Revised version -- Feb. 24,1988 -- Proof5(complete proof)

Appendix

The following eight Lemmas and Theorem adapt a proof given in Gul/Sonnenschein/Wilson (GSW) to show that in the one-sided offer bargaining game where the buyer is uncertain about the quality of the good but the seller is perfectly informed, if the buyer's discount factor is not less than the seller's the equilibrium path is characterized by a determinate sequence of price offers. The theorem is preceded by eight Lemmas which establish necessary conditions of any sequential equilibrium to the bargaining game.

The Model

A single buyer makes offers to a single seller for the purchase of a good of uncertain quality. The seller is perfectly informed about the quality of the good, the buyer is not. A strategy for the buyer is to choose a price to offer at each period given the past history of price offers (and rejections). A strategy for the seller is a function from past histories including the outstanding price to a set of only two values -- accept or reject. An acceptance ends the game and the good is transferred at the price accepted. Players choose strategies to maximize expected utility. Utility is determined for the buyer by

$$u(b;(p,t)) = b^t(v - p),$$

where p is the price agreed on, v is the buyer's valuation of the good (in money terms) and b is his discount factor. Similarly, the seller's utility is expressed as

$$u(s;(p,t)) = s^t(p - f),$$

Throughout the model it will be required that $b \geq s$, or the buyer is no less patient than the seller.

Let q be uniformly distributed over the unit interval. The seller has a valuation function dependent on q which can be described by a left-continuous, non-decreasing function, f(q). It will be required that f(q) also satisfy a Lipschitz condition at one. That is, there is a $q^* < 1$ and a $k > 0$ such that

$$f(1) - f(q) \leq k(1 - q) \text{ for } q \in [q^*,1].$$

The buyer's valuation is represented by the function, v(q), where $v(q) = v'(f(q))$. v' is also left-continuous and non-decreasing. Note that the existence of v' implies that if the seller can observe q, he also knows the buyer's true valuation.To ensure that there are always strictly positive gains from trade, it is required that v(q) satisfy

$$v(q) > f(q) \text{ for all } q.$$

A further condition is also required to ensure that the potential gains from trade do not become arbitrarily small. It is assumed that v(q) and f(q) satisfy

$$\lim_{q' \, q} v(q') > \lim_{q' \, q} f(q') \text{ for all } q.$$

This conditon is used to prove a fact that is used throughout the proof. Lemma 0 shows that if we integrate v(q) from q to some r a small distance from q then this value exceeds the integral of the constant function f(r) over the same range.

Lemma 0: For any $q < 1$, there is an r* close enough to q such that

$$\int^{r} v(q) - f(r) \, dq > (r - q)\epsilon, \text{ for some } \epsilon > 0.$$

$$\int_q$$

Proof: Define $z = \lim_{q' \, q} v(q')$.

Since $v(q)$ is non-decreasing there exists an $r^* > q$ such that $z >$ $f(r)$ for all $r \leq r^*$. Define $\epsilon = z - f(r)$. Then

$$\int_q^r v(x) - f(r)dx \geq \int_q^r z - f(r)dx = (r - q)\epsilon, \text{ for } r \in (q,r^*).$$

This fact is used throughout the proof to show the possibility of profitable deviations.

The next Lemma shows that, if he so wishes, a buyer can offer a price of $p = f(1)$ after any history in the game and be assured acceptance in equilibrium. Thus the expected value of any equilibrium strategy, σ, at state q_i is bounded from below by

$$R^{\sigma}(q_i) \geq \int_{q_i}^{1} v(x) - f(1)dx.$$

Lemma 1:(From GSW) The infimum of prices accepted by all sellers in any sequential equilibrium σ, after any history, h_t, is $f(1)$.

Proof: Let c be the infimum of prices accepted by all sellers after any history. Clearly $c \geq f(1)$. Suppose $c > f(1)$. A buyer will never offer a price greater than c since he can offer c and be sure of acceptance. Suppose at any history, a buyer offers $p = (1 - s)f(1) + sc + \epsilon$ such that $[p - f(1)] > s[c - f(1)]$ but $c > p$. Since all sellers prefer to accept p now rather than to wait for at most c in the continuation game, it must be the case that $c > f(1)$ is not the infimum of prices accepted by all sellers.

The next Lemma shows that there are histories of the game at which for any sequential equilibrium, σ, the buyer does, in fact, prefer to offer $p = f(1)$ right away rather than to continue bargaining.

Lemma 2: There exists a $q^* < 1$ such that for any sequential equilibrium, σ, and after any history, h_t, if the beliefs of the buyer are that $q \in [q^*, 1]$ then the buyer offers $p = f(1)$ immediately.

Proof: By Lemma 1 we know that for any sequential equilbrium strategy, σ, the expected value of σ must be at least

$$(1)\quad R^{\sigma}(q_i) \geq \int_{q_i}^{1} v(x) - f(1)\, dx = \int_{q_i}^{q_{i+1}} v(x)dx + \int_{q_{i+1}}^{1} v(x)dx - (1 - q_i)f(1).$$

The value of following σ is bounded from above by

$$R^{\sigma}(q_i) \leq \int_{q_i}^{q_{i+1}} v(x) - f(q_{i+1})dx + b\int_{q_{i+1}}^{1} v(x) - f(q_{i+1})dx$$

$$= \int_{q_i}^{q_{i+1}} v(x)dx + b\int_{q_{i+1}}^{1} v(x)dx - ((q_{i+1} - q_i) + b(1 - q_{i+1}))f(q_{i+1}).$$

But using the Lipschitz condition gives us

$$-f(q_{i+1}) \leq k(1 - q_{i+1}) - f(1).$$

Combining this with the above equation yields

$$(2)\quad R^{\sigma}(q_{i+1}) \leq \int_{q_i}^{q_{i+1}} v(x)dx + b\int_{q_{i+1}}^{1} v(x)dx + ((q_{i+1} - q_i) + b(1 - q_{i+1}))(f(1) - k(1 - q_{i+1})).$$

But

$$(1 - q_i)f(1) = ((1 - q_{i+1}) + (q_{i+1} - q_i))f(1),$$

Collecting terms and combining with (1) yields

$$\begin{aligned}(3)\quad 0 &\geq (1 - b)\int_{q_{i+1}}^{1} v(x) - f(1)dx - k(1 - q_{i+1})((q_{i+1} - q_i) + b(1 - q_{i+1})) \\ &\geq (1 - b)(1 - q_{i+1})[v(q_{i+1}) - f(1) - k((q_{i+1} - q_i) + b(1 - q_{i+1}))]\end{aligned}$$

As q_i approaches one, the term in the square brackets is strictly positive since $v(q)$ is left-continuous and $v(1) > f(1)$. Thus, for the inequality to be valid we must have $q_{i+1} = 1$. That is, the buyer will prefer to make an offer immediately which all sellers would accept when $q_i > q^*$, for some $q^* < 1$. q^* is chosen to be the smallest q for which this is true in any SE to the bargaining game.

The next Lemma uses this fact to describe what the behaviour of the seller must be, given that this is the equilibrium behaviour of the buyer. The Lemma uses the concept of a reservation price strategy pair, (r,P). The pair describes how a buyer will come to believe that $q \in [q^*,1]$ and it characterizes the prices a seller with valuation f(q) will accept when $q \in [q^*,1]$. The Lemma is a direct application of Lemma 3 in GSW.

Define a reservation price strategy pair (rpsp) by an $r < 1$ and a function P: $[r,1]$ to R^+ such that:

i) P(.) is left-continuous and non-decreasing;

In any sequential equilibrium, σ, if at period i the buyer's beliefs are $q_i \in [0,1]$ and the buyer offers p_i then

ii) if $p_i > P(q)$, then $q_{i+1} \geq q$;

If $q_i \in [r,1]$ then

iii) for $p_i < P(q)$, $q_{i+1} < q$.

Lemma 3: A rpsp exists.

Proof:(From GSW) Let $r = q^*$ from Lemma 2 and define P by

$$P(q) = (1 - s)f(q) + sf(1), \quad q \in [r,1].$$

From the assumptions on f(q), i) is satisfied. ii) can be shown as follows: Suppose $p_i > P(q)$ is offered but that $q_{i+1} < q$, or q rejects the offer p_i. Then

$$\begin{aligned} s^i(p_i - f(q)) &> s^i(P(q) - f(q)) \\ &= s^i((1 - s)f(q) + sf(1) - f(q)) \\ &= s^{i+1}(f(1) - f(q)). \end{aligned}$$

But f(1) is the most the seller of good q can expect in any SE after any history, thus q should accept p_i and P(q) satisfies ii).

Now suppose $p_i < P(q)$ and $q_{i+1} \geq q$.

$$\begin{aligned} s^i(p_i - f(q)) &< s^i(P(q) - f(q)) \\ &= s^{i+1}(f(1) - f(q)). \end{aligned}$$

Since $q_{i+1} \geq q^*$, the next equilibrium offer is $p = f(1)$ and f(q) prefers waiting for f(1) than accepting p_i immediately. Since f(q) is left-continuous, this fact is also true for some q' close to q. Hence $p_i < P(q)$ implies

$$q_{i+1} < q \text{ for } q_i \geq r.$$

The next set of Lemmas characterize a buyer's equilibrium behaviour when he has beliefs $[q_i,1]$ in period i, given that the sellers have a rpsp (r,P) and given that the buyer is constrained to make a first offer $\pi_i \geq P(r)$. Define the function

$$L(Q;\{Q_j,\pi_j\}_{j=0}^{\infty}) = \sum_{j=0}^{\infty} b^j \int_{Q_j}^{Q_{j+1}} v(x) - \pi_j dx.$$

The constrained maximization program can now be defined as

(A) $Z(Q;(r,P)) = \max_C L(Q;\{Q_j,\pi_j\})$

where the constraint set C is defined as

$$C = \{\{Q_j,\pi_j\} | Q_o = Q, Q_1 \geq r,\ Q_{j+1} \geq Q_j,\ \pi_j \geq P(Q_{j+1})\}.$$

Note that since P(q) is left-continuous and non-decreasing and since L is continuous, this is a well-defined problem. Furthermore, Z is continuous in Q. The next Lemma shows that there is a rpsp for which the value of Z(r;(r,P)) is strictly positive. Since v(q) may be less than f(q') for some values of q' > q, Z(q;(r,P)) is not in general positive.

Lemma 4: For the rpsp from Lemma 3, Z(r;(r,P)) > 0.

Proof: Suppose not. That is, Z(r;(r,P)) = 0 and so $Q_{i+1} = Q_i = r$ for all i is a solution to A. Since $q_i = r$ implies that $q_{i+1} = 1$ in any SE and since the expected value of a SE strategy, σ, must take the same value as Z at $q_i = r$ we have

$$Z(r) = \int_r^1 v(x) - f(1)dx = 0$$

or $Q_1 = 1$ must also be a solution to (A) for $Q = q^* = r$.

Define r* relative to r as in Lemma 0 and choose $r' \in [r,r^*]$. Consider the following deviation. Choose $Q_1 = r'$, $\pi_o = P(r')$ and $Q_2 = 1$, $\pi_1 = f(1)$. The expected value of the deviation is

$$D = \int^{r'} v(x) - P(r')dx + b\int^1 v(x) - f(1)dx.$$

$$r \qquad\qquad r'$$

But $Z(r) = 0 = \int_r^1 v(x) - f(1)dx$

$$= \int_r^{r'} v(x) - f(1)dx + \int_{r'}^1 v(x) - f(1)dx$$

so $b\int_{r'}^1 v(x) - f(1)dx = -b\int_r^{r'} v(x) - f(1)dx.$

Using $P(r') = (1 - s)f(r') + sf(1)$,

$$D = \int_r^{r'} v(x) - (1 - s)f(r') - sf(1)dx - b\int_r^{r'} v(x) - f(r')dx$$

$$+ b(r' - r)(f(r' - f(1))$$

$$= \int_r^{r'} (1 - b)(v(x) - f(r')) + s(f(r') - f(1)) + b(f(1) - f(r'))dx$$

$$=(1 - b)\int_r^{r'} v(x) - f(r')dx + (b - s)(f(1) - f(r')(r' - r)$$

$$\geq(1 - b)(r' - r)\epsilon + (b - s)(r' - r)(f(1) - f(r')).$$

For $b \geq s$, this quantity is strictly positive and the deviation would have been profitable. Thus $Z(r;(r,P)) > 0$ for the rpsp given in Lemma 3. Furthermore, $Q_{j+1} > Q_j$ unless $Q_j = 1$ for all solutions to $Z(Q;(r,P))$ and $\pi_j = P(Qj_{+1})$. As r is extended over the unit interval it will be clear that as long as $Z(r) > 0$ for all reservation price strategy pairs this condition will still hold. That is, the buyer will never break off bargaining if $q_i < 1$.

The next Lemma shows that if $Z(r;(r,P) > 0$, then we can always find an $r' < r$ such that the expected value to the buyer of any SE strategy, σ, is the same as $Z(r';(r,P))$.

Lemma 5:(An adaptation of Lemma 4i in GSW). Suppose $Z(r) > 0$, $Q_{i+1} > Q_i$ unless $Q_i = 1$. Then there exists an $r' < r$ such that in any equilibrium σ, if $q_i \geq r'$ is the belief, the expected value of σ satisfies

$$R^{\sigma}(q_i) = Z(Q_o;(r,P)), \quad Q_o = q_i \geq r'.$$

Proof: Let $\{Q_j,\pi_j\}$ be a solution to (A) for $Q = q_i$. Suppose $R^{\sigma} < Z$ or $R^{\sigma}(q_i) = Z(q_i) - \epsilon$, $\epsilon > 0$. Consider the deviation from σ in which the buyer offers p_{i+j} close to and no less than π_j. In Lemma 8 it is shown that for all Q_j, there does not exist a $q > Q_j$ such that $P(q) = P(Q_j)$, for $Q_j < 1$.
p_{j+i} is chosen as follows.
Choose $p_{i+j} > \pi_j$ such that $p_{i+j} - \pi_j \leq \epsilon/2$, $p_{i+j} \leq f(1)$,

and the new path $\{q_{i+j}\}_{j=0}^{\infty}$ satisfies $q_{i+j} - Q_j < \epsilon/(2f(1))$ for all j. The expected value of offering the sequence p_{i+j} at period i is

$$D = \sum_{j=0}^{\infty} b^j \int_{q_{i+j}}^{q_{i+j+1}} v(x) - p_{i+j}dx, \text{ with } q_i = Q_o.$$

$$= \sum_{j=0}^{\infty} b^j \left[\int_{Q_j}^{Q_{j+1}} v(x) - p_{i+j}dx + \int_{Q_{j+1}}^{q_{i+j+1}} v(x) - p_{i+j}dx - \int_{Q_j}^{q_{i+j}} v(x) - p_{i+j}dx \right.$$

$$\geq Z(q_i) - \sum_{j=0}^{\infty} b^j (Q_{j+1} - Q_j)\epsilon/2 + \sum_{j=0}^{\infty} b^j [(1 - b)\int_{Q_{j+1}}^{q_{i+j+1}} v(x)dx$$

$$+ (q_{i+j+1} - Q_{j+1})(bp_{i+j+1} - p_{i+j})]$$

$$\geq Z(q_i) - (1 - q_i)\epsilon/2 - \Sigma b^j(1 - b)(q_{i+j+1} - Q_{j+1})p_{i+j}$$

Now using $q_{i+j} - Q_j \leq \epsilon/(2f(1))$ and $p_{i+j} \leq f(1)$,

$$\geq Z(q_i) - (1 - q_i)\epsilon/2 - (1 - b)\Sigma b^j \epsilon/(2f(1))(f(1))$$

$$\geq Z(q_i) - ((1 - q_i) + 1)\epsilon/2 > Z(q_i) - \epsilon = R^\sigma(q_i).$$

which is a contradiction. So $R^\sigma(q_i) \geq Z(q_i)$.

Now suppose that for all $r' < r$, $R^\sigma(q_i) > Z(q_i)$, for $q_i > r'$. If $q_{i+1} > r$, this can not be the case since it would violate the definition of Z. Suppose $q_i < r$ and along σ, $q_{i+1} < r$.

$$R^\sigma(q_i) = \int_{q_i}^{q_{i+1}} v(x) - p_i dx + bR^\sigma(q_{i+1}).$$

But if $q_{i+1} < r$,

$$R^\sigma(q_i) < (r - q_i)(v(r) - f(q_i)) + bZ(r).$$

$Z(.)$ is continuous so for any $\epsilon > 0$ we can find a $\delta > 0$ such that $Z(r) - Z(q_i) < \epsilon$ if $r - q_i < \delta$. Note that if $Z(q_i) > Z(r)$, we can use the proof directly from GSW. Let $\epsilon \leq (1 - b)Z(r)$. Then there is a $\delta > 0$ such that for all q_i such that $r - q_i < \delta$, we have $Z(q) > bZ(r)$. Thus for $r - q_i < \delta$

$$R^\sigma(q_i) < (r - q_i)(v(r) - f(q_i)) + bZ(r)$$
$$< (r - q_i)(v(r) - f(q_i)) + Z(q_i).$$

Therefore, $q_{i+1} < r$ implies $R^\sigma(q_i) < Z(q_i)$. Combined with the earlier result we get

$$R^\sigma(q_i) = Z(q_i).$$

Lemma 6: (Direct from GSW, Lemma 4ii),iii). Let

Q

$$M = \{Q| \int_{q_i} v(x) - P(Q)dx + bZ(Q) = Z(q_i)\}.$$

Then i) $\inf M(q_i) \in M(q_i)$.

ii) $q_i > r'$ implies $q_{i+1} \in M(q_i)$.

iii) $\hat{P}(q) = P(\inf M(q))$ is non-decreasing and left-continuous. In particular, if

$$Q' > Q,\ p_1 \in M(Q),\ p_2 \in M(Q'),\ \text{then } p_2 \geq p_1.$$

Proof: Let $x_t \in M(Q)$ for all t and choose the decreasing sequence $\{x_t\}_{t=0}^{\infty}$ such that it converges to $m = \inf M(Q)$. This generates a non-increasing sequence $\{P(x_t)\}_{t=0}^{\infty}$ converging to some $p^* \geq P(m)$.

Since Z(.) is continuous,

$$Z(Q) = \int_Q^{x_t} v(x) - P(x_t)dx + bZ(x_t)$$
$$= \int_Q^{m} v(x) - p^*dx + bZ(m)$$

But $p^* > P(m)$ contradicts the definition of Z(.) so

$$p^* = P(m) \text{ and } m \in M(Q).$$

ii) That $q_{i+1} \geq r$, for $q_i \geq r'$ is a result of Lemma 5. Since

$$Z(q_i) = R^{\sigma}(q_i), \quad q_{i+1} \in M(q_i).$$

iii) Let $Q' > Q$, $p_1 \in P(M(Q))$, $p^2 \in P(M(Q'))$ and suppose that $p^1 > p^2$. Then there exists an $x_1 \in M(Q)$ and $x_2 \in M(Q')$ such that $x_1 > x_2$ and

$$Z(Q) \geq \int_Q^{x_2} v(x) - P(x_2)dx + bZ(x_2)$$

$$= \int_{Q}^{Q'} v(x) - P(x_2)dx + \int_{Q'}^{x_2} v(x) - P(x_2)dx + bZ(x_2)$$

$$= \int_{Q}^{Q'} v(x) - P(x_2)dx + Z(Q'). \qquad (1)$$

Similarly,

$$Z(Q') \geq - \int_{Q}^{Q'} v(x) - P(x_1)dx + Z(Q). \qquad (2)$$

Adding (1) and (2) gives

$$0 \geq (Q' - Q)(P(x_1) - P(x_2)) \text{ or } P(x_1) \leq P(x_2).$$

Since $m = \inf M(Q) \leq x_1 \epsilon\ M(Q)$ and $P(x_1) \leq P(x_2)$ for all $x_1 \epsilon\ M(Q)$, $x_2 \epsilon\ M(Q')$ such that $Q' > Q$, then $\hat{P}(Q) = P(\inf M(Q)) \leq \hat{P}(Q')$ for $Q' > Q$ or $\hat{P}$ is non-decreasing. Now let $\{x_t\}_{t=0}^{\infty}$ be an increasing sequence converging to Q. Let $y_t = \inf M(x_t)$. Let $p* = \lim \hat{P}(x_t) = \lim P(y_t)$. In general, $\{y_t\}_{t=0}^{\infty}$ need not be a convergent sequence but since it is bounded, it is possible to choose a convergent subsequence converging to some y_* and so redefine $\{x_t\}_{t=0}^{\infty}$ to be the corresponding subsequence of the original sequence.

$$Z(x_t) = \int_{x_t}^{y_t} v(x) - \hat{P}(x_t)dx + bZ(y_t)$$

Since Z(.) is continuous,

$$Z(Q) = \int_{Q}^{y} v(x) - p{*}dx + bZ(y).$$

Note that $p* \leq P(y)$ since P(.) is left-continuous and non-decreasing. If $p* < P(y)$ this violates the definition of Z, so so $p^* = P(y)$. Also $y \epsilon M(Q)$, so $y \geq \inf M(Q)$, $P(y) \geq \hat{P}(Q)$ or $p^* \geq \hat{P}(Q)$ which gives us $p^* = \hat{P}(Q)$ and $\hat{P}(.)$ is left-continuous.

We now use what is learned in Lemmas 5 and 6 about the buyer's behaviour given a rpsp to extend the (r,P) to (r',P'), with $r' < r$ so an iterative process is derived by which to describe the seller's behaviour over the whole unit interval.

Lemma 7: Let (r,P) be a rpsp satisfying

i) $P(q) = (1 - s)f(q) + s\hat{P}(q;(r,P))$,

and

ii) $Z(r) > 0$, $r > 0$.

Then there exists a rpsp such that $P'(q) = P(q)$ for $q \geq r$ amd P' satisfies i) and ii) also. over $(r',1)$.

Proof:(The first steps are directly from GSW Lemma 5). From Lemma 5 we know there is an $r' < r$ such that in every SE, σ, $q_i \geq r'$ implies $q_{i+1} \epsilon M(q_i)$. Define

$$P'(q) = (1 - s)f(q) + s\hat{P}(q;(r,P)),\ q \geq r'.$$

Clearly P' satisfies left-continuity, non-decreasing and for $q \geq r$, condition i). Also $P = P'$ for $q \geq r$. Since $q_{i+1} \epsilon M(q_i;(r,P))$, then $M(q_i;(r,P)) = M(q_i;(r',P'))$ for $q_i \geq r'$. Thus $\inf M(q_i;(r,P)) = \inf M(q_i;(r',P'))$ so

$$\hat{P}(q_i;(r',P')) = P(\inf M(q_i;(r,P))$$
$$= \hat{P}(q_i;(r,P)),\quad q_i \geq r'.$$

Thus P' satisfies i) for all $q_i \geq r'$.

We now need to show that P' is a rpsp. Suppose q_i is the state and $p_i > P'(q)$ is offered. Suppose now that $q_{i+1} < q$. Without loss of generality, assume $q_{i+1} > r'$. Then $q_{i+2} \in M(q_{i+1};(r,P))$, $q_{i+2} \geq r$.

$$s^i(p_i - f(q)) > s^i(P'(q) - f(q))$$
$$= s^{i+1}(\hat{P}(q;(r,P)) - f(q)).$$

But since $q > q_{i+1}$,

$$s^i(p_i - f(q)) \geq s^{i+1}(p - f(q)) \text{ for all } p \in P(M(q_{i+1};(r,P))$$

including p_{i+1} so q should accept p_i. (By Lemma 6 (iii)).

Now suppose $p_i < P'(q)$ and $q_{i+1} \geq q$ or q accepts p_i. Then $q_{i+1} \in M(q_{i+1};(r,P))$ and $p_{i+1} \geq \hat{P}(q;(r,P))$ so

$$s^i(p_i - f(q)) < s^i(P'(q) - f(q))$$
$$= s^{i+1}(\hat{P}(q;(r',P') - f(q))$$
$$\leq s^{i+1}(p_{i+1} - f(q))$$

or the seller q would prefer to wait as would some q′ less than but close to q.

Thus P′ satisfies the condition of a rpsp. Finally, it must be shown that at r′, $Z(r') > 0$. To do so, the following fact is required:

Proposition: Let $\{x_t\}_{t=0}^{\infty}$ be a decreasing sequence converging to r′ and let $y_t = \inf M(x_t)$ be a (sub)sequence converging to y. Then $y \in M(r')$ and there exists a sequence of prices, $\{\hat{P}(x_t)\}_{t=0}^{\infty}$ converging to P(y).

Proof: Define $p^* = \lim \hat{P}(x_t) \geq P(y)$. Using an argument similar to Lemma 6 it can be shown that Z(r′) can not be a maximum unless $p^* = P(y)$.

Now suppose that $Z(r') = 0$. Choose $y \in M(r')$ as above. Then

$$Z(r') = 0 = \int_{r'}^{y} v(x) - P(y)dx + bZ(y).$$

Choose $\tilde{r}$ such that $\tilde{r} > r'$ and $\int_{r'}^{\tilde{r}} v(x) - f(\tilde{r})dx > 0$. Note that

Lemma 0 assures us that this is also true for any r'' between r' and $\tilde{r}$. So

$$0 = Z(r') = \int_{r'}^{r''} v(x) - P(y)dx + \int_{r''}^{y} v(x) - P(y)dx + bZ(y)$$

$$\leq \int_{r'}^{r''} v(x) - P(y)dx + Z(r'').$$

Or

$$b(Z(r'') \geq -b\int_{r'}^{r''} v(x) - P(y)dx$$

Now consider the deviation which involves choosing $Q_1 = r''$ and $\pi_o = P'(r'')$. The expected value of this deviation is

$$D = \int_{r'}^{r''} v(x) - P'(r'')dx + bZ(r'')$$

$$\geq \int_{r'}^{r''} v(x) - (1 - s)f(r'') - \hat{P}(r'')dx - b\int_{r'}^{r''} v(x) - P(y)dx$$

$$= \int_{r'}^{r''} v(x) - f(r'')dx + s(r'' - r')(f(r'') - \hat{P}(r''))$$

$$- b\int_{r'}^{r''} v(x) - f(r'') + f(r'') - P(y)dx$$

$$= (1 - b)\int_{r'}^{r''} v(x) - f(r'')dx + (r'' - r')[(b - s)(\hat{P}(r'') - - f(r'')) + b(P(y) - \hat{P}(r''))].$$

But since $b \geq s$ and since $r'' \leq \tilde{r}$ implies

$$\int_{r'}^{r''} v(x) - f(r'')dx > 0$$

we have

$$D \geq (r'' - r')[\epsilon + b(P(y) - \hat{P}(r''))].$$

As r'' goes to r', $P(y) - \hat{P}(r'')$ gets arbitrarily small by Proposition 1 so $D > 0$; the deviation would have been profitable.

Thus $Z(r') > 0$.

The last Lemma before the main theorem shows that if a buyer can get more types of sellers at the same price, he will prefer to do so. This result is not trivial (as it is in the private values model) since it could be that a buyer prefers to leave some sellers until later if buying from them would entail a loss. Lemma 8 shows that under the conditions assumed, this is not the case.

Lemma 8: Let $Q_j \geq r$ be an element of a solution to A, $\{Q_j, \pi_j\}_{j=0}^{\infty}$

If $q > Q_j$, then $P(q) > P(Q_j)$.

Proof: Without loss of generality, let $Q_j = Q_1$. Then

$$Z(Q) = \int_{Q}^{Q_1} v(x) - P(Q_1)dx + bZ(Q_1)$$

or

$$Z(Q) = \int_{Q}^{Q_1} v(x) - P(Q_1)dx + b\int_{Q_1}^{r'} v(x) - \hat{P}(Q_1)dx + b\int_{r'}^{\inf M(Q_1)} v(x) - \hat{P}(Q_1)dx$$
$$+ b^2 Z(\inf M(Q_1)),$$

$$(1)\ Z(Q) \leq \int_{Q}^{Q_1} v(x) - P(Q_1)dx + b\int_{Q_1}^{r'} v(x) - \hat{P}(Q_1)dx + bZ(r')$$

for some $r' < q$.

Suppose $P(q) = P(Q_1)$. Since $P(q) = (1 - s)f(q) + s\hat{P}(q)$ and $\hat{P}(q)$ and $f(q)$ are non-decreasing, then we must have for all

$r' \in [Q_1, q]$, $\hat{P}(q) = \hat{P}(r') = \hat{P}(Q_1)$, $f(q) = f(r') = f(Q_1)$.

Consider the deviation to r' instead of Q_1 in the maximization problem, (A). Then

$$Z(Q) \geq \int_{Q}^{Q_1} v(x) - P(Q_1)dx + \int_{Q_1}^{r'} v(x) - P(Q_1)dx + bZ(r')$$

Using the result in (1) above yields

$$\begin{aligned} 0 &\geq \int_{Q_1}^{r'} v(x) - P(Q_1)dx - b\int_{Q_1}^{r'} v(x) - \hat{P}(Q_1)dx \\ &= \int_{Q_1}^{r'} v(x) - (1 - s)f(r') - s\hat{P}(r')dx - b\int_{Q_1}^{r'} v(x) - f(r') + f(r') - \hat{P}(Q_1)dx \\ &= (1 - b)\int_{Q_1}^{r'} v(x) - f(r')dx + s(r' - Q_1)(f(r') - \hat{P}(Q_1)) \\ &\qquad + b(r' - Q_1)(\hat{P}(Q_1) - f(r')), \\ &\geq (r' - Q_1)[(1 - b)\epsilon + (b - s)(\hat{P}(Q_1) - f(r')). \end{aligned}$$

But $\hat{P}(Q_1) = \hat{P}(r') \geq f(r')$, $b \geq s$, $r' > Q_1$. So for r' close to Q_1 this deviation would have increased the value of Z(Q). Thus P(q) > $P(Q_1)$. To state the result somewhat differently, if $Q > Q_j$, $Q_j \in M(q;(r,P))$, then $P(Q) > P(Q_j)$. This is a result shown for a different case in GSW Lemma 4ii).

These eight Lemmas can now be used to describe necessary characteristics of the equilibrium behaviour of both the buyer and all types of sellers. Use the fact that a rpsp (r,P) exists and the result in Lemma 7 that shows we can always extend it to r' < r, with Z(r') > 0 to define the unique rpsp over [0,1]. Call it (0,P). A buyer then is faced with this equilibrium behaviour by the sellers and the Theorem shows that the equilibrium behaviour of the buyer at any q_i is always to choose

$q_{i+1} = \inf M(q_i)$ by offering $\hat{P}(q_i)$.

Theorem 1: (Identical to GSW, Theorem 1) Suppose

I) f(.) and v(.) are left-continuous and non-decreasing.

II) $v(q) = v'(f(q))$, $v(q) > f(q)$ for all q and

$$\lim_{q'\; q} v(q') > \lim_{q'\; q} f(q') \text{ for all } q.$$

III) f(q) satisfies a Lipschitz condition at 1.

IV) $b \geq s$.

Let (0,P) represent the unique rpsp (up to a set of measure zero) over $q \in [0,1]$. $\{q_i, \pi_i\}$ for i=0 to ∞ is a sequential equilibrium path iff $q_1 \in M(0;(0,P))$, $q_{i+1} = \inf M(q_i)$ for $i \geq 1$, and $p_i = P(q_{i+1}) = \hat{P}(q_i)$ for $i \geq 1$.

Proof: (only if) Let σ be any SE. $R^{\sigma}(0) = Z(0)$ by Lemma 5 and so $q_1 \in M(0)$ and $p_o = P(q_1)$. Now consider $p_{i+1} = \hat{P}(q_i)$, $i \geq 1$. Again $R^{\sigma}(q_i) = Z(q_i)$, $q_{i+1} \in M(q_i)$, $p_i = P(q_{i+1})$ and $q_{i+1} > q_i$ (or $q_i = 1$). Since P satisfies CE

$$P(q_{i+1}) = p_i = (1 - s)f(q_{i+1}) + \hat{P}(q_{i+1})$$

or

$$s^i(p_i - f(q_{i+1})) = s^{i+1}(\hat{P}(q_{i+1}) - f(q_{i+1})).$$

≤ ?

≤ seems correct 3/00.

Since $q_{i+1} > q_i$, it must be the case that $p_{i+1} \geq \hat{P}(q_{i+1})$ or else $q \in [q', q_{i+1}]$ for some q' would not accept p_i. But $\hat{P}(q_{i+1}) \leq p$, for all $p \in P(M(q_{i+1}))$, so $\hat{P}(q_{i+1}) \leq p_{i+1}$. Thus $p_{i+1} = \hat{P}(q_{i+1})$.

Now let $x = \inf M(q_{i+1}) \in M(q_{i+1})$ so $q_{i+2} \geq x$ since $q_{i+2} \in$

$M(q_{i+1})$ also. But

$$P(x) = \hat{P}(q_{i+1}) = p_{i+1} = P(q_{i+2}).$$

Since $P(q_{i+2}) = P(x)$, by Lemma 8, $x = q_{i+2}$. Thus $q_{i+2} = x = \inf M(q_{i+1})$ for all $i \geq 0$.

(If) It is now necessary to show that the path described above can be supported by a sequential equilibrium. Before doing so, a word about mixed strategies for the sellers is in order. If the set $T(f) = \{q \mid f(q) = f\}$ has measure 0, the influence of a mixed strategy on the part of a seller has no effect on the payoffs of the game. If, however, $T(f)$ is of positive probability it is necessary to consider mixed strategies in general. In the proof this is done by requiring all $q \in T(f)$ to follow a pure strategy but having different q's use different strategies so that the net effect is as if a seller of type f mixed. Thus for example, if $T(f) = [q, q']$ a strategy for a seller of type f of the form accept with probability 1/2 can be represented as the pure strategies, accept if $q \leq (q' + q'')/2$ and reject if $q > (q' + q'')/2$ for $q \in T(f)$.

Let $q_1 \in M(0)$ and $p_o = P(q_1)$. Define the player's strategies as:

For the seller: For any state q_i, a seller of type q accepts p_i if and only if $p_i \geq P(q)$.

For the buyer: For any state q_i, if $p_{i-1} \geq \hat{P}(q_{i-1})$ offer $p_i = \hat{P}(q_i)$. If $p_{i-1} < \hat{P}(q_{i-1})$ offer $P(q_i)$ with probability β and p_1 with probability $1 - \beta$ where

1) $p_1 = \lim \hat{P}(q)$,

$q \; q_i$

2) $x = \lim_{q \; q_i} f(q)$

3) $\beta = \begin{cases} 1 \text{ if } p_{i-1} - x \geq s[\hat{P}(q_i) - \hat{x}] \\ \beta \text{ such that } p_{i-\bar{1}} \; x = s[\beta(\hat{P}(q_i) - x) + (1 - \beta)(p_i - x)]. \end{cases}$

Note that since $\hat{P}$ is non-decreasing, $p_1 \geq \hat{P}(q_i)$.

The first step shows that $\beta \in [0,1]$ since $\beta \neq 1$ implies

$p_{i-1} - x \leq s(\hat{P}(q_i) - x)$ and CE gives us

$P(q) = (1 - s)f(q) + s\hat{P}(q)$ or

$P(q) - f(q) = s(\hat{P}(q) - f(q)$ and

$\lim_{q \; q_i}(P(q) - f(q)) = \lim_{q \; q_i}(P(q) - f(q))s$ so

$\lim_{q \; q_i} P(q) - x = s(p_1 - x)$

Since $p_{i-1} \leq \hat{P}(q_{i-1}) \leq P(q_i) \leq P(q)$, for $q \geq q_i$

$s(p_1 - x) \geq (p_{i-1} - x)$.

Therefore, $p_{i-1} - x = s(\beta(\hat{P}(q_i) - x) + (1 - \beta)(p_1 - x))$

yields $\beta \in [0,1]$. So β is a probability distribution.

The next step shows that the strategy described above is optimal for the buyer. By definition $\hat{P}(.)$ is optimal. If it can be shown that $p_i \in M(q_i)$ then so is the mixed strategy for the buyer using β. Let $\{x_t\}$ be a decreasing sequence to q_i and let $y_t = \inf M(x_t)$, $y = \lim y_t$.

Since $y \geq x \geq q_i$, then $p \geq P(y)$.

$$Z(x \;) = \int_{x_t}^{y_t} v(x_t) - P(y_t)dx + bZ(y_t)$$

So $Z(q_i) = \int_{q_i}^{y} v(x) - p_1 dx + bZ(y)$

which implies that $p_1 \leq P(y)$ or $p_1 = P(y)$. Therefore p_1 is also an optimal price in state q_i and any randomization between p_1 and $\hat{P}(q_i)$ is optimal.

Now show that the seller's strategy is optimal. That is, seller q never regrets rejecting $p < P(q)$ or accepting $p \geq P(q)$. Suppose $p_i < P(q)$ for $q \in [q_i,1]$. Then $q_{i+1} < q$ and p_{i+1} is such that

$$p_i - f(q_{i+1}) = s(E\tilde{p}_{i+1} - f(q_{i+1})).$$

But $f(q) \geq f(q_{i+1})$ so

$$p_i - f(q) \leq s(E\tilde{p}_{i+1} - f(q))$$

and since the p_j's are non-decreasing in j q does not regret rejecting p_i.

Suppose $p_i \geq P(q)$ and suppose (p_j, q_j) are the subsequent states (non-random for now).

$$s^j(p_j - f(q_{j+1})) = s^{j+1}(p_{j+1} - f(q_{j+1})) \text{ for all } j \geq i$$
$$q_{j+1} \geq q \text{ for } j \geq i \text{ and } f(q_{j+1}) \geq f(q) \text{ gives}$$
$$(p_i - f(q)) \geq s^j(p_{j+1} - f(q))$$

so q should accept. If p_i is followed by a random variable $\tilde{p}_{i+1}$ we have $(p_i - f(q_{i+1})) = s(E\tilde{p}_{i+1} - f(q_{i+1}))$

But again $q_{i+1} \geq q$ so $f(q_{i+1}) \geq f(q)$ and

$$s^i(p_i - f(q)) \geq s^{i+1}(E\tilde{p}_{i+1} - f(q)$$

so he weakly prefers accepting now.