Supplemental Appendix for "Jackknife Estimation of a Cluster-Sample IV Regression Model with Many Weak Instruments" [1]

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Abstract

This Supplemental Appendix is comprised of two sub-appendices. Appendix S1 provides proofs for Theorems 2 and 3 of the main paper. Appendix S2 states additional supporting lemmas used to prove the main theorems of the paper. Proofs for these additional lemmas are reported in a separate Online Appendix which can be viewed at the URL:
http://econweb.umd.edu/~chao/Research/research_files/Additional_
Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

Appendix S1: Proof of Theorems 2 and 3

Proof of Theorem 2: Define $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1,Q)} \varepsilon/\sqrt{n} + D_\mu^{-1} \underline{U}' A\varepsilon$. Note that, by the result of Lemma S2-9 given in Appendix S2 below, we have that $D_\mu^{-1}\widehat{\Delta}(\delta_0) = \Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}'A\varepsilon + o_p(1) = \mathcal{Y}_n + o_p(1)$.
We now establish the asymptotic normality of $\mathcal{Y}_n$, upon appropriate standardization, in the case where $K_{2,n}/\left(\mu_n^{\min}\right)^2 = O(1)$. To proceed, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and define $b_{1n} = \Sigma_n^{-1/2} a$. and $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1} \Sigma_n^{-1/2} a$, where $\Sigma_n = VC\left(\mathcal{Y}_n | \mathcal{F}_n^Z\right) = \Sigma_{1,n} + \Sigma_{2,n}$, with $\Sigma_{1,n} = VC\left(\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^Z\right)$ and $\Sigma_{2,n} = VC\left(D_\mu^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^Z\right)$. Now, let $\mathcal{L}_{(i,t),n}$ $= b_{1n}'\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\varepsilon_{(i,t)}/\sqrt{n}$ and $\mathcal{N}_{(i,t),n} = K_{2,n}^{-1/2}\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}\left[\underline{u}_{2,(i,t),n}\varepsilon_{(j,s)} + \underline{u}_{2,(j,s),n}\varepsilon_{(i,t)}\right]$, where $\underline{u}_{2,(i,t),n} = b_{2n}'\underline{U}_{(i,t)}$, with $\underline{u}_{2,(j,s),n}$ similarly defined, and where $e_{(i,t)}$ denotes an $m_n \times 1$ elementary vector whose $(i,t)^{th}$ component is 1 and all other components are 0. Using these notations, note that we can write $a'\Sigma_n^{-1/2}\mathcal{Y}_n = \mathcal{L}_{(1,1),n} + \sum_{(i,t)=2}^{m_n}\left\{\mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}\right\}$. Next, observe that,

$$
\begin{aligned}
E\left[\mathcal{L}_{(1,1),n}^2|\mathcal{F}_n^Z\right] &= E\left[\varepsilon_{(1,1)}^2|\mathcal{F}_n^Z\right]\frac{\left[a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)} e_{(1,1)}\right]^2}{n} \\
&\le E\left[\varepsilon_{(1,1)}^2|\mathcal{F}_n^Z\right] a'\Sigma_n^{-1} a\left(\frac{\left\|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(1,1)}\right\|_2}{\sqrt{n}}\right)^2 \quad \text{(by CS inequality)}
\end{aligned}
$$

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$$
\begin{aligned}
&\leq \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^Z_n\right]\right) a'\Sigma_n^{-1}a\left(\frac{\max_{1\leq(i,t)\leq m_n}\left\|\Upsilon' Z_2' M^{(Z_1,Q)}e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2 \\
&= o_p(1)
\end{aligned}
$$

in light of Assumptions 2(i) and 7 and part (d) of Lemma S2-3[2]. Moreover, under Assumptions 2 and 3(iii), there exists a positive constant C^* such that

$$
\begin{aligned}
E_Z\left\{\left(E\left[\mathcal{L}^2_{(1,1),n}|\mathcal{F}^Z_n\right]\right)^2\right\} &= \frac{E_Z\left\{\left[a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)}e_{(1,1)}\right]^4\left(E\left[\varepsilon^2_{(1,1)}|\mathcal{F}^Z_n\right]\right)^2\right\}}{n^2} \\
&\leq \frac{C}{n^2}E\left(\left[a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)}e_{(1,1)}\right]^4\right) \text{ (by Assumption 2(i))} \\
&\leq CE\left(\frac{a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a}{n}\right)^2 \text{ (by CS inequality)} \\
&\leq C\overline{C} = C^* < \infty \text{ (by Assumption 3(iii) and Lemma S2-3(d))}
\end{aligned}
$$

Since the upper bound above does not depend on n, we further deduce that $\sup_n E_Z\left\{\left(E\left[\mathcal{L}^2_{(1,1),n}|\mathcal{F}^Z_n\right]\right)^2\right\} < \infty$. It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that $E\left(\mathcal{L}^2_{(1,1),n}\right) = E_Z\left(E\left[\mathcal{L}^2_{(1,1),n}|\mathcal{F}^Z_n\right]\right) \to 0$. Application of Markov's inequality then allows us to deduce that $\mathcal{L}_{(1,1),n} = b_{1n}'\Upsilon' Z_2' M^{(Z_1,Q)}e_{(1,1)}\varepsilon_{(1,1)}/\sqrt{n}$ $= o_p(1)$, from which we obtain the representation $a'\Sigma_n^{-1/2}\mathcal{Y}_n = \mathcal{V}_n + o_p(1)$, where $\mathcal{V}_n = \sum_{(i,t)=2}^{m_n}\mathcal{V}_{(i,t),n}$ with $\mathcal{V}_{(i,t),n} = \mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}$. Note we can also write $\mathcal{V}_n = \mathcal{L}_n + \mathcal{N}_n$, where $\mathcal{L}_n = \sum_{(i,t)=2}^{m_n}\mathcal{L}_{(i,t),n}$ and $\mathcal{N}_n = \sum_{(i,t)=2}^{m_n}\mathcal{N}_{(i,t),n}$.

Next, define the σ-fields $\mathcal{F}_{(i,t),n} = \sigma\left(\left\{\varepsilon_{(k,v)}, U_{(k,v)}\right\}_{(k,v)=1}^{(i,t)}, Z\right)$ for $(i,t) = 1,2,...,m_n$, note that by construction $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$ for $(i,t) = 2,...,m_n$ and $\mathcal{V}_{(i,t),n}$ is $\mathcal{F}_{(i,t),n}$-measurable. Note also that, under Assumption 1, it is easily seen that $E\left[\mathcal{V}_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] = 0$. In addition, note that, by part (d) of Lemma S2-3 and Lemma S2-6, and Assumption 2(i);

$$
\begin{aligned}
E\left[\underline{u}^2_{2,(i,t),n}|\mathcal{F}^Z_n\right] &\leq \left(b_{2n}'b_{2n}\right)\max_{1\leq(i,t)\leq m_n}E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2|\mathcal{F}^Z_n\right] \\
&\leq \frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2}a'\Sigma_n^{-1}a\max_{1\leq(i,t)\leq m_n}E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2|\mathcal{F}^Z_n\right] = O_{a.s.}(1) \qquad (1)
\end{aligned}
$$

since, for this theorem, we assume that $K_{2,n}/\left(\mu_n^{\min}\right)^2 = O(1)$. It follows then from straightforward calculations, from applying the triangle and CS inequalities, as well as from expression (1), part

[2] Lemma S2-3 is stated in Appendix S2 below. A proof of this lemma is provided in section 1 of the Additional Online Appendix which can be viewed at the URL:
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(d) of Lemma S2-1, part (d) of Lemma S2-3, and Assumptions 2(i) and 3(iii) that

$$
\begin{aligned}
& Var\left(\mathcal{V}_{(i,t),n}|\mathcal{F}_n^Z\right) \\
= \; & E\left[\mathcal{L}_{(i,t),n}^2|\mathcal{F}_n^Z\right] + E\left[\mathcal{N}_{(i,t),n}^2|\mathcal{F}_n^Z\right] \\
\leq \; & \left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\right) a'\Sigma_n^{-1}a\lambda_{\max}\left(\frac{\Upsilon' Z_2' Z_2 \Upsilon}{n}\right) \\
& + \frac{4}{K_{2,n}}\left(\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\right)\left(\max_{1\leq(i,t)\leq m_n} E\left[\underline{u}_{2,(i,t),n}^2|\mathcal{F}_n^Z\right]\right)\left(\max_{1\leq(i,t)\leq m_n}\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2\right) \\
= \; & O_{a.s.}(1) + O_{a.s.}\left(\frac{K_{2,n}}{\left(\mu_n^{\min}\right)^2 n}\right) = O_{a.s.}(1)
\end{aligned}
$$

By the law of iterated expectations and Theorem 16.1 of Billingsley (1995), there exists a constant $\overline{C}$ such that $Var\left(\mathcal{V}_{(i,t),n}\right) = E\left(\mathcal{V}_{(i,t),n}^2\right) = E_Z\left[E\left(\mathcal{V}_{(i,t),n}^2|\mathcal{F}_n^Z\right)\right] \leq \overline{C} < \infty$ for all n sufficiently large. These results show that $\left\{\mathcal{V}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n, \, n \geq 1\right\}$ forms a square-integrable martingale difference array.

To show the asymptotic normality of $\mathcal{V}_n$, we verify the conditions of the central limit theorem for martingale difference arrays given in Lemma S2-15. To proceed, first consider condition (22), which, as noted in the remark following Lemma S2-15, is a sufficient condition for condition (20) of Lemma S2-15. We shall verify (22) for the case where $\delta = 2$. Note first that, by applying Loève's c_r inequality, we get

$$
\sum_{(i,t)=2}^{m_n} E\left[\mathcal{V}_{(i,t),n}^4\right] = \sum_{(i,t)=2}^{m_n} E\left[\left(\mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}\right)^4\right] \leq 8\sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}_{(i,t),n}^4\right] + 8\sum_{(i,t)=2}^{m_n} E\left[\mathcal{N}_{(i,t),n}^4\right]
$$

Hence, to verify condition (22), it suffices to show that $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}_{(i,t),n}^4\right] = o(1)$ and $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{N}_{(i,t),n}^4\right] = o(1)$. To do this, we first focus on a conditional expectation analogue of $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}_{(i,t),n}^4\right]$. Note that

$$
\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}_{(i,t),n}^4|\mathcal{F}_n^Z\right] \\
= \; & \frac{1}{n^2}\sum_{(i,t)=2}^{m_n}\left[a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\right]^4 E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right] \\
\leq \; & a'\Sigma_n^{-1}a\frac{1}{n}\sum_{(i,t)=2}^{m_n}\left[a'\Sigma_n^{-1/2}\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\right]^2\left(\frac{\left\|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2 E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right] \\
& \text{(by CS inequality)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq a'\Sigma_n^{-1}a\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right]\right)\left(\frac{\max_{1\leq(i,t)\leq m_n}\left\|\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2\\
&\quad\times\frac{1}{n}a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}\sum_{(i,t)=1}^{m_n}e_{(i,t)}e_{(i,t)}'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a\\
&\leq a'\Sigma_n^{-1}a\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right]\right)\left(\frac{\max_{1\leq(i,t)\leq m_n}\left\|\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2\\
&\quad\times\frac{a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a}{n}\\
&\leq \left(a'\Sigma_n^{-1}a\right)^2\left(\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right]\right)\left(\frac{\max_{1\leq(i,t)\leq m_n}\left\|\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2\lambda_{\max}\left(\frac{\Upsilon'Z_2'Z_2\Upsilon}{n}\right)\\
&\leq C\left(\frac{\max_{1\leq(i,t)\leq m_n}\left\|\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right\|_2}{\sqrt{n}}\right)^2=o_p(1)
\end{aligned}
$$

where the last line above follows from Assumptions 2(i), 3(iii), and 7 and by Lemma S2-3(d). Next, note that, under Assumptions 2 and 3(iii), there exists a positive constant C^* such that

$$
\begin{aligned}
&E_Z\left(\sum_{(i,t)=2}^{m_n}E\left[\mathcal{L}_{(i,t),n}^4|\mathcal{F}_n^Z\right]\right)^2\\
&=\frac{1}{n^4}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=2}^{m_n}E_Z\left(\left[a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right]^4\left[a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}e_{(j,s)}\right]^4\right.\\
&\qquad\left.\times E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right]E\left[\varepsilon_{(j,s)}^4|\mathcal{F}_n^Z\right]\right)\\
&\leq\frac{C}{n^4}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=2}^{m_n}E_Z\left(\left[a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}e_{(i,t)}\right]^4\left[a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}e_{(j,s)}\right]^4\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq\frac{C}{n^4}E_Z\left\{a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}\sum_{(i,t)=1}^{m_n}e_{(i,t)}e_{(i,t)}'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}aa'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}\right.\\
&\qquad\left.\times\sum_{(j,s)=1}^{m_n}e_{(j,s)}e_{(j,s)}'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a\left(a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a\right)^2\right\}\\
&=CE_Z\left(\frac{a'\Sigma_n^{-1/2}\Upsilon'Z_2'M^{(Z_1,Q)}Z_2\Upsilon\Sigma_n^{-1/2}a}{n}\right)^4\\
&\leq C\overline{C}=C^*<\infty\ \ \text{(by Assumption 3(iii) and Lemma S2-3(d))}
\end{aligned}
$$

where the second inequality above follows from applying the CS inequality. Since the upper bound above does not depend on n, we further deduce that

$\sup_n E_Z \left(\sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}^4_{(i,t),n} | \mathcal{F}^Z_n \right] \right)^2 < \infty$. It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{L}^4_{(i,t),n} \right] = \sum_{(i,t)=2}^{m_n} E_Z \left(E\left[\mathcal{L}^4_{(i,t),n} | \mathcal{F}^Z_n \right] \right) \to 0$.

Turning our attention to the bilinear term, note that by Loève's c_r inequality we have $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{N}^4_{(i,t),n} | \mathcal{F}^Z_n \right] \leq \mathcal{R}_1 + \mathcal{R}_2$, where
$\mathcal{R}_1 = \sum_{(i,t)=2}^{m_n} \left(8/K^2_{2,n} \right) E\left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}^Z_n \right]$ and
$\mathcal{R}_2 = \sum_{(i,t)=2}^{m_n} \left(8/K^2_{2,n} \right) E\left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right)^4 | \mathcal{F}^Z_n \right]$. Focusing first on the term $\mathcal{R}_1$, note that, by straightforward calculations as well as by making use of Assumptions 2(i) and 5(ii), parts (b) and (c) of Lemma S2-1, part (d) of Lemma S2-3, and Lemma S2-6; we deduce that, there exists a positive constant $\overline{C}$ such that

$$\begin{aligned}
\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}} \mathcal{R}_1 &\leq 24n \left(a' \Sigma^{-1}_n a \right)^2 \left(\max_{1 \leq (i,t) \leq m_n} E\left[\left\| \underline{U}_{(i,t)} \right\|^4_2 | \mathcal{F}^Z_n \right] \right) \left(\max_{1 \leq (i,t) \leq m_n} E\left[\varepsilon^4_{(i,t)} | \mathcal{F}^Z_n \right] \right) \\
&\quad \times \left[\frac{1}{K^2_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A^4_{(i,t),(j,s)} + \frac{1}{K^2_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} \right] \\
&\leq \overline{C} n \left[\frac{1}{K^2_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A^4_{(i,t),(j,s)} + \frac{1}{K^2_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} \right] \\
&= O_{a.s.} \left(\frac{K_{2,n}}{n} \right) + O_{a.s.}(1) = O_{a.s.}(1).
\end{aligned}$$

Applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we then have

$$\begin{aligned}
&\frac{\left(\mu^{\min}_n\right)^4 n}{K^2_{2,n}} E_Z(\mathcal{R}_1) \\
&\leq \overline{C} n E_Z \left[\frac{1}{K^2_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A^4_{(i,t),(j,s)} + \frac{1}{K^2_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A^2_{(i,t),(j,s)} A^2_{(i,t),(k,v)} \right] \\
&= O(1)
\end{aligned}$$

from which we further deduce that

$$E_Z(\mathcal{R}_1) = \sum_{(i,t)=2}^{m_n} \frac{8}{K^2_{2,n}} E\left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 \right] = O\left(\frac{K^2_{2,n}}{\left(\mu^{\min}_n\right)^4 n} \right) = o(1)$$

In a similar way, we can also show that
$E_Z(\mathcal{R}_2) = \left(8/K_{2,n}^2\right) E\left[\sum_{(i,t)=2}^{m_n}\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(j,s),(i,t)}\underline{u}_{2,(j,s)}\varepsilon_{(i,t)}\right)^4\right] = o(1)$. It follows that $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{N}_{(i,t),n}^4\right] \leq E_Z(\mathcal{R}_1) + E_Z(\mathcal{R}_2) = o(1)$. This verifies condition (22).

Next, we verify condition (21) of Lemma S2-15. To proceed, first let $s_Z^2 = Var\left[\mathcal{V}_n|\mathcal{F}_n^Z\right] = Var\left(\sum_{(i,t)=2}^{m_n}\mathcal{V}_{(i,t),n}|\mathcal{F}_n^Z\right)$, and note that

$$s_Z^2 = Var\left(\frac{b_{1n}'\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + \frac{b_{2n}'\underline{U}'A\varepsilon}{\sqrt{K_{2,n}}}|\mathcal{F}_n^Z\right) + o_p(1) = a'\Sigma_n^{-1/2}\Sigma_n\Sigma_n^{-1/2}a + o_p(1) = 1 + o_p(1) \quad (2)$$

On the other hand, by straightforward calculation, we can write

$$\begin{aligned}
s_Z^2 &= \frac{1}{n}\sum_{(i,t)=2}^{m_n}\left[b_{1n}'\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\right]^2 E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right] \\
&+ \frac{1}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2\left\{E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right] + E\left[\underline{u}_{2,(j,s)}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\right\} \\
&+ \frac{2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right] E\left[\underline{u}_{2,(j,s)}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right] \quad (3)
\end{aligned}$$

Making use of expression (3), we obtain, after some further calculations,

$$\begin{aligned}
&\sum_{(i,t)=2}^{m_n} E\left[\mathcal{V}_{(i,t),n}^2|\mathcal{F}_{(i,t)-1,n}\right] - s_Z^2 \\
=\ & \frac{2}{\sqrt{n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\left[b_{1n}'\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\right]\frac{A_{(i,t),(j,s)}}{\sqrt{K_{2,n}}}\left\{\varepsilon_{(j,s)}E\left[\varepsilon_{(i,t)}\underline{u}_{2,(i,t)}|\mathcal{F}_n^Z\right] + \underline{u}_{2,(j,s)}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\right\} \\
&+ \sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\varepsilon_{(j,s)}^2 - E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right]\right) E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right] \\
&+ \sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s)}^2 - E\left[\underline{u}_{2,(j,s)}^2|\mathcal{F}_n^Z\right]\right) E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right] \\
&+2\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\varepsilon_{(j,s)}\underline{u}_{2,(j,s)} - E\left[\underline{u}_{2,(j,s)}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]\right) E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right] \\
&+2\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,v)=1}^{(j,s)-1}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}}{K_{2,n}} E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]\left\{\underline{u}_{2,(j,s)}\varepsilon_{(k,v)} + \varepsilon_{(j,s)}\underline{u}_{2,(k,v)}\right\}
\end{aligned}$$

$$+2\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,v)=1}^{(j,s)-1}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}}{K_{2,n}}\varepsilon_{(j,s)}\varepsilon_{(k,v)}E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right]$$
$$+2\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,v)=1}^{(j,s)-1}\frac{A_{(i,t),(j,s)}A_{(i,t),(k,v)}}{K_{2,n}}\underline{u}_{2,(j,s)}\underline{u}_{2,(k,v)}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]$$
$$= \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5+\mathcal{T}_6+\mathcal{T}_7,\ \ (say)$$

Note first that, by applying parts (a)-(c) of Lemma S2-14, we have $\mathcal{T}_1 \xrightarrow{p} 0$, $\mathcal{T}_2 \xrightarrow{p} 0$, and $\mathcal{T}_3 \xrightarrow{p} 0$. Consider next the term

$$\mathcal{T}_4 = 2\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}\frac{A_{(i,t),(j,s)}^2}{K_{2,n}}\left(\underline{u}_{2,(j,s)}\varepsilon_{(j,s)} - E\left[\underline{u}_{2,(j,s)}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]\right)E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right].$$

In this case, we apply part (a) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$, $\overline{\psi}_{(j,s)} = E\left[\underline{u}_{2,(j,s)}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]$, and $\phi_{(i,t)} = E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]$. Note that, in this case, $\left\{\left(\underline{u}_{2,(i,t)},\varepsilon_{(i,t)}\right)\right\}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z$, and $\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right] \leq C$ *a.s.* by Assumptions 1 and 2(i), respectively. Moreover, note that Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that $K_{2,n}/\left(\mu_n^{\min}\right)^2 = O(1)$ in this case together imply that there exists a constant $C \geq 1$ such that $E\left[\underline{u}_{2,(i,t)}^4|\mathcal{F}_n^Z\right] \leq \left[K_{2,n}^2/\left(\mu_n^{\min}\right)^4\right]E\left[\left\|\underline{U}_{(i,t)}\right\|_2^4|\mathcal{F}_n^Z\right]\left(a'\Sigma_n^{-1}a\right)^2 \leq C < \infty$ *a.s.* for all $(i,t)\in\{1,2,...,m_n\}$ and for all n sufficiently large, so that $\max_{1\leq(i,t)\leq m_n}E\left[\underline{u}_{2,(i,t)}^4|\mathcal{F}_n^Z\right] \leq C$ *a.s.n.* Finally, using the upper bound derived in expression (29) in the proof of part (a) of Lemma S2-14[3], we obtain $\max_{1\leq(i,t)\leq m_n}\left|\phi_{(i,t)}\right| \leq \max_{1\leq(i,t)\leq m_n}E\left[\left|\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}\right||\mathcal{F}_n^Z\right] \leq C$ *a.s.n.* and $\max_{1\leq(j,s)\leq m_n}\left|\overline{\psi}_{(j,s)}\right| \leq \max_{1\leq(i,t)\leq m_n}E\left[\left|\underline{u}_{2,(j,s)}\varepsilon_{(j,s)}\right||\mathcal{F}_n^Z\right] \leq C$ *a.s.n.* It follows by part (a) of Lemma S2-8 that $\mathcal{T}_4 \xrightarrow{p} 0$.

Now, consider $\mathcal{T}_5$. Here, we apply part (b) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$ and $\phi_{(i,t)} = E\left[\underline{u}_{2,(i,t)}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]$. Note again that $\left\{\left(\underline{u}_{2,(i,t)},\varepsilon_{(i,t)}\right)\right\}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z$, and $\max_{1\leq(i,t)\leq m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right] \leq C$ *a.s.* by Assumptions 1 and 2(i), respectively. Moreover, previously, we have shown that $E\left[\underline{u}_{2,(i,t)}^4|\mathcal{F}_n^Z\right] \leq C$ *a.s.n.* and $\max_{1\leq(i,t)\leq m_n}\left|\phi_{(i,t)}\right| \leq C$ *a.s.n.* Hence, applying part (b) of Lemma S2-8, we deduce that $\mathcal{T}_5 \xrightarrow{p} 0$.

Turning our attention to $\mathcal{T}_6$, we note that, for this term, we can apply part (c) of Lemma S2-8 with $\phi_{(i,t)} = E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right]$. From (1), there exists a positive constant C such that $E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right] \leq C < \infty$ *a.s.* for all $(i,t)\in\{1,2,...,m_n\}$ and for all n sufficiently large, so that $\max_{1\leq(i,t)\leq m_n}\left|\phi_{(i,t)}\right| = \max_{1\leq(i,t)\leq m_n}E\left[\underline{u}_{2,(i,t)}^2|\mathcal{F}_n^Z\right] \leq C$ *a.s.n.* Hence, applying part (c) of Lemma

[3] A proof of Lemma S2-14 is given in section 1 of the Additional Online Appendix which can be viewed at the URL:
http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

S2-8, we obtain $\mathcal{T}_6 \xrightarrow{p} 0$.

Finally, consider $\mathcal{T}_7$. In this case, we apply part (d) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$, $u_{(k,v)} = \underline{u}_{2,(k,v)}$, and $\phi_{(i,t)} = E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^Z_n\right]$. Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain $E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^Z_n\right] \leq \left(E\left[\varepsilon^4_{(i,t)}|\mathcal{F}^Z_n\right]\right)^{1/2} \leq C < \infty$ $a.s.$ for all $(i,t) \in \{1, 2, ..., m_n\}$ and for all n, so that $\max_{1\leq(i,t)\leq m_n}\left|\phi_{(i,t)}\right|$
$= \max_{1\leq(i,t)\leq m_n} E\left[\varepsilon^2_{(i,t)}|\mathcal{F}^Z_n\right] \leq C$ $a.s.$ Moreover, as noted previously, Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that $K_{2,n}/\left(\mu^{\min}_n\right)^2 = O(1)$ together imply that
$\max_{1\leq(i,t)\leq m_n} E\left[\underline{u}^4_{2,(i,t)}|\mathcal{F}^Z_n\right] \leq C$ $a.s.n.$ It follows by applying part (d) of Lemma S2-8 that $\mathcal{T}_7 \xrightarrow{p} 0$.

The above argument shows that $\sum^{m_n}_{(i,t)=2} E\left[\mathcal{V}^2_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] - s^2_Z = \sum^7_{k=1}\mathcal{T}_k = o_p(1)$. On the other hand, expression (2) above implies that $s^2_Z - 1 = o_p(1)$. Putting these two results together, we obtain $\sum^{m_n}_{(i,t)=2} E\left[\mathcal{V}^2_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] - 1 = o_p(1)$, which establishes condition (21) of Lemma S2-15.

It now follows from Lemma S2-15 that
$\mathcal{V}_n = \sum^{m_n}_{(i,t)=2}\left\{b'_{1n}\Upsilon' Z'_2 M^Q e_{(i,t)}\varepsilon_{(i,t)}/\sqrt{n} + \sum^{(i,t)-1}_{(j,s)=1} A_{(i,t),(j,s)}\left[\underline{u}_{2,(i,t)}\varepsilon_{(j,s)} + \underline{u}_{2,(j,s)}\varepsilon_{(i,t)}\right]\right\} \xrightarrow{d} N(0,1)$.
Since, previously, we have shown that $a'\Sigma^{-1/2}_n\mathcal{Y}_n = \mathcal{V}_n + o_p(1)$, this further implies that $a'\Sigma^{-1/2}_n\mathcal{Y}_n \xrightarrow{d} N(0,1)$. Given that this result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we can then apply the Cramér-Wold device to obtain

$$\Sigma^{-1/2}_n\mathcal{Y}_n = \Sigma^{-1/2}_n\left(\frac{\Upsilon' Z'_2 M^{(Z_1,Q)}\varepsilon}{\sqrt{n}} + D^{-1}_\mu \underline{U}' A\varepsilon\right) \xrightarrow{d} N(0, I_d) \quad (4)$$

Next, let $H_n = \Upsilon' Z'_2 M^{(Z_1,Q)} Z_2\Upsilon/n$, $\Lambda_{I,n} = H^{-1}_n\Sigma_n H^{-1}_n$, and $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D^{-1}_\mu\underline{U}' A\varepsilon$, as given above. Consider first $\widehat{\delta}_{L,n}$. Theorem 1 has already shown that $\widehat{\delta}_{L,n} \xrightarrow{p} \delta_0$. To show asymptotic normality of $\widehat{\delta}_L$, note first that, by Lemma S2-11, $\widehat{\delta}_{L,n}$ satisfies the set of (normalized) first-order conditions $\widehat{\Delta}\left(\widehat{\delta}_{L,n}\right) = 0$, where
$\widehat{\Delta}(\delta) = -\left[(y - X\delta)' M^{(Z_1,Q)}(y - X\delta)/2\right]\left[\partial\widehat{Q}_{FELIM}(\delta)/\partial\delta\right]$. Applying the mean-value theorem to each component of $\widehat{\Delta}(\delta)$ and expanding it around the point $\delta = \delta_0$, we obtain $0 = \widehat{\Delta}\left(\widehat{\delta}_{L,n}\right) = \widehat{\Delta}(\delta_0) + \left(\partial\widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right)\left(\widehat{\delta}_{L,n} - \delta_0\right)$, with $\overline{\delta}_n$ lying on the line segment between $\widehat{\delta}_{L,n}$ and δ_0. Multiplying both sides of this equation by D^{-1}_μ, we further obtain

$$0 = D^{-1}_\mu\widehat{\Delta}(\delta_0) + D^{-1}_\mu\frac{\partial\widehat{\Delta}\left(\overline{\delta}_n\right)}{\partial\delta'}\left(\widehat{\delta}_{L,n} - \delta_0\right) = D^{-1}_\mu\widehat{\Delta}(\delta_0) + D^{-1}_\mu\frac{\partial\widehat{\Delta}\left(\overline{\delta}_n\right)}{\partial\delta'}D^{-1}_\mu D_\mu\left(\widehat{\delta}_{L,n} - \delta_0\right) \quad (5)$$

From the result of Lemma S2-10, we have $-D^{-1}_\mu\left(\partial\widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right)D^{-1}_\mu = H_n + o_p(1)$, where $H_n = \Upsilon' Z'_2 M^{(Z_1,Q)} Z_2\Upsilon/n$ is a positive definite matrix $a.s.n.$ by Assumption 3(iii), which, in turn, implies that $D^{-1}_\mu\left(\partial\widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right)D^{-1}_\mu$ is nonsingular and, thus, invertible w.p.a.1. It follows that, for all n

sufficiently large, we can solve for $D_{\mu}\left(\widehat{\delta}_{L,n}-\delta_0\right)$ in (5) above to get

$$\begin{aligned} D_{\mu}\left(\widehat{\delta}_{L,n}-\delta_0\right) &= -\left[D_{\mu}^{-1}\left(\frac{\partial\widehat{\Delta}\left(\overline{\delta}_n\right)}{\partial\delta'}\right)D_{\mu}^{-1}\right]^{-1}D_{\mu}^{-1}\widehat{\Delta}\left(\delta_0\right) \\ &= H_n^{-1}\left(\frac{\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon}{\sqrt{n}}+D_{\mu}^{-1}\underline{U}'A\varepsilon\right)\left[1+o_p\left(1\right)\right], \end{aligned} \tag{6}$$

where the last equality follows by applying Lemma S2-9. By part (d) of Lemma S2-3, Σ_n is positive definite $a.s.n.$, so that Σ_n^{-1} is well-defined for all n sufficiently large, and both $\Sigma_n^{1/2}$ and $\Sigma_n^{-1/2}$ can be taken to be symmetric matrices. Since H_n is also symmetric, it further follows that $\Lambda_{I,n}=H_n^{-1}\Sigma_n H_n^{-1}$ is symmetric and positive definite $a.s.n.$, and both $\Lambda_{I,n}^{-1}=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1}$ and $\Lambda_{I,n}^{-1/2}=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1/2}$ are well-defined for all n sufficiently large. Multiplying both sides of the equation above by $\Lambda_{I,n}^{-1/2}$, we then get $\Lambda_{I,n}^{-1/2}D_{\mu}\left(\widehat{\delta}_{L,n}-\delta_0\right)=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1/2}H_n^{-1}\mathcal{Y}_n\left[1+o_p\left(1\right)\right]$, where $\mathcal{Y}_n=\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n}+D_{\mu}^{-1}\underline{U}'A\varepsilon$. Let $R_{W,n}=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1/2}H_n^{-1}\Sigma_n^{1/2}$, and note that $R_{W,n}R_{W,n}'=I_d$ for all n sufficiently large. It, thus, follows from the result given in (4) above and the continuous mapping theorem that $\Lambda_{I,n}^{-1/2}D_{\mu}\left(\widehat{\delta}_{L,n}-\delta_0\right)\overset{d}{\to}N\left(0,I_d\right)$, as $n\to\infty$, as required.

Turning our attention now to $\widehat{\delta}_{F,n}$, note that we can write this estimator, appropriately standardized, as

$$D_{\mu}\left(\widehat{\delta}_{F,n}-\delta_0\right)=\left(D_{\mu}^{-1}X'\left[A-\widehat{\ell}_{F,n}M^{(Z_1,Q)}\right]XD_{\mu}^{-1}\right)^{-1}D_{\mu}^{-1}X'\left[A-\widehat{\ell}_{F,n}M^{(Z_1,Q)}\right]\left(y-X\delta_0\right) \tag{7}$$

so that, multiplying by $\Lambda_{I,n}^{-1/2}=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1/2}$ and applying Lemmas S2-12 and S2-13, we obtain $\Lambda_{I,n}^{-1/2}D_{\mu}\left(\widehat{\delta}_{F,n}-\delta_0\right)=\left(H_n^{-1}\Sigma_n H_n^{-1}\right)^{-1/2}H_n^{-1}\mathcal{Y}_n\left[1+o_p\left(1\right)\right]$. It follows from the result given in (4) above and the continuous mapping theorem that $\Lambda_{I,n}^{-1/2}D_{\mu}\left(\widehat{\delta}_{F,n}-\delta_0\right)\overset{d}{\to}N\left(0,I_d\right)$, as $n\to\infty$, as required. □

Proof of Theorem 3: To proceed, note that, in this case, $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}}=o\left(1\right)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2\to 0$, so that, by the result given in Lemma S2-9, we have

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}}D_{\mu}^{-1}\widehat{\Delta}\left(\delta_0\right)=\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}}D_{\mu}^{-1}\underline{U}'A\varepsilon+o_p\left(1\right) \tag{8}$$

where $\underline{U}=U-\varepsilon\rho'$. Again, let $H_n=\Upsilon' Z_2' M^{(Z_1,Q)}Z_2\Upsilon/n$, and $\Sigma_{2,n}=VC\left(D_{\mu}^{-1}\underline{U}'A\varepsilon|\mathcal{F}_n^Z\right)$ $=D_{\mu}^{-1}VC\left(\underline{U}'A\varepsilon|\mathcal{F}_n^Z\right)D_{\mu}^{-1}$. Now, by assumption, $\widetilde{L}_n$ can be any sequence of bounded $(l\times d)$ nonrandom matrices such that $\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2\widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)\geq\underline{C}$ $a.s.n.$ for some constant $\underline{C}>0$. It follows that $\left(\mu_n^{\min}\right)^2\widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}$ is positive definite $a.s.n.$, so that, with probability one, $\left(\left(\mu_n^{\min}\right)^2\widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)^{-1/2}$ is well-defined for all n sufficiently large. Hence, we can let
$\widetilde{\mathcal{N}}_n=\left(\left(\mu_n^{\min}\right)^2\widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)^{-1/2}\widetilde{L}_n H_n^{-1}\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right)D_{\mu}^{-1}\underline{U}'A\varepsilon$ and construct the

linear combination $\mathcal{J}_n = a'\widetilde{\mathcal{N}}_n$ for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$. Next, define $\underline{u}_{(i,t),n} = a'\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n} H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)^{-1/2} \widetilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$, with $\underline{u}_{(j,s),n}$ similarly defined, and we can write $\mathcal{J}_n = \left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[\underline{u}_{(i,t),n}\varepsilon_{(j,s)} + \underline{u}_{(j,s),n}\varepsilon_{(i,t)}\right]$ $= \sum_{(i,t)=2}^{m_n} \mathcal{J}_{(i,t),n}$, where $\mathcal{J}_{(i,t),n} = \left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[\underline{u}_{(i,t),n}\varepsilon_{(j,s)} + \underline{u}_{(j,s),n}\varepsilon_{(i,t)}\right]$. Again, define the σ-fields $\mathcal{F}_{(i,t),n} = \sigma\left(\left\{\varepsilon_{(k,v)}, U_{(k,v)}\right\}_{(k,v)=1}^{(i,t)}, Z\right)$ for $(i,t) = 1, 2, ..., m_n$, noting that by construction $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$ for $(i,t) = 2, ..., m_n$ and $\mathcal{J}_{(i,t),n}$ is $\mathcal{F}_{(i,t),n}-$measurable. In addition, note that, using Assumption 1, it is easily seen that $E\left[\underline{u}_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] = 0$ and $E\left[\varepsilon_{(i,t)}|\mathcal{F}_{(i,t)-1,n}\right] = 0$, from which it follows that $E\left[\mathcal{J}_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] =$ $\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left\{\varepsilon_{(j,s)} E\left[\underline{u}_{(i,t),n}|\mathcal{F}_{(i,t)-1,n}\right] + \underline{u}_{(j,s),n} E\left[\varepsilon_{(i,t)}|\mathcal{F}_{(i,t)-1,n}\right]\right\} = 0$. Moreover, applying the CS inequality and making use of the fact that

$$E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right] \leq \frac{\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2 |\mathcal{F}_n^Z\right] \left\|\widetilde{L}_n\right\|_F^2}{\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n} H_n^{-1}\widetilde{L}_n'/K_{2,n}\right) \left[\lambda_{\min}\left(H_n\right)\right]^2} \left(\frac{1}{\mu_n^{\min}}\right)^2 = O_{a.s.}\left(\frac{1}{\left(\mu_n^{\min}\right)^2}\right) \tag{9}$$

and that $E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right] \leq \overline{C}$ $a.s.$ by Assumption 2(i), we see that

$$\begin{aligned}
& Var\left(\mathcal{J}_{(i,t),n}|\mathcal{F}_n^Z\right) \\
\leq\ & \frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \Bigg(E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right] + E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right] E\left[\underline{u}_{(j,s),n}^2|\mathcal{F}_n^Z\right] \\
& \qquad +2\sqrt{E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]}\sqrt{E\left[\underline{u}_{(j,s),n}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right]}\Bigg) \\
\leq\ & \frac{4\overline{C}^2}{\left(\mu_n^{\min}\right)^2} \frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 = \frac{4\overline{C}^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \ a.s.n.
\end{aligned} \tag{10}$$

Hence, applying the law of iterated expectations, part (d) of Lemma S2-1, and Theorem 16.1 of Billingsley (1995), we further deduce that $Var\left(\mathcal{J}_{(i,t),n}\right) = E_Z\left[E\left(\mathcal{J}_{(i,t),n}^2|\mathcal{F}_n^Z\right)\right]$ $\leq \left(4\overline{C}^2/K_{2,n}\right) \sum_{(j,s)=1}^{(i,t)-1} E_Z\left[A_{(i,t),(j,s)}^2\right] \leq C$ for some positive constant C for all n sufficiently large. These results show that $\left\{\mathcal{J}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n,\, n \geq 1\right\}$ forms a square-integrable martingale difference array.

Next, we verify condition (22) of the central limit theorem for martingale difference arrays given in Lemma S2-15 below. By Loève's c_r inequality we have

$$
\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E\left[\left(\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}\left[\underline{u}_{(i,t),n}\varepsilon_{(j,s)}+\underline{u}_{(j,s),n}\varepsilon_{(i,t)}\right]\right)^4 |\mathcal{F}_n^Z\right] \\
\leq\ & 8\sum_{(i,t)=2}^{m_n} \frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} E\left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}\underline{u}_{(i,t),n}\varepsilon_{(j,s)}\right)^4 |\mathcal{F}_n^Z\right] \\
& +8\sum_{(i,t)=2}^{m_n} \frac{\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} E\left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}\underline{u}_{(j,s),n}\varepsilon_{(i,t)}\right)^4 |\mathcal{F}_n^Z\right] \\
=\ & \mathcal{E}_1+\mathcal{E}_2,\ \ (say)\,. \qquad (11)
\end{aligned}
$$

Focusing first on $\mathcal{E}_1$, it is easy to see that there exists some positive constant C such that

$$
\begin{aligned}
& \mathcal{E}_1 \\
=\ & \frac{8\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} E\left[\sum_{(i,t)=2}^{m_n}\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}\underline{u}_{(i,t),n}\varepsilon_{(j,s)}\right)^4 |\mathcal{F}_n^Z\right] \\
\leq\ & \frac{8\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s)=1\\(j,s)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^4 E\left[\underline{u}_{(i,t),n}^4|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(j,s)}^4|\mathcal{F}_n^Z\right] \\
& +\frac{24\left(\mu_n^{\min}\right)^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq(i,t),(k,v)\neq(i,t)\\(j,s)\neq(k,v)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 E\left[\underline{u}_{(i,t),n}^4|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right] E\left[\varepsilon_{(k,v)}^2|\mathcal{F}_n^Z\right] \\
\leq\ & \frac{C}{K_{2,n}}\left[\frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1\\(j,s)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq(i,t),(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2\right]
\end{aligned}
$$

where the second inequality above follows from Assumption 2(i) and from an upper bound on the conditional fourth moment of
$\underline{u}_{(i,t),n}=a'\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)^{-1/2}\widetilde{L}_n H_n^{-1} D_\mu^{-1}\underline{U}_{(i,t)}$ given by

$$
\begin{aligned}
E\left[\underline{u}_{(i,t),n}^4|\mathcal{F}_n^Z\right] \leq\ & \frac{1}{\left(\mu_n^{\min}\right)^4}\left(\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^4|\mathcal{F}_n^Z\right]\right)\frac{1}{\left[\lambda_{\min}\left(H_n\right)\right]^4} \\
& \times\left\|\widetilde{L}_n\right\|_F^4\left(\frac{1}{\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)}\right)^2 \\
\leq\ & \frac{C^*}{\left(\mu_n^{\min}\right)^4}\ \ a.s.n.,\text{ for some constant } C^*>0. \qquad (12)
\end{aligned}
$$

Note also that, in deriving the upper bound given in (12), we have applied Assumption 3(iii), Lemma S2-6, the boundedness of $\left\|\widetilde{L}_n\right\|_F^2$, and the assumption that $\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2\widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)\geq \underline{C}>0$ $a.s.n.$ Moreover, by parts (b) and (c) of Lemma S2-1, we have that $K_{2,n}^{-1}\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}\left(K_{2,n}^2/n^2\right)$ and $K_{2,n}^{-1}\sum_{(i,t)=1}^{m_n}\sum_{(j,s),(k,v)=1,(j,s)\neq(i,t),(k,v)\neq(i,t)}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 = O_{a.s.}\left(K_{2,n}/n\right)$. from which it follows that $n\mathcal{E}_1 = O_{a.s.}(1)$ in light of Assumption 5(ii). Hence, by applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we obtain, for all n sufficiently large,

$$\begin{aligned}
& nE_Z\left[\mathcal{E}_1\right] \\
= & \frac{8n\left(\mu_n^{\min}\right)^4}{K_{2,n}^2}E_Z\left\{E\left[\sum_{(i,t)=2}^{m_n}\left(\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}\underline{u}_{(i,t),n}\varepsilon_{(j,s)}\right)^4|\mathcal{F}_n^Z\right]\right\} \\
\leq & \frac{Cn}{K_{2,n}}\left\{E_Z\left[\frac{1}{K_{2,n}}\sum_{\substack{(i,t),(j,s)=1\\(i,t)\neq(j,s)}}^{m_n}A_{(i,t),(j,s)}^4+\frac{1}{K_{2,n}}\sum_{(i,t)=1}^{m_n}\sum_{\substack{(j,s),(k,v)=1\\(j,s)\neq(i,t),(k,v)\neq(i,t)}}^{m_n}A_{(i,t),(j,s)}^2A_{(i,t),(k,v)}^2\right]\right\} \\
= & O(1),
\end{aligned}$$

which shows that $E_Z\left[\mathcal{E}_1\right]=O(1/n)=o(1)$. In a similar way, we can also show that $E_Z\left[\mathcal{E}_2\right] = 8\left[\left(\mu_n^{\min}\right)^4/K_{2,n}^2\right]E\left[\sum_{(i,t)=2}^{m_n}\left(\sum_{(j,s)=1}^{(i,t)-1}A_{(j,s),(i,t)}\underline{u}_{(i,t),n}\varepsilon_{(i,t)}\right)^4\right] = o(1)$. Condition (22) of Lemma S2-15 then follows from these calculations since

$$\sum_{(i,t)=2}^{m_n}E\left[\left(\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}\left[\underline{u}_{(i,t),n}\varepsilon_{(j,s)}+\underline{u}_{(j,s),n}\varepsilon_{(i,t)}\right]\right)^4\right]\leq E_Z\left[\mathcal{E}_1\right]+E_Z\left[\mathcal{E}_2\right]=o(1)$$

Next, we verify condition (21) of Lemma S2-15. Note first that, by construction, $Var\left(\mathcal{J}_n|\mathcal{F}_n^Z\right)$ $= a'\left(\widetilde{L}_n\Lambda_{II,n}\widetilde{L}_n'\right)^{-1/2}\widetilde{L}_n\Lambda_{II,n}\widetilde{L}_n'\left(\widetilde{L}_n\Lambda_{II,n}\widetilde{L}_n'\right)^{-1/2}a=1$, with $\Lambda_{II,n}=\left[\left(\mu_n^{\min}\right)^2/K_{2,n}\right]H_n^{-1}\Sigma_{2,n}H_n^{-1}$. This, in turn, implies that $Var\left(\mathcal{J}_n\right)=E_Z\left[E\left(\mathcal{J}_n^2|\mathcal{F}_n^Z\right)\right]=E_Z\left[Var\left(\mathcal{J}_n|\mathcal{F}_n^Z\right)\right]=1$. On the other hand, by direct calculation, we obtain

$$\begin{aligned}
1 &= Var\left(\mathcal{J}_n|\mathcal{F}_n^Z\right) \\
&= \frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right]E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right] \\
&\quad+\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2E\left[\underline{u}_{(j,s),n}^2|\mathcal{F}_n^Z\right]E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]
\end{aligned}$$

$$+2\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2 E\left[\underline{u}_{(j,s),n}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]E\left[\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right] \quad (13)$$

Making use of expression (13), we obtain, after some further calculations,

$$\begin{aligned}
&\sum_{(i,t)=2}^{m_n}E\left[\mathcal{J}_{(i,t),n}^2|\mathcal{F}_{(i,t)-1,n}\right]-1\\
=\;&\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\varepsilon_{(j,s)}^2-E\left[\varepsilon_{(j,s)}^2|\mathcal{F}_n^Z\right]\right)E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right]\\
&+\frac{\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\underline{u}_{(j,s),n}^2-E\left[\underline{u}_{(j,s),n}^2|\mathcal{F}_n^Z\right]\right)E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\\
&+\frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\underline{u}_{(j,s),n}\varepsilon_{(j,s)}-E\left[\underline{u}_{(j,s),n}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]\right)E\left[\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]\\
&+\frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,\upsilon)=1}^{(j,s)-1}A_{(i,t),(j,s)}A_{(i,t),(k,\upsilon)}E\left[\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]\left\{\underline{u}_{(j,s),n}\varepsilon_{(k,\upsilon)}+\varepsilon_{(j,s)}\underline{u}_{(k,\upsilon),n}\right\}\\
&+\frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,\upsilon)=1}^{(j,s)-1}A_{(i,t),(j,s)}A_{(i,t),(k,\upsilon)}\varepsilon_{(j,s)}\varepsilon_{(k,\upsilon)}E\left[\underline{u}_{(i,t),n}^2|\mathcal{F}_n^Z\right]\\
&+\frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,\upsilon)=1}^{(j,s)-1}A_{(i,t),(j,s)}A_{(i,t),(k,\upsilon)}\underline{u}_{(j,s),n}\underline{u}_{(k,\upsilon),n}E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]\\
=\;&\mathcal{TT}_1+\mathcal{TT}_2+\mathcal{TT}_3+\mathcal{TT}_4+\mathcal{TT}_5+\mathcal{TT}_6 \qquad (14)
\end{aligned}$$

To analyze the terms $\mathcal{TT}_k$ $(k=1,..,6)$, note first that, by applying parts (b) and (a) of Lemma S2-16, we obtain $\mathcal{TT}_1\xrightarrow{p}0$ and $\mathcal{TT}_2\xrightarrow{p}0$, respectively. Consider now the term

$$\mathcal{TT}_3=\frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=2}^{m_n}\sum_{(j,s)=1}^{(i,t)-1}A_{(i,t),(j,s)}^2\left(\underline{u}_{(j,s),n}\varepsilon_{(j,s)}-E\left[\underline{u}_{(j,s),n}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]\right)E\left[\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]$$

In this case, we apply part (a) of Lemma S2-8 with $u_{(j,s),n}=\left(\mu_n^{\min}\right)\underline{u}_{(j,s),n}$, $\overline{\psi}_{(j,s)}=E\left[\left(\mu_n^{\min}\right)\underline{u}_{(j,s),n}\varepsilon_{(j,s)}|\mathcal{F}_n^Z\right]$, and $\phi_{(i,t)}=E\left[\left(\mu_n^{\min}\right)\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]$. Note that, in this case, $\left\{\left(u_{(i,t),n},\varepsilon_{(i,t)}\right)\right\}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z=\sigma\left(Z\right)$, and $\max_{1\le(i,t)\le m_n}E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right]\le C$ $a.s.$ by Assumptions 1(i) and 2(i), respectively. Moreover, the upper bound given by (12) implies that there exists a constant $C^*>0$ such that $\max_{1\le(i,t)\le m_n}E\left[u_{(i,t),n}^4|\mathcal{F}_n^Z\right]=\max_{1\le(i,t)\le m_n}\left(\mu_n^{\min}\right)^4E\left[\underline{u}_{(i,t),n}^4|\mathcal{F}_n^Z\right]\le\left(\mu_n^{\min}\right)^4C^*/\left(\mu_n^{\min}\right)^4=C^*$ $a.s.n.$ Finally, note that, by using the fact that $\underline{u}_{(i,t),n}=a'\left(\left(\mu_n^{\min}\right)^2\widetilde{L}_nH_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)^{-1/2}\widetilde{L}_nH_n^{-1}D_\mu^{-1}\underline{U}_{(i,t)}$ and by applying Assumption

2(i), Lemma S2-6, and the assumption that $\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right) \geq \underline{C} > 0 \;\; a.s.n.$; we can show that there exists a constant $C > 0$ such that

$$\begin{aligned}
& E\left[\left|\left(\mu_n^{\min}\right)\underline{u}_{(i,t),n}\varepsilon_{(i,t)}\right| |\mathcal{F}_n^Z\right] \\
= \; & \left(\mu_n^{\min}\right) E\left[\left|\varepsilon_{(i,t)}\underline{U}_{(i,t)}' D_\mu^{-1} H_n^{-1}\widetilde{L}_n'\left(\frac{\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'}{K_{2,n}}\right)^{-1/2} a\right| |\mathcal{F}_n^Z\right] \\
\leq \; & \left(\mu_n^{\min}\right)\sqrt{E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]}\left[a'\left(\frac{\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'}{K_{2,n}}\right)^{-1/2}\widetilde{L}_n H_n^{-1} D_\mu^{-1}\right. \\
& \left.\times E\left[\underline{U}_{(i,t)}\underline{U}_{(i,t)}'|\mathcal{F}_n^Z\right] D_\mu^{-1} H_n^{-1}\widetilde{L}_n'\left(\frac{\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'}{K_{2,n}}\right)^{-1/2} a\right]^{1/2} \text{ (by CS inequality)} \\
\leq \; & \left(\mu_n^{\min}\right)\sqrt{E\left[\varepsilon_{(i,t)}^2|\mathcal{F}_n^Z\right]}\frac{1}{\left(\mu_n^{\min}\right)}\left(\sqrt{\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^2|\mathcal{F}_n^Z\right]}\right) \\
& \times\frac{1}{\lambda_{\min}\left(\Upsilon' Z_2' M^{(Z_1,Q)} Z_2\Upsilon/n\right)}\left\|\widetilde{L}_n\right\|_F\left(\frac{1}{\sqrt{\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1}\Sigma_{2,n}H_n^{-1}\widetilde{L}_n'/K_{2,n}\right)}}\right) \\
\leq \; & C < \infty \;\; a.s. \text{ for all } (i,t) \in \{1,2,...,m_n\} \text{ and for all } n \text{ sufficiently large} \qquad (15)
\end{aligned}$$

from which we further deduce that $\max_{(i,t)}\left|\phi_{(i,t)}\right| \leq \max_{(i,t)} E\left[\left|\left(\mu_n^{\min}\right)\underline{u}_{(i,t),n}\varepsilon_{(i,t)}\right| |\mathcal{F}_n^Z\right] \leq C$ $a.s.n.$and also that $\max_{(j,s)}\left|\overline{\psi}_{(j,s)}\right| \leq \max_{(j,s)} E\left[\left|\left(\mu_n^{\min}\right)\underline{u}_{(j,s),n}\varepsilon_{(j,s)}\right| |\mathcal{F}_n^Z\right] \leq C$ $a.s.n.$ Hence, applying part (a) of Lemma S2-8, we have $\mathcal{TT}_3 \xrightarrow{p} 0$.

Next, consider the term

$$\mathcal{TT}_4 = \frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}}\sum_{(i,t)=3}^{m_n}\sum_{(j,s)=2}^{(i,t)-1}\sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)}A_{(i,t),(k,v)}E\left[\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]\left\{\underline{u}_{(j,s),n}\varepsilon_{(k,v)} + \varepsilon_{(j,s)}\underline{u}_{(k,v),n}\right\}$$

Here, we apply part (b) of Lemma S2-8 with $u_{(j,s),n} = \left(\mu_n^{\min}\right)\underline{u}_{(j,s),n}$ and $\phi_{(i,t)} = E\left[\left(\mu_n^{\min}\right)\underline{u}_{(i,t),n}\varepsilon_{(i,t)}|\mathcal{F}_n^Z\right]$. Note that $\left\{\left(u_{(i,t),n},\varepsilon_{(i,t)}\right)\right\}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z = \sigma(Z)$, and $\max_{1\leq(i,t)\leq m_n} E\left[\varepsilon_{(i,t)}^4|\mathcal{F}_n^Z\right] \leq C$ $a.s.$ by Assumptions 1 and 2(i), respectively. Moreover, from calculations given previously, we have $\max_{1\leq(i,t)\leq m_n}\left(\mu_n^{\min}\right)^4 E\left[\underline{u}_{(i,t),n}^4|\mathcal{F}_n^Z\right] \leq C$ $a.s.n.$ and $\max_{(i,t)}\left|\phi_{(i,t)}\right| \leq C$ $a.s.n.$ Hence, by applying part (b) of Lemma S2-8, we deduce that $\mathcal{TT}_4 \xrightarrow{p} 0$.

Turning our attention to the term

$$\mathcal{TT}_5 = \frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E\left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z\right]$$

For this term, we apply part (c) of Lemma S2-8 with $\phi_{(i,t)} = E\left[u_{(i,t),n}^2 | \mathcal{F}_n^Z\right]$ and $u_{(i,t),n} = \left(\mu_n^{\min}\right) \underline{u}_{(i,t),n}$. From (9), there exists a positive constant C such that $E\left[u_{(i,t),n}^2 | \mathcal{F}_n^Z\right]$ $= \left(\mu_n^{\min}\right)^2 E\left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z\right] \leq C < \infty$ *a.s.* for all $(i,t) \in \{1,2,...,m_n\}$ and for all n sufficiently large, so that $\max_{(i,t)} \left|\phi_{(i,t)}\right| = \max_{1 \leq (i,t) \leq m_n} E\left[u_{(i,t),n}^2 | \mathcal{F}_n^Z\right] \leq C$ *a.s.n.* Hence, applying part (c) of Lemma S2-8, we obtain $\mathcal{TT}_5 \xrightarrow{p} 0$.

Finally, consider the term

$$\mathcal{TT}_6 = \frac{2\left(\mu_n^{\min}\right)^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \underline{u}_{(j,s),n} \underline{u}_{(k,v),n} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z\right]$$

In this case, we apply part (d) of Lemma S2-8 with $u_{(j,s)} = \left(\mu_n^{\min}\right) \underline{u}_{(j,s),n}$, $u_{(k,v)} = \left(\mu_n^{\min}\right) \underline{u}_{(k,v),n}$, and $\phi_{(i,t)} = E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z\right]$. Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain $E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z\right] \leq \left(E\left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z\right]\right)^{1/2} \leq C < \infty$ *a.s.* for all $(i,t) \in \{1,2,...,m_n\}$ and for all n sufficiently large, so that $\max_{(i,t)} \left|\phi_{(i,t)}\right| = \max_{(i,t)} E\left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z\right] \leq C$ *a.s.n.* Moreover, the upper bound in (12) implies that $\max_{1 \leq (i,t) \leq m_n} E\left[u_{(i,t),n}^4 | \mathcal{F}_n^Z\right] = \max_{1 \leq (i,t) \leq m_n} \left(\mu_n^{\min}\right)^4 E\left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z\right]$ $\leq C$ *a.s.n.* It follows by applying part (d) of Lemma S2-8 that $\mathcal{TT}_6 \xrightarrow{p} 0$.

It follows from the above calculations that the terms $\mathcal{TT}_k \xrightarrow{p} 0$ for each $k \in \{1,...,6\}$, which in light of equation (14) implies that $\sum_{(i,t)=2}^{m_n} E\left[\mathcal{J}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n}\right] - 1 = o_p(1)$. This establishes condition (21) of Lemma S2-15. It now follows from Lemma S2-15 that $\mathcal{J}_n$ $= \left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) a' \left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n}\right)^{-1/2} \widetilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0,1)$. Since this result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, applying the Cramér-Wold device, we further deduce that

$$\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) \left(\widetilde{L}_n \Lambda_{II,n} \widetilde{L}_n'\right)^{-1/2} \widetilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0, I_d), \tag{16}$$

where $\Lambda_{II,n} = \left(\mu_n^{\min}\right)^2 H_n^{-1} \Sigma_{2,n} H_n^{-1} / K_{2,n}$ with $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon / n$. Next, recall that $\widehat{\Delta}(\delta) = -\left[(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)/2\right] \left[\partial \widehat{Q}_{FELIM}(\delta)/\partial\delta\right]$; and note that, by Lemma S2-10, we have $-D_\mu^{-1} \left(\partial \widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right) D_\mu^{-1} = H_n + o_p(1)$, with H_n being positive definite in light of Assumption 3(iii), so that upon inverting the expansion given in expression (5) above and multiplying by $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}}$, we obtain

$$\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) D_\mu \left(\widehat{\delta}_{L,n} - \delta_0\right) = \left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) H_n^{-1} D_\mu^{-1} \widehat{\Delta}(\delta_0) \left[1 + o_p(1)\right]$$

$$= \left(\mu_n^{\min}/\sqrt{K_{2,n}}\right) H_n^{-1} D_\mu^{-1} \underline{U}' A\varepsilon \left[1 + o_p(1)\right],$$

where the last equality comes from applying expression (8). It follows by multiplying both sides of the equation above by $\left(\widetilde{L}_n \Lambda_{II,n} \widetilde{L}_n\right)^{-1/2} \widetilde{L}_n$ and applying the result given in expression (16) that $\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right)\left(\widetilde{L}_n \Lambda_{II,n} \widetilde{L}_n\right)^{-1/2} \widetilde{L}_n D_\mu \left(\widehat{\delta}_{L,n} - \delta_0\right) \xrightarrow{d} N(0, I_d)$.

Turning our attention now to $\widehat{\delta}_{F,n}$, note that, using expression (7) above, we can write

$$\frac{\left(\mu_n^{\min}\right) D_\mu \left(\widehat{\delta}_{F,n} - \delta_0\right)}{\sqrt{K_{2,n}}}$$

$$= \frac{\left(\mu_n^{\min}\right)\left(D_\mu^{-1} X' \left[A - \widehat{\ell}_{F,n} M^{(Z_1,Q)}\right] X D_\mu^{-1}\right)^{-1} D_\mu^{-1} X' \left[A - \widehat{\ell}_{F,n} M^{(Z_1,Q)}\right](y - X\delta_0)}{\sqrt{K_{2,n}}}$$

It follows by applying Lemmas S2-12 and S2-13 that

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu \left(\widehat{\delta}_{F,n} - \delta_0\right) = \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} H_n^{-1} D_\mu^{-1} \underline{U}' A\varepsilon + o_p(1), \tag{17}$$

noting that, in this case, $\left(\mu_n^{\min}\right)/\sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)^2 \to 0$. It follows by multiplying both sides of equation (17) above by $\left(\widetilde{L}_n \Lambda_{II,n} \widetilde{L}_n\right)^{-1/2} \widetilde{L}_n$ and applying the result given in expression (16) that
$\left(\mu_n^{\min}/\sqrt{K_{2,n}}\right)\left(\widetilde{L}_n \Lambda_{II,n} \widetilde{L}_n\right)^{-1/2} \widetilde{L}_n D_\mu \left(\widehat{\delta}_{F,n} - \delta_0\right) \xrightarrow{d} N(0, I_d)$. □

Appendix S2: Key Lemmas Used in Proving the Main Theorems

In this appendix, we state a number of lemmas that are used in the proofs of the main theorems of the paper. Proofs for these lemmas are available in a separate online appendix which can be viewed at the URL: http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

Lemma S2-1: Let $A = P^{\perp} - M^{(Z,Q)} D_{\widehat{\vartheta}} M^{(Z,Q)}$. Then, under Assumptions 2-6, the following statements hold as $K_{2,n}$, $n \to \infty$.

(a) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 = O_{a.s.}(K_{2,n})$.

(b) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}\left(K_{2,n}^3/n^2\right)$.

(c) $\sum_{(j,s)=1}^{m_n} \sum_{(i,t),(k,v)=1,(i,t)\neq(j,s),(k,v)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 A_{(j,s),(k,v)}^2 = O_{a.s.}\left(K_{2,n}^2/n\right)$.

(d) $\max_{1\leq(i,t)\leq m_n}\left(\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2\right) = O_{a.s.}(K_{2,n}/n)$.

(e) $\sum_{i_1,i_2=1}^{n} \sum_{j=1,j\neq i_1,i_2}^{n} \sum_{t_1=1}^{T_{i_1}} \sum_{t_2=1}^{T_{i_2}} \sum_{s_1,s_2=1}^{T_j} A_{(i_1,t_1),(j,s_1)}^2 A_{(i_2,t_2),(j,s_2)}^2 = O_{a.s.}\left(K_{2,n}^2/n\right)$ and $\sum_{i=1}^{n} \sum_{j_1=1,j_1\neq i}^{n} \sum_{j_2=1,j_2\neq i}^{n} \sum_{t_1,t_2=1}^{T_i} \sum_{s_1=1}^{T_{j_1}} \sum_{s_2=1}^{T_{j_2}} A_{(i,t_1),(j_1,s_1)}^2 A_{(i,t_2),(j_2,s_2)}^2 = O_{a.s.}\left(K_{2,n}^2/n\right)$

(f) $\sum_{i=1}^{n}\sum_{t=1}^{T_i}\sum_{s=1}^{T_i} A_{(i,t),(i,s)}^2 = O_{a.s.}\left(K_{2,n}^2/n\right)$.
(g) $\sum_{i=1}^{n}\sum_{s_1,t_1=1,s_1\neq t_1}^{T_i}\sum_{s_2,t_2=1,s_2\neq t_2}^{T_i} A_{(i,t_1),(i,s_1)}^2 A_{(i,t_2),(i,s_2)}^2 = O_{a.s.}\left(K_{2,n}^4/n^3\right)$.

Lemma S2-2: Let Assumptions 1-6 be satisfied. Then, the following statements are true: (a) $D_\mu^{-1}X'M^{(Z_1,Q)}XD_\mu^{-1} = O_p\left(n\left(\mu_n^{\min}\right)^{-2}\right)$; (b) $D_\mu^{-1}X'AXD_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon/n = O_p(1)$.

Lemma S2-3: Let $\underline{U} = U - \varepsilon\rho'$ and $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$ and let $VC\left(X|\mathcal{F}_n^Z\right)$ denote the conditional covariance matrix of the random vector X given $\mathcal{F}_n^Z$. Under Assumptions 1-2, 5-6, and 8; there exists positive constants $0 < \underline{C} \leq \overline{C} < \infty$ such that the following statements are true.
(a) $\lambda_{\max}\left[VC\left(\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^Z\right)\right] \leq \overline{C}$ $a.s.$ and $\lambda_{\min}\left[VC\left(\Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n}|\mathcal{F}_n^Z\right)\right] \geq \underline{C}$ $a.s.$ for all n sufficiently large.
(b) $VC\left(\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^Z\right) \geq \underline{C}I_d > \underset{d\times d}{0}$ $a.s.$, for all n sufficiently large.
(c) $\lambda_{\max}\left(VC\left[\underline{U}'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^Z\right]\right) \leq \overline{C}$ $a.s.$, $\lambda_{\max}\left(VC\left[\underline{U}'A\varepsilon/\sqrt{K_{2,n}}\right]\right) \leq \overline{C}$, $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}|\mathcal{F}_n^Z\right]\right) \leq \overline{C}$ $a.s.$, and $\lambda_{\max}\left(VC\left[U'A\varepsilon/\sqrt{K_{2,n}}\right]\right) \leq \overline{C}$, for all n sufficiently large.
(d) For any $a \in \mathbb{R}^d$ with $\|a\|_2 = 1$ and for all n sufficiently large, $\lambda_{\min}(\Sigma_n) \geq \underline{C} > 0$ $a.s.$ and $a'\Sigma_n^{-1}a \leq \overline{C} < \infty$ $a.s.$, where $\Sigma_n = VC\left(\mathcal{Y}_n|\mathcal{F}_n^Z\right) = \Sigma_{1,n} + \Sigma_{2,n}$, as defined in section 4 of the main paper, and where $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}'A\varepsilon$.

Lemma S2-4: Under Assumptions 1-6, $D_\mu^{-1}X'A\varepsilon = \Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}U'A\varepsilon$
$= O_p\left(\max\left\{1, \sqrt{K_{2,n}}/\left(\mu_n^{\min}\right)\right\}\right)$

Lemma S2-5: Under Assumptions 1-6, $D_\mu^{-1}X'M^{(Z_1,Q)}\varepsilon = O_p\left(n/\mu_n^{\min}\right)$.

Lemma S2-6: If Assumptions 2 and 8 are satisfied; then, for $1 \leq p \leq 8$ and for all n, there exists a positive constant C such that $\max_{1\leq(i,t)\leq m_n} E\left[\left\|\underline{U}_{(i,t)}\right\|_2^p|\mathcal{F}_n^Z\right] \leq C < \infty$ $a.s.$, where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$.

Lemma S2-7: Under Assumptions 1-6, the following results hold: (a) $\widehat{\ell}_{L,n} = o_p\left(\left[\mu_n^{\min}\right]^2/n\right)$; (b) $\widehat{\ell}_{F,n} = o_p\left(\left[\mu_n^{\min}\right]^2/n\right)$.

Lemma S2-8: Let A be as defined above. Assume that i) $\left(u_{(1,1),n}, \varepsilon_{(1,1)}\right), ..., \left(u_{(1,T_1),n}, \varepsilon_{(1,T_1)}\right),$ $\left(u_{(2,1),n}, \varepsilon_{(2,1),n}\right), ..., \left(u_{(2,T_2),n}, \varepsilon_{(2,T_2),n}\right), ..., \left(u_{(n,1),n}, \varepsilon_{(n,1),n}\right) ...., \left(u_{(n,T_n),n}, \varepsilon_{(n,T_n),n}\right)$ are independent conditional on $\mathcal{F}_n^Z = \sigma(Z)$; ii) there exists a constant C such that, almost surely for all n sufficiently large, $\max_{1\leq(i,t)\leq m_n} E\left(u_{(i,t),n}^4|\mathcal{F}_n^Z\right) \leq C$, $\max_{1\leq(i,t)\leq m_n} E\left(\varepsilon_{(i,t),n}^4|\mathcal{F}_n^Z\right) \leq C$, and
$\max_{1\leq(i,t)\leq m_n}\left|\phi_{(i,t),n}\right| \leq C$. In addition, define $\overline{\psi}_{(j,s),n} = E\left[u_{(j,s),n}\varepsilon_{(j,s),n}|\mathcal{F}_n^Z\right]$ for $(j,s) = 1, ..., m_n$. Then, under Assumptions 5 and 6, the following statements are true:
(a) $K_{2,n}^{-1}\sum_{1\leq(j,s)<(i,t)\leq m_n} A_{(i,t),(j,s)}^2\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(j,s),n} - \overline{\psi}_{(j,s),n}\right\} \xrightarrow{p} 0$;
(b) $K_{2,n}^{-1}\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\left\{u_{(j,s),n}\varepsilon_{(k,v),n} + \varepsilon_{(j,s),n}u_{(k,v),n}\right\} \xrightarrow{p} 0$;
(c) $K_{2,n}^{-1}\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}\varepsilon_{(j,s),n}\varepsilon_{(k,v),n} \xrightarrow{p} 0$;
(d) $K_{2,n}^{-1}\sum_{1\leq\ (k,v)<(j,s)<(i,t)\leq\ m_n} A_{(i,t),(j,s)}A_{(i,t),(k,v)}\phi_{(i,t),n}u_{(j,s),n}u_{(k,v),n} \xrightarrow{p} 0$.

Lemma S2-9: Let

$$\widehat{\Delta}\left(\delta_0\right) = -\frac{\left(y - X\delta_0\right)' M^{(Z_1,Q)}\left(y - X\delta_0\right)}{2}\frac{\partial}{\partial\delta}\left\{\frac{\left(y - X\delta\right)' A\left(y - X\delta\right)}{\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)}\right\}\Big|_{\delta=\delta_0}.$$

If Assumptions 1-6 and 8 are satisfied; then, $D_\mu^{-1}\widehat{\Delta}\left(\delta_0\right) = \Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}' A\varepsilon + o_p\left(1\right)$, where $\underline{U} = U - \varepsilon\rho'$ and where $\rho = \lim_{n\to\infty} E\left[U' M^Q \varepsilon\right]/E\left[\varepsilon' M^Q \varepsilon\right]$.

Lemma S2-10: Let Assumptions 1-6 be satisfied, and let $\overline{\delta}_n$ be any estimator such that, as $n \to \infty$, $D_\mu\left(\overline{\delta}_n - \delta_0\right)/\mu_n^{\min} = o_p\left(1\right)$. Then, $-D_\mu^{-1}\left(\partial\widehat{\Delta}\left(\overline{\delta}_n\right)/\partial\delta'\right) D_\mu^{-1} = H_n + o_p\left(1\right)$, where $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon/n$ and where
$\widehat{\Delta}\left(\delta\right) = -\left[\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)/2\right]\left[\partial\widehat{Q}_{FELIM}\left(\delta\right)/\partial\delta\right]$
$= X' A\left(y - X\delta\right) - \widehat{\ell}\left(\delta\right) X' M^{(Z_1,Q)}\left(y - X\delta\right)$, with
$\widehat{\ell}\left(\delta\right) = \left(y - X\delta\right)' A\left(y - X\delta\right)/\left[\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)\right]$. In addition, we also have

$$D_\mu^{-1} X'\left[A - \widehat{\ell}\left(\overline{\delta}_n\right) M^{(Z_1,Q)}\right] X D_\mu^{-1} = H_n + o_p\left(1\right). \tag{18}$$

Lemma S2-11: Let $\widehat{\ell}_L = Q\left(\widetilde{\beta}\right) = \min_{\beta\in\overline{B}} Q\left(\beta\right)$, where $Q\left(\beta\right)$ is as defined in Assumption 9. Then, $\widehat{\ell}_L$ is also the smallest root of the determinantal equation $\det\left[\overline{X}' A\overline{X} - \ell\overline{X}' M^{(Z_1,Q)}\overline{X}\right] = 0$, where $\overline{X} = [y, X]$. Assume in addition that condition (13) in Assumption 9 is satisfied; then, $\widehat{\ell}_L$ has the representation

$$\widehat{\ell}_L = \frac{\left(y - X\widehat{\delta}_L\right)' A\left(y - X\widehat{\delta}_L\right)}{\left(y - X\widehat{\delta}_L\right)' M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)}, \tag{19}$$

where $\widehat{\delta}_L$ denotes the FELIM estimator. Moreover, $\overline{X}' A\left(y - X\widehat{\delta}_L\right) - \widehat{\ell}_L\overline{X}' M^{(Z_1,Q)}\left(y - X\widehat{\delta}_L\right)$
$= 0$. In particular, this implies that $\widehat{\Delta}\left(\widehat{\delta}_L\right) = 0$, where
$\widehat{\Delta}\left(\delta\right) = -\left[\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)/2\right]\left(\partial\widehat{Q}_{FELIM}\left(\delta\right)/\partial\delta\right)$, so that $\widehat{\delta}_L$ satisfies the set of (normalized) first-order conditions for minimizing the variance ratio objective function $\widehat{Q}_{FELIM}\left(\delta\right) = \left(y - X\delta\right)' A\left(y - X\delta\right)/\left[\left(y - X\delta\right)' M^{(Z_1,Q)}\left(y - X\delta\right)\right]$.

Lemma S2-12: If Assumptions 1-6 are satisfied; then,
$D_\mu^{-1} X'\left[A - \widehat{\ell}_{F,n} M^{(Z_1,Q)}\right] X D_\mu^{-1} = H_n + o_p\left(1\right)$, where $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2\Upsilon/n$,
$\widehat{\ell}_{F,n} = \left[\widehat{\ell}_{L,n} - \left(1 - \widehat{\ell}_{L,n}\right)\left(C/m_n\right)\right]/\left[1 - \left(1 - \widehat{\ell}_{L,n}\right)\left(C/m_n\right)\right]$, and $\widehat{\ell}_{L,n}$ is smallest root of the determinantal equation $\det\left\{\overline{X}' A\overline{X} - \ell\overline{X}' M^{(Z_1,Q)}\overline{X}\right\} = 0$, with $\overline{X} = \left[\begin{array}{cc} y & X \end{array}\right]$.

Lemma S2-13: If Assumptions 1-6 and 8-9 are satisfied; then, $D_\mu^{-1} X'\left[A - \widehat{\ell}_{F,n} M^{(Z_1,Q)}\right]\left(y - X\delta_0\right)$
$= \mathcal{Y}_n\left[1 + o_p\left(1\right)\right]$, where $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1,Q)}\varepsilon/\sqrt{n} + D_\mu^{-1}\underline{U}' A\varepsilon$ with $\underline{U} = U - \varepsilon\rho'$ and
$\rho = \lim_{n\to\infty} E\left[U' M^Q\varepsilon\right]/E\left[\varepsilon' M^Q\varepsilon\right]$.

Lemma S2-14: For any $a \in \mathbb{R}^d$ such that $\|a\| = 1$, define $b_{1n} = \Sigma_n^{-1/2} a$, $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1}\Sigma_n^{-1/2} a$, $\underline{u}_{2,(i,t),n} = b_{2n}'\underline{U}_{(i,t)}$

$= \sqrt{K_{2,n}} a' \Sigma_n^{-1/2} D_\mu^{-1} \underline{U}_{(i,t)}$, $\sigma^2_{(i,t),n} = E\left[\varepsilon^2_{(i,t)} | \mathcal{F}_n^Z\right]$, $\widetilde{\psi}_{(i,t),n} = E\left[\underline{u}_{2,(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z\right]$, and $\widetilde{\omega}^2_{(i,t)} = E\left[\underline{u}^2_{2,(i,t),n} | \mathcal{F}_n^Z\right]$. If Assumptions 1-2 and 5-6 are satisfied; then, the following statements are true.

(a) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[b'_{1n} \Upsilon' Z'_2 M^{(Z_1,Q)} e_{(i,t)} / \sqrt{n}\right] \left(A_{(i,t),(j,s)} / \sqrt{K_{2,n}}\right) \left\{\varepsilon_{(j,s)} \widetilde{\psi}_{(i,t),n} + \underline{u}_{2,(j,s)} \sigma^2_{(i,t),n}\right\}$
$= O_p\left(K_{2,n}^{1/4} / \mu_n^{\min}\right) = o_p(1)$.

(b) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left(A^2_{(i,t),(j,s)} / K_{2,n}\right) \left(\varepsilon^2_{(j,s)} - \sigma^2_{(j,s),n}\right) \widetilde{\omega}^2_{(i,t),n} = O_p\left(K_{2,n} \left(\mu_n^{\min}\right)^{-2} n^{-1/2}\right) = o_p(1)$.

(c) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left(A^2_{(i,t),(j,s)} / K_{2,n}\right) \left(\underline{u}^2_{2,(j,s),n} - \widetilde{\omega}^2_{(j,s),n}\right) \sigma^2_{(i,t),n} = O_p\left(K_{2,n} \left(\mu_n^{\min}\right)^{-2} n^{-1/2}\right) = o_p(1)$.

Lemma S2-15 (Gänsler and Stute, 1977): Let $\{X_{i,n}, \mathcal{F}_{i,n}, 1 \le i \le k_n, n \ge 1\}$ be a square integrable martingale difference array. Suppose that for all $\epsilon > 0$

$$\sum_{i=1}^{k_n} E\left[X^2_{i,n} \mathbb{I}\{|X_{i,n}| > \epsilon\} | \mathcal{F}_{i-1,n}\right] \xrightarrow{p} 0 \text{ and} \tag{20}$$

$$\sum_{i=1}^{k_n} E\left[X^2_{i,n} | \mathcal{F}_{i-1,n}\right] \xrightarrow{p} 1. \tag{21}$$

Then, $\sum_{i=1}^{k_n} X_{i,n} \xrightarrow{d} N(0,1)$.

Remark: Note that a sufficient condition for condition (20), which we will verify in lieu of (20) in the proof of Theorems 2 and 3 in Appendix S1, is the following

$$\sum_{i=1}^{k_n} E\left[|X_{i,n}|^{2+\delta}\right] \xrightarrow{p} 0, \text{ for some } \delta > 0. \tag{22}$$

Lemma S2-16: Let $\widetilde{L}_n$ be a sequence of $l \times d$ nonrandom matrices (with $l \le d$) such that $\left\|\widetilde{L}_n\right\|_F^2 \le \overline{C} < \infty$ for some constant $\overline{C}$, and let $\Sigma_{2,n} = VC\left(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z\right)$
$= D_\mu^{-1} VC\left(\underline{U}' A \varepsilon | \mathcal{F}_n^Z\right) D_\mu^{-1}$. Assume that there exists a positive constant $\underline{C}$ such that
$\lambda_{\min}\left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}'_n / K_{2,n}\right) \ge \underline{C} > 0 \; a.s.n.$ Furthermore, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and let $\underline{u}_{a,(i,t),n} = a' \left(\left(\mu_n^{\min}\right)^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}'_n / K_{2,n}\right)^{-1/2} \widetilde{L}_n D_\mu^{-1} \underline{U}_{(i,t)}$. Let Assumptions 1-2 and 5-6 be satisfied and assume that $\left(\mu_n^{\min}\right)^2 / K_{2,n} = o(1)$ but
$\sqrt{K_{2,n}} / \left(\mu_n^{\min}\right)^2 \to 0$. Under these conditions, the following statements are true:

(a) $\left[\left(\mu_n^{\min}\right)^2 / K_{2,n}\right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A^2_{(i,t),(j,s)} \left(\underline{u}^2_{a,(j,s),n} - E\left[\underline{u}^2_{a,(j,s),n} | \mathcal{F}_n^Z\right]\right) E\left[\varepsilon^2_{(i,t)} | \mathcal{F}_n^Z\right]$
$= O_p\left(n^{-1/2}\right) = o_p(1)$;

(b) $\left[\left(\mu_n^{\min}\right)^2 / K_{2,n}\right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A^2_{(i,t),(j,s)} \left(\varepsilon^2_{(j,s)} - E\left[\varepsilon^2_{(j,s)} | \mathcal{F}_n^Z\right]\right) E\left[\underline{u}^2_{a,(i,t),n} | \mathcal{F}_n^Z\right]$
$= O_p\left(n^{-1/2}\right) = o_p(1)$.

Lemma S2-17 Under Assumptions 1-6, $D_\mu^{-1} X' A D (\varepsilon \circ \varepsilon) A X D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1} + o_p(1)$, where $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} Z_2 \Upsilon / n$, $\sigma^2_{(i,t)} = E\left[\varepsilon^2_{(i,t)} | \mathcal{F}_n^Z\right]$, $D_{\sigma^2} = diag\left(\sigma^2_{(1,1)}, ..., \sigma^2_{(n,T_n)}\right)$, and $\Psi_{(j,s)} = E\left[U_{(j,s)} U'_{(j,s)} | \mathcal{F}_n^Z\right]$.

Lemma S2-18 Let Assumptions 1-6 and 8 be satisfied, and let $\left\{\widehat{\delta}_n\right\}$ be any sequence of estimators such that $\left\|\widehat{\delta}_n - \delta_0\right\|_2 \xrightarrow{p} 0$ as $n \to \infty$, as long as $\sqrt{K_{2,n}} / \left(\mu_n^{\min}\right)^2 \to 0$. Also, define the following notations: let $\widehat{\varepsilon} = M^{(Z,Q)}\left(y - X\widehat{\delta}_n\right)$, $J = \left[M^Q \circ M^Q\right]^{-1}$, $S_1 = X' A D (J [\widehat{\varepsilon} \circ \widehat{\varepsilon}]) A X$, $S_2 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J (A \circ A) J \left(\widehat{\varepsilon} \iota_d' \circ M^{(Z,Q)} X\right)$, $\underline{S}_2 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J (A \circ A) J \left(\widehat{\varepsilon} \iota_d' \circ \widehat{\underline{U}}\right)$ with $\widehat{\underline{U}} = M^{(Z,Q)} X - \widehat{\varepsilon} \widehat{\rho}_n'$, $S_3 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J (A \circ A) J (\widehat{\varepsilon} \circ \widehat{\varepsilon})$, $S_4 = \left(\widehat{\varepsilon} \iota_d' \circ M^{(Z,Q)} X\right)' J (A \circ A) J \left(\widehat{\varepsilon} \iota_d' \circ M^{(Z,Q)} X\right)$, $\underline{S}_4 = \left(\widehat{\varepsilon} \iota_d' \circ \widehat{\underline{U}}\right)' J (A \circ A) J \left(\widehat{\varepsilon} \iota_d' \circ \widehat{\underline{U}}\right)$, and $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} Z_2 \Upsilon / n$. In addition, define $\sigma^2_{(i,t)} = E\left[\varepsilon^2_{(i,t)} | \mathcal{F}_n^Z\right]$, $D_{\sigma^2} = diag\left(\sigma^2_{(1,1)}, ..., \sigma^2_{(n,T_n)}\right)$, $\phi_{(i,t)} = E\left[U_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z\right]$, $\Psi_{(i,t)} = E\left[U_{(i,t)} U'_{(i,t)} | \mathcal{F}_n^Z\right]$, $\underline{\phi}_{(i,t)} = E\left[\underline{U}_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z\right]$, and $\underline{\Psi}_{(i,t)} = E\left[\underline{U}_{(i,t)} \underline{U}'_{(i,t)} | \mathcal{F}_n^Z\right]$ where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho \varepsilon_{(i,t)}$ and where for notational convenience we suppress the dependence of $\sigma^2_{(i,t)}$, $\phi_{(i,t)}$, $\Psi_{(i,t)}$, $\underline{\phi}_{(i,t)}$, and $\underline{\Psi}_{(i,t)}$ on $\mathcal{F}_n^Z = \sigma(Z)$. Then, under the above conditions, the following statements are true.

(a) $D_\mu^{-1} S_1 D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1}$
$+ o_p\left(\max\left\{1, K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right\}\right)$.

(b) $S_3 / K_{2,n} - K_{2,n}^{-1} \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} \sigma^2_{(j,s)} = o_p(1)$.

(c) $D_\mu^{-1} S_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} D_\mu^{-1} \phi_{(i,t)} \phi'_{(j,s)} D_\mu^{-1} = o_p\left(K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right)$.

(d) $\left(\mu_n^{\min} / K_{2,n}\right) S_2 D_\mu^{-1} - \left(\mu_n^{\min} / K_{2,n}\right) \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} \phi'_{(j,s)} D_\mu^{-1} = o_p(1)$.

(e) $D_\mu^{-1} \widehat{\rho}_n = O_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$ and $D_\mu^{-1}(\widehat{\rho}_n - \rho) = o_p\left(\left(\mu_n^{\min}\right)^{-1}\right)$, where $\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty}\left(E\left[U' M^Q \varepsilon\right]/n\right) / \left(E\left[\varepsilon' M^Q \varepsilon\right]/n\right)$.

(f) $D_\mu^{-1} \underline{S}_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} D_\mu^{-1} \underline{\phi}_{(i,t)} \underline{\phi}'_{(j,s)} D_\mu^{-1} = o_p\left(K_{2,n}\left(\mu_n^{\min}\right)^{-2}\right)$.

(g) $\left(\mu_n^{\min} / K_{2,n}\right) - \left(\mu_n^{\min} / K_{2,n}\right) \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A^2_{(i,t),(j,s)} \sigma^2_{(i,t)} \underline{\phi}'_{(j,s)} D_\mu^{-1} = o_p(1)$.

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