas Yannelis for useful conversations. This work was partially supported by NSF grants SES-0095524 and SES-0241373 (Manelli), and SES-0095729 and SES-0241173 (Vincent).

1 Introduction

We consider a standard setting. An individual wishes to sell N indivisible objects. A potential buyer has private information about his or her valuations—the maximum amounts the buyer is willing to pay for each object. The seller has prior beliefs about the buyer's valuations and the buyer's preferences are linear.

How to carry out the sale so as to maximize seller's expected revenue, is a classic problem in mechanism design. When there is a single object (i.e. $N = 1$), its solution is well known: The seller posts a price and lets the buyer decide whether to purchase the object. The "same" mechanism solves the seller's problem for any seller's beliefs—beliefs determine the actual price posted but not the general form of the optimal mechanism. This property is largely responsible for the success of mechanism design in numerous applications across various fields. There is a regularity in the form of the optimal mechanism that allows to make predictions independent of the seller's beliefs, typically an unobservable component of the model.

We do for the case of multiple objects (i.e. $N \geq 2$), what standard mechanism design did for the $N = 1$ case. We characterize the set of all mechanisms that maximize the seller's expected revenue for *some* seller's beliefs. While in the $N = 1$ case the optimal mechanism has always the same form, in the $N > 1$ case the form of the optimal mechanism varies significantly with seller's beliefs. Our analysis is based on the following observation. The seller's problem is an optimization program where the mechanism is the optimization variable and the seller's expected revenue is a linear objective function. The set of maximizers of the objective function coincides with a face of the feasible set of mechanisms. In addition, a maximizer can always be found at extreme point of the feasible set (Bauer Maximum Principle). By characterizing the relevant faces and extreme points of the feasible set, we identify all potential solutions to the seller's problem.

Consider first the $N = 1$ case. If a mechanism is represented by a function $p(x)$ that indicates the probability that a buyer with reported valuation x will get the object, extreme points are step functions with at most two steps. In the first step, the good is not traded ($p(x) = 0$); in the second step the good is traded for certain ($p(x) = 1$). Extreme-point mechanisms can be implemented by a simple and familiar institution: the seller posts an appropriate price and consumer types separate themselves into types who purchase the object and those who do not. Thus, whatever the beliefs of the seller, posting the appropriate price is a revenue-maximizing mechanism. There is no loss in restricting the optimization problem solely to this class of mechanisms. Note, in particular, that to maximize expected revenue, it is never necessary to randomize in the assignment of the object.

When $N \geq 2$, posting prices—for instance, a price for each individual good and a price for each possible bundle—no longer suffices to implement all the extreme-point mechanisms. Posting prices maximizes expected revenue for some prior beliefs but for many other beliefs, revenue-maximization requires the use of other mechanisms. We find that the set of extreme points contains, in addition to price-posting, many "novel" mechanisms. In particular, extreme points need not be simple functions (Example 2), and even when they are, they may randombly assign objects to consumers (Examples 1 and 3). In contrast to the one-good case, the *form* of the optimal mechanism is determined by the prior distribution of buyer valuations.

We offer two main contributions: A procedure to determine whether a proposed mechanism is an extreme point of the feasible set; and a characterization of the mechanisms that maximize expected revenue for some seller's beliefs.

The procedure that we introduce is based on a characterization of some relevant faces of the feasible set (Theorem 19 and Subsection 6.3). Determining whether a mechanism is an extreme point is equivalent to determining if a consistent, linear system of finitely many equations has a unique solution. If the coefficient matrix has full rank the mechanism is an extreme point. The "novel" extreme points illustrated by our examples are generically so, in the sense that small changes will not alter their status as extreme points. One might have hoped that "novel" extreme points might be peculiar, in the sense that they are not plentiful. This is not the case.

Identifying extreme points of the feasible set is not sufficient for our purposes because there are extreme points that minimize rather than maximize the seller's expected revenue. For instance, the mechanism that never sells the goods and the mechanism that always sells all the goods are extreme points but generate no revenue. One may conjecture that the "novel" extreme points are within the class of mechanisms that never maximize expected revenue. We show that this is not the case. A mechanism specifies the dollar amount $t(x)$ that a buyer of valuation x must transfer to the seller; i.e. $t(x)$ is the seller's revenue of dealing with a buyer of valuation x under the mechanism. We say a mechanism is undominated if there is no alternative mechanism that generates at least as much seller's revenue for all buyer's valuations (and strictly more for some). We prove that *every* undominated mechanism—not just the extreme points—maximizes expected seller's revenue for some independent distribution of valuations (Theorem 9). This describes the relevant portion of the boundary of the feasible set. We also show that all our "novel" examples of extreme points are, indeed, undominated (Lemma 12 and Remark 22).

Finally, we note that our methods have a strong geometric quality.

We conclude the introduction with a brief review of related literature. Our primary concern is with the theory of mechanism design in multi-dimensions, i.e. when the uncertainty variables has more than one dimension. The application on which we focus, that of monopoly pricing, is of independent interest and has a long history in economics. We mention only two works in the area. Adams and Yellen (1976) showed by example that if the buyer's valuations are negatively correlated, the monopolist may obtain higher revenue by bundling—posting a bundle price in addition to prices for the individual goods. McAfee, McMillan, and Whinston

(1989) provide sufficient conditions under which bundling dominates individual sales, and note that when the buyer's valuations are independently distributed those conditions are satisfied. The papers mentioned do not pose a full mechanism design question; they restrict a priori the seller's available instruments.

The multi-dimensional mechanism design literature is not as extensive. We will present a very brief summary here. (Rochet and Stole (2003) offer a very readable survey.) Different authors use slightly different models and assumptions; the interested reader should consult the original sources.

McAfee and McMillan (1988) propose a generalized "single crossing property" to pursue global optima. They use this condition to extend the results of Laffont, Maskin, and Rochet (1985) in a model with a single good but where consumers are differentiated by a two-dimensional parameter.

Wilson (1993) derives first order-conditions for the optimality of a mechanism. In general this approach does not yield a description of the optimal mechanism. Wilson also uses computational methods to obtain particular solutions. Armstrong (1996) extends the methodology used in the one-good case. He obtains a general and useful principle, his "exclusion" principle. He proves that when there are at least two objects to sell, provided the support of the prior density of buyer's valuations is strictly convex, the optimal mechanism will assign no goods to a group of buyers of positive measure. In addition, Armstrong finds closed-form solutions in some environments where the only binding incentive compatibility constraints are along rays from the origin. Rochet and Choné (1998) argue that the assumptions necessary to obtain these environments make them the exception rather than the rule. Armstrong (1999) studies how to find an approximately optimal mechanism in certain models when the number of objects to be sold is large.

Rochet and Choné (1998) analyze a general multi-dimensional screening model. They show that, in general, the monopolist will use mechanisms in which there is bunching, i.e., different consumer-types will be treated equally. Rochet and Choné develop a methodology, their sweeping technique, for dealing with bunching in multiple dimensions. Basov (2001) extends Rochet and Choné's sweeping technique using a Hamiltonian approach. Basov (2005) summarizes the existing literature, illustrates the usefulness of the Hamiltonian approach, and presents many new developments. He analyses a variety of problems including cases where the number of instruments and the number of goods do not coincide.

Thanassoulis (2004) studies a model with two perfectly substitutable goods and shows, among other things, that randomization in the assignment of goods typically dominates deterministic assignments. Thanassoulis (2004) shows that conditions on prior beliefs, previously believed to guarantee that zero-one mechanisms maximize seller's expected revenue, do not do so. (Manelli and Vincent (2004) independently provided another example in this regard.)

One may hope that restricting the class of prior distributions may yield some general results. Our work suggests that if the class of prior densities considered is sufficiently rich, so will be the variety of solutions to the seller's problem. As a point of methodology, we think it may be more promising to proceed in the opposite direction, that is to say, to propose a class of mechanisms and then to find the prior densities under which those mechanisms solve the seller's problem. This is what we do in a companion paper (Manelli and Vincent (2004)). There we identify conditions on the prior distributions under which the zero-one mechanisms, i.e. the posting of prices for bundles, solve the seller's problem.

There have also been some recent contributions to the question of optimal multiple-object auctions. We mentioned only two, Kazumori (2001) and Zheng (2000). (The interested reader should consult the references listed by them.) Kazumori applies the Rochet and Choné's sweeping procedure. Zheng adapts many of the ideas in Armstrong (1996). He also obtains an explicit formula for the non-linear pricing mechanism in his setting.

Section 2 presents the basic notation and describes the model. Section 3 describes the optimization program in terms of the buyer's indirect utility. Section 4 contains examples, both single and multi-dimensional. Section 5 describes the class of mechanisms that maximize the seller's expected revenue. Section 6 characterizes the faces and extreme points of the feasible set. It also introduces a procedure to determine whether a mechanism is an extreme point. An Appendix contains some technical results; lemmas whose label begins with the letter A are located in the Appendix.

2 Preliminaries

2.1 Notation

Sequences are denoted by $\{x_n\}$; when confusion is unlikely we may use x_n to denote both the sequence and its n^{th} element. Given a subset E of a topological space X, $int\, E$ is the interior of E and $\overline{E}$ is the closure of E.

We let I represent the interval $[0, 1]$. For any positive integer N, a *ray* from the origin through an element $x \in I^N$ is defined as $\mathcal{R}_x = \{\delta x : \delta \in [0, \infty)\}$. We denote by $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$ the weakly positive and weakly negative orthants of $\mathbb{R}^N$. We write $\mathbf{0}$ to denote the null element in $\mathbb{R}^N$, and $\mathbf{1}$ to denote $(1, 1, \ldots, 1)$ in $\mathbb{R}^N$. The i^{th} component of any vector $x \in \mathbb{R}^N$ is denoted by x_i; x_{-i} is the vector obtained by removing x_i from x; and (y, x_{-i}) is the vector constructed by replacing x_i in the vector x with $y \in \mathbb{R}$.

Given $A \subset \mathbb{R}^N$, $\mathtt{1}_A$ is the indicator function of A.

The Lebesgue measure is denoted by λ. For $1 \leq p \leq \infty$, $L^p(I^N)$ is the classical Banach space of equivalent classes of real-valued functions f on I^N with finite norm $\|f\|_p$. We will often write simply L^p. If $f \in L^p$ and g is an element of its dual L^q, then the bilinear dual operation is denoted by $\langle f, g\rangle = \int_{I^N} f(x)g(x)\, d\lambda$. By $C^0(I^N)$ we denote the space of continuous functions on I^N.

Let u be a real-valued function defined on a subset E of $\mathbb{R}^N$. Then, for all x in E, $u_+(x) = \max\{u(x), 0\}$, and $u_-(x) = \max\{-u(x), 0\}$. If u is differentiable at x, its gradient at x is denoted by $\nabla u(x)$ in $\mathbb{R}^N$; $\nabla_i u(x)$ is its i^{th} component.

2.2 Model

A seller with N different objects faces a single buyer. The buyer's preferences over consumption and money transfers are given by $U(x, p, t) = x \cdot p - t$, where $x \in I^N$ is the N-vector of buyer's valuations, p is the N-vector of quantities consumed of each good, and $t \in \mathbb{R}$ is a monetary transfer made to the seller.[1] The buyer's valuations x are only observed by the buyer. A density function $f(x)$ represents the seller's beliefs about the buyer's private information x.

The seller's problem is to design a revenue-maximizing mechanism to carry out the sale. By the revelation principle, the seller may restrict attention to direct revelation mechanisms where each buyer type reports his type truthfully.[2] A direct revelation mechanism is a pair of integrable functions

$$p : I^N \longrightarrow I^N$$
$$t : I^N \longrightarrow \mathbb{R},$$

where, given the buyer's valuation x, $p_i(x)$ (i.e. the i^{th} component of $p(x)$) is the probability that the buyer will obtain good i given her valuation x, and $t(x)$ is the expected transfer made by the buyer to the seller.[3]

The buyer's expected payoff under the direct revelation mechanism (p, t), when the buyer has valuation x and reports x' is $p(x') \cdot x - t(x')$. Truthful reporting of valuation x yields the payoff

$$u(x) = p(x) \cdot x - t(x).$$

The buyer must prefer to reveal its information truthfully—incentive compatibility (IC)—and to participate in the mechanism voluntarily—individual rationality (IR). Thus (p, t) satisfies IC and IR if and only if

(IC) for almost all $x, u(x) \geq p(x') \cdot x - t(x') \; \forall x'$

(**IR**) for almost all $x, u(x) \geq 0$.

The seller's problem is therefore to select the functions (p, t) to maximize expected revenue, $E(t)$, subject to IC and IR.

[1] These preferences rule out the complementarities and substitutabilities that would be capture with the more general preferences $\sum_{A \subset \{1,\ldots,N\}} y_A \, p_A - t$ where y_A is the buyer's valuation for the bundle A.

[2] The possibility of multiple equilibria in the direct revelation mechanism is the basis of a well-known critique to the use of the revelation principle.

[3] In an alternative formulation we could allow for correlation in $p(x)$. Since the buyer's preferences are linear there is no loss of generality in the alternative we adopted in the paper.

3 The Approach

When $N = 1$, the seller's problem is usually simplified using a familiar characterization of incentive compatibility: a mechanism satisfies IC if and only if p is non-decreasing. In turn, integrating a non-decreasing p, one obtains the buyer's expected payoff $u(x) = u(0) + \int_0^x p(y)\, dy$. The seller's problem is then generally stated and solved using only the probability-of-trade function p.

We set up the optimization problem in two alternative forms. In the first one, we use the payoff functions u as the variable of optimization. In the second one, developed in Section 5, we use the transfer functions t as the optimization variables. A useful characterization of incentive compatibility, first noted by Rochet (1985), gives us the choice. We state it without proof as Lemma 1.[4]

Lemma 1. *If (p, t) satisfies IC, the corresponding buyer's expected payoff $u(x)$ is convex with gradient $\nabla u(x)$ in I^N for almost all x and $\nabla u(x) = p(x)$ almost everywhere.*

If $u(x)$ is a convex function with gradient $\nabla u(x)$ in I^N for almost all $x \in I^N$, then there exist functions (p, t) satisfying IC such that u represents the corresponding buyer's payoffs. The direct revelation mechanism is defined by $p(x) = \nabla u(x)$ almost everywhere, and $t(x) = \nabla u(x) \cdot x - u(x)$.

The lemma states that, roughly, a mechanism is IC if and only if the corresponding buyer's payoff is convex, with partial derivatives between zero and one.

Lemma 1 characterizes incentive compatibility. To satisfy individual rationality, u must be non-negative. Since the objective is to find an optimal policy for the seller, and since the buyer's expected payoff is non-decreasing, any mechanism that maximizes expected seller's revenue will yield payoff $u(\mathbf{0}) = 0$ to buyers with valuation $x = \mathbf{0}$. This discussion prompts the following definition.

Definition 2. The feasible set in the seller's problem is

$$W = \left\{ u \in C^0(I^N) \mid u(x) \text{ is convex, } \nabla u(x) \in I^N \text{ } a.e., \text{ and } u(\mathbf{0}) = 0 \right\}.$$

Abusing terminology, we refer to any u in W as a feasible mechanism. (A mechanism however is the pair (p, t) yielding u.)

Given a feasible mechanism u, a buyer with type x receives $u(x) = \nabla u(x) \cdot x - t(x)$. The seller's revenue from a buyer of type x when using mechanism u is $t(x) = \nabla u(x) \cdot x - u(x)$. The seller's expected revenue is therefore $E[t(x)] = E\left[\nabla u(x) \cdot x - u(x)\right]$. Hence, the seller's problem is

$$\max_{u \in W} E[\nabla u(x) \cdot x - u(x)]. \tag{1}$$

[4] This characterization has been extensively used in the literature. See, for instance, Armstrong (1996), and Jehiel, Moldovanu, and Stacchetti (1998).

The objective function of the seller's problem is an expectation and is linear on the optimization variable, the function u in problem (1).

Note that any u obtained as a convex combination of elements of W is convex, non-negative, and satisfies the bounds on partial derivatives (its gradient takes values in I^N). Hence, W is itself a convex set. It is also simple to verify that W is compact with respect to the sup-norm topology (Lemma A.1). Thus, the seller wishes to maximize a linear function on a convex compact set.

If this maximization took place on the plane, the solution would be at a point where a hyperplane representing a level set of the objective function is tangent to the feasible set. The solution set may be a singleton or it may include a segment. If the feasible set were a polygon, the solution set would always include a corner point but may also include an entire face of the polygon. This intuition carries over to the infinite dimensional optimization problem. The corresponding definitions are as follows.

Definition 3. Let V be a subset of a linear space X. A set $E \subset V$ is an extreme set of V if

$$[(x = \alpha y + (1-\alpha)z) \in E,\ \alpha \in (0,1),\ z, y \in V] \implies y, z \in E.$$

A face is an extreme set of V that is also convex. An extreme set of V consisting of a single point is an extreme point of V.

Thus a point $u \in V$ is an extreme point of V if and only if for every $g \in X$ with $g \neq 0$, $u + g$ does not belong to V or $u - g$ does not belong to V. Alternatively, $u \in V$ is an extreme point of V if and only if u is not the midpoint of any segment included in V.

The Bauer Maximum Principle (Lemma A.2) implies that *the set of maximizers in the seller's problem must be a face of the feasible set; and that the maximum is achieved at an extreme point of the feasible set.* To study the seller's problem, we characterize extreme points and some relevant faces of W.

4 Examples

This Section illustrates the differences between the single and the multiple-good monopoly. We first characterize the extreme points of W when $N = 1$. We then show, by example, that when $N > 1$, the extreme points of W have very different characteristics.

4.1 A single good, $N = 1$

The following lemma characterizes the extreme points of W.

Lemma 4. *If the seller has a single good, a mechanism $u \in W$ is an extreme point if and only if for almost all buyer's valuations $x \in I^N$, the object is assigned either with probability one or with probability zero, i.e., $p(x) = \nabla u(x) \in \{0, 1\}$.*

Proof. Let $u \in W$ be such that $\nabla u(x) \in \{0,1\}$ for almost all $x \in I$. Then u is absolutely continuous. Let g be any continuous real-valued function defined on I^N. If g is not absolutely continuous, or a.e. differentiable, then $u+g$ is not in W. If $\nabla g(x) \neq 0$ a.e., then $u+g$ or $u-g$ are not in W. Hence, $\nabla g(x) = 0$ a.e, and therefore, since g is absolutely continuous, $g = 0$. We conclude u is an extreme point of W.

To establish the converse select any $u \in W$ that is not a zero-one mechanism. Then, there is a set of positive measure $B \subset [0,1]$ such that $\epsilon < \nabla u(x) < 1-\epsilon$. Let

$$\nabla g(x) = \begin{cases} 1 - \nabla u(x) & \text{if } \nabla u(x) > 0.5 \\ \nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Let $g(x) = \int_0^x \nabla g(z)dz$; $g(x)$ is an absolutely continuous function. We now verify that both $u+g$ and $u-g$ are in W. First, the gradient of $u+g$ is in $[0,1]$:

$$\nabla(u(x) + g(x)) = \begin{cases} 1 & \text{if } \nabla u(x) > 0.5 \\ 2\nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Second, since $\nabla(u(x) + g(x))$ is increasing a.e. in x, $u+g$ is convex. Third, $g(0) = 0$ by construction. Thus $u+g$ is in W. A similar argument applies to $u-g$. Q.E.D.

Lemma 4's characterization of the extreme points of W immediately provides an alternative and simple proof of various well-known results which we summarize below (Myerson 1981).

1. A take-it-or-leave-it offer is the mechanism that maximizes expected seller's revenue among all feasible bargaining mechanisms. (Given the optimal mechanism, p, the offer is $\inf\{x \in [0,1] : p(x) = 1\}$.)

2. Randomization (i.e. "haggling" as in Riley and Zeckhauser (1984)) is not necessary to maximize expected seller's revenue.

3. The revenue-maximizing mechanism is piecewise linear. The buyer's expected payoff u is a piecewise linear function. There is always an optimal mechanism where the transfer t and the probability of trade p are step functions with at most two steps.

The next Subsection illustrates with several examples that these well-known results do not extend to higher dimensions.

4.2 Some Examples with two goods $N = 2$

Two examples demonstrate that if $N \geq 2$ the set of extreme points of W becomes much more varied. The first example identifies an extreme point of W that is piecewise linear but involves randomization. The second example identifies an extreme point that is not piecewise

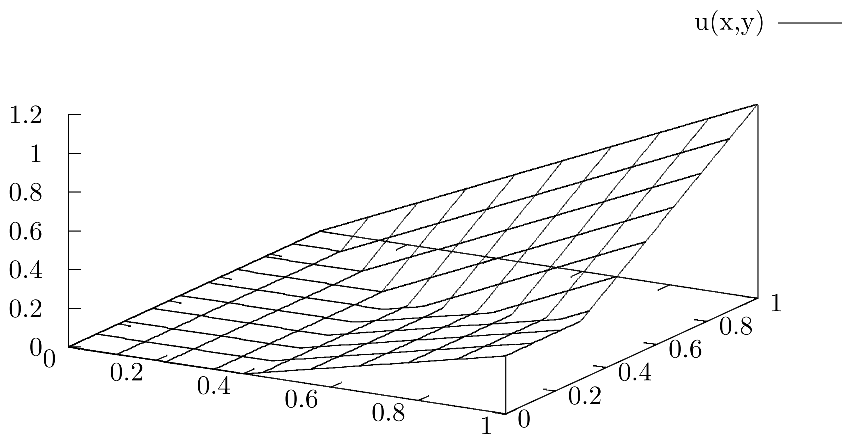

Figure 1: $u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}$

linear. In Section 5 we show that these extreme points are the optimal mechanisms for some well-behaved prior distributions of bidders' types.

Example 1. An extreme point with random assignments. Let $N = 2$ and let $u \in W$ be

$$u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}.$$

The graph of u is depicted in Figure 1. The mechanism u has three linear pieces. Each piece defines a group of consumer-types: $A^0 = \{x \in I^N : \nabla u(x) = (0,0)\}$, $A^1 = \{x \in I^N : \nabla u(x) = (\frac{1}{2}, 0)\}$, and $A^2 = \{x \in I^N : \nabla u(x) = (1,1)\}$. These market segments are depicted in Figure 2. (All three sets A^i are open.) Buyers with valuations in any given set A^i are treated similarly. Consider, for instance, a buyer with valuation $x \in A^1$. The corresponding probabilities of trade are $\nabla u = (\frac{1}{2}, 0)$. The buyer never receives good two. The toss of a fair coin determines whether the buyer receives good one.[5]

We now show that u is indeed an extreme point of W. If u is not an extreme point, then there is a function $g \neq 0$ such that both $u + g$ and $u - g$ are in W. Since both $u + g$ and $u - g$ are convex, they are continuous, and a.e. differentiable. Thus, g must be continuous, and a.e. differentiable. Then for $x \in A^0 \cup A^2$, ∇g must be identically zero; otherwise either $\nabla(u+g)$ or $\nabla(u-g)$ is not in I^N. Therefore $g(x) = 0$ for all $x \in A^0 \cup A^2$.[6] Suppose $g(x) > 0$

[5]The direct mechanism can also be implemented with an indirect mechanism that consists of a menu of choices offering a fifty percent chance at good 1 alone for a price of 0.2 or the full bundle for sure for a price of 1.

[6]This follows, for instance, from Krishna-Maenner (2001) who prove that if a function is convex, the function can be recovered through line integration of any measurable selection of its subdifferential.

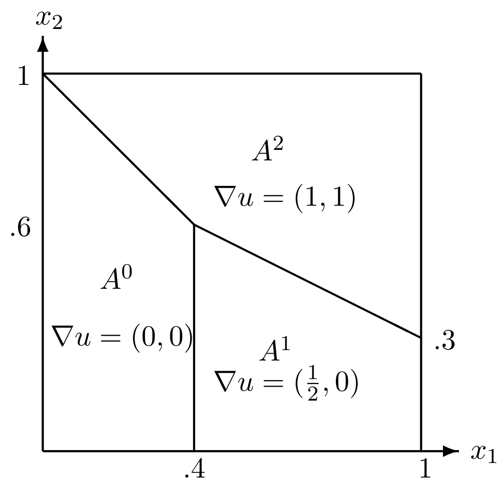

Figure 2: Market Segments

for some $x = (x_1, x_2) \in A^1$. Since $u + g$ is in W, $u + g$ is non-decreasing. Hence $g(x') > 0$ for some $x' \in A^2$. (Simply find $r > 1$ so that $x' = (x'_1, x'_2) = (x_1, rx_2)$.) This is a contradiction.

A similar argument holds using $u - g$ if $g(x) < 0$ for some $x \in A^1$.

With $N = 1$ all extreme-point mechanisms are deterministic; there is no randomization in the assignment of the good. If, however, expected seller's revenue achieves its maximum at more than one extreme point, any randomization between those mechanisms will also maximize seller's revenue. Thus, although randomization may maximize expected revenue, the expected revenue can always be achieved with a deterministic mechanism. More formally, if a mixture of deterministic mechanisms is optimal, it must belong to the relative interior of a non-trivial face of W. The same expected revenue, however, is obtained at any vertex of that face.

Example 1 shows that when $N \geq 2$, there are extreme points that involve randomization. Thus, randomization is a "robust" feature.

Example 2. A non-piecewise linear extreme point. Let $N = 2$ and let $u \in W$ be defined by

$$u(x) = \max\{0, (0.25x_1^2 + x_2 - 0.5), (x_1 + x_2 - 1.01)\}.$$

The mechanism u is piecewise differentiable but not piecewise linear. The function u and its corresponding pieces are depicted in Figure 3.

The mechanism u, determines three pieces, $A^0 = \{x \in I^N : \nabla u(x) = (0,0)\}$, $A^1 = \{x \in I^N : \nabla u(x) = (\frac{1}{2}x_1, 1)\}$, and $A^2 = \{x \in I^N : \nabla u(x) = (1,1)\}$. The boundary between A^0 and A^1 is $\overline{A^0} \cap \overline{A^1} = \{x \in I^N : x_1 \in [0, 0.6],$ and $x_2 = \frac{1}{2} - \frac{1}{4}x_1^2\}$.

Suppose u is not an extreme point. Then there is a function $g(x) \neq 0$ such that both $u + g$ and $u - g$ are in W. For any x in $A^0 \cup A^2$, $\nabla g(x)$ is $(0,0)$. (As in our previous example, $g(x)$ can be recovered by line integration of any measurable selection of its subdifferential.)

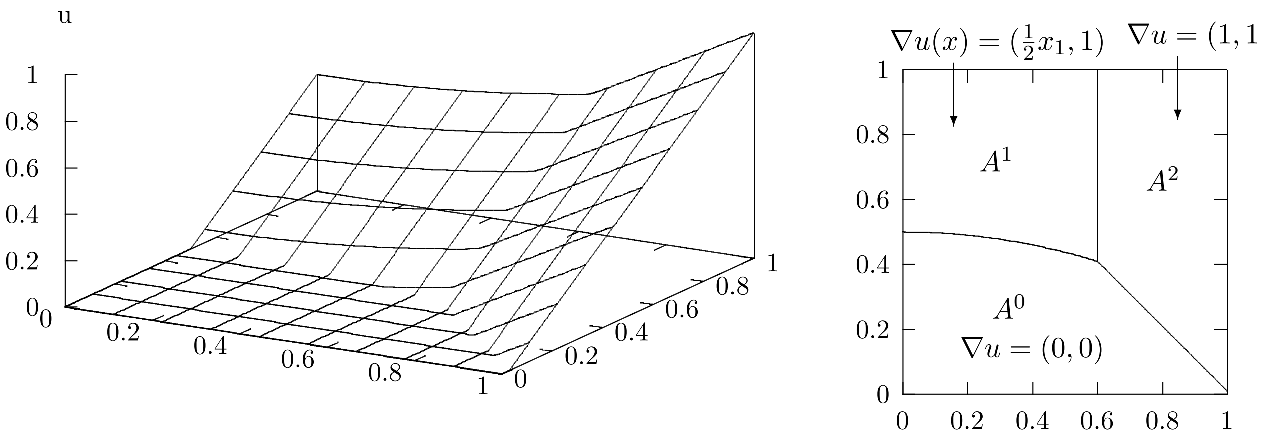

Figure 3: $u(x) = \max\{0,\ (0.25x_1^2 + x_2 - 0.5), (x_1 + x_2 - 1.01)\}$

It thus follows that $g(x)$ must also be 0. If $g \neq 0$, there must be an element $x' = (x'_1, x'_2)$ in A^1 such that $g(x') \neq 0$. Suppose without loss of generality that $g(x') > 0$. (If $g(x') < 0$, a similar argument will apply to $u - g$.) Let $y' = \frac{1}{2} - \frac{1}{4}x_1'^2$; thus (x'_1, y') is the point on the boundary between A^0 and A^1, directly below (x'_1, x'_2). On that boundary both g and u must be zero; thus $g(x'_1, y') = 0 = u(x'_1, y')$. Because $u + g$ must be convex and its gradient is in $[0,1]^2$,

$$u(x'_1, x'_2) + g(x'_1, x'_2) \leq u(x'_1, y') + g(x'_1, y') + (x'_1 - x'_1) + (x'_2 - y').$$

Since $u(x'_1, x'_2) = u(x'_1, y') + (x'_2 - y')$, we obtain, $g(x'_1, x'_2) \leq g(x'_1, y') = 0$, a contradiction.

Both examples illustrate that extreme points may include randomization in the assignment of goods to customers. Example 1 demonstrates that randomization can occur even with piecewise linear mechanisms. Example 2 demonstrates that an extreme point need not be piecewise linear. In both examples, randomization takes place on the assignment of a single good. Example 3 in Subsection 6.2 presents a piecewise linear extreme point with randomization over all goods.

5 Revenue Maximization vs Revenue Minimization

In this section we characterize the mechanisms that maximize seller's revenue for some prior distribution of buyer's valuations. The examples discussed so far are shown to be within this class.

There are extreme points of W that are never a best choice for the seller. Two such extreme points are the mechanism in which no buyer ever gets an object (i.e., $\nabla \bar{u} = \mathbf{0}$), and the mechanism in which buyers always get the object (i.e., $\nabla \bar{u} = \mathbf{1}$). That these mechanisms are extreme points follows easily from the definition noting that the vector of probabilities of trade, $\nabla u(x)$, equals $\mathbf{0}$ and $\mathbf{1}$ respectively. Both mechanisms, however, always yield zero revenue to the seller, $t = 0$. Clearly, the seller will not use the mechanisms described. There are alternative mechanisms, for instance the mechanism $u'(x) = \max\left\{0, (1 \cdot x - \frac{N}{2})\right\}$, which always yield at least as much revenue.

In order to identify mechanisms that are the solution to the seller's problem for some prior density of buyer's valuations, we restate the program so that the optimization variable is the transfer function t. We do so for two reasons. First, the characterization of the mechanisms that maximize seller's revenue for some prior density of valuations is very natural in terms of transfers. Second, transfers underline the geometric quality of our arguments. Both points are developed throughout this section.

Definition 5. The feasible set of transfer functions in the seller's problem is

$$T = \{t \,:\, t(x) = \nabla u(x) \cdot x - u(x) \; a.e., u \in W\}.$$

Since feasible mechanisms $u \in W$ need only be differentiable almost everywhere in I^N, their corresponding transfers t are only defined for almost all x in I^N.

Remark 6. It is simple to verify that T is convex, L^1-compact (Lemma A.3), and that for any extreme point $\bar{u} \in W$, its corresponding transfer function $\bar{t}$ is an extreme point of T. For any $u \in W$ there is a $t \in T$. However, for some $t \in T$ there may be many $u \in W$ that generate it.

In terms of transfers, the seller's problem is

$$\max_{t \in T} E[t]. \tag{2}$$

Although the two forms of the seller's problem—program (1) in terms of payoffs u and program (2) in terms of transfers t—are equivalent, the latter has a more transparent geometric interpretation. The expectation in the objective function of both problems is taken with respect to a density of buyer's valuations, the seller's prior beliefs. If $f \in L^\infty(I^N)$ is such density function, then

$$E[t] = \int_{I^N} t(x) f(x)\, dx = \langle t, f \rangle.$$

The latter notation highlights the bilinear relationship between the density f and the transfer t; f may be seen as a linear function with t as argument, and t is a linear function with f as argument. For any real number r, the set $\{g \in L^1(I^N) : \langle g, f \rangle = r\}$ represents a "hyperplane" in the space L^1—each "hyperplane" corresponds to a level set of the seller's objective function for the given beliefs f.

Intuitively, a mechanism is undominated if there is no alternative mechanism yielding always at least as much revenue to the seller and strictly more in some cases. (A formal definition is provided below.) We prove that, for any undominated mechanism $\bar{t} \in T$, there is a density over valuations f for which $\bar{t}$ is a revenue-maximizing mechanism. That is to say, $\langle \bar{t}, f \rangle \geq \langle t, f \rangle$ for all $t \in T$.

Definition 7. A mechanism $t \in T$ is dominated if there is an alternative mechanism $t' \in T$ such that $t'(x) \geq t(x)$ a.e. in I^N, with strict inequality in a set of positive Lebesgue measure. A mechanism t is undominated if it is not dominated. (We will say a mechanism $u \in W$ is undominated if its corresponding transfer $t(x) = \nabla u(x) \cdot x - u(x)$ is undominated.)

Definition 8. An integrable function $f : I^N \longrightarrow \mathbb{R}_+$ is a density function if $\int_{I^N} f(x)dx = 1$. In addition f satisfies independence if $f(x) = f_1(x_1) \times \ldots \times f_N(x_N)$, where for $i = 1, \ldots N$, $f_i(x_i) = \int f(x_i, x_{-i})\, dx_{-i}$.

The following theorem shows that any mechanism $\bar{t}$ which is undominated is optimal for some seller beliefs. Furthermore, the result holds even if we restrict attention to the narrower class of densities where the buyer's valuations for each good are distributed independently. We briefly describe its proof; the same approach may apply to other classes of prior densities. First, the set $\mathcal{F}$ from which the supporting density will be obtained is defined. (In our case the set of essentially bounded densities satisfying independence.) Arguing by contradiction, suppose that $\bar{t}$ is not the solution to the seller's problem for any relevant density. For each $f \in \mathcal{F}$ there is a mechanism t_f that yields higher expected revenue than the proposed $\bar{t}$. (Otherwise the claim is established.) Any density sufficiently close to f will also yield a higher expected revenue under t_f than under t. Compactness of $\mathcal{F}$ implies that we can select finitely many mechanisms $\{t_f\}$, so that under any density, one of those t_f will give higher revenue than $\bar{t}$. A convex combination $\tilde{t}$ of those finitely many transfers $\{t_f\}$ is constructed using a finite-dimensional separating hyperplane argument to obtain the weights. It is shown that for any density in $\mathcal{F}$, $\tilde{t}$ yields higher expected revenue than $\bar{t}$. Only the compactness of $\mathcal{F}$ has been used so far. To prove that $\tilde{t}$ dominates $\bar{t}$, the set $\mathcal{F}$ must be sufficiently rich. Let E be the set of buyer types where $\bar{t}(x) > \tilde{t}(x)$. The set of possible densities $\mathcal{F}$ must include some density with support in E. Then E must have zero measure or the separation established earlier would be violated. In summary, since $\mathcal{F}$ is weak* compact, and it includes sufficiently many densities, the argument holds.

Theorem 9. *Let $\bar{t} \in T$ be undominated. Then there is a density function $f \in L^\infty$ satisfying independence for which $\bar{t}$ maximizes expected revenue.*

Proof. Let $\mathcal{F}$ be the set of independent density functions $f \in L^\infty(I^N)$. The set $\mathcal{F}$ is weak* compact.

For each $f \in \mathcal{F}$, select $t_f \in T$ such that $\langle t_f, f \rangle > \langle \bar{t}, f \rangle$. If, for some $f \in \mathcal{F}$, no such t_f exists, then for that f, $\langle \bar{t}, f \rangle \geq \langle t, f \rangle \, \forall t \in T$ and the proof is complete.

By continuity, there is a weak* open neighborhood $O_f \ni f$ such that

$$f' \in O_f \implies \langle t_f, f' \rangle > \langle \bar{t}, f' \rangle. \tag{3}$$

The collection $\{O_f : f \in \mathcal{F}\}$ is an open cover of $\mathcal{F}$; by compactness it has a finite subcover $\{O_m : m = 1, \ldots, M\}$. Denote by $\{t_1, t_2, \ldots, t_M\}$ the corresponding transfer functions identified in (3). The identified transfer functions are now used to construct a weakly dominant strategy t' using a finite-dimensional separating hyperplane argument.

Let

$$G = \{\langle t_1 - \bar{t}, f\rangle, \langle t_2 - \bar{t}, f\rangle, \ldots, \langle t_M - \bar{t}, f\rangle : f \in \mathcal{F}\}$$

The set G is a convex subset of $\mathbb{R}^M$ and $G \cap \mathbb{R}^M_- = \emptyset$. Therefore, there is a separating hyperplane $\alpha \in \mathbb{R}^M_+$ such that $\alpha \cdot y > 0 \; \forall y \in G$. Without loss of generality, we may normalize α so that $\sum_{i=1}^M \alpha_i = 1$. Let

$$\tilde{t} = \alpha \cdot (t_1, \ldots t_M).$$

Since T is convex, $\tilde{t} \in T$. Observe that

$$\forall f \in \mathcal{F} \; \langle \tilde{t}, f\rangle - \langle \bar{t}, f\rangle = \alpha \cdot (\langle t_1 - \bar{t}, f\rangle, \ldots, \langle t_M - \bar{t}, f\rangle) > 0. \tag{4}$$

Since f is arbitrary within $\mathcal{F}$, it must be the case that $\tilde{t}$ dominates $\bar{t}$. To see this, let $E = \{x \in I^N : \bar{t}(x) > \tilde{t}(x)\}$. This set is measurable. Suppose $\lambda(E) > 0$. Let $\mathcal{D} = \{A \subset I^N : \langle \tilde{t}, \mathtt{1}_A\rangle \geq \langle \bar{t}, \mathtt{1}_A\rangle\}$. Note that $\mathcal{D}$ is a $\pi - class$, and a $\lambda - class$. Then $\mathcal{D}$ is a sigma field. Since f comes from the class of independent densities, (4) implies that all measurable rectangles in I^N are in $\mathcal{D}$ and, therefore, $\mathcal{D}$ must include the Borel sigma field in I^N. Thus $E \in \mathcal{D}$. This proves that $\tilde{t} \geq \bar{t}$ a.e. in I^N. If the two functions were equal, the separation in (4) would not be strict.

Q.E.D.

Remark 10. Theorem 9 applies to *every* undominated t in T, not just its extreme points. The supporting density function identified need not have full support in I^N.

Lemma 11 below presents a property of undominated mechanisms that links domination, defined on transfers, with the behavior of the corresponding payoff functions. We use this property to show, among other things, that the extreme points in our Examples are undominated. According to the lemma, if a mechanism $t_{u'}$ dominates a mechanism t_u, and if $u'(x)$ exceeds $u(x)$ for some x, then u' must remain above u for all points farther out along the ray through the origin containing x.

Lemma 11. *Let u and u' be two mechanisms in W and let t and t' denote their corresponding transfer functions. Suppose t' dominates t and let x be any element of I^N. Then,*

1. $u'(x) > u(x) \implies u'(\delta x) > u(\delta x)$ *for all* $\delta x \in I^N$, $\delta > 1$, *and*
2. $u'(x) \geq u(x) \implies u'(\delta x) \geq u(\delta x)$ *for all* $\delta x \in I^N$, $\delta > 1$.

Proof. Part 1. Let $u'(x) > u(x)$ and suppose that for some $\delta > 1$, $u'(\delta x) \leq u(\delta x)$. Let $\delta' = \inf\{\delta > 1 : u'(\delta x) \leq u(\delta x)\}$. By continuity, $u'(\delta' x) - u(\delta' x) = 0$ and $\delta' > 1$. Furthermore, $u'(\delta x) > u(\delta x)$ for all $\delta \in (1, \delta')$.

By definition, $t' = \nabla u' \cdot x - u'$ and $t = \nabla u \cdot x - u$ almost everywhere. Since t' dominates t, $(\nabla u'(x) - \nabla u(x)) \cdot x - (u'(x) - u(x)) \geq 0$ for almost all $x \in I^N$.

We will prove the theorem under two additional assumptions and show later that the two assumptions are always satisfied. Suppose for the moment that

$$\forall \delta \in (1, \delta'), \ \ -[u'(\delta x) - u(\delta x)] = \int_{\delta}^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma, \text{ and that} \tag{5}$$

$$\nabla u'(x) \cdot x - u'(x) \geq \nabla u(x) \cdot x - u(x), \ \forall x \in I^N. \tag{6}$$

(Note that (5) holds immediately if u and u' are differentiable everywhere. Assumption (6) simply states that $t'(x) \geq t(x)$ everywhere in I^N. When u and u' are differentiable, t and t' are defined everywhere and (6) holds, provided t' dominates t.)

Using (5) and our observation that $u'(\delta x) - u(\delta x) > 0$ for $\delta \in (1, \delta')$, we obtain

$$\forall \delta \in (1, \delta'), \ \ -[u'(\delta x) - u(\delta x)] = \int_{\delta}^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma < 0.$$

From (6), it follows in particular, that for all γ in (δ, δ'), we have that $(\nabla u'(\gamma x) - \nabla u(\gamma x)) \cdot \gamma x \geq u'(\gamma x) - u(\gamma x) > 0$. This implies that $[\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x > 0$, which contradicts (5) and proves Part 1 under our two additional assumptions.

That the two extra assumptions are unnecessary follows from Lemma A.4 in the Appendix. There we construct selections from the subdifferential of u and u' satisfying both assumptions.

Part 2. A similar argument to that used in Part 1 suffices; we sketch it in the following lines. Suppose in this case that $u'(x) < u(x)$ and for some $\delta < 1$, $u'(\delta x) \geq u(\delta x)$. Let $\delta' = \sup\{\delta < 1 : u'(\delta x) \geq u(\delta x)\}$. By continuity, $u'(\delta' x) - u(\delta' x) = 0$ and $\delta' < 1$. Furthermore, $u'(\delta x) < u(\delta x)$ for all $\delta \in (\delta', 1)$. The proof continues as in Part 1. Q.E.D.

The following lemma illustrates the usefulness of Lemma 11 in identifying undominated mechanisms. It also highlights that, depending on priors, the type of revenue-maximizing mechanism varies significantly.

Lemma 12. *The mechanisms described in Examples 1 and 2 are undominated, and hence they maximize expected seller's revenue for some prior density of buyer's valuations.*

Proof. Before considering each example individually, we highlight the following consequence of Lemma 11 for later use:

Let u and $\bar{u}$ be feasible mechanisms with associated transfers t and $\bar{t}$ respectively. Suppose t dominates $\bar{t}$. Then

$$[u \geq \bar{u} \text{ and for some } x \in I^N, u(x) = \bar{u}(x)] \implies u(\delta x) = \bar{u}(\delta x) \ \forall \delta \in [0, 1]. \tag{7}$$

We now consider the examples individually. In both examples we suppose, arguing by contradiction, that there is a mechanism $u \in W$ with transfer t, and that t dominates $\bar{t}$, the transfer associated with $\bar{u} \in W$. In each example $\bar{u}$ represents the candidate optimum.

Example 1. It is useful to revisit Figure 2.

First, we establish that $u(x) \geq \bar{u}(x)$ for all x. Note that if $x \in A^0$, $u(x) \geq \bar{u}(x) = 0$. Lemma 11 then implies that $u(y) \geq \bar{u}(y)$ for all y in $\mathcal{R}_x \cap I^N$. The set I^N is a subset of $\bigcup_{x \in A^0} \mathcal{R}_x$.

Second, we show that $u(x) = \bar{u}(x)$ for any $x \in A^2$. Since t dominates $\bar{t}$, we have $t(x) \geq \bar{t}(x)$, or equivalently $\nabla u(x) \cdot x - u(x) \geq \nabla \bar{u}(x) \cdot x - \bar{u}(x)$ a.e. in I^N. We also have $\nabla \bar{u}(x) = \mathbf{1}$ a.e. in A^2. Therefore $0 \geq (\nabla u(x) - \mathbf{1}) \cdot x \geq u(x) - \bar{u}(x) \geq 0$ a.e. in A^2. It follows that $u(x) = \bar{u}(x)$ a.e in A^2. Continuity implies the desired result.

Third, we prove that $u(y) = \bar{u}(y)$ for any $y \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. This follows from (7).

Fourth, we show that $u(y) = \bar{u}(y)$, for any $y \in \left[I^N \setminus \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Pick any such $y = (y_1, y_2)$ and suppose by way of contradiction that $u(y) > \bar{u}(y)$. It must be the case that $y_2 < 0.3y_1$; otherwise y would belong to $\left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Let $y_2' = 0.3y_1$. Then, $u(y) > \bar{u}(y) = \bar{u}(y_1, y_2') = u(y_1, y_2')$. Hence u is not monotone. It follows that $u \notin W$, a contradiction.

We have demonstrated that $u = \bar{u}$. This implies $t = \bar{t}$ and therefore t does not dominate $\bar{t}$.

Example 2. Figure 4 is useful in following the proof.

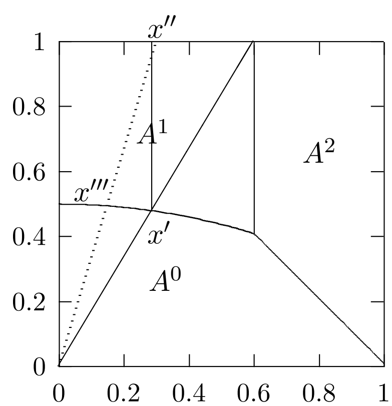

Figure 4: Proof of Lemma 12

Similar arguments to those used in the previous example establish that first, $u \geq \bar{u}$; second, $u(x) = \bar{u}(x)$ for all $x \in A^2$; and third, $u(y) = \bar{u}(y)$ for all $y \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Note that the area underneath the solid line in the figure, $I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)$, is $\{(y_1, y_2) \in I^N : \frac{3}{5}y_2 \leq y_1\}$.

Fourth, let $x' = (x_1', x_2')$ be the intersection of the line $x_2 = \frac{5}{3}x_1$, and the line defining the boundary between A^0 and A^1, $x_2 = \frac{1}{2} - \frac{1}{4}x_1^2$. We show that $u(y) = \bar{u}(y)$ for any y with $y_1 \in [x_1', 0.6]$ and $y_2 > \frac{5}{3}y_1$. Pick any such $y = (y_1, y_2)$, and suppose $u(y) > \bar{u}(y)$. Note that $(y_1, \frac{5}{3}y_1) \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$, and therefore $u(y_1, \frac{5}{3}y_1) = \bar{u}(y_1, \frac{5}{3}y_1)$. Using the fact that $0 < \nabla u_2 < 1$, that $\nabla \bar{u}_2 = 1$, and that $u(y) > \bar{u}(y)$ we generate the following contradiction, $\left(y_2 - \frac{5}{3}y_1\right) + u(y_1, \frac{5}{3}y_1) \geq u(y) > \bar{u}(y) = \bar{u}(y_1, \frac{5}{3}y_1) + \left(y_2 - \frac{5}{3}y_1\right)$. We have proved that $u'(y) = u(y)$ for any $y \in E$, where $E \subset I^N$ is the union of the convex hull of $\{(0,0), x', (1,0)\}$ and the convex hull of $\{x', x'', (1,1), (1,0)\}$.

Fifth, we prove that $u'(y) = u(y)$ for any $y \in (I^N \cap \bigcup_{x \in E} \mathcal{R}_x)$. This follows from (7).

Note that the proof proceeds by showing that $u = \bar{u}$ in a given area, and then that in the rays defined by that area u must also equal $\bar{u}$. We continue with this procedure. The intersection of the segment $[\mathbf{0}, x'']$ with the boundary $A^0 \cap A^1$, defines a point x'''. In turn the points $(x''', (x_1''', 1), x'', x')$ define a new area. Arguments similar to those used in point four, establish that $u = \bar{u}$ in that region. Continuing with this process establishes that $u = \bar{u}$ for all $x \in I^N$. Q.E.D.

In Section 6.1 we prove that any undominated, piecewise-linear mechanism must include a market segment where all goods are traded with certainty, and a market segment where there is no trade at all (Theorem 16).

6 The Structure of Potential Solutions

Section 6 contains our main results. It outlines a procedure to determine whether a proposed mechanism is an extreme point. The procedure is based on the structure of the feasible set.

Before presenting our findings, we summarize them, although not in the order in which they are derived. First, we show that restricting attention to piecewise linear mechanisms is, essentially, without loss of generality. We have already shown that non-piecewise linear mechanisms can be extreme points (Example 2), and they can even maximize expected revenue (Lemma 12). That piecewise linear mechanisms are dense in W is a straightforward observation (Lemma A.5). We demonstrate that, in addition, the set of *piecewise linear extreme points*, is dense in the set of *all* extreme points (Theorem 21). Since expected seller's revenue is always maximized at an extreme point (Bauer Maximum Principle), there is little loss in restricting attention to *piecewise linear* extreme points.

Second, we show that it is comparatively simple to verify whether a *piecewise linear* mechanism is an extreme point. Generally, a mechanism $\bar{u}$ in W is an extreme point when it is *not* possible to move from $\bar{u}$ in any direction g and in the opposite direction $-g$, remaining in both cases within the feasible set W, i.e. $\bar{u} + g$ or $\bar{u} - g$ must be outside W. Thus, to determine whether a given mechanism is an extreme point, the number of directions g to check is quite large. The situation is simpler, however, when $\bar{u}$ is piecewise linear. Piecewise linear mechanisms partition the set of buyers in finitely many pieces or subsets such that consumer types in each piece are treated similarly by the mechanism. We demonstrate that to verify whether a piecewise linear mechanism $\bar{u}$ is an extreme point, it suffices to check directions g that are also piecewise linear, and that define *the same pieces* or subsets as $\bar{u}$ does (Theorem 17). This observation is fundamental in the sense that all other results in the section rely on it.

Finally, we describe an algebraic method to identify extreme points. Determining whether a piecewise linear mechanism is an extreme point is, essentially, equivalent to determining if a consistent, linear system of equations has a unique solution. To obtain this procedure we characterize first some useful faces of W. Pick any piecewise linear mechanism $\bar{u}$ and its implicit partition of buyer's types. We define a face relative to $\bar{u}$ and more importantly, relative to the partition defined by $\bar{u}$. More precisely, the collection of all piecewise linear mechanisms with coarser partitions of buyer's types than the partition defined by $\bar{u}$ is a face $F_{\bar{u}}$ of W (Theorem 19).

We present our result in three subsections. The first one contains the theoretical results. The second consists of an example of an undominated extreme point that involves randomization for all goods. The example also illustrates the use of some of the results developed in the first subsection. The last subsection describes how to use our characterization of faces to determine if a piecewise linear mechanism is an extreme point. We use the example to illustrate this methodology.

6.1 Piecewise Linear Mechanisms

A function u is piecewise linear if it consists of finitely many linear pieces. Because of incentive compatibility, feasible mechanisms are the pointwise supremum of linear functions with gradient in the N-dimensional unit cube (see Lemma 1 and the discussion surrounding it). A piecewise linear mechanism must, therefore, be the pointwise maximum of *finitely* many linear functions. Because of individual rationality, one of those linear functions is the null map. These observations establish the following remark.

Remark 13. The mechanism u is piecewise linear and feasible if and only if there is a finite family of linear functions, $a^j \cdot x - b^j$ with $a^j \in I^N$ and $b^j \in \mathbb{R}$ for $j = 0, 1, \ldots, J$, such that for every x in I^N, $u(x) = \max\{a^j \cdot x - b^j : j = 0, 1, \ldots, J, \text{ and } a^0 = \mathbf{0}, b^0 = 0\}$.

A piecewise linear mechanism "partitions" the set I^N of consumer types into finitely many groups. Types within each group are treated equally, in the sense that they all face the same probabilities of trade and pay the same transfer. We refer to those groups as market segments. Market segments are the effective domains of the different linear pieces forming the mechanism.

Definition 14. Let u be a piecewise linear mechanism in W, and let $\{a^j \cdot x - b^j\}_{j=0}^J$ be the smallest (by set inclusion) family of linear functions such that $u(x) = \max\{a^j \cdot x - b^j : j =$

$0, 1 \ldots, J\}$. We say that $A^j = \{x \in I^N : a^j \cdot x - b^j > a^k \cdot x - b^k \ \forall k \neq j\}$ is a market segment of the mechanism u. We denote by $m(u)$ the collection of all such market segments, by ∇u^{A^j} the gradient of u in A^j (i.e., $\nabla u^{A^j} = \nabla u(x)$ for every x in A^j), and by t^{A^j} the transfer from members of A^j to the seller (i.e., $t^{A^j} = t(x)$ for every $x \in A^j$ where t is the transfer associated with u). For ease of notation, we will use ∇u^j instead of ∇u^{A^j} when no confusion is possible.

A market segment is a collection of buyer types x satisfying finitely many, linear, strict inequalities. Redundant pieces, such as those that are never a maximum or those that are, at best, a weak maximum, are eliminated from the definition. From this consideration we derive the following remark.

Remark 15. Given a piecewise linear, feasible mechanism u, its market segments are convex, and relatively open subsets of I^N with full dimension. Given any two market segments A^j and A^k, $k \neq j$, then $\nabla u^{A^j} \neq \nabla u^{A^k}$.

The following Theorem states that any undominated, piecewise-linear mechanism must include a market segment where all goods are traded with certainty, and a market segment where there is no trade at all. We use it in Section 6.3 in conjunction with Theorem 20 to identify extreme points.

Theorem 16. *Let u be an undominated, piecewise linear mechanism in W. Then there are market segments A^0 and A^J such that no good is assigned if the buyer's type is in A^0, and all goods are assigned with certainty if the buyer's type is in A^J, i.e. $\nabla u^{A^0} = \mathbf{0}$ and $\nabla u^{A^J} = \mathbf{1}$.*

We provide the proof in the Appendix.

Theorem 17 is the fundamental building block of this section. To determine that a mechanism u is *not* an extreme point of W it suffices to find a single function g such that moving from u in the direction g yields a feasible mechanism $u + g \in W$, and moving in the opposite direction also yields a feasible mechanism $u - g \in W$. To determine that a mechanism $\bar{u}$ *is* an extreme point, however, involves verifying that $u + g$ or $u - g$ are not feasible for every possible direction g. Theorem 17 reduces significantly the number of directions that must be verified when dealing with piecewise linear mechanisms. It states that if $\bar{u}$ is piecewise linear, it suffices to verify only the piecewise linear, continuous functions g whose pieces have, as effective domain, the market segments of $\bar{u}$.

Theorem 17. *Let $\bar{u}$ be a piecewise linear mechanism. The mechanism $\bar{u}$ is an extreme point of W if and only if $\bar{u} + g \notin W$ or $\bar{u} - g \notin W$, for every continuous, piecewise linear function $g : I^N \longrightarrow \mathbb{R}$ such that $A \in m(\bar{u})$ implies g is linear on A.*

Proof. By definition of extreme point, necessity is obvious.

We prove sufficiency. If $\bar{u}$ is not an extreme point of W, then there is a function g such that $\bar{u} + g \in W$ and $\bar{u} - g \in W$. This implies that g must be continuous for otherwise $\bar{u} + g$ is not continuous.

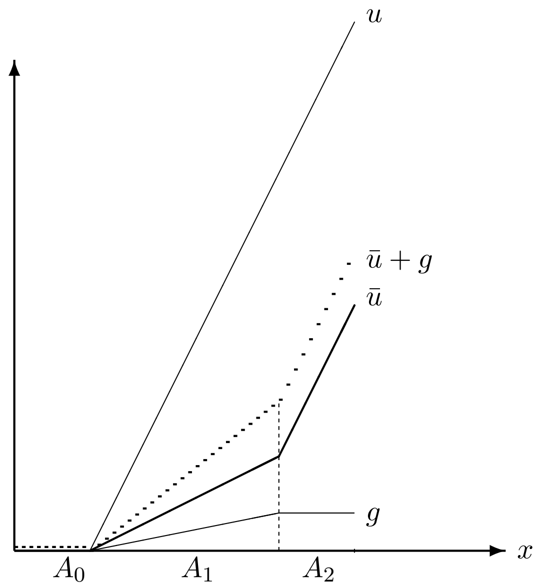

Figure 5: Identifying Extreme Points

Pick any market segment A in $m(\bar{u})$. The restriction of u to A is linear. Both $\mathtt{1}_A(\bar{u}+g)$ and $\mathtt{1}_A(\bar{u}-g)$ are convex when restricted to the domain A. Therefore, g must be linear within A. Q.E.D.

Figure 5 illustrates Theorem 17. The mechanism $\bar{u}$ determines three market segments, A^0, A^1, A^2. To determine if $\bar{u}$ is an extreme point of W, it suffices to check whether $\bar{u}+g$ and $\bar{u}-g$ are in W for functions g that are linear on the market segments of $\bar{u}$.

We use Theorem 17 repeatedly in this and the next Sections. We also use it to show that the mechanism in Example 3 is an extreme point. In that example *all goods* are assigned randomly within a market segment.

We now characterize some very useful faces of W. Theorem 19 states, roughly, that given a piecewise linear mechanism $\bar{u}$, the set of piecewise linear mechanisms in W with the same market segments as $\bar{u}$ is a face of W. Theorem 20 shows that the market segments determine whether a candidate piecewise linear mechanism is an extreme point.

Definition 18. Given any piecewise linear mechanism $\bar{u}$ in W, define the set

$$F_{\bar{u}} = \{u \in W : \forall A \in m(\bar{u}),\ u \text{ is linear on } A; \text{ and } [\nabla_i \bar{u}(x) \in \{0,1\} \implies \nabla_i u(x) = \nabla_i \bar{u}(x)]\}.$$

The set $F_{\bar{u}}$ contains the mechanisms u that have three properties: (i) u must be linear on every market segment of $\bar{u}$; (ii) whenever $\bar{u}(x)$ assigns an object i with probability zero, so does $u(x)$; and (iii) whenever $\bar{u}(x)$ assigns an object i with probability one, so does $u(x)$. Note that because of (i), consumers in two different market segments of $\bar{u}$, may be treated equally by some u in $F_{\bar{u}}$; for instance, in Figure 5, consumers in A^1 and A^2 in $m(\bar{u})$ are treated equally by u.

Theorem 19. *Let $\bar{u}$ be any piecewise linear mechanism in W. The set $F_{\bar{u}}$ is a face of W.*

Proof. Let u be any element of $F_{\bar{u}}$. Suppose $u = 1/2u' + 1/2u''$, for some $u', u'' \in W$, $u' \neq u''$. Pick any A in $m(u)$, and suppose u' is not linear on A. Then, since u' is convex, there are $x', x'' \in A$ such that $u'(\bar{x}) < \frac{u'(x')+u'(x'')}{2}$ where $\bar{x} = \frac{x'+x''}{2}$. Note that

$$\begin{aligned} u''(\bar{x}) - \frac{1}{2}[u''(x') + u''(x'')] &= 2u(\bar{x}) - u'(\bar{x}) - [u(x') - \frac{1}{2}u'(x')] - [u(x'') - \frac{1}{2}u'(x'')] \\ &= [2u(\bar{x}) - u(x') - u(x'')] - [u'(\bar{x}) - \frac{1}{2}u'(x') - \frac{1}{2}u'(x'')] \\ &= -[u'(\bar{x}) - \frac{1}{2}u'(x') - \frac{1}{2}u'(x'')] > 0, \end{aligned}$$

which implies that u'' is not convex and therefore u'' is not an element of W, a contradiction. We conclude that u' must be linear on A. A symmetric argument shows that u'' must also be linear on A. Since A is arbitrary, both u' and u'' must be linear on each A. This proves that $F_{\bar{u}}$ is an extreme set of W. Noting that $F_{\bar{u}}$ is also convex, completes the proof. Q.E.D.

The definition of $F_{\bar{u}}$ includes a restriction on the gradient of its members. If we did not impose the restriction on gradients, the resulting set of mechanisms would still be a face of W. However, not all faces of W are useful for our problem. For instance, the entire set W is a face, and the singleton containing any extreme point is a face. The faces we defined are useful to identify extreme points. Pick any piecewise linear mechanism $\bar{u}$ and consider the face $F_{\bar{u}}$ described earlier. Theorem 20 below demonstrates that $\bar{u}$ is an extreme point if and only if $F_{\bar{u}}$ is the singleton $\{\bar{u}\}$.

Theorem 20. *Let $\bar{u}$ be a piecewise linear element of W. Then $\bar{u}$ is an extreme point of W if and only if $F_{\bar{u}} = \{\bar{u}\}$.*

Proof. One direction is trivial: Theorem 19 demonstrates that $F_{\bar{u}}$ is a face of W; therefore if $F_{\bar{u}} = \{\bar{u}\}$, $\bar{u}$ is an extreme point of W.

We prove the converse. Suppose $\bar{u}$ is a piecewise linear element of W and suppose there is $u' \in F_{\bar{u}}$, $u' \neq \bar{u}$. We will show that $\bar{u}$ is not an extreme point.

Let $\{A^j\}_{j=0}^J$ be the family of all market segments $m(\bar{u})$. It follows from the definition of $F_{\bar{u}}$, that both $\nabla\bar{u}$ and $\nabla u'(x)$ are constant in any market segment A^j in $m(\bar{u})$. For simplicity, we denote those constants as $\nabla\bar{u}^j$ and $\nabla u'^j$ respectively.

For $r \in [0, 1]$, define functions mapping I^N into $\mathbb{R}$ by by

$$\begin{aligned} v_r &= (1-r)\bar{u} + ru', \\ w_r &= (1-r)\bar{u} + r[2\bar{u} - u']. \end{aligned}$$

The functions v_r and w_r are piecewise linear, indeed they are linear on each market segment A^j in $m(\bar{u})$. For any such $A^j \in m(\bar{u})$, denote by ∇v_r^j and ∇w_r^j the gradients of v_r and w_r respectively (evaluated at any $x \in A^j$), and denote by $t_{v_r}^j$ and $t_{w_r}^j$ the corresponding intercepts. For any r, both ∇v_r and ∇w_r take at most $J+1$ values, the number of market segments defined by $\bar{u}$.

Pick any r. The function v_r is in W because it is the convex combination of elements of W. By construction $\bar{u}$ is the midpoint of the interval $[w_r, v_r]$. Hence, it suffices to show that for some $r \in (0,1)$, the function w_r is in W to prove that $\bar{u}$ is not an extreme point. We must prove that (i) $w_r(\mathbf{0}) = 0$; (ii) ∇w_r is in I^N; and (iii) w_r is convex. Point (i) is obvious from the definition of w_r.

The proofs of (ii) and (iii) follow from an observation: w_r is piecewise linear, defines the same pieces as $\bar{u}$, and converges uniformly to $\bar{u}$. Since the gradient ∇w_r takes only finitely many values, ∇w_r also converges uniformly to the gradient $\nabla \bar{u}$. The details are below.

We verify (ii). We prove that there is an $r' \in (0,1)$ such that for every $r \in (0, r')$ and every j, ∇w_r^j is in I^N.

If for some good i and market segment $A^j \in m(\bar{u})$, $\nabla_i \bar{u}^j \in \{0,1\}$, then $\nabla_i u'^j = \nabla_i \bar{u}^j$. Thus, $\nabla_i w_r^j$ is in $\{0,1\}$ for any r.

If for some good i and market segment j, $0 < \nabla_i \bar{u}^j < 1$, let

$$\epsilon = \min_{i,j} \left\{ \min\{(1 - \nabla_i \bar{u}^j), \nabla_i \bar{u}^j\} : 0 < \nabla_i \bar{u}^j < 1 \right\}.$$

The minimum is taken over finitely many values. As r tends to zero, the functions w_r converge uniformly to u. It follows from Lemma A.3 that ∇w_r converges pointwise to $\nabla \bar{u}$. Hence there is $r_i^j \in (0,1)$ such that $|\nabla_i w_r^j - \nabla_i \bar{u}^j| < \epsilon$ and therefore, $0 < \nabla_i w_r^j < 1$. Letting $r' = \min_{j,i}\{r_i^j\}$, the claim (ii) is established.

We verify (iii). We will prove that there is $r'' \in (0,1)$ such that $r \in [0, r'')$ implies that w_r is convex.

For $x \in \mathbb{R}^N$, denote by

$$w_r^j(x) = \nabla w_r^j \cdot x - t_{w_r}^j.$$

The function w_r^j is the extension to the entire space $\mathbb{R}^N$ of the linear piece forming w_r on A^j. Similarly, we denote by $\bar{u}^j$ and u'^j the extensions of the linear pieces forming $\bar{u}$ and u' on A^j respectively. Thus, $w_r^j = (1+r)\bar{u}^j - r u'^j$.

Fix any A^j and A^k in $m(\bar{u})$ such that $\dim(\bar{A}^j \cap \bar{A}^k) = N - 1$. For any $x \in A^j$,

$$w_r^j(x) - w_r^k(x) = (1+r)[\bar{u}^j(x) - \bar{u}^k(x)] - r[u'^j(x) - u'^k(x)]. \tag{8}$$

Since A^j and A^k share an $(N-1)$-dimensional boundary and since u' is linear on A^j and A^k, we obtain that

$$\exists \alpha \in \mathbb{R} : u'^j - u'^k = \alpha[\bar{u}^j - \bar{u}^k].$$

(This follows because $(u'^j - u'^k)$ and $(\bar{u}^j - \bar{u}^k)$ are affine operators with the same kernel of dimension $N-1$.) Replacing the last expression in (8), we obtain that for any $x \in A^j$

$$w_r^j(x) - w_r^k(x) = (1 + r - r\alpha)[\bar{u}^j(x) - \bar{u}^k(x)].$$

The second factor is non-negative because $\bar{u}$ is convex and therefore $\bar{u}$ is the maximum of the linear functions forming it (Remark 13). There is $r_{j,k}$ in $(0,1]$ such that for each $r \in [0, r_{j,k}]$ the first factor is strictly positive thus making the entire expression non-negative:

$$\forall x \in A^j, w_r^j(x) - w_r^k(x) \geq 0. \tag{9}$$

The value $r_{j,k}$ depends on the chosen market segments $A^j, A^k \in m(\bar{u})$. Let $r'' = \min\{r_{j,k} : A^j, A^k \in m(\bar{u}), \dim(\bar{A^j} \cap \bar{A^k}) = N-1\}$. Since there are finitely many market segments, $r'' > 0$. Hence, using (9), we have proved that for every $r \in [0, r'']$, for every market segments $A^j, A^k \in m(\bar{u})$ with $\dim(\bar{A^j} \cap \bar{A^k}) = N-1$, and for every $x \in A^j \cup A^k$,

$$w_r(x) = \max\{w_r^j(x), w_r^k(x)\}.$$

We now prove that w_r is convex. For any $(x,y) \in I^N \times I^N$, let

$$f(x,y) = \frac{1}{2}[w_r(x) + w_r(y)] - w_r\left(\frac{x+y}{2}\right).$$

Suppose by way of contradiction that w_r is not convex. Then there are market segments A^j, A^k in $m(\bar{u})$ and points $x' \in A^j$, $y' \in A^k$ such that $f(x', y') < 0$. Since f is continuous, there is an $\epsilon > 0$ such that for any $x \in B(x', \epsilon)$ and $y \in B(y', \epsilon)$, the function $f(x,y) < 0$.

Denote by $[x,y] = \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$. Let $C = \{[x,y] : x \in B(x', \epsilon), y \in B(y', \epsilon)\}$. Then C is an N-dimensional cylinder.

There is $[x,y]$ in C such that any element $z \in [x,y]$ belongs to the closure of, at most, two market segments: if $z \in \overline{D'} \cap \overline{D''}$ for $D', D'' \in m(\bar{u})$, then $z \notin \overline{D}$ for any $D \in m(\bar{u})$, $D' \neq D \neq D''$.

The proof of this fact is based on the following observation. Let $B(0,\epsilon) \subset \mathbb{R}^{N-1}$. Define the N-dimensional cylinder

$$C' = \{z \in \mathbb{R}^N : z = (x,d), \text{ where } x \in B(0,\epsilon), d \in \mathbb{R}_+\}.$$

For $h = 1, \ldots, H$, let S_h be an affine subspace of $\mathbb{R}^N$ with $\dim(S_h) \leq N-2$. We will show that there is an $x \in B(0,\epsilon)$ such that $\{(x,d) : d \in \mathbb{R}_+\} \cap S_h = \emptyset$ for every h. To see this, let s_h be the projection of S_h into $B(0,\epsilon)$. Then $\dim(s_h) \leq N-2$ and therefore has measure zero in $B(0,\epsilon)$. The countable union of set of measure zero, has measure zero. Thus there exists $x \in B(0,\epsilon)$ such that $x \notin s_h$ for all h. Then, the set $\{(x,d) : d \in \mathbb{R}_+\}$ is the desired path. Q.E.D.

Theorem 20 reduces the process of verifying whether a piecewise linear mechanism u is an extreme point to determining if there are other mechanisms in the face F_u. In turn this is equivalent to determining whether a consistent system of linear equations has multiple solutions. We expand and illustrate this statement in Subsection 6.3, where we analyze another example.

Before turning to that pursuit, we discuss another application of the faces F_u identified above. Although some extreme points are not piecewise linear, there is no great loss in restricting attention to piecewise linear extreme points of W. This is the content of the following theorem.

Theorem 21. *The set of feasible mechanisms W is the closed convex hull of the set of its piecewise linear, extreme points.*

Proof. Let $\bar{u}$ be an extreme point of W that is not piecewise linear, and let $\bar{t}(x) = \nabla\bar{u}(x) \cdot x - \bar{u}(x)$ be its corresponding transfer function. Let $I_n = \{0, 1/n, 2/n, \ldots, n/n\}$. Thus, $(I_n)^N$ is a discretization of the set I^N, increasingly finer as n tends to infinity. For each $x \in I^N$, define $v^n(x) = max_{z\in(I_n)^N}[\nabla\bar{u}(z) \cdot x - \bar{t}(z)]$. It is routine to verify that v^n belongs to W, and that

$$\sup_{x\in I^N} |v^n(x) - \bar{u}(x)| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{10}$$

The mechanism v^n belongs to F_{v^n}, the face of W defined earlier. Note that if e_n^k is an extreme point of F_{v^n}, then it is also an extreme point of W. (To see this, assume by way of contradiction that e_n^k is not an extreme point of W. Then $e_n^k = (1/2)e' + (1/2)e''$ for some e', e'' in W, $e' \neq e^n \neq e''$. Since F_{v^n} is a face, however, e', e'' must then be in F_{v^n}. But then e_n^k is not an extreme point of F_{v^n}.)

The face F_{v^n} is convex and compact; therefore F_{v^n} is the closure of the convex hull of its extreme points (Krein-Milman Theorem). Hence, for each v^n, there is

$$w^n = \sum_{k=1}^{K^n} \alpha_n^k e_n^k$$

where $\alpha_n^k \in (0, 1]$; $\sum_{k=1}^{K^n} \alpha_n^k = 1$; for each $1 \leq k \leq K^n$, e_n^k is a piecewise linear extreme point of W; and

$$\|v^n - w^n\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{11}$$

Combining (10) and (11) it follows that $\|w^n - \bar{u}\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$. Q.E.D.

Since the closure of the set of extreme points of W is the minimal closed subset of W whose convex closure equals W (Schaefer (1966), Corollary to Theorem 10.5, page 68); we have the following result.

Corollary 21.1. *The set of piecewise linear extreme points of W is norm dense in the set of extreme points of W.*

6.2 Another Example

The following example identifies an extreme point in which randomization occurs over all goods for all consumers within a market segment.

Example 3. Mixing on all goods. Let $N = 2$ and let $u \in W$ be defined by

$$u(x) = \max\{0, \, (0.4x_1 + 0.6x_2 - \frac{1}{5}), \, (x_1 + x_2 - \frac{3}{5})\}.$$

The graph of u and its market segments $\{A^j\}_{j=0}^2$ are depicted in Figure 6.

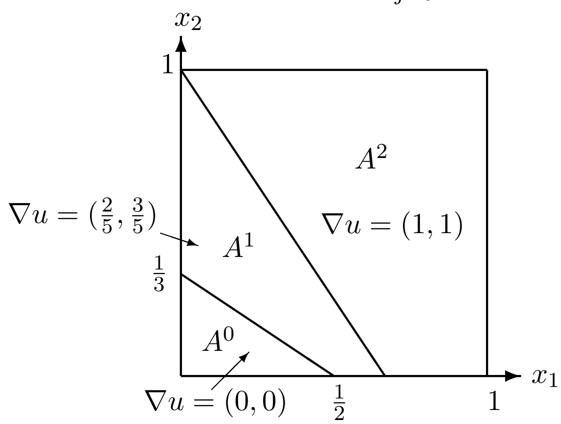

Figure 6: Mixing in all Goods

To see that u is indeed an extreme point, suppose temporarily that it is not. By Theorem 17, $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$ where u_1 and u_2 are piecewise linear and belong to W. Furthermore, the market segments $\{A^j\}_{j=0}^2$ determined from u suffice to define the linear pieces of u_1 and u_2. Note also that ∇u must be the average of ∇u_1 and ∇u_2. Thus, for $i = 1, 2$, ∇u_i must be $(0, 0)$ in A^0, and ∇u_i must be $(1, 1)$ in A^2. Pick any i in $\{1, 2\}$. It follows that

$$u_i(x) = \begin{cases} (0,0) \cdot (x_1, x_2) - 0, & \text{if } x \in A^0 \\ (c_1, c_2) \cdot (x_1, x_2) - c_0, & \text{if } x \in A^1 \\ (1,1) \cdot (x_1, x_2) - b_0, & \text{if } x \in A^2 \end{cases}, \tag{12}$$

for some $b_0, c_0 \in [0, \infty)$ and $c_1, c_2 \in [0, 1]$. The value of these unknowns is determined by the boundaries of the market segments, i.e., $A^0 \cap A^1$ and $A^1 \cap A^2$. From u, it follows that $A^0 \cap A^1 = \{x \in I^2 : x_2 = \frac{1}{3} - \frac{2}{3}x_1\}$, and $A^1 \cap A^2 = \{x \in I^2 : x_2 = 1 - \frac{3}{2}x_1\}$. From u_i, the boundaries in question are $A^0 \cap A^1 = \left\{x \in I^2 : x_2 = \frac{c_0}{c_2} - \frac{c_1}{c_2}x_1\right\}$ and $A^1 \cap A^2 = \left\{x_2 = \frac{c_0 - b_0}{c_2 - 1} - \frac{c_1 - 1}{c_2 - 1}x_1\right\}$. We thus obtain the following system of four equations and four unknowns: $\frac{c_0}{c_2} = \frac{1}{3}$, $\frac{c_1}{c_2} = \frac{2}{3}$, $\frac{c_0 - b_0}{c_2 - 1} = 1$, $\frac{c_1 - 1}{c_2 - 1} = \frac{3}{2}$. The unique solution to the system is $c_0 = \frac{3}{5}$, $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, and $b_0 = 1$. Thus, u_i equal u, a contradiction that proves u is an extreme point.

Remark 22. The mechanism $\bar{u}$ in the example is undominated. To see this, suppose $\bar{t}$ is dominated by t derived from a mechanism u. Since $I^2 \subset \bigcup_{x \in A^0} \mathcal{R}_x$, then $u \geq \bar{u}$ (Lemma 11). For every $x \in A^2$, $\nabla \bar{u}(x) = \mathbf{1}$; hence, $u(x) = \bar{u}(x)$. By (7), $u(\delta x)$ must be equal to $\bar{u}(\delta x)$ for every $\delta \in [0, 1]$.

The example shows that mixing in all goods may be a feature of the optimal mechanism. The argument used to prove that $\bar{u}$ is an extreme point is an application of Theorem 17.

6.3 Identifying Extreme Points

We construct an algebraic procedure to determine whether any proposed piecewise linear mechanism is an extreme point, and argue, based on that procedure, that piecewise linear extreme points with randomization are plentiful. The procedure is based on properties of faces of the feasible set and can be implemented numerically.

For the remainder of the section let $\bar{u}$ be a piecewise linear mechanism in W.

The face $F_{\bar{u}}$ is the set of all mechanisms u that have the same market segments as $\bar{u}$ and satisfy a gradient restriction (Definition 18). The mechanism $\bar{u}$ is an extreme point if and only if its face $F_{\bar{u}}$ has $\bar{u}$ as its unique element (Theorem 20). Therefore, determining whether $\bar{u}$ is an extreme point is roughly "equivalent" to determining whether there is another mechanism u that generates the same market segments as $\bar{u}$. In turn, market segments are defined by finitely many linear inequalities (Definition 14). These inequalities, when satisfied as equalities, determine the boundaries between adjacent market segments. The collection of those boundaries constitute a system of linear equations. Mechanisms that generate the same market segments must solve the same system of linear equations. These ideas will be developed presently.

Market segments are subsets of I^N. Neighboring market segments share an $(N-1)$-dimensional boundary (See Remark 15). We make this precise with a definition.

Definition 23. Two market segments A and A' of a piecewise linear mechanism are adjacent if their common boundary $\overline{A} \cap \overline{A'}$ is an $(N-1)$-dimensional set.

The market segments $\{A^0, A^1\}$ and $\{A^1, A^2\}$ in Examples 1 and 3 are adjacent. In Example 1, the market segments $\{A^0, A^2\}$ are also adjacent.

Let

$$B_{\bar{u}} = \{\{A, A'\} : A, A' \in m(\bar{u}), \ \ A \text{ and } A' \text{ are adjacent.}\}.$$

The set $B_{\bar{u}}$ contains all pairs of adjacent market segments. Its elements are used to index boundaries. For instance $\{A, A'\} \in B_{\bar{u}}$ refers to the boundary $\overline{A} \cap \overline{A'}$ between A and A'.

Pick any $\{A, A'\}$ in $B_{\bar{u}}$. If a piecewise linear mechanism u defines the same boundary between A and A' as $\bar{u}$ does, then

$$\forall x, \ \ \left[(\nabla u^A - \nabla u^{A'}) \cdot x = t_u^A - t_u^{A'}\right] \iff \left[(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \cdot x = t_{\bar{u}}^A - t_{\bar{u}}^{A'}\right].$$

This holds if and only if there is a number $\alpha^{A,A'}$ such that

$$\nabla u^A - \nabla u^{A'} = \alpha^{A,A'}(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \text{ and } t_u^A - t_u^{A'} = \alpha^{A,A'}(t_{\bar{u}}^A - t_{\bar{u}}^{A'}).$$

To every pair of adjacent market segments $\{A, A'\}$, we associate the equation

$$z^A - z^{A'} - \alpha^{A,A'}(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) = \mathbf{0}.$$

The collection of all such equations, one per boundary, constitutes a system of linear equations:

$$z^A - z^{A'} - \alpha^{A,A'}(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) = \mathbf{0} \qquad \forall \{A, A'\} \in B_{\bar{u}}. \tag{13}$$

Each unknown z^A is an N-dimensional vector; it will be used to construct the gradient ∇u^A of a mechanism u in market segment A. Each unknown $\alpha^{A,A'}$ is a real number. Therefore system (13) has $N \times |m(\bar{u})|$ real-valued unknowns z_i^A, and $|B_{\bar{u}}|$ real-valued unknowns $\alpha^{A,A'}$.[7] For each boundary $\{A, A'\} \in B_{\bar{u}}$, expression (13) represents N equations, one for each component z_i^A. Thus there are $N|B_{\bar{u}}|$ equations in total.

The number of equations will exceed the number of unknowns for some mechanisms but not for others. System (13), however, is always consistent. One solution is $z^A = \nabla \bar{u}^A$ for every $A \in m(\bar{u})$, and $\alpha^{A,A'} = 1$ for every $\{A, A'\} \in B_{\bar{u}}$. We refer to this solution as the *trivial solution*.

The following theorem summarizes the algebraic procedure.

Theorem 24. *Let $\bar{u}$ in W, $\bar{u} \neq 0$, be a piecewise linear mechanism with market segments $m(\bar{u})$, and let $B_{\bar{u}}$ identify the boundaries of its adjacent market segments. The following statements are equivalent.*

(a) The mechanism $\bar{u}$ is an extreme point of W.

(b) There is a unique, non-negative solution to the system of equations (13) such that $\forall A \in m(\bar{u})$, $z^A \leq \mathbf{1}$, and

$$\forall A \in m(\bar{u}), \forall 1 \leq i \leq N, \ \left[\nabla_i \bar{u}^A \in \{0, 1\}\right] \implies \left[z_i^A = \nabla_i \bar{u}^A\right]. \tag{14}$$

The unique solution is the trivial one.

Theorem 24 is essentially a corollary to Theorem 20. It is based on three observations. First, a non-negative solution (other than the trivial one) to system (13) exists if and only if there is a mechanism u that generates the same market segments as $\bar{u}$. (The solution is the mechanism's gradient.) Second, a mechanism u with the same market segments as $\bar{u}$ satisfies the gradient restriction (14) if and only if u is a member of the face $F_{\bar{u}}$. (Note that (14) is precisely the gradient restriction used in the definition of $F_{\bar{u}}$ (Definition 18).) Finally, by Theorem 20, $\bar{u}$ is an extreme point if and only if $F_{\bar{u}}$ is the singleton $\{\bar{u}\}$. The proof of the theorem makes these observations precise.

Proof. We show that "not (a)" implies "not (b)." If $\bar{u}$ is not an extreme point, there is $u' \neq \bar{u}$, u' in $F_{\bar{u}}$. Let $u = (1/2)u' + (1/2)\bar{u}$. Then u is in $F_{\bar{u}}$ and $m(u)$ is equal to $m(\bar{u})$. (The average is carried out because the market segments of u' may be coarser than those of $\bar{u}$.)

[7] Given finite set D, $|D|$ is its cardinality.

Since u is in W, ∇u^A is in I^N for every $A \in m(\bar{u})$. Since u is in $F_{\bar{u}}$, its gradient $\{\nabla u^A\}_{A\in m(\bar{u})}$ satisfies (14) (Definition 18). Since $m(u) = m(\bar{u})$, for every $\{A, A'\} \in B_{\bar{u}}$, there is $\alpha^{A,A'} \in \mathbb{R}$ such that

$$\nabla u^A - \nabla u^{A'} = \alpha^{A,A'}(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \text{ and } t_u^A - t_u^{A'} = \alpha^{A,A'}(t_{\bar{u}}^A - t_{\bar{u}}^{A'}).$$

Thus, (13) is satisfied.

We now show that $\alpha^{A,A'} > 0$. For every x in A, $(\nabla u^A - \nabla u^{A'}) \cdot x > t_u^A - t_u^{A'}$. This is equivalent to writing $\alpha^{A,A'}(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \cdot x > \alpha^{A,A'}(t_{\bar{u}}^A - t_{\bar{u}}^{A'})$. Since it must be the case that for $x \in A$, $(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \cdot x > t_{\bar{u}}^A - t_{\bar{u}}^{A'}$, we conclude $\alpha^{A,A'} > 0$.

Since $u \neq \bar{u}$, $\nabla u^A \neq \nabla \bar{u}^A$ for some $A \in m(\bar{u})$. For every $A \in m(\bar{u})$, define $z^A = \nabla u^A$. We have constructed an alternative solution.

We prove that "not (b)" implies "not (a)." Let $\{z^A\}_{A\in m(\bar{u})}$, $\{\alpha^{A,A'} \geq 0\}_{\{A,A'\}\in B_{\bar{u}}}$ be the alternative (non-trivial) solution mentioned in (b).

First note that there is no loss of generality in assuming that $\alpha^{A,A'} > 0$ for all $\{A, A'\} \in B_{\bar{u}}$. To see this, note that in the trivial solution $\alpha^{A,A'} = 1$ for all $\{A, A'\} \in B_{\bar{u}}$. Since a convex combination of solutions is a solution, there will be a solution to the system that has strictly positive variables $\alpha^{A,A'}$. We thus assume, without loss of generality, that $\alpha^{A,A'} > 0$ for all $\{A, A'\} \in B_{\bar{u}}$.

We will construct a function u and show that $u \neq \bar{u}$ and u is in $F_{\bar{u}}$. This implies $\bar{u}$ is not an extreme point. For every $A \in m(\bar{u})$, define $\nabla u^A = z^A$. To construct the corresponding transfers t_u , we proceed as follows. If $\mathbf{0} \in \overline{A}$, then $t_u^A = 0$. For every $\{A, A'\} \in B_{\bar{u}}$, let $t_u^A - t_u^{A'} = \alpha^{A,A'}(t_{\bar{u}}^A - t_{\bar{u}}^{A'})$.

For $x \in I^N$, define $u(x) = \max_{A\in m(\bar{u})} z^A \cdot x - t_u^A$. By construction $u(\mathbf{0}) = 0$ and u is convex. Also u defines the same market segments as $\bar{u}$: for every $\{A, A'\} \in B_{\bar{u}}$, $(z^A - z^{A'}) \cdot x \geq (t_u^A - t_u^{A'})$ if and only if $(\nabla \bar{u}^A - \nabla \bar{u}^{A'}) \cdot x \geq (t_{\bar{u}}^A - t_{\bar{u}}^{A'})$. Thus u belongs to $F_{\bar{u}}$. Theorem 20 implies $\bar{u}$ is not an extreme point, a contradiction. Q.E.D.

Theorem 24 may be applied as an algebraic procedure to determine whether any piecewise linear mechanism is an extreme point. A candidate mechanism $\bar{u}$ is proposed. Its gradient $\{\nabla \bar{u}^A\}_{A\in m(\bar{u})}$ defines the system of linear equations (13). The proposed mechanism $\bar{u}$ is an extreme point if and only if the only non-negative solution satisfying the gradient restriction (14) is the trivial solution.

We will illustrate a different application of Theorem 24 than the one described in the previous paragraph. Instead of proposing a single mechanism $\bar{u}$, we propose a class of mechanisms, and we take advantage of the structure of system (13)—every equation has few non-zero coefficients—to determine the elements within the class that are extreme points. We summarize our findings in Corollary 24.1 and the remark following it. Before stating the corollary, we explain the ideas leading up to it for they may prove useful in obtaining similar results.

A piecewise linear mechanism $\bar{u}$ is in the proposed class if it has two defining characteristics,

$$\exists A^0 \in m(\bar{u}) \quad : \quad A^0 = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{0}\} \tag{15}$$

$$\exists A^J \in m(\bar{u}) \quad : \quad A^J = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{1}\}. \tag{16}$$

Candidate mechanisms have a market segment A^0 where no goods are assigned, and a market segment A^J where all goods are assigned for certain. Since every undominated, piecewise linear mechanism has these characteristics (Theorem 16), it is unlikely that the seller will choose a mechanism without them. The mechanisms in Examples 1 and 3 are within the class considered.

Given any mechanism $\bar{u}$ in the proposed class, the system of equations (13) can be rewritten so that every vector of unknowns z^A is expressed solely in terms of the real-valued unknowns $\{\alpha^{A,A'}\}_{\{A,A'\}\in B_{\bar{u}}}$. To see this, two observations are useful. First, in any solution to (13) compatible with the gradient restriction (14), z^0 must be $\mathbf{0}$ and z^J must be $\mathbf{1}$ (because $\nabla \bar{u}^0 = \mathbf{0}$ and $\nabla \bar{u}^J = \mathbf{1}$ respectively). Second, for any A in $m(\bar{u})$, there is a path of market segments from A^0 to A where each element of the path is adjacent to the previous one. In other words, there is a collection of market segments that includes both the null assignment set A^0 and the target set A, and whose members can be conveniently labeled so that contiguous segments (according to their label) are adjacent:
$\forall A \in m(\bar{u}), \exists \{A^k\}_{k=0}^K \subset m(\bar{u})$ such that

$$\begin{cases} \text{(i)} & A^0 = \{x \in I^N : \nabla \bar{u}^0 = \mathbf{0}\}, \\ \text{(ii)} & A^K = A, \text{ and} \\ \text{(iii)} & \text{for } 1 \leq k \leq K,\ A^k \text{ and } A^{k-1} \text{ are adjacent.} \end{cases} \tag{17}$$

The market segments in Examples 1 and 3 have been labeled to illustrate this condition: For instance, let A^2 in Example 1 be the target market segment. The collection of all market segments $\{A^0, A^1, A^2\}$ satisfies (17-(i)) to (17-(iii)). The collection consisting only of A^0 and A^2 also satisfies the requirements when properly relabeled.

The equations in (13) derived from contiguously labeled adjacent market segments in $\{A^k\}_{k=0}^K$ are

$$z^{A^k} = z^{A^{k-1}} + \alpha^{k,k-1}(\nabla \bar{u}^{A^k} - \nabla \bar{u}^{A^{k-1}}), \text{ for } k = 1, 2, \ldots, K.$$

Reordering terms and substituting repeatedly, the system may be written as

$$z^{A^k} - z^{A^0} - \sum_{j=1}^{k} \alpha^{j,j-1}(\nabla \bar{u}^{A^j} - \nabla \bar{u}^{A^{j-1}}) = \mathbf{0}, \text{ for } k = 1, 2, \ldots, K.$$

Since $z^{A^0} = \mathbf{0}$, every vector z^{A^k} may be expressed solely in terms of the unknowns $\alpha^{j,j-1}$. In other words, the values of $\alpha^{j,j-1}$ for $j = 1, \ldots, k$ determine the value of the unknown z^{A^k}:

$$z^{A^k} = \sum_{j=1}^{k} \alpha^{j,j-1}(\nabla \bar{u}^{A^j} - \nabla \bar{u}^{A^{j-1}}). \tag{18}$$

Of the two characteristics (15) and (16) defining the proposed class, we have used so far only (15). We will now describe how the second one is used. Every equation in (18), i.e., the equation corresponding to each k, is itself a system of N linear equations—one for each object to be sold—with $N+k$ real valued unknowns—the N values z_i^k, and the values $\alpha^{j,j-1}$ for $j=1,\dots,k$. If, in addition, $A^K = \{x \in I^N : \nabla \bar{u}^K = \mathbf{1}\}$, the gradient restriction (14) implies that z^K must be $\mathbf{1}$. Then the K^{th} equation in (18) becomes

$$\sum_{j=1}^{K} \alpha^{j,j-1}\left(\nabla \bar{u}^{A^j} - \nabla \bar{u}^{A^{j-1}}\right) = \mathbf{1}.$$

We thus have a system of equations with only $\{\alpha^{j,j-1}\}_{j=0}^{K}$ as unknowns. Its solutions may be used to construct solutions to (13). The corollary below summarizes this discussion.

Corollary 24.1. *Let $\bar{u} \in W$, $\bar{u} \neq 0$, be a piecewise linear mechanism with market segments $m(\bar{u}) = \{A^j\}_{j=0}^{J}$. Suppose*

(a) $A^0 = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{0}\}$,

(b) $A^J = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{1}\}$, and

(c) for $1 \le j \le J$, $(A^j, A^{j-1}) \in B_{\bar{u}}$.

If $\alpha^{j,j-1} = 1$ for $j = 1, \dots, J$ is the unique solution to

$$\sum_{j=1}^{J} \alpha^{j,j-1}\left(\nabla \bar{u}^{A^j} - \nabla \bar{u}^{A^{j-1}}\right) = \mathbf{1}$$

then $\bar{u}$ is an extreme point.

Remark 25. The set of piecewise linear extreme points satisfying (a)-(c) is relatively open in the space of piecewise linear mechanisms satisfying (a)-(c). If in addition, the number of market segments is no larger than the number of goods plus one (i.e., $J \le N$), then the above-mentioned set is also dense.

The remark summarizes two genericity results that follow from the corollary and that apply to mechanisms satisfying the conditions of the corollary. The first one states that small perturbations of a piecewise linear extreme point yield additional extreme points. The second one states that, roughly, most piecewise linear mechanisms that have fewer market segments than the number of goods are extreme points; if one such mechanism is not an extreme point then it is arbitrarily close to one.

We illustrate the discussion in this section and the genericity statements with an example. Consider the two-good case ($N = 2$), and a mechanism $\bar{u}$ with three market segments (i.e.,

$J = 2$). The reader may think of the mechanisms in Examples 1 and 3. In these cases, system (18) becomes

$$\begin{pmatrix} 1 & 0 & -\nabla_1 \bar{u}^{A^1} & 0 \\ 0 & 1 & -\nabla_2 \bar{u}^{A^1} & 0 \\ 0 & 0 & \nabla_1 \bar{u}^{A^1} & \left(1 - \nabla_1 \bar{u}^{A^1}\right) \\ 0 & 0 & \nabla_2 \bar{u}^{A^1} & \left(1 - \nabla_2 \bar{u}^{A^1}\right) \end{pmatrix} \cdot \begin{pmatrix} z_1^{A^1} \\ z_2^{A^1} \\ \alpha^{1,0} \\ \alpha^{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

where subindices indicate goods. The first two rows correspond to the equation $k = 1$ in (18) and the last two rows represent the equation $k = 2$ in (18).

If the coefficient matrix has full rank, then the trivial solution is unique and, by Theorem 24, $\bar{u}$ is an extreme point. The rank of the coefficient matrix is fully determined by its last two rows, the last equation in (18). Corollary 24.1 shows that this is the case in general.

The last two equations can be written as

$$\begin{pmatrix} \nabla_1 \bar{u}^{A^1} & \left(1 - \nabla_1 \bar{u}^{A^1}\right) \\ \nabla_2 \bar{u}^{A^1} & \left(1 - \nabla_2 \bar{u}^{A^1}\right) \end{pmatrix} \cdot \begin{pmatrix} \alpha^{1,0} \\ \alpha^{2,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

It is immediate that for the mechanism $\bar{u}$ in Examples 1 and 3, the coefficient matrix above has full rank. Thus, the mechanisms in both examples are extreme points; we have just provided another proof.

The matrix has full rank except when the rows $\left(\nabla_1 \bar{u}^{A^1}, \left(1 - \nabla_1 \bar{u}^{A^1}\right)\right)$ and $\left(\nabla_2 \bar{u}^{A^1}, \left(1 - \nabla_2 \bar{u}^{A^1}\right)\right)$ are not linearly independent. Linear dependence can only arise if $\nabla_1 \bar{u}^{A^1} = \nabla_2 \bar{u}^{A^1}$. Therefore when $N = 2$, generically, the piecewise linear mechanisms $\bar{u}$ with three market segments, $\nabla \bar{u}^0 = \mathbf{0}$, and $\nabla \bar{u}^2 = \mathbf{1}$ are extreme points. Geometrically, one such mechanism $\bar{u}$ is *not* an extreme point if and only if the boundaries between market segments are parallel.

The examples discussed are canonical in the following sense. For each market segment k there are N equations, one for each good. The last N equations correspond to the boundary between market segments A^J and A^{J-1}. Since $\nabla u^{A^J} = \nabla \bar{u}^{A^J} = \mathbf{1}$, the last N equations constitute a linear system with the J unknowns $\{\alpha^{j,j-1}\}_{j=1}^J$ as the only unknowns. The same arguments made in the previous paragraph hold whenever $J \leq N$. Note that J is the number of market segments minus one, or the number of market segments where some assignment is made, according to the mechanism $\bar{u}$. Thus, provided J is no larger than the number of goods, extreme points are abundant. This is summarized in the remark following Corollary 24.1.

When the number of market segments is larger than the number of goods plus one, even if the assumptions of Corollary 24.1 hold, the system of equations in the corollary typically has more than one solution. We may not conclude, however, that $\bar{u}$ is not an extreme point. For $\bar{u}$ to fail to be an extreme point, a non-trivial solution $\{\alpha^{j,j-1}\}_{j=1}^J$ must yield a feasible mechanism u, i.e a mechanism u in W.

7 Odds and Ends

1. Much of our analysis can incorporate some form of production costs.[8] The objective function in the seller's problem, $E[\nabla u(x) \cdot x - u(x)]$, is linear on u. Let $C : I^N \longrightarrow \mathbb{R}$ be concave. Then, $E[\nabla u(x) \cdot x - u(x) - C(\nabla u(x))]$ is convex as a function of u and it achieves a maximum at an extreme point of W.[9]

   The introduction of the cost function C is more natural in a reinterpretation of the formal model: the seller produces a single good but must decide on the good's characteristics, an N-dimensional vector; the buyer buys at most one unit of the commodity and the buyer's private information x is the buyer's valuation for the different characteristics. In this context, $p(x) = \nabla u(x)$ is the characteristics selected by the seller given the buyer's reported valuations x. The function $C(\nabla u(x))$ thus represents the cost of providing the level of characteristics $\nabla u(x)$.

   The inclusion of a cost function $C(\cdot)$ facilitates comparisons with Rochet and Choné (1998). They consider essentially the following problem:

   $$\max_{u \in W} E[\nabla u(x) \cdot x - u(x) - C(\nabla u(x))], \tag{19}$$

   and assume that $C(\cdot)$ is twice-continuously differentiable and strictly convex. Under their assumptions the optimization problem has a unique solution.

   Our case is complementary to theirs. We study essentially the same problem but we assume that the cost function $C(\cdot)$ is concave. This assumption is consistent with a common justification for natural monopoly, namely that the cost function has non-convexities.

2. The algebraic procedure described in Section 6 can be used to determine whether any proposed piece-wise linear mechanism is an extreme point. To determine whether the identified extreme point maximizes expected revenue for some seller's beliefs, one must show, in addition, that the proposed mechanism is undominated (Section 5). Lemma A.4 and Theorem 16 are useful in this regard.

   Identifing extreme points, even dominated ones, is potentially useful. While dominated extreme points will not solve the seller's problem, they will be the solution to some program with a convex objective function.

3. We have shown in Section 6 that the mechanisms in Examples 1 and 3 are generic extreme points. Thus, nearby mechanisms are also extreme points. Lemma 12 and Remark 22) show that the mechanisms in Examples 1 and 3 are undominated. Similar arguments show that nearby extreme points are undominated.

---

[8] We are grateful to Kim Border for this observation.

[9] In fact, we only need that the objective function $E[\nabla u(x) \cdot x - u(x) - C(\nabla u(x))]$ be quasiconvex. Assuming that $C(\cdot)$ is concave suffices to obtain the quasiconvexity of the objective function.

8 References

Aliprantis, C. D., and K. C. Border (1999): *Infinite Dimensional Analysis: a hitchhiker's guide*. Second edition. Berlin: Springer.

Armstrong, M. (1996): "Multiproduct Nonlinear Pricing," *Econometrica*, 64(1), pp. 51-75.

Armstrong, M. (1999): "Price Discrimination by a Many-Product Firm," *Review of Economic Studies*, 66(1), pp. 151-168.

Basov, S. (2005): *Multidimensional Screening*. Studies in Economic Theory v22, Berlin: Springer Verlag.

Basov, S. (2001): "Hamiltonian Approach to Multi-dimensional Screening," *Journal of Mathematical Economics*, 36, pp. 77-94.

Border, K. (2002): "Notes on Product Differentiation," Working Paper, Caltech.

Jehiel, Ph., B. Moldovanu, and E. Stacchetti (1999): "Multidimensional Mechanism Design for Auctions with Externalities," *Journal of Economic Theory*, 85, pp. 258-293.

Kazumori, E. (2001), "Optimal Auction Design for Heterogeneous Assets," photocopy, Caltech.

Krishna, V., and E. Maenner (2001): "Convex-Potentials with an Application to Mechanism Design," *Econometrica*, 69(4), pp. 1113-1119.

Laffont, J.-J., E. Maskin, and J.-C. Rochet (1985): "Optimal Non-Linear Pricing with Two Dimensional Characteristics," in *Information, Incentives, and Economic Mechanisms*, ed. by T. Groves, R. Radner, and S. Reiter. Oxford: Basil Blackwell.

Manelli, A. and R. Vincent (2004), "Bundling as an Optimal Mechanism by a Multiple-Good Monopolist," *Journal of Economic Theory*, forthcoming.

McAfee, R.P. and J. McMillan (1988): "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46(2), pp. 335-354.

McAfee, R.P., J. McMillan, and M. Whinston (1989): "Multiproduct Monopoly, Commodity Bundling, and the Correlation of Values," *Quarterly Journal of Economics*, 103, pp. 371-383.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, pp. 58-73.

Riley, J., and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Gaggle, When to Hold Firm," *Quarterly Journal of Economics*, 98, pp. 267-289.

Rochet, J.-C. (1985): “The Taxation Principle and Multi-Time Hamilton-Jacobi equations,” *Journal of Mathematical Economics*, 14(2), pp. 113-128.

Rochet, J.-C. and L. Stole (2003): “The Economics of Screening,” in *Advances in Economics and Econometrics*, ed. by Dewatripont, Hansen, and Turnovsky. Cambridge University Press.

Rochet, J.-C. and P. Choné (1998): “Ironing, Sweeping, and Multidimensional Screening,” *Econometrica*, , pp. 783-826.

Rockafellar, R. (1970): *Convex Analysis.* Number 28 in Princeton Mathematical Series. Princeton, NJ: Princeton University Press.

Schaefer, H. (1966): *Topological Vector Spaces.* Macmillan, New York.

Thanassoulis, X. (2004): “Haggling Over Substitutes,” *Journal of Economic Theory*, , pp. 217-245.

Wilson, R. (1993): *Nonlinear Pricing.* Oxford: Oxford University Press.

Zheng, C. (2000): “Optimal Auction in a Multidimensional World,” Working Paper working paper, Northwestern University.

Appendix

Lemma A.1. *Let* $W = \{u \in C^0(I^N) \mid u(x) \text{ is convex, } \nabla u(x) \in I^N \text{ a.e., and } u(\mathbf{0}) = 0\}$. *The set* W *is compact with respect to the sup norm.*

Proof. The family of functions W is equicontinuous and uniformly bounded. The Arzela-Ascoli Theorem implies the desired result. Q.E.D.

We provide without proof the following well-known result.

Lemma A.2. *Let* X *be a locally convex, topological vector space,* W *be a non-empty compact, convex subset of* X*, and* $S : W \longrightarrow \mathbb{R}$ *be a continuous linear function. Then the set* F *of maximizers of* S *over* W *is a face of* W*. Furthermore,* F *contains an extreme point of* W*.*

Lemma A.3. *Let* $W = \{u \in C^0(I^N) \mid u(x) \text{ is convex, } \nabla u(x) \in I^N \text{ a.e., and } u(\mathbf{0}) = 0\}$. *Let* $u \in W$ *and let* t *map* $x \mapsto \nabla u(x) \cdot x - u(x)$. *For* $n = 1, 2, \ldots$, *let* u^n *be an element of* W *and* t^n *map* $x \mapsto \nabla u^n(x) \cdot x - u^n(x)$. *If the sequence* $\{u^n\}$ *converges uniformly to* $u \in W$*, then,*
(i) $\{\nabla u^n\} \xrightarrow{\lambda - a.e.} \nabla u$ *and therefore* $\{\nabla u^n(x) \cdot x\} \xrightarrow{\lambda - a.e.} \nabla u(x) \cdot x$*; and*
(ii) $\{t^n\} \xrightarrow{L_1} t$*, and*
(iii) T *is compact in the* L^1 *norm.*

Proof. (i) For $n = 1, 2, \ldots$, let D^n be the set of x in the interior of I^N where $u^n(x)$ is differentiable, and let D' be similarly defined for u. The sets $D^n, \forall n$ and D' are dense in I^N and have λ-measure one (Rockafellar (1970), Theorem 25.5, page 246). The set $D = (\bigcap_{n \geq 1} D^n) \cap D'$ has full measure.

Pick any $x \in D$. Since u^n is differentiable at x, $\nabla u^n(x)$ equals the unique subgradient at x (Rockafellar (1970), Theorem 25.1, page 242). Therefore

$$\forall y \in \mathbb{R}^N, \frac{u^n(x - \delta y) - u^n(x)}{\delta} \leq \nabla u^n(x) \cdot y \leq \frac{u^n(x + \delta y) - u^n(x)}{\delta},$$

for all $\delta \in (0, \bar{\delta}]$ such that $(x + \bar{\delta} y) \in I^N$ and $(x - \bar{\delta} y) \in I^N$. (Such $\bar{\delta}$ exists because x is in the interior of I^N.)

It follows that for any $\epsilon > 0$ and y, there is $\bar{n}$ such that $n > \bar{n}$ implies

$$\frac{u(x - \delta y) - u(x)}{\delta} - \epsilon \leq \nabla u^n(x) \cdot y \leq \frac{u(x + \delta y) - u(x)}{\delta} + \epsilon. \tag{20}$$

To see this, note that given any two sequence of real numbers r^n, s^n, with $r^n \geq s^n, \forall n$, and $s^n \longrightarrow s$, the following inequalities hold: $r^n - s \geq s^n - s \geq -\|s^n - s\|$. Since for any $\epsilon > 0$, there is $\bar{n}$ such that $n > \bar{n}$ implies $-\|s^n - s\| \geq -\epsilon$, it follows that $[n > \bar{n} \implies r^n - s \geq -\epsilon]$. The same argument can be used to obtain both inequalities in (20).

Finally letting $\delta \downarrow 0$ in (20) and using the definition of a gradient, it follows that $n > \bar{n}$ implies

$$\nabla u(x) \cdot y - \epsilon \leq \nabla u^n(x) \cdot y \leq \nabla u(x) \cdot y + \epsilon.$$

Since y and ϵ are arbitrary, the proof of (i) is complete.

(ii) By (i), $|t^n - t| \xrightarrow{\lambda-a.e.} 0$. By construction $|t^n - t|$ is bounded. The Lebesgue Dominated Convergence Theorem implies that $\int |t^n - t|\, d\lambda \longrightarrow 0$. This completes the proof.

(iii) It follows from *(ii)* and Lemma A.1. Q.E.D.

Lemma A.4. *Let u and u' be two mechanisms in W and let t and t' denote their respective transfer functions. Suppose t' dominates t. Then, there exist measurable functions $\nabla u'$ and ∇u both defined from I^N into I^N, such that*

(i) $\nabla u'(x) \in \partial u'(x)$ *and* $\nabla u(x) \in \partial u(x)$, *(where $\partial u(x)$ is the subdifferential of u at x) and*

(ii) $[\nabla u'(x') - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]$ *for all* $x \in I^N$,

(iii) $\forall \delta \in (1, \delta'),\ \ -[u'(\delta x) - u(\delta x)] = \int_{\delta}^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma$,

(iv) $\nabla u'(x) \cdot x - u'(x) \geq \nabla u(x) \cdot x - u(x),\ \forall x \in I^N$.

Proof. Let $D' = \{x \in I^N : \nabla u'(x) \text{ exists}\}$, $D = \{x \in I^N : \nabla u(x) \text{ exists}\}$ and $D'' = \{x \in I^N : [\nabla u'(x) - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]\}$. Since $\lambda(D'') = \lambda(D') = \lambda(D) = 1$, then $\lambda(D'' \cap D' \cap D) = 1$.

Let $E = \{(x, \nabla u'(x), \nabla u(x)) : x \in D'' \cap D' \cap D\}$. Then $E \subset I^N \times I^N \times I^N$ and $\overline{E}$ is compact.

Let $\texttt{proj}_{I^N}(\overline{E}) = \{x \in I^N : (x, y, z) \in \overline{E}\}$; $\texttt{proj}_{I^N}(\overline{E})$ is the projection of $\overline{E}$ on its first coordinate. The set $\overline{E}$ is the graph of the correspondence $\varphi : \texttt{proj}_{I^N}(\overline{E}) \longrightarrow I^N \times I^N$ defined by $\varphi(x) = \{(y, z) \in I^N \times I^N : (x, y, z) \in \overline{E}\}$. By the selection Theorem of Kuratowsky and Ryll-Nardzewski (see for instance Hildenbrand (1974)), there is a measurable selection g of φ.

We first show that $\texttt{proj}_{I^N}(\overline{E}) = I^N$. Suppose not. Then there is $x \in I^N$ and $x \notin \texttt{proj}_{I^N}(\overline{E})$. Since $\texttt{proj}_{I^N}(\overline{E})$ is closed, there is $\epsilon > 0$ such that $B(x, \epsilon) \cap \texttt{proj}_{I^N}(\overline{E}) = \emptyset$ where $B(x, \epsilon) = \{x' \in I^N : \|x' - x\| < \epsilon\}$. Thus $\lambda(\texttt{proj}_{I^N}(\overline{E})) < 1$. By construction $(D'' \cap D' \cap D) \subset \texttt{proj}_{I^N}(\overline{E})$, and therefore $\lambda(\texttt{proj}_{I^N}(\overline{E})) = 1$, a contradiction.

Let $\partial u'$ and ∂u denote the subdifferential correspondence of u' and u respectively. Since both u' and u are convex, for all $x \in I^N$, $\partial u(x)$ is non-empty, and closed.

It is a matter of verifying definitions to show that $g(x) \in (\partial u'(x), \partial u(x))$ for all $x \in I^N$. Abusing notation slightly, we will denote $(\nabla u'(x), \nabla u(x)) = g(x)$ for all $x \in I^N$.

Finally, note that by construction $[\nabla u(x') - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]$ for all $x \in I^N$.

Then, from Krishna and Maenner (2001), Theorem 1, it follows that the integral (5) is valid for any measurable functions satisfying (i) above. Condition (ii) states that $t' \geq t$ everywhere in I^N. Q.E.D.

Lemma A.5. *The set of piecewise linear mechanisms in W is dense in W with the sup norm.*

Proof. Sketch. Pick any $u \in W$ and let $t = \nabla u \cdot x - u$ be its corresponding transfer function. Let $I_n = \{0, 1/n, 2/n, \dots, n/n\}$; I_n^N is a discretization of the set I^N. For each $z \in I_n^N$, define the linear function of $x \in I^N$, $\nabla u(z) \cdot x - t(z)$, and consider the function $v^n(x) = max_{z \in I_n^N} \nabla u(z) \cdot x - t(z)$. It is routine to check that $\sup_{x \in I^N} |v^n(x) - u(x)|$ tends to zero as n tends to infinity. Q.E.D.

Proof of Theorem 16. For any $A^j \in m(u)$, let ∇u^{A^j} denote the gradient of u evaluated at any $x \in A^j$, and t^{A^j} be the transfer for every $x \in A^j$. Therefore,

$$u(x) = \max\{\nabla u^{A^j} \cdot x - t^{A^j} \; : \; A^j \in m(u)\}.$$

First, suppose that for every $A^j \in m(u)$, $\nabla u^{A^j} \neq \mathbf{0}$. We will show u is dominated.

Let $\mathcal{M} = \{A^j \in m(u) : \mathbf{0} \in \overline{A^j}\}$. and define

$$\begin{aligned} v(x) &= \max\{\nabla u^{A^j} \cdot x - t^{A^j} \; : \; A^j \in m(u) \setminus \mathcal{M}\}, \\ w(x) &= \max\{\nabla u^{A^j} \cdot x - t^{A^j} \; : \; A^j \in \mathcal{M}\}. \end{aligned}$$

The set $\mathcal{M}$ is non-empty because $u(\mathbf{0}) = 0$. Suppose momentarily that $m(u) \setminus \mathcal{M}$ is not empty; we will show later in the proof that the alternative case is trivial.

We use the functions v and w to define a new function u', and to express u. For every $x \in I^N$, let

$$u'(x) = \max\{v(x), 0\}. \tag{21}$$

We will show that u' dominates u. Note that for every $x \in I^N$,

$$u(x) = \max\{v(x), w(x), 0\}. \tag{22}$$

The mechanism u has three components and its corresponding transfer t is strictly positive only on the effective domain of v. More precisely, we will show that

$$\begin{aligned} A^j \in \mathcal{M} \quad &\Longrightarrow \quad t^{A^j} = 0, \text{and} \\ A^k \in m(u) \setminus \mathcal{M} \quad &\Longrightarrow \quad t^{A^k} > 0. \end{aligned}$$

To see this, pick any $A^j \in \mathcal{M}$. Observe that $x \in A^j$ and $\mathbf{0} \in \overline{A^j}$ implies that $\alpha x \in A^j$ for any $\alpha \in (0, 1]$. By definition, $u(\alpha x) = \nabla u^{A^j} \cdot \alpha x - t^{A^j}$. If $t^{A^j} > 0$, then there is $\alpha \in (0, 1]$ such that $u(\alpha x) < 0$, a contradiction. We have thus shown that $t^{A^j} = 0$.

Pick then any $A^k \in m(u) \setminus \mathcal{M}$ and $A^j \in \mathcal{M}$. By definition of market segments,

$$\nabla u^{A^j} \cdot x - t^{A^j} > \nabla u^{A^k} \cdot x - t^{A^k}, \forall x \in A^j.$$

Let x tend to $\mathbf{0}$ which we may do because $\mathbf{0} \in \overline{A^j}$. Then

$$\nabla u^{A^j} \cdot \mathbf{0} - t^{A^j} \geq \nabla u^{A^k} \cdot \mathbf{0} - t^{A^k}.$$

If the expression above is satisfied as equality, then $\mathbf{0} \in \overline{A^k}$, and this is a contradiction since $A^k \in m(u) \setminus \mathcal{M}$. Thus, we conclude that

$$-t^{A^j} > -t^{A^k}.$$

Since $A^j \in \mathcal{M}$, $t^{A^j} = 0$ and thus, we have $t^{A^k} > 0$.

Consider now the mechanism u'. It has two components and its corresponding t' is strictly positive also on the effective domain of v.

Observe that, by construction, $\max\{w(x), 0\} \geq 0$ for every $x \in I^N$ and it is strictly positive for any $x \in \mathit{int}\, A^j$, $A^j \in \mathcal{M}$ because the gradient $\nabla u^{A^j} \neq 0$. Hence $w(x) > 0$ in a set of positive measure. Thus, the effective domain of v in the definition (21) of u', $\{x \in I^N : v(x) > 0\}$, strictly contains the effective domain of v in the definition (22) of u, $\{x \in I^N : v(x) > \max\{w(x), 0\}\}$. This completes the proof under the assumption that $m(u) \setminus \mathcal{M}$ is non-empty.

If $m(u) \setminus \mathcal{M} = \emptyset$, then u yields revenue $t = 0$ because $A^j \in \mathcal{M}$ implies $t^{A^j} = 0$. Hence u is clearly dominated. This completes the proof of the first part.

Second, suppose that $\nabla u^{A^j} \neq \mathbf{1}$ for every $A^j \in m(u)$. For any $r \geq 0$ and $x \in I^N$ define the functions

$$\begin{aligned} v_r(x) &= \mathbf{1} \cdot x - [N - u(\mathbf{1}) - r] \\ u_r(x) &= \max\{v_r(x), u(x)\}. \end{aligned}$$

We will prove that for sufficiently small r, u_r dominates u.

Define $K_r = \{x \in I^N : v_r(x) \geq u(x)\}$. For all $x \in I^N \setminus K_r$, $u_r(x) = u(x)$ and therefore both mechanisms generate the same transfer.

Pick any $x \in K_r$. The transfer generated by u_r is

$$\begin{aligned} t_r &= N - u(\mathbf{1}) - r = \mathbf{1} \cdot \mathbf{1} - u(\mathbf{1}) - r, \\ &= [\mathbf{1} - \nabla u(\mathbf{1})] \cdot \mathbf{1} + \nabla u(\mathbf{1}) \cdot \mathbf{1} - u(\mathbf{1}) - r. \end{aligned} \tag{23}$$

If $x \in K_r$ belongs to $A^j \in m(u)$, then the transfer generated by u is

$$\begin{aligned} t^{A^j} &= \nabla u(x) \cdot x - u(x) \\ &\leq \nabla u(x) \cdot x - u(\mathbf{1}) - \nabla u(\mathbf{1}) \cdot (x - \mathbf{1}), \\ &= [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x + \nabla u(\mathbf{1}) \cdot \mathbf{1} - u(\mathbf{1}), \end{aligned} \tag{24}$$

where the inequality follows because of the convexity of u. Subtracting (24) from (23) we obtain

$$\begin{aligned} t_r - t^{A^j} &\geq [\mathbf{1} - \nabla u(\mathbf{1})] \cdot \mathbf{1} - [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x - r. \\ &= [\mathbf{1} - \nabla u(\mathbf{1})] \cdot (\mathbf{1} - x + x) - [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x - r. \\ &= [\mathbf{1} - \nabla u(\mathbf{1})] \cdot (\mathbf{1} - x) + [\mathbf{1} - \nabla u(x)] \cdot x - r. \end{aligned}$$

The combination of the first two terms is strictly positive. Therefore, we conclude that

$$\exists\, r_j > 0 : [0 < r < r_j] \implies t_r > t^{A^j}.$$

Let $r' = \min\{r_j : A^j \in m(u)\}$.

We have thus proved that for $0 < r < r'$, t_r dominates t. A contradiction. Q.E.D.