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Sequentially Optimal Auctions

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Abstract: In auctions where a seller can post a reserve price but if the object fails to sell cannot commit never to attempt to resell it, revenue equivalence between repeated first price and second price auctions without commitment results. When the time between auctions goes to zero, seller expected revenues converge to those of a static auction with no reserve price. With many bidders, the seller equilibrium reserve price approaches the reserve price in an optimal static auction. An auction in which the simple equilibrium reserve price policy of the seller mirrors a policy commonly used by many auctioneers is computed.

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1. Introduction

Regular participants in the now defunct Christies' auctions of fine wines in Chicago often experienced deja vu. The same bottles of rare wine seemed to appear auction after auction. Similar phenomena occur in government auctions of lumber tracts, oil tracts and real estate, though with somewhat less mystery -- by policy, properties that failed to sell at earlier auctions were put up for bids at later auctions. Either implicitly or by explicit policy, auctioneers were acknowledging the impossibility of resisting the temptation to try to resell an object that failed to meet a reserve price in an earlier auction.

It has long been recognized in the bargaining literature that sequential rationality imposes constraints on the behavior of agents. Although in many environments, bargainers would like to impose take-it-or-leave-it offers, they often cannot credibly commit never to attempt to renegotiate in the event that no sale occurs. This inability often prevents a trader from extracting much surplus from the transaction, a phenomenon called the "Coase conjecture." Solutions to dynamic bargaining games, therefore, frequently impose, as an additional constraint, some form of sequential rationality. This constraint has been ignored in the literature on optimal auctions (McAfee and McMillan (1987) survey this extensive literature) which shows that in many circumstances, sellers maximize expected profits by withholding the item from the market, even when it is common knowledge that the buyer's willingness to pay exceeds the seller's value.

In this paper, we wed the literature on one-sided offer sequential bargaining (see for example, Gul, Sonnenschein and Wilson (1986) or Fudenberg, Levine and Tirole(1986)) with that of optimal auctions to characterize the dynamic path of reserve prices in auctions in which a seller can commit not to sell only for an exogenously fixed period of time. We show that if bidder types are independently and identically distributed such that the value of the lowest possible bidder type exceeds the seller's use value, then in a game consisting of repeated second price auctions with reserve prices, there is a unique perfect Bayesian equilibrium path of reserve prices which decline deterministically over time. We also show that there is an equilibrium in the repeated first price auctions with reserve prices which generates the same reserve prices and expected revenue for the seller as the sequentially optimal repeated second price auction. In both cases, as the length of time which the seller can commit to keeping the object off the market goes to zero, her revenue converges to her expected revenue from an auction with no reserve price. In contrast to the dynamic monopoly case, however, as the time between auctions shrinks to zero, the initial reserve price remains bounded above the lowest possible bidder valuation. As the number of bidders becomes large, the reserve prices converge to the static optimal reserve price.

In a recent study of auction mechanisms by Bulow and Klemperer (1994) it is shown that an auctioneer may opt to seek more bidders and impose no reserve price rather than attempt to impose an optimal reserve price. Our results in Section 4 provide a complementary explanation. A seller may just as well forgo any attempt to post reserve prices since the gain in expected revenue is small. We begin the analysis with an illustrative "no gap" example in which an equilibrium path of an auction game has the characteristic that reserve prices fall in fixed proportion -- a feature of sequential reserve price policies actually followed by some auctioneers.

2. A Linear Example

We begin with a parametrized example of an infinite horizon auction game. Suppose a seller with one object for which her value is normalized to zero faces n bidders each with valuations for the good which are drawn independently and identically from the Uniform [0,1] distribution. The seller wishes to sell the object via a second price auction with a reserve price. It is well-known that if the seller can commit to a reserve price, the optimal reserve price in this environment is one-half for any number of bidders. If the reserve is not met, however, the seller is now faced with the temptation to reauction the good. Furthermore, if she is not able to resist this temptation, then it is clear that in the first period, bidders with valuations close to but above one-half, will not submit bids and will wait, instead, for a chance at a later auction at a lower reserve price. In this example, we will show that a stationary equilibrium exists which is characterized by two simple constants, r and γ. In any period, if the seller believes that the support of the bidder types she is facing lies in the interval, $[0,v_t]$, she will post a reserve price such that only bidders with valuations above γv_t submit bids above the reserve price. And, in any period, a bidder with valuation v will submit a bid above the reserve price only if the reserve price is rv or lower. (Thus the equilibrium reserve price in any period is $r\gamma v_t$.)

This example differs somewhat from the general class we will analyze later since the bottom of the support of the bidders is not bounded away from the seller's marginal cost. If there is only one bidder, ($n = 1$), Ausubel and Deneckere (1989) show that as well as the Coasian stationary equilibria in which the initial seller price approaches zero as the discount factor, δ, approaches one, there also exist supergame-like equilibria in which the seller is able to support high initial prices which decline slowly over time. This price path is supported by a non-stationary equilibrium involving a threat to revert to the low-profit Coasian price path. However, with more than one bidder, such equilibria are less likely to be supportable. The difference between the two cases arises because even if the reserve price were to approach zero, seller profits do not go to zero. Thus the threat which supports the Ausubel and Deneckere path is not as severe when $n \geq 2$. Observe that the equilibrium we characterize here, as well as the unique equilibrium we find in the general model, are both stationary.

To construct the stationary, linear equilibrium, suppose that whenever the seller believes the bidder types lie in the interval $[0,v_t]$, the seller's best response cutoff function is given by a constant fraction of v_t, γ. Assume, as well, that the function determining the maximal reserve price for which a bidder of type submits a serious bid is also a constant fraction of his valuation, denote it by r. Because the auction is a second price auction, it is straightforward to show, that if the bidder submits a bid, bidding his true valuation is a best response.

Throughout the paper, we use the notation, X_1, to denote the random variable which is the highest of the n bidders' valuations and Y_1 to denote the random variable which is the highest of n-1 bidders' valuations. The corresponding distribution and density functions of Y_1 are

$$F_{Y_1}(Y_1)=F^{n-1}(Y_1),\ \ dF_{Y_1}(Y_1)=f_{Y_1}(Y_1)=(n-1)F^{n-2}(Y_1)f(Y_1).$$

For any reserve price, R, if a bidder of type $x \geq R/r$, submits a bid and if other bidders and the seller follow the assumed behavior, his expected payoff is

$$xF_{Y_1}(x)-RF_{Y_1}(R/r)-\int_{R/r}^{x}Y_1 dF_{Y_1}.$$

That is, he will win only if all the other bidders have valuations below x, will pay the reserve price if all other bidders' valuations are below v' such that $rv' < R$ and otherwise will pay the second highest bid. If, on the other hand, he waits but expects to bid in the next period, in the event of no sale, he will obtain

$$*(xF_{Y_1}(R/r)-r(R/rF_{Y_1}((R/r)-\int_{(R/r}^{R/r}Y_1 dF_{Y_1}).$$

Since $x \geq R/r$, if no sale occurs in the current period, he will win for sure in the next period and pay either the second highest price of the other bidders or the next period reserve price which by assumption will be $r\gamma R/r$ since given the equilibrium strategies, the seller believes only bidder with valuations below R/r would fail to submit bids in the current period. Similar computations can be performed for $x \leq R/r$. Notice that given that the lowest type of bidder to submit a bid is strictly monotonic in the reserve price and bids are strictly monotonic in bidder type, in equilibrium, if x is the lowest type to submit a bid with reserve price R, then x will only win if no other bidder submits a bid and therefore if he wins he must win at exactly the reserve price. Thus, for a reserve price, R, the lowest type bidder to bid is x such that $rx = R$. Combining the equations above yields that r must satisfy

$$(x-rx)F_{Y_1}(x) = *(xF_{Y_1}(x) - r(xF_{Y_1}((x) - \int_{(x}^{x} Y_1 dF_{Y_1}) . \tag{1}$$

Using the uniform distribution, this implies

$$r = 1 - \frac{*}{n}\frac{1-(^{n}}{1-*(^{n}} . \tag{2}$$

The stationary character of the equilibrium implies that we should be able to represent the expected payoff of a seller who is facing bidders with types in the interval, $[0,v_t]$ as a time independent function of v_t alone, $\Pi(v_t)$. Also, since for any reserve price, R, there is a unique lowest type bidder who submits a bid, we can write the seller's choice problem as if she were choosing the lowest type, or cutoff type, rather than the reserve price, R. In any given period, then, for any cutoff level, x, selected by the seller, her payoff is

$$g(v_t,x)=rxnF_{Y_1}(x)[F(v_t)-F(x)] + \int_x^{v_t}\int_x^{X_1} nY_1 dF_{Y_1} dX_1 + *\mathrm{A}(x). \quad \textbf{(3)}$$

$\Pi(v_t)$ must yield the maximized value of this expression for every $v_t \in [0,1]$. Therefore, we can use the envelope theorem to get

$$\frac{\partial \mathrm{A}(x)}{\partial x} = nr(xF_{Y_1}(\,(x) + n\int_{(x}^{x} Y_1 dF_{Y_1}.$$

An optimal choice of cutoff level x given beliefs v_t must satisfy

$$\frac{\partial g(v_t,x)}{\partial x} = 0 = -nrxx^{n-1} + n(v_t - x)nrx^{n-1} - n(n-1)(v_t - x)x^{n-1} + *(nr(x(\,(x)^{n-1} + n\int_{(x}^{x}(n-1)Y_1^{n-1}dY_1).$$

Using (1), and the assumption that $\gamma(v_t) = \gamma\, v_t = x$, then this implies γ and r must satisfy

$$r = 1 + ((1-*)\frac{(}{1-(} - 1)/n \quad \textbf{(4)}$$

Equations (4) and (2) together define the linear solution to the stationary equilibrium. They combine to yield $2\gamma - 1 = \delta\gamma^{n+1}$. The graphs illustrated in Figure 1 show how the reserve price, $r\gamma$, and the cutoff value, γ, vary with selected values of δ and n.

Comments: These equations imply

i) As δ rises, γ increases. The limit of these equations as δ approaches zero approaches the static solution

$$\lim_{*\to 0}(= \frac{1}{2} \text{ and } \lim_{*\to 0} r = 1.$$

Simulations indicate that the reserve price, $r\gamma$, decreases in δ.

ii) As n rises, γ falls and the limit as n approaches infinity also approaches the static solution

$$\lim_{n\to\infty} (= \frac{1}{2} \text{ and } \lim_{n\to\infty} r = 1.$$

iii) As δ approaches one, γ is the solution of $\gamma(2-\gamma^{n+1}) = 1$. For $n = 1$, the unique solution is $\gamma = 1$, for $n > 1$, the correct solution is less than one. The cutoff reserve price constant r approaches $(n-1)/n$. For $n = 1$, then, this implies that the initial price is arbitrarily close to zero. This is the standard Coase like equilibrium price path. (See, for example, Ausubel and Deneckere (1989)). If $n > 1$, since $\gamma < 1$, the reserve price begins strictly positive but must fall arbitrarily quickly as δ approaches one.[3]

iv) Simulation of the equations indicates that γ falls with n, and r increases with n and the reserve price, $r\gamma$ increases with n (as indicated in the last of the four graphs in Figure 1).

The US Forest Service uses a reserve price policy of a form that very closely matches that illustrated in the above example. If the tract fails to sell at a current reserve price, the property is re-auctioned at a reserve price that is ten percent below the previous reserve[4]. That is, the Forest Service has adopted a policy that involves a linearly decreasing reserve price. However, at a real interest rate of anywhere from three percent to ten percent, and assuming that the US Forestry Service reauctions tracts every six months, such a policy would be optimal only if the number of bidders is essentially one. While this is evidently counterfactual, the policy could be interpreted as a concern about collusive behavior by bidders, a possibility ruled out exogenously in this analysis.

The closed form equilibrium strategies allows a more precise determination of the value of

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[3] We are grateful to an associate editor who pointed out this feature.

[4] We are grateful to Robert Marshall for drawing our attention to this fact.

posting reserve prices with limited commitment. Assuming an annual interest rate of 5%, if the auctioneer can commit to keeping the object off the market for as long as a year each time it fails to sell, his gain is at most 10% of the increment earned in the case of full commitment The 10% gain is computed with 2 bidders, and falls to 4% in the case of 5 bidders. If the auctions are spaced only six months apart, the corresponding increments are 5% and 3% of the extra revenues earned in the auction with full commitment. These results reinforce the conclusions of Bulow and Klemperer (1994) that the very small benefits from imposing reserve prices may often be swamped by other considerations.

3. Equilibria in Two Sequentially Optimal Auction Games

The example in Section Two provides some suggestive comparative statics. In this section, we provide a general characterization of equilibria in sequentially optimal reserve price auction games for the case of both first and second price auctions. As mentioned earlier, in this section we focus on the case in which the bidder types valuations are bounded above the valuation of the seller. This is primarily for tractability reasons. The "no-gap" case poses substantial difficulties as a general analysis. So far as we know, little is known about the full equilibrium set even in the case with $n = 1$. The reason is that in the case where the lowest possible bidder valuation is not bounded above the seller's valuation, there is no finite number T after which the game ends with probability one. The proofs of the existence and uniqueness of equilibria in this section show how the equilibria can be constructed by iterating from informationally "small" games (games with the support limited to the bidders with low types and which will always end immediately) to larger games.

The seller of a single good for which she has zero use-value attempts to sell it to a market of n potential buyers. Each buyer values the object in monetary units, v which is ex ante independently and identically distributed according to the distribution function, $F(!)$. It is assumed that $F(!)$ has a strictly positive density *f(!)* on $[1,v_H]$, $v_H < \infty$.[5]. The seller can commit in any given period to sell the good via a second price auction with a reserve price or minimum accepted bid. A bid exceeding the reserve price will be called a "serious bid". The seller cannot commit to withholding the object from sale one period later if bids fail to meet the reserve price in the current period. A sequential auction trading game thus emerges consisting of a potentially infinite sequence of second price auctions with reserve prices. In any period t = 0,1,2,..., if the seller obtains the price, p_t, her payoff is given by $\delta^t p_t$; similarly, if a bidder with valuation, v obtains the object and pays p_t in period t, his payoff is $\delta^t(v - p_t)$, otherwise he receives zero. All agents are risk neutral. (For an analysis of repeated auctions with risk averse bidders, see McAfee and Vincent (1993)) Incorporating both the demand for sequential rationality and for sophisticated learning by the seller, the solution concept we focus on is perfect Bayesian equilibrium (pBe).[6]

Often the phrase "beliefs v_t" will be used as shorthand for the state of a game in which the seller believes that all remaining bidder valuations lie in $[1,v_t)$ in period t. The skimming behavior this terminology implies is justified by the following lemma.

Lemma 0:*i) In any pBe, if a bidder submits a bid above the posted reserve price, R_t, his unique weakly dominant strategy is to bid $\beta(v) = v$.*

ii) (Successive skimming) In any pBe following any history h_t with posted reserve price, R_t, for any bidder, if it is a best response to submit a serious bid for a bidder with valuation v, then it is a strict best response for a bidder with valuation v' > v to submit a serious bid.

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[5] The assumption that the bottom of the support is one has no further substantial consequence beyond the implication that we are in what is known as the "gap" case.

[6] For a definition of perfect Bayesian equilibrium, see Freixas, Guesnerie and Tirole (1985).

Proof: Proofs are provided in the Appendix.

Remark: Second price auctions also possess asymmetric equilibria in which one bidder bids very high and all others bid low. These equilibria involve the use of weakly dominated strategies. In what follows, we restrict attention to equilibria with the feature that if a serious bid is submitted, it satisfies $\beta(v) = v$.

We begin by iteratively defining a sequence of optimization problems. The idea (similar to that of Fudenberg,Levine and Tirole(1986)) is to consider games which artificially must end after at most i periods with the imposition of a reserve price of one.[7] We show that there is a strictly increasing sequence of numbers, $\{z_i\}$, with the feature that for seller beliefs v_t, $v_t \leq z_i$, in all equilibria, the game will end in at most i periods and yield outcomes equivalent to the solution of the artificially constrained optimization problem.

Fix

$$(_0 \equiv (^*_0 \equiv (^*_{-1} \equiv 1, \quad A_0(v) \equiv \int_1^v \int_1^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1, \quad r_0 \equiv 1.$$

Define the sequences,[8]

$$\{(_j\}_{j=0}^{i-1}, \quad \{(^*_j\}_{j=0}^{i-1}, \quad \{r_j\}_{j=0}^{i-1}, \quad \{A_j\}_{j=0}^{i-1}, \quad \{g_j\}_{j=1}^{i-1},$$

[7] We restrict attention to reserve prices of at least one in order to include the case of a single seller facing a single bidder -- the one-sided offer bargaining situation. If there are two or more bidders, since the lowest possible bidder type is assumed to be one, all serious bids will be at least as high as one. Thus, any reserve price from zero to one would have the same consequence. However, if $n = 1$, the price is solely determined by the price posted by the seller. In this case, she would definitely prefer to set a reserve price no lower than one.

[8] The optimization problem is stated in terms of choosing bidder types who submit serious bids in a given period rather than choosing reserve prices. Since it will be shown that for each reserve price there is a unique partition of bidder types who submit serious bids, this behavior will correspond to equilibrium behavior.

iteratively in the following manner.

The sequence of functions, $r_j(v,w)$, denotes the lowest reserve price which would induce a bidder of type v to submit a serious bid:

$$r_j(x,w)F_{Y_1}(x)=(1-*)xF_{Y_1}(x)+*(r_{j-1}(w,(^{*}_{j-2}(w))F_{Y_1}(w)+\int_{w}^{x}Y_1 dF_{Y_1}).$$

The definition corresponds to equation (1) given in the linear example where, in that case, $r(v,w) = rv$. The equation is derived from the comparison made by a bidder who is just indifferent between bidding this period and bidding in the next period. Since this is the lowest type of bidder who will bid, if he wins in this period, it will only be at the reserve price. If he wins in the following period, it may be at the next period reserve price or at a price submitted by a serious bidder next period.

The second argument, w, requires some further explanation. By virtue of Lemma 0, the strategy choice of the seller can be expressed in terms of a selection of the lowest type of bidders who would submit bids in any period instead of as a choice of reserve prices. In some histories of the game, it may be the case that there exist more than one choice of a lowest type that maximizes the seller's expected profits in the next period. Along the equilibrium path, it will be shown that the seller will always select the highest of these possible optimizers (see Lemma Three in the Appendix). However, to show this result we need a device that will illustrate how bidders would behave if they anticipated a different selection by the seller in the subsequent period. Therefore, whether a bidder of type x submits a serious bid depends also on the marginal type, w, that is expected to be indifferent between submitting a bid in the subsequent period and waiting one period more. Observe that, assuming this function is increasing, the lowest bidder type to win at the current reserve price trades off winning at the reserve price this period against the probability weighted sum of the next period reserve price and the second highest bid.

The next sequence of functions, $g_j(v_t,x,w)$ characterize the return to a seller when the possible bidder types lie in the interval, $[1,v_t]$, the lowest bidder type who submits a bid this period is x and the lowest type to submit a bid next period if the game continues would be w.

$$g_j(v_t,x,w) = r_j(x,w)\,nF_{Y_1}(x)[F(v_t) - F(x)] + \int_x^{v_t}\int_x^{X_1} nY_1 f(X_1)\,dF_{Y_1}\,dX_1 + {*}\mathrm{A}_{j-1}(x).$$

This expression is the analog to equation (3) in the linear example. The seller may obtain a sale in the current period, either at the reserve price (with probability $nF^{n-1}(x)[F(v_t)-F(x)]$), or at the second highest valuation if there are two or more bidders. Otherwise, the seller learns that no bidder had a value as great as x and she obtains a discounted continuation value, $\Pi_{j-1}(x)$.

The functions, $\Pi_j(v_t)$, are the maximized values of the seller's continuation payoff in any period with beliefs $[1,v_t]$ assuming the game must end after j periods and subject to what will be the sequential rationality constraint on subsequent choices of bidder cut-offs (selections from γ_{j-1}).

$$\mathrm{A}_j(v_t) = \max_{x \le v_t,\, w \in (_{j-1}(x)} g_j(v_t,x,w).$$

Finally, the sequence of correspondences, $\gamma_j(v_t)$, are the set of maximizers from the same seller optimization problem and determine the seller's sequentially optimal cutoff bidder type when her beliefs are such that the remaining types lie in the interval $[1,v_t]$. As mentioned above, in general γ_j might be set-valued. The function $\gamma^*_j(v_t)$ is constructed by choosing the maximum from $\gamma_j(v_t)$ for every v_t.

$$\gamma_j(v_t) = argmax_{x \le v_t}\{g_j(v_t, x, w) \mid for\ some\ w \in \gamma_{j-1}(x)\},$$

$$\gamma_j^*(v_t) = sup\{\gamma_j(v_t)\}.$$

Whenever γ_j is single-valued, the two coincide. In general, γ^* is a complicated function of v_t but it is analogous to the constant, γ, from the linear example.

For any *i*, assume that this sequence is defined up to *i-1* and make the following induction hypotheses for all $j < i$:

(**H1**) Π_j increasing and continuous.

(**H2**) $\gamma_j(x) < x$ and γ_j is compact-valued, increasing and upper hemi-continuous (implying γ^*_j is increasing and upper semi-continuous).

(**H3**) $r_j(x,w)$ is strictly increasing in both of its arguments, continuous in *x* and upper semi-continuous in *w* and satisfies $r_j(x,w) < x$ for $w < x$ and, where defined,

$$\frac{d[r_j(x,\gamma^*_{j-1}(x))F_{Y_1}(x)]}{dx} \ge x f_{Y_1}(x). \qquad \textbf{(5)}$$

Observe that (**H1-H3**) hold for $i\text{-}1 = 1$.

Lemma One: *If (**H1-H3**) hold for i-1, (**H1-H3**) also hold for i.*

Define

$$z_1 = sup\{v_t \mid \gamma_1^*(v_t) = 1\}$$
$$z_i = \min\{sup\{v_t \mid \gamma_i^*(v_t) \le z_{i-1}\}, v_H\}.$$

The artificial optimization problem represented by the maximizers γ_i determine the seller's optimal choice of a cutoff bidder type this period given that the game must end in at most $i-1$ periods if no sale occurs this period, that is, counting from the next period on. The terms, z_i, denote the largest interval $[1,v_t]$ of possible bidder types such that if the seller believed bidder types lay in this interval, she would be willing to end the game in at most $i-1$ periods counting from the current period -- that is, the constraint that the game end in at most i periods is not binding. The next lemma indicates that for some interval, $[1,v_t]$, $v_t > 1$, the seller would prefer to post a trivial reserve price this period, $(R_t = 1)$ and gain a sale for sure rather than wait until the next period and offer the trivial reserve price. It also shows that as we define higher z_i's, eventually we must cover the whole possible interval of bidder types.

Lemma Two: *There exists an $\epsilon > 0$ such that for all δ and for all n, $z_1 \geq 1 + \epsilon$ and there exists an $T < \infty$, such that $z_T = v_H$.*

Observe that for any $v_t < z_i$, $\gamma_i(v_t) = \gamma_{i-1}(v_t)$, $\Pi_i(v_t) = \Pi_{i-1}(v_t)$ and $r_i(x,\gamma^*_{i-1}(x)) = r_{i-1}(x,\gamma^*_{i-2}(x))$ for $x \leq v_t$. Thus, by Lemma Two, we can define some γ, γ^*, Π and r independent of i over $[1,v_H]$. Fix a v_H and define v_t so that $v_1 \in \gamma(v_H)$, $v_t = \gamma^*(v_{t-1})$ and R_t so that $R_t = r(v_{t+1},\gamma^*(v_{t+1}))$. Observe that since $\gamma^*(\bullet)$ is increasing, such a sequence is generically unique in v_H. Theorem One shows that the solution to the inductively defined optimization problem described above yields the unique pBe path of the sequentially optimal second price auction game.

Theorem One: *In the sequential second price auction game, in any perfect Bayesian equilibrium, in any period $t > 1$, if the belief is v_t, the seller's best response reserve price is $R_t = r(v_{t+1},\gamma^*(v_{t+1}))$ for*

$v_{t+1} \in \gamma(v_t)$. All bidders with type $x \geq \gamma^*(v_t)$ submit bids equal to their own value. No other bidder type submits a serious bid. In period $t = 1$, any reserve price $R_1 = r(v_2,\gamma^*(v_2))$ for $v_2 \in \gamma(v_H)$ is an equilibrium reserve price offer. Along the equilibrium path, for $t \geq 2$, the unique equilibrium reserve price is $R_t = r(\gamma^*(v_t),\gamma^*(\gamma^*(v_t)))$.

Corollary One: *For any seller belief, v_t, let $\{v_{t+i}\}$, $i = 1,2,..,$ be the subsequent seller beliefs along the (unique) equilibrium continuation. The seller's expected equilibrium revenue from this period onward can be expressed as*

$$n\sum_{i=0}^{\infty} *^i \int_{v_{t+i+1}}^{v_{t+i}} \left(v - \frac{F(v_t)-F(v)}{f(v)}\right) F^{n-1}(v) f(v)\,dv. \qquad \textbf{(6)}$$

Now consider a sequential auction game in which the seller conducts first price auctions in every period with reserve prices. In this game, given a reserve price, R_t, in period t, if the highest bid exceeds R_t, the bidder submitting the bid obtains the object and pays the amount bid. If no bid exceeds the reserve, then the game moves to the next period and the seller names a new reserve price. Theorem Two shows that in this game, there is a pBe which is very similar to that of the sequentially optimal second-price auction.

Theorem Two *(Revenue Equivalence): There exists a perfect Bayesian equilibrium of the sequential first price auction such that along the equilibrium path, for every seller belief $[1,v_t]$, the equilibrium reserve price and the seller's expected revenue along the equilibrium is the same as the Sequentially Optimal Second Price Auction.*

Theorem Two demonstrates that there is a pBe for the first-price sealed bid auction which replicates the payoffs of the equilibrium in the second price auction. Thus, the well-known revenue equivalence theorem for one-shot auctions with independent private values (see Milgrom and Weber (1982)) extends to the dynamic auction environment. Furthermore, the reserve prices in the two auctions coincide.

4. Comparative Statics

When a single seller faces a single buyer and has the strategic power to make take-it-or-leave-it offers in every period, Gul, Sonnenschein and Wilson (1986) prove, formally, a conjecture of Coase that as the time costs of waiting until the next period go to zero, the expected profits of the seller converge to the profits she would enjoy against only the buyer with the lowest valuation. That environment, of course, is a special case of the environment analyzed here and, not surprisingly, a generalized version of the Coase Conjecture also holds. Theorem Three shows that as the time costs go to zero, the expected seller revenues converge to the expected revenues from an auction with a reserve price set at the lowest valuation.[9] In the case of more than one bidder, this corresponds to the revenues earned in a no-reserve price auction.

Theorem Three: *(Coase Conjecture)*

$$\lim_{\delta \to 1} A(v_H) = \int_1^{v_H} \int_1^{X_1} Y_1 n f(X_1) \, dF_{Y_1} \, dX_1$$

That is, as δ approaches one, the expected revenues of the seller is the same as in a game with no reserve price.

The next result uses Theorem Three to provide a bound on seller revenues when she cannot

[9] In a continuous time version of this game where the seller's valuation equals the lowest bidders valuation, Milgrom (1987) shows the existence of a perfect equilibrium with the same feature.

commit to keeping the object off the market in the event the reserve price is not met.

Corollary Two: *Let* $\hat{P}$ *denote the seller's expected revenue in a static optimal auction beginning from any period with beliefs,* $[1,v_t]$*. Let* P_δ *denote the expected revenue in the sequential second price auction, and let P denote the expected revenue in an auction with no reserve price. For any* $\delta \geq 1/2$*,*

$$\frac{P_{*}-P}{P} \leq \frac{\hat{P}-P}{P} * (1-*^{T})$$

where T is approximately

$$T \approx 2*\left(\frac{n(v_t-1)}{F(z_1)^n}+1\right). \qquad \textbf{(7)}$$

Corollary Two bounds the gains from using the sequentially optimal auctions relative to a one-shot auction without reserve. It depends on the discount factor, the number of bidders, the range of possible values, and the likelihood that any given bidder has a low enough valuation that he would trade only in the last period of the auction game, $(F(z_1))$. By Lemma Two, and the assumption that $f(\bullet) > 0$, $F(z_1)$ is bounded above zero for all δ and n.

Corollary Three: *For* $v_t \geq z_1$ *and* $n > 1$*, as* δ *approaches one, there exists an* $\epsilon > 0$*, independent of* δ *such that the first period equilibrium reserve price exceeds* $1 + \epsilon$*.*

By Theorem Three, as agents become more patient, seller expected revenue converges to expected revenue from the no-reserve auction. In the standard Coase situation, with $n = 1$, a consequence of this result, is an initial price that is very close to the final price of one. If $n > 1$, so the situation is one of a non-trivial auction, Corollary Three shows that this does not imply a trivial reserve price in general. The initial reserve price remains above one. Furthermore, the next Theorem illustrates that as the number of bidders becomes large, seller revenue approaches that achievable in an auction in which the seller can commit to a static auction with a reserve price. This result is somewhat obvious since even in the one-shot case, as *n* becomes large, the reserve price tends not to add much to expected revenues. More significantly, for the case in which the equilibrium solution is differentiable, it shows that the equilibrium reserve price approaches the optimal reserve price in a static auction.

Theorem Four: *If for all n, there is a number M such that* $\partial\gamma^{*}(v)/\partial v \leq M$*, then as n becomes large, the sequentially optimal reserve prices in each period approach the static optimal reserve price.*

Whether or not the condition of Theorem Four is satisfied will depend on how well-behaved is the sequence of seller optimization problems corresponding to equation (3). Typically, we might expect it to fail if γ turns out not to be singleton-valued for some cutoff bidder type, v_t along the equilibrium path. In the linear case, since γ is a constant, the condition is trivially satisfied. If the objective function $g(v_t,x,w)$ is concave in x for all v_t, then a version of the theorem of the maximum would imply the differentiability of γ.

Theorem Four implies a monotonicity of the reserve prices in the limit as *n* becomes large. One might expect that as δ approaches one, the equilibrium reserve price falls, however, analytic comparative statics in δ do not appear to be available. Theorem Five yields some information on the behavior of the reserve prices and cutoff bidder types for informationally "small" games (which end within two periods).

Theorem Five: *Let $v_H \in [1,z_2)$. For all n and δ such that $v_H \le z_2(n',\delta')$ for any n',δ' in a neighborhood of n and δ,*[10] *in the unique pBe of the sequential auction game,*

i) the first period reserve price R_1 falls as δ increases and rises as n increases.

ii) the second period equilibrium reserve price R_2 is the same independent of δ and n.

iii) there is a number v satisfying $0 = F(v_H) - F(v) - vf(v)$ such that the probability that trade occurs in the first period is given by $1- F^n(v)$. In particular the probability trade occurs in the first period is independent of δ and depends on n only as $1- F^n(v)$ depends on n.

Recall that in optimal static auctions with independent private values, the optimal reserve price is independent of the number of bidders. Theorem Five illustrates that this result does not extend to auctions in which the seller cannot commit to keeping the good off the market. There is a good intuition for this difference. With the possibility of future auctions, along any equilibrium path, the opportunity cost to a bidder of failing to trade in a given auction is determined by the continuation value from subsequent auctions. That is, in any period, a bidder's net value of trading is an induced value determined in part by the continuation path of the equilibrium. A bidder's expected utility from an auction is determined in part by the degree of competition. Thus rises in *n* increase the opportunity cost of a failure to trade. In the second to last period, this is the only effect at work since in the last period, the seller's reserve price is, by assumption, independent of *n*. In longer games, though, there is the additional effect that the seller alters her reserve price as well in response to changes in the profile of induced bidder valuations brought on by changes in *n*.

[10] Note that z_2 will in general vary with δ and *n*. In this sense, the results of Theorem Five are to be thought of as "local" results. We are grateful to an associate editor who pointed out this partial dependence.

The reader acquainted with literature on mechanism design might wonder why an assumption on distributions commonly used in the analysis of reserve price auction, the so-called inverse hazard rate condition, is not needed here. There are two reasons. First, by construction, we restrict attention to the smaller class of mechanisms which is the class of reserve price auctions. Thus, in a full sequentially optimal mechanism game, where the strategy choice of the seller may range across the whole class of implementable mechanisms, the equilibrium path is likely to be different in the absence of this assumption. However, in a different environment where many sellers compete in mechanisms, McAfee (1993) shows that, in fact, seller choices of reserve price auctions are a necessary feature of equilibria. Second, the assumption of an inverse hazard rate condition is often used to ensure the concavity of the seller's static optimization problem and thus the uniqueness of a solution. As is evident in the proof, we do not require the seller's best response correspondence to be singleton-valued in order to obtain uniqueness of the equilibrium path. In periods where the seller's best response correspondence may be multiple-valued, self-interest on the part of the seller ensures that actions are taken in early periods to ensure that the highest element of this set is selected. (See Lemma Three and the discussion in Footnote 10.)

5. Conclusion

The results of our analysis confirm natural conjectures about the ability of sellers to impose reserve prices. As in the case of sequential bargaining, the ability to impose a credible reserve price hinges on the seller's ability to commit to either destroy the product in the event of no sale or keep it for herself. Excess rents are derived from this commitment power. The paper also suggests testable implications of the theory of sequentially optimal auctions. Suppose data which tracks objects for sale at a sequence of auctions and records the number of bidders and\or the length of time between auctions were available, Theorem Four and the example in Section 4 provide predictions about the response of reserve prices to changes in interest rates, auction frequency and the number of bidders. A note of caution must be voiced though. The practice of many auctioneers may frustrate the attempt to gather such data. Ashenfelter (1989) remarks on the tendency of auctioneers to keep reserve prices secret. One possible explanation of this behavior involves common values which we rule out in our model. Thus, the phenomenon of secret reserve prices, themselves, may be treated as evidence that the current private model is not appropriate. (See Vincent (1994)). However, some auctioneers do post explicit reserve prices sometimes as a matter of policy and in other cases, effective reserve prices may be derivable from other data such as suggested minimum bids. Unless bidders are required to apply for eligibility before bidding (as happens in many government auctions), it may also be extremely difficult to extract exact information on the number of bidders. Thus, the positive applications of the theory of sequentially optimal auctions, are limited as are many results from the theory of auctions, by the availability of the appropriate data. Nevertheless, as the analysis of Section 4 illustrates, there remain normative applications of the theory that may be useful either in providing guidance to policymakers or deriving information about other, hidden, aspects of the auction environment.

Appendix

Proof of Lemma 0: i) Fix a reserve price R_t and any bidder and let dB_1 be the density of the highest bid of the other $n-1$ bidders. Conditional on submitting a serious bid, trade will occur in the current period with probability one. The expected return from any bid b is

$$(v-r_t)\int_0^{r_t} dB_1 + \int_{r_t}^{b} (v-B_1)\, dB_1$$

For any bidding behavior of the other bidders, a bid of $b = v$ maximizes this expression.

ii) Observe that if a bidder bids seriously against R_t then by i) he bids $\beta(v) = v$ and will never bid if $v < R_t$. Let dB_1 be the density of the highest of the other *n-1* bids in the current period and consider the expected utility from the equilibrium continuation to a bidder of type *v* when the history is h_t, the bidder has not submitted a bid in the current period and the game has continued to the next period. If *v* submits a bid then

$$(v-R_t)\int_0^{R_t} dB_1 + \int_{R_t}^{v} (v-B_1)\, dB_1 \geq {*V_B}(v,h_t) Prob[B_1 < R_t] \tag{8}$$

Suppose there is a type $v' > v$ who does not submit a bid. Then

$$\int_0^{R_t} dB_1 + \int_{R_t}^{v} (v'-B_1)\, dB_1 + \int_{v}^{v'} (v'-B_1)\, dB_1 \leq {*V_B}(v',h_t) Pro \tag{9}$$

Subtracting (8) from (9) and rearranging yields

$$v'-v \leq \frac{Prob[B_1 \leq R_t]}{Prob[B_1 \leq v]} * (V_B(v',h_t) - V_B(v,h_t)) \tag{10}$$

Observe that a bidder of type *v* can always mimic the behavior of bidder of type *v'*. Let $\alpha_{t+j}(v',h_t)$ be the probability a bidder who behaves as if he were *v'* obtains the object in period *t+j* in the pBe following history h_t (calculated from period *t*) and let $p_{t+j}(v',h_t)$ be the expected price paid conditional on obtaining the good. By definition

$$V_B(v,h_t) \geq \sum_{j=0}^{\infty} *^{t+j} "_{t+j}(v',h_t)(v-p_{t+j}(v',h_t)) \quad \textbf{(11)}$$

while

$$V_B(v',h_t) = \sum_{j=0}^{\infty} *^{t+j} "_{t+j}(v',h_t)(v'-p_{t+j}(v',h_t)). \quad \textbf{(12)}$$

Subtracting (11) from (12) and combining with (10) yields

$$v'-v \leq (v'-v) * \frac{Prob[B_1 \leq R_t]}{Prob[B_1 \leq v]} \sum_{j=0}^{\infty} *^{j} "_{t+j}(v',h_t).$$

a contradiction since the sum of the α's must be one or less.

Proof of Lemma One: (**H3**) $r_i(x,w) < x$ for $x > w$ since it is a convex combination of x and values strictly less than x. To see how it changes with w,

$$\frac{\partial r_i(x,w)F_{Y_1}(x)}{\partial w} = *\left(\frac{d[r_{i-1}(v,(^{*}_{i-2}(w))F_{Y_1}(w)]}{dw} - wf_{Y_1}(w)\right)$$

This term is positive by (**H3**) for $i\text{-}1$. Furthermore,

$$\frac{\partial[r_i(x,w)F_{Y_1}(x)]}{\partial x} = \frac{\partial r_i(x,w)}{\partial x}F_{Y_1}(x) + r_i(x,w)f_{Y_1}(x)$$
$$= xf_{Y_1}(x) + (1-*)F_{Y_1}(x)$$
$$or$$
$$\frac{\partial r_i(x,w)}{\partial x}F_{Y_1}(x) = (x-r_i(x,w))f_{Y_1}(x) + (1-*)F_{Y_1}(x)$$

so $r_i(x,w)$ is increasing in x for $x \geq w$. Since r_i is also increasing in w and since γ^*_{i-1} is increasing, equation (5) is satisfied for i.

(**H1**) Since g_i is continuous in v_t and x and increasing and continuous in r_i, since $r_i(x,w)$ is increasing and upper semi-continuous, g_i is upper semi-continuous. A version of the theorem of the maximum (exploiting the fact that $w > \gamma_j(w)$) then implies that $\Pi_i(v_t)$ is continuous and $\gamma_i(v_t)$ is upper hemi-continuous.

(**H2**) To see that $\gamma_i(v_t)$ is increasing, let $y < y'$ and $x \in \gamma_i(y)$, $x' \in \gamma_i(y')$ and suppose that $x' < x$. To save on notation, let $m = r_i(x,w)F_{Y1}(x)$ for some $w \in \gamma_{i-1}(x)$ and $m' = r_i(x',w')F_{Y1}(x')$ for some $w' \in \gamma_{i-1}(x')$. By the induction hypothesis, $w' < w$. By definition,

$$\begin{aligned} g_i(y,x,w) + \int_y^{y'}\int_x^{X_1} nY_1 dF_{Y_1} f(X_1)dX_1 \\ + m\,n[F(y') - F(y)] = g_i(y',x,w) \end{aligned} \quad \textbf{(13)}$$

and

$$\begin{aligned} g_i(y,x',w') + \int_y^{y'}\int_{x'}^{X_1} nY_1 dF_{Y_1} f(X_1)dX_1 \\ + m'\,n[F(y') - F(y)] = g_i(y',x',w') \end{aligned} \quad \textbf{(14)}$$

Subtracting equation (14) from (13) yields

$$\begin{aligned} n(F(y') - F(y))(m - m' - \int_{x'}^{x} Y_1 dF_{Y_1}) \\ g_i(y',x,w) - g_i(y',x',w') - (g_i(y,x,w) - g_i(y,x',w') \end{aligned} \quad \textbf{(15)}$$

The right side of equation (15) is non-positive by definition of x,y,x' and y'. Since $w' \leq w$ and we have shown that r_i is increasing in w, by definition of r_i,

$$\begin{aligned} m &= r_i(x,w)F_{Y_1}(x) \\ &\geq r_i(x,w')F_{Y_1}(x) \\ &= m' + (1-*)(xF_{Y_1}(x) - x'F_{Y_1}(x')) + *\int_{x'}^{x} Y_1 dF_{Y_1} \end{aligned}$$

We can rewrite the left side of (15) therefore as greater than

$$n(1-*)(F(y')-F(y))\{(x-x')F_{Y_1}(x')+(F_{Y_1}(x)-F_{Y_1}(x'))(x-E[Y_1|x'\leq Y_1\leq x])\}$$

Since $x > x'$ this expression is strictly positive -- a contradiction. Therefore, $\gamma_i(x)$ is increasing and $\gamma^*_i(x)$ is increasing and upper semi-continuous. ■

Proof of Lemma Two: Observe that

$$\frac{\partial g_1(v_t,x,1)}{\partial x}=n(1-*)[F(v_t)-F(x)-xf(x)]F_{Y_1}(x)$$

Since $f(x) > 0$, there is an $\epsilon > 1$ such that this expression is strictly less than zero for all $v_t \in [1,\epsilon)$. Fix i-1. By definition, for $x \in \gamma_i(v_t)$, $x \leq z_{i-1}$ and therefore $\Pi_i(x) = \Pi_{i-1}(x)$. Since

$$\begin{aligned}\mathrm{A}_i(v_t)-*\mathrm{A}_i(x)&=n(F(v_t)-F(x))r_i(x,(_{i-1}(x))F_{Y_1}(x)+\\&\quad n\int_x^{v_t}\int_x^{X_1}Y_1f(X_1)dF_{Y_1}dX_1+*(\mathrm{A}_{i-1}(x)-\mathrm{A}_i(x))\\&=n(F(v_t)-F(x))r_i(x,(_{i-1}(x))F_{Y_1}(x)+n\int_x^{v_t}\int_x^{X_1}Y_1f(X_1)dF_{Y_1}dX_1\end{aligned}$$

and

$$\mathrm{A}_i(v_t)-*\mathrm{A}_i(x)\geq(1-*)\mathrm{A}_i(v_t)\geq n\int_1^{v_t}\int_1^{X_1}Y_1f(X_1)dF_{Y_1}dX_1(1-*)$$

we have that for all $x \in \gamma_i(v_t)$, there exists an $\nu > 0$ independent of i such that $x \leq v_t - \nu$. Since γ_i is increasing and upper hemicontinuous and satisfies $\gamma_i(x) \leq x - \nu$, the convex hull of γ_i has an inverse which is increasing and upper hemicontinuous defined over $[1,v_H]$ and lies above the line, $y - \nu = x$.

Thus $z_i = \max\{v \mid \gamma_i(v) = z_{i-1}\}$ exists and satisfies $z_i \geq z_{i-1} + \nu$. This procedure extends the definition of z_i over the interval $[1,v_H]$. ■

Proof of Theorem One: The proof proceeds by defining necessary conditions of bidder and seller strategies iteratively over the support of bidder types via two lemmas. Let $R^\sigma(v_t,h_t)$ the seller's best response reserve price in some pBe, σ, following a history, h_t and with beliefs that bidder types lie in $[1,v_t]$ and let $P^\sigma(v_t,h_t)$ be her expected payoff. Condition C1(a,j) partly characterizes the strategy of the seller.

$$C1(a,j): \forall F, \forall v_t<a, \forall h_t,\ R^{F}(v_t,h_t)=r_j(x,(^*_j(x)),\ for\ x\in(_j(v_t)).$$

Condition C2(j) characterizes strategies of bidder types below z_j and partially for types above z_j.

$$C2(j): \forall F, \forall h_t, \forall v_t<z_j,\ if\ R_t>r_j(v_t,(^*_j(v_t)),\ No\ Bid$$
$$\forall v_t,\ if\ R_t \leq r_j(v_t, \min(_j(v_t)),\ Bid\ B(v_t)=v_t.$$

Lemma Three[11]: *If C1, C2 hold for j= i-1 and a =* z_{i-1}*, then C1 holds for j = i and a =* z_i.

Proof of Lemma Three: Let a denote the supremum of a's such that C1 holds for $j = i$ and a'. Since for $v_t \leq z_{i-1}$, $r_{i-1} = r_i$ and $\gamma_{i-1} = \gamma_i$ for $v_t < z_{i-1}$, then $a \geq z_{i-1}$. Observe that for $a > 1$, since $\Pi(v_t)$ is bounded above zero, and $f(v)$ is positive, there is always an $\epsilon > 0$ such that $(F(v)-F(a))v_H n + \delta\Pi_i(a) < \Pi_i(v)$, for all v in $[a,a+\epsilon)$. Suppose $a < z_i$. Then there exists a σ and h_t and v_t in $(a,a+\epsilon)$ such that the seller's best response reserve price exceeds $r_i(z_{i-1},z_{i-2})$ and since a reserve price below that level

[11] Observe that the optimization problems as stated are "as if" the seller can also choose in period t her most favorable cutoff level (among her optimal responses) in period t+1 if the object fails to sell. This is not true in general since $\gamma(v_{t+1})$ may not be single valued. Lemma Three illustrates that since if there were a possibility that the future belief is unfavorable (that is, v too low) then the upper hemicontinuity and monotonicity of the optimal choice function would have allowed the seller to do better by selecting a slightly higher cutoff level this period. This would yield only first order costs in the probability of a sale but increase the reserve price by an amount bounded above zero. Therefore, for $t > 2$, the equilibrium reserve price is $R_t(v_t) = r(\gamma^*(v_t),\gamma^*(\gamma^*(v_t)))$.

generates payoffs determined by C2, the payoff from σ is bounded from above and below by

$$A_i(v_t) \leq P^F(v_t, h_t) \leq (F(v_t) - F(a))v_H + {}^*A_i(a) < A_i(v_t)$$

Which is a contradiction. Therefore, $a \geq z_i$.

Now, suppose that $R_t = r(v_{t+1}, y)$ for $y < \gamma^*(v_{t+1}), y \in \gamma(v_{t+1})$. Since $r(x,v)$ is strictly increasing in v, for every $v' > v_{t+1}$, and every $y' \in \gamma(v')$, there is an $\epsilon > 0$ such that $r(v',y') \geq r(v_{t+1},y) + \epsilon$. A reserve price of r' instead of R_t, yields the seller an expected revenue of

$$g(v_t, v', y') \geq n(F(v_t) - F(v'))(r_t + ,)F_{Y_1}(v') + \int_{v'}^{v_t}\int_{v'}^{v_t} Y_1 dF_{Y_1} f(X_1) dX_1 + {}^*A_i(v')$$

Since this function is continuous in v'

$$\lim_{v' \downarrow v_{t+1}} g(v_t, v', y') - g(v_t, v_{t+1}, y) \geq n, (F(v_t) - F(v_{t+1}))F_{Y_1}(v_{t+1}) > 0$$

the seller could have improved on R_t by offering a slightly higher reserve price contradicting the assumption that R_t was an equilibrium reserve price. ■

Lemma Four: *If C2 holds for j = i-1 and C1 holds for j = i and* $a = z_i$, *C2 holds for j = i.* **Proof of Lemma Four**: Since $r(x,w)$ is strictly increasing in both its arguments and γ is an increasing correspondence, the correspondence, $r(x,\gamma^*(x)),\ x \in \gamma(v)$, has a unique inverse, call it $\rho(r)$. By Lemma 0, for any reserve price, R_t, there is a v_{t+1} such that only bidder types above v_{t+1} submit serious bids. Suppose $R_t > r(z_i,\gamma^*(z_i))$ and $v_{t+1} < z_i$. By C1, the next period reserve price is $R_{t+1} \leq r(\gamma_{*(v}t+1), \gamma^*(\gamma^*(v_{t+1}))$. By bidding in period t, v_{t+1} receives

$$(v_{t+1} - R_t)F_{Y_1}(v_{t+1}) \qquad \textbf{(16)}$$

while by waiting until the next period, he would get no worse than

$$*[(v_{t+1}-R_{t+1})F_{Y_1}(v_{t+2})+\int_{v_{t+2}}^{v_{t+1}}(v-Y_1)dF_{Y_1}] \tag{17}$$

By definition of r, equation (17) strictly exceeds equation (16) so all types in the neighborhood of v_{t+1} do better not to bid when the reserve price is R_t. Now suppose that $R_t < r(z_i, min\gamma(z_i))$ and $v_{t+1} > \rho(R_t) = v$. Let τ be the smallest number such that along the equilibrium continuation path, $z_i > v_{t+\tau+1}$. (If the equilibrium involves mixed strategies, then the following argument can be made using distributions over continuation paths). If $v_{t+\tau} \geq v$, then $R_{t+\tau} > r(\gamma^*(v),\gamma^*\gamma^*(v))$ (by C1) and bidder type v would have done better to bid when the reserve price was R_t. If $v_{t+\tau} < v$, since $v_{t+\tau-1} \geq z_i$, $R_{t+\tau-1} \geq r(z_i, min\gamma(z_i)) \geq r(z_{i-1},\gamma^*(z_{i-1}))$ and

$$R_{t+J-1}=(1-*)v_{t+J-1}+\frac{*}{F_{Y_1}(v_{t+J-1})}(R_{t+J}F_{Y_1}(v_{t+J-1})+\int_{v_{t+J}}^{v_{t+J-1}}Y_1 dF_{Y_1})$$

we must have $R_{t+\tau} \geq r(z_{i-1},\gamma^*(z_{i-1}))$ and $v_{t+\tau} \geq z_{i-1}$. But this violates the optimality of type v's decision not to bid when the reserve price is R_t since

$$R_t=(1-*)v+\frac{*}{F_{Y_1}(v)}(r(y,(^*(y))F_{Y_1}(v)+\int_{(^*(v)}^{v}Y_1 dF_{Y_1}),$$

for some $y \in Convexhull\gamma(v)$. ■

A simple adaptation of Lemmas Three and Four show that (**C1**) and (**C2**) hold for $i = 1$. Therefore, we can now apply Lemmas Three and Four iteratively to specify necessary conditions of equilibrium behavior over the whole interval, $[1,v_H]$. Sufficiency is not quite shown. The result would follow simply if $\gamma(v_t)$ was known to be single-valued. In general it is not, but the argument given in Gul, Sonnenschein, Wilson (1986) is easily adapted to show sufficiency as well. Suppose the seller posts an out of equilibrium reserve price R', $r(v,\min\gamma(v)) < R' < r(v,\gamma^*(v))$, for some v. Subsequent randomization of the reserve price off the equilibrium path is a characteristic of the equilibrium in order to convexify the correspondence, γ(•). All bidder types $v' > v$ submit bids and all bidder types $v'' < v$ do not submit bids. In the next period, the seller randomizes between her two best response choices of v_{t+1}, $\gamma^*(v)$ and min γ(v) by offering either $r(\min\gamma(v),\gamma^*(\min\gamma(v)))$ or $r(\gamma^*(v),\gamma^*(\gamma^*(v)))$ so as to make bidder types v and higher willing to submit bids in the current period. ■

Proof of Corollary One: An adaptation of an argument in Myerson and Satterthwaite (1983) shows that, if $U(v)$ is the expected utility of a buyer in any Bayesian equilibrium, then $dU(v)/dv = \delta^{t(v)}F^{n-1}(v)$ almost everywhere, where $t(v)$ is the equilibrium period of trade of a bidder of type v. By Theorem One, this period is deterministic (up to a selection of v_{t+1}). Integrating by parts yields

$$n\int_1^{v_t} U(v)f(v)dv = \sum_{i=0}^{T} *^i \int_{v_{t+1+i}}^{v_{t+i}} F^{n-1}(v)(1-F(v))dv$$

Using the definition of total expected surplus as the sum of seller's expected revenue and total expected buyer surplus and rearranging terms yields (6). ■

Proof of Theorem Two: The proof proceeds by characterizing strategies and showing that they comprise a pBe of the sequential first price auction. Let $r(v,\gamma^*(v))$ and $\gamma(v)$ be as in the proof of Theorem One.

<u>Seller Strategies</u>: For every seller belief, $[1,v_t)$, if $R_{t-1} = r(v_t,\gamma^*(v))$, post a reserve price $R_t = r(\gamma^*(v_t),\gamma^*(\gamma^*(v_t)))$. If $r_* = r(v_t,\min\gamma(v_t)) \leq R_{t-1} < r(v_t,\gamma^*(v_t)) = r^*$ post reserve price $R_t = r_*$ with probability β and $R_t = r^*$ with probability 1 - β where β satisfies

$$R_{t-1} = \$ * r(\min((v_t),(^*(\min((v_t))) + (1-\$)r((^*(v_t),(^*((^*(v_t)))$$

Seller Beliefs: For any beliefs $[1,v_{t-1})$ in period $t\text{-}1$, if $r(v_t,\min\gamma(v)) \leq R_{t-1} \leq r(v_t,\gamma^*(v))$, no bid is submitted, beliefs in period t are $[1,v_t]$. If $1 > R_{t-1}$, and no bids are submitted, beliefs in period t are the same as in period $t\text{-}1$. If $R_t > r(v_{t-1},\gamma^*(v_{t-1}))$, beliefs are $[1,v_{t-1})$.

Buyer Strategies: In every period, if $v_{t+1} = \rho(R_t)$ where ρ is the inverse of $r(v,\gamma^*(v))$, all $v < v_{t+1}$ do not submit bids. All $v \geq v_{t+1}$ submit bids, $B(v;R_t)$, where

$$\begin{gathered} B(v;R_t)F_{Y_1}(v) = \int_1^v x dF_{Y_1} + c, \\ B(v_{t+1};R_t) = R_t \end{gathered} \tag{18}$$

First suppose that this profile of strategies comprise a pBe. In this case, then for any seller belief, $[1,v_t]$, the sequence of bidder types who bid in all subsequent periods is the same as in the second price equilibrium, that is, trade occurs with the same bidder type in the same period as the second price auction. A simple adaptation of the Myerson-Satterthwaite reasoning then implies that the expected seller revenues are the same. The definition of the seller strategies implies that the equilibrium reserve prices are the same. Thus as long as we can show that the strategies form a pBe, the theorem is proved.

Note that the seller beliefs satisfy Bayes' rule given the buyers' strategies.

To show the optimality of buyer strategies, let the seller beliefs be $[1,v_t)$ and the reserve price be R_t and $v_{t+1} = \rho(R_t)$. Let $r_* = r(\min\gamma(v_{t+1}),\gamma^*(\min\gamma(v_{t+1})))$ and $r(\gamma^*(v_{t+1}),\gamma^*(\gamma^*(v_{t+1}))) = r^*$. Finally suppose that buyers follow the proposed strategies in all later periods. Then, by definition of the bidding functions next period, if the reserve price is r^*, bidder type v_{t+1} bids

$$B(v_{t+1})F_{Y_1}(v_{t+1}) = \int_1^{v_{t+1}} x dF_{Y_1} + c$$
$$= \int_{(^*(v_{t+1})}^{v_{t+1}} x dF_{Y_1} + F_{Y_1}((^*(v_{t+1}))r^*$$

where the constant term is determined by (18). A similar equation holds for a next period reserve price of r_*. Therefore, given the strategies of the other bidders, a bidder of type v_{t+1} who bids when the current reserve is R_t will only bid R_t and receives expected utility

$$(v_{t+1} - R_t)F_{Y_1}(v_{t+1}) \qquad \textbf{(19)}$$

If v_{t+1} waits until next period, his expected utility is

$$\begin{aligned} &*(v_{t+1} - \$B(v_{t+1};r_*) - (1-\$)B(v_{t+1};r^*))F_{Y_1}(v_{t+1}) \\ &= *(v_{t+1} - \$(\int_{\min((v_{t+1})}^{v_{t+1}} x dF_{Y_1} + r_* F_{Y_1}(\min((v_{t+1}))) \\ &\quad - (1-\$)(\int_{(^*(v_{t+1})}^{v_t+1} x dF_{Y_1} + r^* F_{Y_1}((^*(v_{t+1})))) \end{aligned} \qquad \textbf{(20)}$$

By definition of v_{t+1}, and β, (19) equals (20) so bidder type v_{t+1} is just indifferent between bidding this period and next. Since this period utility increases faster in bidder type than next period utility, that means that all bidder types above v_{t+1} strictly prefer bidding and those below, strictly prefer not to bid. Finally, standard arguments from first price auctions illustrate that the bid function (18) is a best response for bidders who bid in period *t*.

Finally, to show the sequential optimality of the seller's strategy, suppose that the seller's strategy is sequentially rational for all beliefs $[1,v]$ with $v \leq z_i$. Then, for any beliefs satisfying this restriction, a further application of Myerson-Satterthwaite illustrates that the expected payoff for the seller from the pBe is the same as $\Pi(v)$. An argument similar to that of Lemma Three for Theorem

One, then shows that there is an $\epsilon > 0$, such that for all $v \leq z_i + \epsilon$, a reserve price such that $v_{t+1} \leq z_i$ is optimal and therefore, given proposed bidder behavior, expected equilibrium payoffs are again given by $\Pi(v)$. The same argument then is applied to $v \leq z_i + 2\epsilon$ and so on. That the proposed seller behavior is optimal for $v \leq z_1$ is straightforward to show. ■

Proof of Theorem Three: Part of this proof follows Tirole (1988). We show first that for any $\epsilon > 0$ there is an T such that all equilibria of games with $1 > \delta > \epsilon$ end with probability one after T periods, independent of δ. For any δ, Theorem One shows there exists a unique equilibrium in which the decision of bidders whether to participate given a current period reserve price is time independent and the seller's profits depends only on the current state, v_t. For any δ and pBe, let v_t, v_{t+1}, v_{t+2} be the equilibrium cutoff levels of participating bidders in periods $t,t+1$ and $t+2$ respectively with $v_{t+2} \geq z_1$ and define $F_{t+i} \equiv F(v_{t+i})$. Note that given the current state, v_t, since bidder strategies are stationary, a seller could always have chosen a reserve price to induce a next period state v_{t+2} instead of v_{t+1} so we must have

$$
\begin{gathered}
R_t F_{t+1}^{n-1} n[F_t - F_{t+1}] + \int_{v_{t+1}}^{v_t}\int_{v_{t+1}}^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 + \\
* R_{t+1} F_{t+2}^{n-1} n[F_{t+1} - F_{t+2}] + * \int_{v_{t+2}}^{v_{t+1}}\int_{v_{t+2}}^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 + *^2 \mathrm{A}(v_{t+2}) \\
\geq R_{t+1} F_{t+2}^{n-1} n[F_t - F_{t+2}] + \int_{v_{t+2}}^{v_{t+1}}\int_{v_{t+2}}^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 + \\
\int_{v_{t+1}}^{v_t}\int_{v_{t+2}}^{v_{t+1}} nY_1 f(X_1)\, dF_{Y_1} dX_1 + \int_{v_{t+1}}^{v_t}\int_{v_{t+1}}^{X_1} nY_1 f(X_1)\, dF_{Y_1} dX_1 + * \mathrm{A}(v_{t+2})
\end{gathered}
$$

where the second integral has been broken to facilitate rearranging terms. This can be written as

$$
\begin{aligned}
&[R_t F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1}] n [F_t - F_{t+1}] \\
&\geq (1-\delta) R_{t+1} F_{t+2}^{n-1} n [F_{t+1} - F_{t+2}] + (1-\delta) \int_{v_{t+2}}^{v_{t+1}} \int_{v_{t+2}}^{X_1} n Y_1 f(X_1) dF_{Y_1} dX_1 + \\
&\int_{v_{t+1}}^{v_t} \int_{v_{t+2}}^{v_{t+1}} n Y_1 f(X_1) dF_{Y_1} dX_1 + \delta (1-\delta) \mathrm{A}(v_{t+2})
\end{aligned}
$$

By definition of R and v_{t+i},

$$
R_t F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1} = (1-\delta)(v_{t+1} F_{t+1}^{n-1} - R_{t+1} F_{t+2}^{n-1}) + \delta \int_{v_{t+2}}^{v_{t+1}} Y_1 dF_{Y_1}
$$

so substituting into the above inequality, rearranging terms (by subtracting the last term in the above equation and dropping the last integral in the right side of the inequality) and dividing by 1-δ > 0 yields

$$
[v_{t+1} F_{t+1}^{n-1} n [F_t - F_{t+1}] - R_{t+1} F_{t+2}^{n-1} n [F_t - F_{t+2}] - \int_{v_{t+1}}^{v_t} \int_{v_{t+2}}^{v_{t+1}} n Y_1 f(X_1) dF_{Y_1} dX_1 \geq \delta \mathrm{A}(v_{t+2})
$$

Rearranging once more gives

$$
\begin{aligned}
&n [F_t - F_{t+1}] [v_{t+1} (F_{t+1}^{n-1} - F_{t+2}^{n-1}) - \int_{v_{t+2}}^{v_{t+1}} Y_1 dF_{Y_1}] \\
&+ [v_{t+1} - R_{t+1}] F_{t+2}^{n-1} n [F_t - F_{t+2}] \geq \delta \mathrm{A}(v_{t+2})
\end{aligned}
$$

The first term is positive and $F_{t+2} \leq F_{t+1}$, so the inequality can be written

$$
n [F_t - F_{t+2}] [v_{t+1} (F_{t+1}^{n-1} - F_{t+2}^{n-1}) - \int_{v_{t+2}}^{v_{t+1}} n Y_1 dF_{Y_1} + [v_{t+1} - R_{t+1}] F_{t+2}^{n-1}] \geq \delta \mathrm{A}(v_{t+2})
$$

Since $v_H - 1 \geq v_{t+1} - p_{t+1}$ for any price paid by a bidder in any equilibrium, this implies

$$
n [F_t - F_{t+2}] [v_H - 1] F_{t+1}^{n-1} \geq \delta \mathrm{A}(v_{t+2}) \geq \delta F_{t+2}^{n}
$$

The last inequality comes from the fact that the seller can always post a reserve price of one in any period. By Lemma 2, $F(z_1) > 0$ for any $v_{t+2} > z_1$, (for $v_t < z_3$ the game ends in two periods). Rewrite the inequality as

$$F_t - F_{t+2} \geq \frac{*F(z_1)^n}{n(v_H - 1)F_{t+1}^{n-1}} \geq \frac{*F(z_1)^n}{n(v_H - 1)}$$

Since the inequality holds for all t such that $v_{t+2} > z_1$, define $\hat{i}$ to be the smallest integer exceeding

$$\frac{n(v_H - 1)}{*F(z_1)^n}$$

Suppose that along the equilibrium path, there is a $v_{t+2} \geq z_1$ such that $t+2 \geq 2\hat{i}$. Since

$$1 \geq \sum_{k=0}^{t/2} (F_{2k} - F_{2k+2}) = 1 - F_{t+2} \geq (\frac{t}{2} + 1)\frac{*F(z_1)^n}{n(v_H - 1)} > \hat{i}\frac{*F(z_1)^n}{n(v_H - 1)} \geq 1,$$

which is a contradiction. Observe that $\hat{i}$ is decreasing in δ. Combined with Corollary One, this implies that as δ approaches one, the seller's expected revenue approaches the expected revenue in a game with no reserve price. ■

Proof of Corollary Two: By Myerson (1981), P and $\hat{P}$ satisfy

$$P = n\sum_{i=0}^{T} \int_{v_{t+i+1}}^{v_{t+i}} (v - \frac{F(v_t) - F(v)}{f(v)}) F^{n-1}(v) f(v) dv \tag{21}$$

and

$$\hat{P} = n\int_{\hat{v}}^{v_t} (v - \frac{F(v_t) - F(v)}{f(v)}) F^{n-1}(v) f(v) dv \tag{22}$$

where $\hat{v}$ is the optimal reserve price for the static auction. By Corollary One and by definition of $\hat{v}$,

$$*\leq\int_{\hat{v}}^{v_t}\left(v-\frac{F(v_t)-F(v)}{f(v)}\right)nF^{n-1}(v)f(v)dv+*^T\int_1^{\hat{v}}\left(v-\frac{F(v_t)-F(v)}{f(v)}\right)nF^{n-1}(v)f(v)dv$$

where T is defined in (7) replacing v_H with v_t. Rearranging terms yields the result. ■

Proof of Corollary Three: Let $r^\delta(v)$ denote the function determining the maximum reserve price for which a bidder with valuation v will submit a serious bid. Then

$$\lim_{*\to 1} r^*(v)=E[Y_1|Y_1\leq v]$$

To see this, note that for any equilibrium corresponding to δ, let $\{v_{t+1+i}\}$ denote the expected sequence of cutoff levels along the equilibrium path when a reserve price $R_t = r^\delta(v_{t+1})$ is posted. Using the proof of Theorem One, r^δ satisfies

$$r^*(v_{t+1})F_{Y_1}(v_{t+1})=\sum_{i=0}^{T(*)} *^i\left((1-*)v_{t+1+i}F_{Y_1}(v_{t+1+i})+*\int_{v_{t+2+i}}^{v_{t+1+i}} Y_1 dF_{Y_1}\right)$$

By Theorem Three, since the number of terms in the summation term is bounded by T, this limit is computed by replacing $T(\delta)$ with T and letting δ go to one. Theorem Five (iii) illustrates that for $v_t \geq z_1$, the equilibrium cutoff type of the next serious bidder is bounded above 1 independent of δ. By the above argument, the minimal reserve price needed to induce his participation is also bounded above 1. ■

Proof of Theorem Four: Fix δ and current seller beliefs, v_t. From Corollary One, the expected revenue of a seller with beliefs v_t, can be expressed solely as a function of her beliefs in subsequent periods, $\{v_{t+i}\}$. Let v_{t+1} be the seller's next period beliefs and $\{v_{t+i}\}$, i = 2,3,...T, be the subsequent beliefs determined by the unique equilibrium continuation.

$$g(v_t, v_{t+1}) = n\sum_{i=0}^{N-1} *^i \int_{v_{t+i+1}}^{v_{t+i}} (v - \frac{F(v_t) - F(v)}{f(v)}) F^{n-1}(v) f(v) dv$$

$$_1 (v - \frac{F(v_t) - F(v)}{f(v)}) dF_{X_1}(v) + \sum_{i=0}^{N-2} *^i \int_{v_{t+2+i}}^{v_{t+1+i}} (v - \frac{F(v_{t+1}) - F(v)}{f(v)} - \frac{F(v_t) - F(v_{t+1})}{f(v)}) dF$$

Differentiating with respect to v_{t+1},

$$\frac{\partial g(v_t, v_{t+1})}{\partial v_{t+1}} = (1-*)(F(v_t) - F(v_{t+1}) - v_{t+1} f(v_{t+1})) +$$

$$(1-*)*\sum_{i=0}^{N-2} *^i (F(v_t) - F(v_{t+1})) \frac{F^{n-1}(v_{t+2+i})}{F^{n-1}(v_{t+1})} \mathrm{A}_{j=0}^{i} \frac{\partial v_{t+2+j}}{\partial v_{t++1+j}}$$

$$+ * \frac{\partial v_{t+2}}{\partial v_{t+1}} \frac{\partial g(v_{t+1}, v_{t+2})}{\partial v_{t+2}} \frac{1}{nF^{n-1}(v_{t+1})}$$

The last term is zero for all n since v_{t+2} must be chosen optimally when the seller beliefs are v_{t+1} and the second term goes to zero as n becomes large if the condition on the derivative of γ is satisfied. The first term is the same as the necessary condition for the static optimization problem. ■

Proof of Theorem Five: By Theorem One, trade occurs with probability one by the second period, for $v_H < z_1$ the initial reserve price is one and trade occurs immediately. For $v_H \in [z_1, z_2)$, the optimal cutoff level of the seller is a $v_2 < w_1$ and bidders in this range submit serious bids only if R_1 is less than $r_1(v,1)$ given by

$$r_1(v,1) F_{Y_1}(v) = (1-*) v F_{Y_1}(v) + * \int_1^v Y_1 dF_{Y_1}$$

The expected utility for a seller who chooses cutoff level x is

$$g_1(v_t, x, 1) = r_1(x,1) nF_{Y_1}(x) [F(v_t) - F(x)]$$

$$+ \int_x^{v_t} \int_x^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1 + * \int_1^x \int_1^{X_1} nY_1 f(X_1) dF_{Y_1} dX_1$$

A necessary condition for x to be chosen optimally is that the derivative of this expression be zero or that $F(v_H) - F(x) - xf(x) = 0$ independent of δ and n. ■

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