Technical Appendices to
"Instrumental Variable Estimation with Heteroskedasticity and Many Instruments"
Proof of Theorem 5

Here, we provide a proof of the existence of moments of the HFUL estimator (Theorem 5 in the paper). The main skeleton of the proof is given first, but the proof draws upon the results of a number of preliminary lemmas which have been organized into two additional appendices (Appendix B and C).

Before proceeding, we first define some notations. In the sequel, let $\mathbb{I}_{\mathcal{A}}$ denote the indicator function of the set $\mathcal{A}$; let $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$ denote, respectively, the minimal and maximal eigenvalue of the matrix B; and let $\|\cdot\|$ denote the Euclidean norm, or the Frobenius norm when applied to matrices so that $\|A\| = \sqrt{tr\{A'A\}}$. Also, the notation $a_n \sim b_n$ means that $\lim_{n\to\infty}(a_n/b_n) = c$ for some constant $c \neq 0$. In addition, M, T and CS denote, respectively, Markov's inequality, the Triangle inequality and the Cauchy-Schwarz inequality.

Finally, the proof given below also makes use of the following notations. Let

$$\begin{aligned}
\mathcal{A}_1 &= \left\{\left\|\frac{z'[P-D_P]\overline{V}}{\mu_n\sqrt{n}}\right\| < \eta_1\right\},\ \mathcal{A}_2 = \left\{\left\|\frac{\overline{V}'[P-D_P]\overline{V}}{\mu_n^2}\right\| < \eta_2\right\} \\
\mathcal{A}_3 &= \left\{\left\|\frac{\left(\overline{V}'[M+D_P]\overline{V} - E\left[\overline{V}'[M+D_P]\overline{V}\right]\right)}{n}\right\| < \eta_3\right\},\ \mathcal{A}_4 = \left\{\left\|\frac{z'[M+D_P]\overline{V}}{\mu_n\sqrt{n}}\right\| < \eta_4\right\}, \\
\mathcal{A} &= \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4 \qquad (1)
\end{aligned}$$

where $\eta_j > 0$ $(j = 1,2,3,4)$. In the proof below, we shall choose the η_j's as follows:

$$0 < \eta_1 \le \min\left\{\frac{C_5}{24\left(\sqrt{C_2}+\sqrt{C_4}\right)}, \frac{C_5\sqrt{GC_7}}{12\sqrt{C_2}\left(2\sqrt{GC_7}+1\right)}, \frac{C_8}{4\sqrt{C_4}}, \frac{C_8C_6}{8\left[C_6\sqrt{C_4}+2G\sqrt{C_2}\left(C_6+2C_3C_4\right)\right]}, \frac{C_8\sqrt{GC_7}}{4\sqrt{C_4}}\right\} \qquad (2)$$

$$\begin{aligned}
0 < \eta_2 \le \min\Bigg\{&\frac{C_5}{6\left[\sqrt{C_2}+\sqrt{C_4}\right]^2}, \frac{C_5\sqrt{GC_7}}{12\sqrt{C_2C_4}}, \frac{C_5C_6}{48\left[C_2\left(C_6+C_3C_4\right)+C_1C_4\right]}, \\
&\frac{C_5\sqrt{GC_7}}{12C_2\sqrt{GC_7}+12C_2+6\sqrt{C_2C_4}}, \frac{C_8C_6}{16G\sqrt{C_2C_4}\left(C_6+C_3C_4\right)}, \frac{C_8\sqrt{GC_7}}{2C_4}, \\
&\frac{C_8\sqrt{GC_7}}{2\left(2\sqrt{C_2C_4}+C_4\right)}, \frac{C_8C_6}{8G\sqrt{C_2}\left[C_6\sqrt{C_4}+\left(2\sqrt{C_2}+\sqrt{C_4}\right)\left(C_6+2C_3C_4\right)\right]}\Bigg\} \qquad (3)
\end{aligned}$$

$$0 < \eta_3 \le \min\left\{\frac{C_6}{2C_4}, \frac{C_6}{4}\right\}, \qquad (4)$$

$$0 < \eta_4 \le \frac{C_6}{8\sqrt{GC_4C_7}}, \qquad (5)$$

where $C_1, C_2, .., C_7$ are constants to be specified in the proof of Lemma C2 below and where C_8 is some positive constant. Also, define

$$\mathcal{B} = \left\{\left\|\frac{D_\mu \widetilde{S}_n'\left(\delta_0 - \widehat{\delta}\right)}{\mu_n}\right\| \ge 1\right\}, \mathcal{C} = \left\{\widehat{\kappa}_{HLIM} \ge 0\right\},$$

where $\widetilde{\kappa}_{HLIM}$ is the smallest root of the equation

$$\det\left\{\overline{X}'[P-D_P]\overline{X} - \kappa\overline{X}'[M+D_P]\overline{X}\right\} = 0,$$

with $\overline{X} = \begin{bmatrix} y & X \end{bmatrix}$. Finally, let $\mathcal{B}^C$ and $\mathcal{C}^C$ be the complement of the events $\mathcal{B}$ and $\mathcal{C}$, respectively.

Proof of Theorem 5: To begin, let $\mathcal{A}$ be the event defined in (1), and we can write

$$E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p\right] = E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}}\right] + E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right].$$

It follows from Lemma C7 that there exists a constant $\overline{C}_2$ such that for all n sufficiently large

$$E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] \leq \overline{C}_2/2 < \infty. \tag{6}$$

Next, note that $HFUL$ can be written in the alternative form

$$\begin{aligned}\widehat{\delta}_{HFUL} &= \left(X'\left[P - D_P\right]X - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]X\right)^{-1} \\ &\quad \times \left(X'\left[P - D_P\right]y - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]y\right),\end{aligned} \tag{7}$$

where $D_P = diag\left(P_{11}, ..., P_{nn}\right)$. (7) is equivalent with probability one to the expression for $HFUL$ given in equation (2.1) of the paper, as will be shown in Lemma B1 below. Now, substitute $y = X\delta_0 + \varepsilon$ into (7), and we get

$$\begin{aligned}\widehat{\delta}_{HFUL} &= \delta_0 + \left(X'\left[P - D_P\right]X - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]X\right)^{-1} \\ &\quad \times \left(X'\left[P - D_P\right]\varepsilon - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]\varepsilon\right) \\ &= \delta_0 + \left(\mu_n S_n'^{-1}\right)\left(S_n^{-1}X'\left[P - D_P\right]XS_n'^{-1} - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]XS_n'^{-1}\right)^{-1} \\ &\quad \times \left(S_n^{-1}X'\left(P - D_P\right)\varepsilon/\mu_n - \left\{\widehat{\kappa}_{HLIM} - \frac{C}{n}\right\}S_n^{-1}X'\left(M + D_P\right)\varepsilon/\mu_n\right) \\ &= \delta_0 + \widetilde{S}_n'^{-1}\left(\mu_n D_\mu^{-1}\right)\left(S_n^{-1}X'\left[P - D_P\right]XS_n'^{-1} - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]XS_n'^{-1}\right)^{-1} \\ &\quad \times \left(S_n^{-1}X'\left(P - D_P\right)\varepsilon/\mu_n - \left\{\widehat{\kappa}_{HLIM} - \frac{C}{n}\right\}S_n^{-1}X'\left(M + D_P\right)\varepsilon/\mu_n\right),\end{aligned}$$

where the last equality above makes use of the fact that by definition $S_n'^{-1} = \widetilde{S}_n'^{-1}D_\mu^{-1}$ with $D_\mu = diag\left(\mu_{1n}, ..., \mu_{Gn}\right)$. Now, using T and the submultiplicativity of the Euclidean norm, we obtain

$$\begin{aligned}&\left\|\widehat{\delta}_{HFUL}\right\| \\ &\leq \left\|\delta_0\right\| + \left\|\widetilde{S}_n'^{-1}\right\|\left\|\mu_n D_\mu^{-1}\right\|\left\|\left(S_n^{-1}X'\left[P - D_P\right]XS_n'^{-1} - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]XS_n'^{-1}\right)^{-1}\right\| \\ &\quad \times \left\|S_n^{-1}X'\left(P - D_P\right)\varepsilon/\mu_n - \left\{\widehat{\kappa}_{HLIM} - \frac{C}{n}\right\}S_n^{-1}X'\left(M + D_P\right)\varepsilon/\mu_n\right\| \\ &\leq \left\|\delta_0\right\| + \sqrt{GC_2}\left\|\left(S_n^{-1}X'\left[P - D_P\right]XS_n'^{-1} - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]XS_n'^{-1}\right)^{-1}\right\| \\ &\quad \times \left\|S_n^{-1}X'\left(P - D_P\right)\varepsilon/\mu_n - \left\{\widehat{\kappa}_{HLIM} - \frac{C}{n}\right\}S_n^{-1}X'\left(M + D_P\right)\varepsilon/\mu_n\right\|.\end{aligned}$$

Note that the last inequality above holds for all n sufficiently large since $\left\|n^{-1/2}D_{\mu}\right\| \leq \sqrt{G}$ given that $\mu_{jn} \leq \sqrt{n}$ $(j = 1, .., G)$ and since, using the same argument as that for deriving (28) in the proof of Lemma C2, we have

$$\left\|\widetilde{S}_n^{\prime -1}\right\| \leq \sqrt{tr\left\{\left(\widetilde{S}_n^{\prime}\widetilde{S}_n\right)^{-1}\right\}} \leq \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}_n^{\prime}\widetilde{S}_n\right)}} \leq \sqrt{C_2} < \infty$$

for some constant C_2 which exists in light of Assumption 2. Loève's c_r inequality and Theorem 15.2 part (iv) of Billingsley (1986) then imply that

$$\begin{aligned}
& E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}}\right] \\
\leq\ & 2^{p-1}\left\|\delta_0\right\|^p \mathbb{I}_{\mathcal{A}} \\
& +2^{p-1}\left(GC_2\right)^{p/2} E\left\{\left\|\left(S_n^{-1}X'\left[P-D_P\right]XS_n^{\prime -1} - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M+D_P\right]XS_n^{\prime -1}\right)^{-1}\right\|^p \mathbb{I}_{\mathcal{A}}\right. \\
& \qquad \left. \times\left\|S_n^{-1}X'\left(P-D_P\right)\varepsilon/\mu_n - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}S_n^{-1}X'\left(M+D_P\right)\varepsilon/\mu_n\right\|^p \mathbb{I}_{\mathcal{A}}\right\} \\
\leq\ & 2^{p-1}C_4^{p/2}\mathbb{I}_{\mathcal{A}} + 2^{p-1}\left(GC_2\right)^{p/2}\left(\frac{2\sqrt{G}}{C_5}\right)^p C_8^p\mathbb{I}_{\mathcal{A}} \\
\leq\ & \overline{C}_1/2, \qquad (8)
\end{aligned}$$

where

$$\overline{C}_1 := 2^p\left[C_4^{p/2} + (2G)^p\left(\frac{C_8\sqrt{C_2}}{C_5}\right)^p\right]$$

and where the second inequality above follows from Lemmas C2-C3 and from Assumption 7 which implies the existence of a constant C_4 such that $\left\|\delta_0\right\|^2 \leq C_4 < \infty$. Finally, let $\overline{C} = \max\left\{\overline{C}_1, \overline{C}_2\right\}$. It follows from (6) and (8) that for all n sufficiently large

$$\begin{aligned}
E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p\right] &= E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}}\right] + E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] \\
&\leq \frac{\overline{C}_1}{2} + \frac{\overline{C}_2}{2} \\
&\leq \overline{C} < \infty,
\end{aligned}$$

which establishes the desired conclusion. *Q.E.D.*

Appendix B

Lemma B1: Suppose that Assumptions 1, 7, and 9 hold. Then, for n sufficiently large, the formula for $\widehat{\delta}_{HFUL}$ given in (7) is equivalent with probability one to the representation given in equation (2.1) of the paper.

Proof: From equation (2.1), we have

$$\widehat{\delta}_{HFUL} = \left(X'\left[P-D_P\right]X - \widehat{\alpha}_{HFUL}X'X\right)^{-1}\left(X'\left[P-D_P\right]y - \widehat{\alpha}_{HFUL}X'y\right),$$

where

$$\widehat{\alpha}_{HFUL} = \frac{\widetilde{\alpha}_{HLIM} - \left(1-\widetilde{\alpha}_{HLIM}\right)C/n}{1-\left(1-\widetilde{\alpha}_{HLIM}\right)C/n},$$

and where $\widetilde{\alpha}_{HLIM}$ is the smallest eigenvalue of the equation

$$\det\left\{\left(\overline{X}'\overline{X}\right)^{-1}\overline{X}'\left[P-D_P\right]\overline{X}-\alpha I_{G+1}\right\}=0. \tag{9}$$

Note that the non-singularity of $\overline{X}'\overline{X}$ holds with probability one for all n such that $n-K\geq G+1$ in light of part (b) of Lemma B13. Next, observe that the smallest root of (9) is equivalent (with probability one) to the smallest root of

$$\det\left\{\overline{X}'\left[P-D_P\right]\overline{X}-\alpha\overline{X}'\overline{X}\right\}=0. \tag{10}$$

Moreover, observe that, under Assumption 9, $\alpha=1$ is a root of the determinantal equation (10) with zero probability; since if $\alpha=1$, we would have

$$\det\left\{\overline{X}'\left[P-D_P\right]\overline{X}-\overline{X}'\overline{X}\right\}=(-1)^{G+1}\det\left\{\overline{X}'\left[M+D_P\right]\overline{X}\right\}=0,$$

which holds only if $\det\left\{\overline{X}'\left[M+D_P\right]\overline{X}\right\}=0$, but this occurs with probability zero given part (a) of Lemma B13. Hence, with probability one, we can rewrite (10) as

$$\begin{aligned}0&=\det\left\{\overline{X}'\left[P-D_P\right]\overline{X}-\alpha\overline{X}'\overline{X}\right\}\\&=\det\left\{(1-\alpha)\overline{X}'\left[P-D_P\right]\overline{X}-\alpha\overline{X}'\left[M+D_P\right]\overline{X}\right\}\\&=(1-\alpha)^{G+1}\det\left\{\overline{X}'\left[P-D_P\right]\overline{X}-\frac{\alpha}{1-\alpha}\overline{X}'\left[M+D_P\right]\overline{X}\right\}\end{aligned} \tag{11}$$

for $\alpha\neq1$. Now, let $\widetilde{\alpha}_1\leq\cdots\leq\widetilde{\alpha}_{G+1}$ ($\widetilde{\alpha}_g\neq1$ for $g=1,...,G+1$) be the roots of the equation (10); then, by (11),

$$\widetilde{\kappa}_1=\frac{\widetilde{\alpha}_1}{1-\widetilde{\alpha}_1},....,\widetilde{\kappa}_{G+1}=\frac{\widetilde{\alpha}_{G+1}}{1-\widetilde{\alpha}_{G+1}}$$

must be the roots of the equation

$$\det\left\{\overline{X}'\left[P-D_P\right]\overline{X}-\kappa\overline{X}'\left[M+D_P\right]\overline{X}\right\}=0. \tag{12}$$

In addition, note that since , for $\kappa=\alpha/(1-\alpha)$,

$$\frac{d\kappa}{d\alpha}=\frac{1}{(1-\alpha)^2}>0.$$

It follows that we must have the ordering

$$\widetilde{\kappa}_1\leq\cdots\leq\widetilde{\kappa}_{G+1},$$

so that, setting $\widetilde{\alpha}_{HLIM}=\widetilde{\alpha}_1$ and letting $\widetilde{\kappa}_{HLIM}$ $(=\widetilde{\kappa}_1)$ denote the smallest root of the determinantal equation (12), it must be that

$$\widetilde{\kappa}_{HLIM}=\frac{\widetilde{\alpha}_{HLIM}}{1-\widetilde{\alpha}_{HLIM}}.$$

Next, write

$$\begin{aligned}\widehat{\delta}_{HFUL}&=\left((1-\widehat{\alpha}_{HFUL})X'\left[P-D_P\right]X-\widehat{\alpha}_{HFUL}X'\left[M+D_P\right]X\right)^{-1}\\&\quad\times\left((1-\widehat{\alpha}_{HFUL})X'\left[P-D_P\right]y-\widehat{\alpha}_{HFUL}X'\left[M+D_P\right]y\right)\\&=\left(X'\left[P-D_P\right]X-\frac{\widehat{\alpha}_{HFUL}}{1-\widehat{\alpha}_{HFUL}}X'\left[M+D_P\right]X\right)^{-1}\\&\quad\times\left(X'\left[P-D_P\right]y-\frac{\widehat{\alpha}_{HFUL}}{1-\widehat{\alpha}_{HFUL}}X'\left[M+D_P\right]y\right)\end{aligned} \tag{13}$$

Note further that

$$\begin{aligned}
\frac{\widehat{\alpha}_{HFUL}}{1-\widehat{\alpha}_{HFUL}} &= \left[1-\frac{\widetilde{\alpha}_{HLIM}-(1-\widetilde{\alpha}_{HLIM})\,C/n}{1-(1-\widetilde{\alpha}_{HLIM})\,C/n}\right]^{-1}\frac{\widetilde{\alpha}_{HLIM}-(1-\widetilde{\alpha}_{HLIM})\,C/n}{1-(1-\widetilde{\alpha}_{HLIM})\,C/n} \\
&= \left[\frac{1-(1-\widetilde{\alpha}_{HLIM})\,C/n-\widetilde{\alpha}_{HLIM}+(1-\widetilde{\alpha}_{HLIM})\,C/n}{1-(1-\widetilde{\alpha}_{HLIM})\,C/n}\right]^{-1}\frac{\widetilde{\alpha}_{HLIM}-(1-\widetilde{\alpha}_{HLIM})\,C/n}{1-(1-\widetilde{\alpha}_{HLIM})\,C/n} \\
&= \frac{\widetilde{\alpha}_{HLIM}-(1-\widetilde{\alpha}_{HLIM})\,C/n}{1-\widetilde{\alpha}_{HLIM}} \\
&= \frac{\widetilde{\alpha}_{HLIM}}{1-\widetilde{\alpha}_{HLIM}}-\frac{C}{n} \\
&= \widetilde{\kappa}_{HLIM}-\frac{C}{n}, \qquad (14)
\end{aligned}$$

so that $\widehat{\alpha}_{HFUL}/(1-\widehat{\alpha}_{HFUL})$ is well-defined provided that $\widetilde{\alpha}_{HLIM}\neq 1$, which occurs with probability one. Substituting (14) into (13) above, and we obtain

$$\begin{aligned}
\widehat{\delta}_{HFUL} &= \left(X'\left[P-D_P\right]X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left[M+D_P\right]X\right)^{-1} \\
&\quad\times\left(X'\left[P-D_P\right]y-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left[M+D_P\right]y\right),
\end{aligned}$$

which establishes the desired conclusion. *Q.E.D.*

Lemma B2: (Poincaré's Separation Theorem)

Let A be a symmetric $m\times m$ matrix with eigenvalues $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$ and let H be a semi-orthogonal $m\times r$ matrix $(1\leq r\leq m)$, so that $H'H=I_r$.Then, the eigenvalues $\mu_1\leq\mu_2\leq\cdots\leq\mu_r$ of $H'AH$ satisfy

$$\lambda_i\leq\mu_i\leq\lambda_{m-r+i}\quad(i=1,2,...,r)$$

Proof: See pages 209-210 of Magnus and Neudecker (1988).

Lemma B3: Let A $(m\times m)$ be a symmetric, positive semidefinite matrix and let B $(m\times m)$ be a symmetric, positive definite matrix. Moreover, let $\widehat{\lambda}$ be the smallest root of the determinantal equation

$$\det\left\{A-\lambda B\right\}=0.$$

Now, partition A and B conformably as

$$A=\begin{pmatrix}\underset{m_1\times m_1}{A_{11}} & \underset{m_1\times m_2}{A_{21}'}\\ \underset{m_2\times m_1}{A_{21}} & \underset{m_2\times m_2}{A_{22}}\end{pmatrix}\text{ and }B=\begin{pmatrix}\underset{m_1\times m_1}{B_{11}} & \underset{m_1\times m_2}{B_{21}'}\\ \underset{m_2\times m_1}{B_{21}} & \underset{m_2\times m_2}{B_{22}}\end{pmatrix}.$$

Then, the matrices $A_{11}-\widehat{\lambda}B_{11}$ and $A_{22}-\widehat{\lambda}B_{22}$ are both positive semidefinite.

Proof: We will only prove the positive semi-definiteness of the matrix $A_{22}-\widehat{\lambda}B_{22}$ since the proof for $A_{11}-\widehat{\lambda}B_{11}$ is similar. To proceed, note that in light of the positive definiteness of B, we can decompose B as

$$B=L'L,$$

where

$$L=\begin{pmatrix}B_{11.2}^{1/2} & 0\\ B_{22}^{-1/2}B_{21} & B_{22}^{1/2}\end{pmatrix}.$$

and where $B_{11.2} = B_{11} - B'_{21} B_{22}^{-1} B_{21}$. Moreover, note that the inverse of L can be obtained as

$$L^{-1} = \begin{pmatrix} B_{11.2}^{-1/2} & 0 \\ -B_{22}^{-1} B_{21} B_{11.2}^{-1/2} & B_{22}^{-1/2} \end{pmatrix}.$$

Next, observe that the roots of the equation

$$\det \left\{ L'^{-1} A L^{-1} - \lambda I_m \right\} = 0$$

are the same as those of

$$\det \{A - \lambda B\} = 0.$$

Now, by direct calculation

$$\begin{aligned}
& L'^{-1} A L^{-1} \\
= & \begin{pmatrix} B_{11.2}^{-1/2} & -B_{11.2}^{-1/2} B'_{21} B_{22}^{-1} \\ 0 & B_{22}^{-1/2} \end{pmatrix} \begin{pmatrix} A_{11} & A'_{21} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11.2}^{-1/2} & 0 \\ -B_{22}^{-1} B_{21} B_{11.2}^{-1/2} & B_{22}^{-1/2} \end{pmatrix} \\
= & \begin{pmatrix} B_{11.2}^{-1/2} & -B_{11.2}^{-1/2} B'_{21} B_{22}^{-1} \\ 0 & B_{22}^{-1/2} \end{pmatrix} \begin{pmatrix} A_{11} B_{11.2}^{-1/2} - A'_{21} B_{22}^{-1} B_{21} B_{11.2}^{-1/2} & A'_{21} B_{22}^{-1/2} \\ A_{21} B_{11.2}^{-1/2} - A_{22} B_{22}^{-1} B_{21} B_{11.2}^{-1/2} & A_{22} B_{22}^{-1/2} \end{pmatrix} \\
= & \begin{pmatrix} \begin{array}{c} B_{11.2}^{-1/2} \left(A_{11} - A'_{21} B_{22}^{-1} B_{21} - B'_{21} B_{22}^{-1} A_{21} \right) B_{11.2}^{-1/2} \\ + B_{11.2}^{-1/2} B'_{21} B_{22}^{-1} A_{22} B_{22}^{-1} B_{21} B_{11.2}^{-1/2} \end{array} & B_{11.2}^{-1/2} \left(A'_{21} - B'_{21} B_{22}^{-1} A_{22} \right) B_{22}^{-1/2} \\ B_{22}^{-1/2} \left(A_{21} - A_{22} B_{22}^{-1} B_{21} \right) B_{11.2}^{-1/2} & B_{22}^{-1/2} A_{22} B_{22}^{-1/2} \end{pmatrix},
\end{aligned}$$

so that

$$B_{22}^{-1/2} A_{22} B_{22}^{-1/2} = H' L'^{-1} A L^{-1} H$$

with H $(m \times m_2)$ being the semi-orthogonal matrix

$$H = \begin{pmatrix} 0 \\ I_{m_2} \end{pmatrix}.$$

Since the roots $\mu_1 \leq \cdots \leq \mu_{m_2}$ of the equation

$$\det \left\{ B_{22}^{-1/2} A_{22} B_{22}^{-1/2} - \mu I_{m_2} \right\} = 0$$

are clearly also the roots of the equation

$$\det \{A_{22} - \mu B_{22}\} = 0,$$

we have that for any vector $x \in R^{m_2}$

$$\begin{aligned}
& x' A_{22} x - \widehat{\lambda} x' B_{22} x \\
= \; & x' B_{22}^{1/2} \left(B_{22}^{-1/2} A_{22} B_{22}^{-1/2} - \widehat{\lambda} I_{m_2} \right) B_{22}^{1/2} x \\
= \; & x^{*\prime} \left(B_{22}^{-1/2} A_{22} B_{22}^{-1/2} - \widehat{\lambda} I_{m_2} \right) x^* \\
\geq \; & \left(\mu_1 - \widehat{\lambda} \right) x^{*\prime} x^* \\
\geq \; & 0,
\end{aligned}$$

where we have let $x^* = B_{22}^{1/2} x$ and where the last inequality follows from applying Poincaré's Separation Theorem, which allows us to conclude that $\widehat{\lambda} \leq \mu_1$. *Q.E.D.*

Lemma B4:

(a) Suppose that A is a symmetric, positive semidefinite (PSD) matrix and B is a symmetric, PD matrix; then

$$\left\| (A+B)^{-1} \right\| \leq \left\| B^{-1} \right\|$$

(b) Suppose that A and B are symmetric, PSD matrices and write

$$A = E - B$$

for some matrix E. Then,

$$\|A\| \leq \|E\|.$$

Here, $\|\cdot\|$ denotes the Frobenius matrix norm, so that $\|A\| = \sqrt{tr\,(A'A)}$

Proof: To prove (a), let A be $G \times G$ without loss of generality. Now,

$$\begin{aligned}
\left\| (A+B)^{-1} \right\| &= \sqrt{tr\left\{ (A+B)^{-2} \right\}} \\
&= \sqrt{\sum\nolimits_{g=1}^{G} \left(\frac{1}{\lambda_g (A+B)} \right)^2} \\
&\leq \sqrt{\sum\nolimits_{g=1}^{G} \left(\frac{1}{\lambda_g (B)} \right)^2} \quad \text{[using the result of Exercise 12.45 of Abadir and Magnus (2005)]} \\
&= \sqrt{tr\left\{ B^{-2} \right\}} \\
&= \left\| B^{-1} \right\|.
\end{aligned}$$

Here, $\lambda_g(B)$ $(g = 1, ..., G)$ are the eigenvalues of the matrix B, and $\lambda_g(A+B)$ $(g = 1, ..., G)$ are similarly defined.

To show part (b), note that by the result Exercise 12.45 of Abadir and Magnus (2005), we have that

$$\lambda_g(E) \geq \lambda_g(A) \geq 0 \;\text{ for } g = 1, ..., G,$$

from which it follows that

$$\|A\| = \sqrt{tr\left\{ A'A \right\}} = \sqrt{\sum\nolimits_{g=1}^{G} \lambda_g^2(A)} \leq \sqrt{\sum\nolimits_{g=1}^{G} \lambda_g^2(E)} = \sqrt{tr\left\{ E'E \right\}} = \|E\|.$$

Q.E.D.

Lemma B5:

(a) If X be a square matrix, then

$$\lambda_{\min}(X + X') \geq -2\|X\|.$$

(b) If X is symmetric matrix, then

$$\lambda_{\min}(X) \geq -\|X\|.$$

(c) If X is symmetric and nonsingular matrix, then

$$\left\| X^{-1} \right\| \leq \frac{\sqrt{G}}{\min_{1 \leq\, g\, \leq G} |\lambda_g(X)|}.$$

Here, G denotes the number of columns (and rows) of the matrix X.

Proof: To show part (a), note first that

$$\begin{aligned}\max_{1\leq\ g\ \leq G}\left|\lambda_g\left(X+X'\right)\right| &\leq \sqrt{\sum_{g=1}^{G}\left[\lambda_g\left(X+X'\right)\right]^2} \\ &= \sqrt{tr\left\{\left(X+X'\right)\left(X+X'\right)\right\}} \\ &\leq \sqrt{4tr\left\{X'X\right\}} \\ &= 2\left\|X\right\|\end{aligned}$$

where the second inequality above follows from applying the trace version of CS (see page 325 of Abadir and Magnus, 2005). It follows that

$$\lambda_{\min}\left(X+X'\right)\geq-\max_{1\leq\ g\ \leq G}\left|\lambda_g\left(X+X'\right)\right|\geq-2\left\|X\right\|.$$

The proof of part (b) follows immediately from that of part (a), since in this case

$$2\lambda_{\min}\left(X\right)=\lambda_{\min}\left(X+X'\right)\geq-2\left\|X\right\|,$$

and the desired conclusion is obtained by dividing through by 2.

To show part (c), note that

$$\begin{aligned}\left\|X^{-1}\right\| &= \sqrt{tr\left[X^{-2}\right]}=\sqrt{\sum\nolimits_{g=1}^{G}\frac{1}{\lambda_g^2\left(X\right)}} \\ &\leq \sqrt{\frac{G}{\left(\min\limits_{1\leq\ g\ \leq G}\left|\lambda_g\left(X\right)\right|\right)^2}} \\ &= \frac{\sqrt{G}}{\min\limits_{1\leq\ g\ \leq G}\left|\lambda_g\left(X\right)\right|}.\end{aligned}$$

Q.E.D.

Lemma B6 (Decoupling Inequality): Let $\chi_i=\left(\xi_i,\zeta_i\right)'$. Assume that $\{\chi_i\}$ is a sequence of independent 2×1 random vectors such that $E\left[\chi_i\right]=0$ and such that $\sup_i E\left\|\chi_i\right\|^\rho\leq C^*<\infty$ for some constant C^* and for some positive integer $\rho\geq1$. Moreover, let $\{\widetilde{\chi}_i\}$ be an independent copy of this sequence. Also, let P_{ij} denote the $(i,j)^{th}$ element of the projection matrix $P=Z\left(Z'Z\right)^{-1}Z'$, with Z satisfying the conditions of Assumption 1. Then,

$$E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\ \neq\ j\ \leq\ n}P_{ij}\xi_i\zeta_j\right|^\rho\leq4^qE\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\ \neq\ j\ \leq\ n}P_{ij}\xi_i\widetilde{\zeta}_j\right|^\rho$$

where $\widetilde{\zeta}_j$ denotes the second component of $\widetilde{\chi}_j=\left(\widetilde{\xi}_j,\widetilde{\zeta}_j\right)'$.

Proof: The proof follows that of Theorem 3.1.1 of de la Peña and Giné (1999). However, we exploit the specialized structure of our problem to obtain a better constant than that of expression (3.1.8) in de la Peña and Giné (1999).

To proceed, let $\{\varepsilon_i\}_{i=1}^n$ be independent Rademacher random variables, i.e.,

$$\varepsilon_i=\begin{cases}1 & \text{with prob } p=\frac{1}{2}\\ -1 & \text{with prob } p=\frac{1}{2}\end{cases},$$

and let $\{\varepsilon_i\}_{i=1}^n$ be independent of $\{\chi_i, \widetilde{\chi}_i\}_{i=1}^n$, and also define υ_i and $\widetilde{\upsilon}_i$ $(i = 1, ..., n)$ as follows:

$$\begin{aligned} \upsilon_i &= \chi_i \mathbb{I}\{\varepsilon_i = 1\} + \widetilde{\chi}_i \mathbb{I}\{\varepsilon_i = -1\}, \\ \widetilde{\upsilon}_i &= \widetilde{\chi}_i \mathbb{I}\{\varepsilon_i = 1\} + \chi_i \mathbb{I}\{\varepsilon_i = -1\}. \end{aligned}$$

Partition υ_i and $\widetilde{\upsilon}_i$ conformably with $\chi_i = (\xi_i, \zeta_i)'$ and $\widetilde{\chi}_i = \left(\widetilde{\xi}_i, \widetilde{\zeta}_i\right)'$, we also have

$$\begin{aligned} \upsilon_{1i} &= \xi_i \mathbb{I}\{\varepsilon_i = 1\} + \widetilde{\xi}_i \mathbb{I}\{\varepsilon_i = -1\}, \\ \upsilon_{2i} &= \zeta_i \mathbb{I}\{\varepsilon_i = 1\} + \widetilde{\zeta}_i \mathbb{I}\{\varepsilon_i = -1\}, \\ \widetilde{\upsilon}_{1i} &= \widetilde{\xi}_i \mathbb{I}\{\varepsilon_i = 1\} + \xi_i \mathbb{I}\{\varepsilon_i = -1\}, \\ \widetilde{\upsilon}_{2i} &= \widetilde{\zeta}_i \mathbb{I}\{\varepsilon_i = 1\} + \zeta_i \mathbb{I}\{\varepsilon_i = -1\}. \end{aligned}$$

Now, note that

$$\begin{aligned} & E^{\chi} \left| \frac{1}{\sqrt{K}} \sum_{1 \le\ i \ne j\ \le\ n} P_{ij} \xi_i \zeta_j \right|^{\rho} \\ = \quad & \frac{1}{K^{\rho/2}} E^{\chi} \left| \sum_{1 \le\ i \ne j\ \le\ n} P_{ij} E^{\widetilde{\chi}} \left[\xi_i \zeta_j |\ \chi_1, .., \chi_n \right] \right|^{\rho} \\ = \quad & \frac{1}{K^{\rho/2}} E^{\chi} \left| \sum_{1 \le\ i \ne j\ \le\ n} P_{ij} E^{\widetilde{\chi}} \left[\xi_i \zeta_j + \widetilde{\xi}_i \zeta_j + \xi_i \widetilde{\zeta}_j + \widetilde{\xi}_i \widetilde{\zeta}_j |\ \chi_1, .., \chi_n \right] \right|^{\rho} \end{aligned} \tag{15}$$

where the second equality follows from the fact that

$$\begin{aligned} & E^{\widetilde{\chi}} \left[\widetilde{\xi}_i \zeta_j + \xi_i \widetilde{\zeta}_j + \widetilde{\xi}_i \widetilde{\zeta}_j |\ \chi_1, .., \chi_n \right] \\ = \quad & \zeta_j E^{\widetilde{\chi}} \left[\widetilde{\xi}_i |\ \chi_1, .., \chi_n \right] + \xi_i E^{\widetilde{\chi}} \left[\widetilde{\zeta}_j |\ \chi_1, .., \chi_n \right] + E^{\widetilde{\chi}} \left[\widetilde{\xi}_i \widetilde{\zeta}_j |\ \chi_1, .., \chi_n \right] \quad a.e. \\ = \quad & \zeta_j E^{\widetilde{\chi}} \left[\widetilde{\xi}_i \right] + \xi_i E^{\widetilde{\chi}} \left[\widetilde{\zeta}_j \right] + E^{\widetilde{\chi}} \left[\widetilde{\xi}_i \right] E^{\widetilde{\chi}} \left[\widetilde{\zeta}_j \right] \\ = \quad & 0 \end{aligned}$$

Next, note that

$$
\begin{aligned}
& E^{\chi}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}E^{\widetilde{\chi}}\left[\xi_i\zeta_j+\widetilde{\xi}_i\zeta_j+\xi_i\widetilde{\zeta}_j+\widetilde{\xi}_i\widetilde{\zeta}_j \middle|\ \chi_1,..,\chi_n\right]\right|^{\rho} \\
\leq\ & E^{(\chi,\widetilde{\chi})}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\left(\xi_i\zeta_j+\widetilde{\xi}_i\zeta_j+\xi_i\widetilde{\zeta}_j+\widetilde{\xi}_i\widetilde{\zeta}_j\right)\right|^{\rho} \quad \text{[by conditional Jensen's inequality]} \\
=\ & 4^{\rho}E^{(\chi,\widetilde{\chi})}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\frac{1}{4}\left(\xi_i\zeta_j+\widetilde{\xi}_i\zeta_j+\xi_i\widetilde{\zeta}_j+\widetilde{\xi}_i\widetilde{\zeta}_j\right)\right|^{\rho} \\
=\ & 4^{\rho}E^{(\chi,\widetilde{\chi})}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}E^{\varepsilon}\left(\upsilon_{1i}\widetilde{\upsilon}_{2j}\middle|\ \chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right)\right|^{\rho} \\
\leq\ & 4^{\rho}E^{(\upsilon,\widetilde{\upsilon})}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\upsilon_{1i}\widetilde{\upsilon}_{2j}\right|^{\rho} \quad \text{[by conditional Jensen's inequality]} \\
=\ & 4^{\rho}E^{(\chi,\widetilde{\chi})}\left|\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\xi_i\widetilde{\zeta}_j\right|^{\rho}, \qquad (16)
\end{aligned}
$$

where the second equality above follows from the fact that

$$
\begin{aligned}
& \frac{1}{4}\left(\xi_i\zeta_j+\widetilde{\xi}_i\zeta_j+\xi_i\widetilde{\zeta}_j+\widetilde{\xi}_i\widetilde{\zeta}_j\right) \\
=\ & \xi_i\zeta_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=1\cap\varepsilon_j=-1\right\}\right]+\widetilde{\xi}_i\zeta_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=-1\cap\varepsilon_j=-1\right\}\right] \\
& +\xi_i\widetilde{\zeta}_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=1\cap\varepsilon_j=1\right\}\right]+\widetilde{\xi}_i\widetilde{\zeta}_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=-1\cap\varepsilon_j=1\right\}\right] \\
& \text{[by definition and independence of } \{\varepsilon_i\}\text{]} \\
=\ & \xi_i\widetilde{\zeta}_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=1\cap\varepsilon_j=1\right\}\middle|\ \chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right] \\
& +\widetilde{\xi}_i\widetilde{\zeta}_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=-1\cap\varepsilon_j=1\right\}\middle|\ \chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right] \\
& +\xi_i\zeta_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=1\cap\varepsilon_j=-1\right\}\middle|\chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right] \\
& +\widetilde{\xi}_i\zeta_jE^{\varepsilon}\left[\mathbb{I}\left\{\varepsilon_i=-1\cap\varepsilon_j=-1\right\}\middle|\chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right] \\
& \text{[by the independence of } \varepsilon_i \text{ and } \left(\chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right) \text{ for all } i\text{]} \\
=\ & E^{\varepsilon}\left(\upsilon_{1i}\widetilde{\upsilon}_{2j}\middle|\ \chi_1,..,\chi_n;\widetilde{\chi}_1,..,\widetilde{\chi}_n\right) \text{ [by definitiion of } \upsilon_{1i} \text{ and } \widetilde{\upsilon}_{2j}\text{]}
\end{aligned}
$$

and where the last equality in (16) above follows from the fact that the joint distribution of $(\upsilon_1,...,\upsilon_n,\widetilde{\upsilon}_1,...,\widetilde{\upsilon}_n)$ is the same as that of $(\chi_1,...,\chi_n,\widetilde{\chi}_1,...,\widetilde{\chi}_n)$. (15) and (16) together implies that

$$
E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\xi_i\zeta_j\right|^{\rho}\leq 4^{\rho}E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq n} P_{ij}\xi_i\widetilde{\zeta}_j\right|^{\rho},
$$

which is the desired conclusion. *Q.E.D.*

Lemma B7 (Moment Inequality for Sums): Let $\{\xi_i\}$ be an independent sequence of mean-zero random variables. Then, for all $\rho>2$, there exists a constant $\overline{C}<\infty$ such that

$$
E\left|\sum_{i=1}^{n}\xi_i\right|^{\rho}\leq\overline{C}^{\rho}\max\left\{\rho^{\rho}E\left(\max_i|\xi_i|^{\rho}\right),\rho^{\rho/2}\left(\sum_{i=1}^{n}E\xi_i^2\right)^{\rho/2}\right\}
$$

Proof: The proof follows as a special case of the proof of Proposition 2.4 of Giné, Latala, and Zinn (2000). *Q.E.D.*

Lemma B8 (Moment Inequality for U-Statistics): Let $\{\chi_i\}$ and $\{\widetilde{\chi}_i\}$ be as defined previously and let these sequences satisfy the same moment conditions as in Lemma B6 above but let the moment conditions be satisfied for $\rho > 2$. Also, let P_{ij} denote the $(i,j)^{th}$ element of the projection matrix $P = Z(Z'Z)^{-1}Z'$, satisfying Assumption 1. Then, there exists some positive constant $\overline{C} < \infty$ such that

$$E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\ \neq\ j\ \leq\ n} P_{ij}\xi_i\widetilde{\zeta}_j\right|^{\rho}$$

$$\leq \overline{C}^{\rho}\max\left\{\rho^{\rho}\left(\frac{1}{K}\sum_{1\leq\ i\ ,\ j\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2},\rho^{3\rho/2}E^{\xi}\max_i\left(\frac{1}{K}\sum_{1\leq\ j\ \leq\ n}P_{ij}^2\xi_i^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2},\right.$$

$$\left.\rho^{3\rho/2}E^{\widetilde{\zeta}}\max_j\left(\frac{1}{K}\sum_{1\leq\ i\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\widetilde{\zeta}_j^2\right)^{\rho/2},\rho^{2\rho}\frac{1}{K^{\rho/2}}E\left[\max_{i,j}\left(|P_{ij}|^{\rho}|\xi_i|^{\rho}\left|\widetilde{\zeta}_j\right|^{\rho}\right)\right]\right\},$$

where $\sigma_{\xi,i}^2 = E\left[\xi_i^2\right]$ and $\sigma_{\widetilde{\zeta},j}^2 = E\left[\widetilde{\zeta}_j^2\right] = E\left[\zeta_j^2\right]$.

Proof: The proof follows as a special case of the proof of Proposition 2.4 of Giné, Latala, and Zinn (2000). *Q.E.D.*

Lemma B9: If Assumptions 1-3, 7, and 8 are satisfied, then, the following results hold

(a)

$$E\left\|\frac{z'P\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} = O\left(\frac{1}{\mu_n^{2pq}}\right);$$

(b)

$$E\left\|\frac{z'D_P\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} = O\left(\frac{1}{\mu_n^{2pq}}\right)$$

(c)

$$E\left\|\frac{z'\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} = O\left(\frac{1}{\mu_n^{2pq}}\right),$$

where p and q are as given in Assumption 8.

Proof:

In the argument given below, we make use of the simple fact that if $\{X_n\}$ is a sequence of $M \times L$ random matrices, where M and L do not depend on n and if $x_{m\ell,n}$ denotes the $(m,\ell)^{th}$ element of X_n; then for $\rho > 0$ to show that

$$E\|X_n\|^{\rho} = O(1),$$

it is sufficient to show that there exists a positive constant C^* such that

$$E|x_{m\ell,n}|^{\rho} = O(1) \quad \text{for } m = 1,..,M; \ell = 1,..,L. \tag{17}$$

To establish this, note that (17) means for any $(m,\ell)^{th}$ element $x_{m\ell,n}$, there exists a positive integer $N(m,\ell)$ such that for all $n \geq N(m,\ell)$,

$$E\left|x_{m\ell,n}\right|^{\rho} \leq C^{*} < \infty.$$

It follows that if we take $N^{*} = \max\{N(1,1), N(1,2), .., N(m,\ell), .., N(M,L)\}$, then for $n \geq N^{*}$

$$\sum_{m=1}^{M}\sum_{\ell=1}^{L} E\left|x_{m\ell,n}\right|^{\rho} \leq MLC^{*} < \infty.$$

Hence,

$$\begin{aligned}
E\left\|X_{n}\right\|^{\rho} &= E\left[tr\left\{X_{n}'X_{n}\right\}\right]^{\rho/2} \\
&= E\left[\sum_{m=1}^{M}\sum_{\ell=1}^{L} x_{m\ell,n}^{2}\right]^{\rho/2} \\
&\leq \max\left\{(ML)^{\frac{\rho}{2}-1}, 1\right\}\sum_{m=1}^{M}\sum_{\ell=1}^{L} E\left|x_{m\ell,n}\right|^{\rho} \quad \text{[by Loève's } c_r \text{ inequality]} \\
&\leq \max\left\{(ML)^{\rho/2}, ML\right\} C^{*} < \infty
\end{aligned}$$

for $n \geq N^{*}$.

Now, to show part (a), note that

$$E\left\|\frac{z'P\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} = \frac{1}{\mu_n^{2pq}} E\left\|\frac{z'P\overline{V}}{\sqrt{n}}\right\|^{2pq},$$

so that we need only to show that $E\left\|z'P\overline{V}/\sqrt{n}\right\|^{2pq} = O(1)$. Moreover, in light of the discussion above, it suffices to show that

$$E\left[\frac{z_{\cdot g}'P\overline{v}_{\cdot h}}{\sqrt{n}}\right]^{2pq} = E\left[\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} z_{ig}P_{ij}\right)\overline{v}_{jh}\right]^{2pq} = O(1) \text{ for } g = 1, ..., G;\ h = 1, .., G+1.$$

To proceed, note that applying the inequality in Lemma B7 above, there exists a constant $\overline{C} < \infty$ such that

$$\begin{aligned}
& E\left[\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} z_{ig}P_{ij}\right)\overline{v}_{jh}\right]^{2pq} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{2pq}\max\left\{(2pq)^{2pq} E\left[\max_{j}\left|\sum_{i=1}^{n} z_{ig}P_{ij}\right|^{2pq}\left|\overline{v}_{jh}\right|^{2pq}\right], (2pq)^{pq}\left(\sum_{j=1}^{n} E\left[\overline{v}_{jh}\sum_{i=1}^{n} z_{ig}P_{ij}\right]^{2}\right)^{pq}\right\} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{2pq}(2pq)^{2pq} E\left[\max_{j}\left|\sum_{i=1}^{n} z_{ig}P_{ij}\right|^{2pq}\left|\overline{v}_{jh}\right|^{2pq}\right] + \frac{1}{n^{pq}}\overline{C}^{2pq}(2pq)^{pq}\left(\sum_{j=1}^{n} E\left[\overline{v}_{jh}\sum_{i=1}^{n} z_{ig}P_{ij}\right]^{2}\right)^{pq} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{2pq}(2pq)^{2pq}\sum_{j=1}^{n} E\left[\left|\sum_{i=1}^{n} z_{ig}P_{ij}\right|^{2pq}\left|\overline{v}_{jh}\right|^{2pq}\right] + \frac{1}{n^{pq}}\overline{C}^{2pq}(2pq)^{pq}\left(\sum_{j=1}^{n} E\left[\overline{v}_{jh}\sum_{i=1}^{n} z_{ig}P_{ij}\right]^{2}\right)^{pq} \quad (18)
\end{aligned}$$

Next, note that

$$\begin{aligned}
\sum_{j=1}^{n} E\left[\left|\sum_{i=1}^{n} z_{ig} P_{ij}\right|^{2pq} |\overline{v}_{jh}|^{2pq}\right] &= \sum_{j=1}^{n} E\left[\left|e_j' P z_{\cdot g}\right|^{2pq} |\overline{v}_{jh}|^{2pq}\right] \\
&= \sum_{j=1}^{n} \left(e_j' P z_{\cdot g}\right)^2 \left|e_j' P z_{\cdot g}\right|^{2(pq-1)} E|\overline{v}_{jh}|^{2pq} \\
&\leq \left(\sup_{1\leq\, j\,\leq\, n} E|\overline{v}_{jh}|^{2pq}\right) \sum_{j=1}^{n} \left(e_j' P z_{\cdot g}\right)^2 \left(z_{\cdot g}' P z_{\cdot g}\right)^{(pq-1)} \\
&= \left(\sup_{1\leq\, j\,\leq\, n} E|\overline{v}_{jh}|^{2pq}\right) \left(z_{\cdot g}' P z_{\cdot g}\right)^{pq} \text{ for } h = 1,..,G+1, \qquad (19)
\end{aligned}$$

where the inequality above follows from CS. Moreover,

$$\begin{aligned}
\sum_{j=1}^{n} E\left[\overline{v}_{jh} \sum_{i=1}^{n} z_{ig} P_{ij}\right]^2 &= \sum_{j=1}^{n} E\left[\overline{v}_{jh}\left(e_j' P z_{\cdot g}\right)\right]^2 \\
&\leq \left(\sup_{1\leq\, j\,\leq\, n} E\left[\overline{v}_{jh}^2\right]\right) z_{\cdot g}' P \sum_{j=1}^{n} e_j e_j' P z_{\cdot g} \\
&= \left(\sup_{1\leq\, j\,\leq\, n} E\left[\overline{v}_{jh}^2\right]\right) z_{\cdot g}' P z_{\cdot g} \text{ for } h = 1,..,G+1. \qquad (20)
\end{aligned}$$

Applying (19) and (20) to the upper bound given by (18), we obtain for $g = 1, ..., G$ and $h = 1, .., G+1$

$$\begin{aligned}
& E\left[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} z_{ig} P_{ij}\right) \overline{v}_{jh}\right]^{2pq} \\
\leq\ & \overline{C}^{2pq} (2pq)^{2pq} \left(\sup_{1\leq\, j\,\leq\, n} E|\overline{v}_{jh}|^{2pq}\right) \left(\frac{z_{\cdot g}' P z_{\cdot g}}{n}\right)^{pq} + \overline{C}^{2pq} (2pq)^{pq} \left(\sup_{1\leq\, j\,\leq\, n} E\left[\overline{v}_{jh}^2\right]\right)^{pq} \left(\frac{z_{\cdot g}' P z_{\cdot g}}{n}\right)^{pq} \\
\leq\ & \overline{C}^{2pq} (2pq)^{2pq} C^* \left(\frac{z_{\cdot g}' P z_{\cdot g}}{n}\right)^{pq} \\
\leq\ & \overline{C}^{2pq} (2pq)^{2pq} C^* \left(\frac{z_{\cdot g}' z_{\cdot g}}{n}\right)^{pq} \\
\leq\ & \overline{C}^{2pq} (2pq)^{2pq} C^* \left(\frac{1}{n} \sum_{i=1}^{n} \|z_i\|^2\right)^{pq} \\
\leq\ & \overline{C}^{2pq} (2pq)^{2pq} C^* \frac{1}{n} \sum_{i=1}^{n} \|z_i\|^{2pq} \quad \text{[by Liapunov's inequality]} \\
=\ & O(1) \quad \text{[by Assumption 8]};
\end{aligned}$$

note that the second inequality above follows from the fact that, under Assumption 8, there exists a constant C^*such that $\left(\sup_{i,h} E\left[\overline{v}_{ih}^2\right]\right)^{pq} \leq \sup_{i,h} E|\overline{v}_{ih}|^{2pq} \leq C^* < \infty$.

For part (b), it suffices to show that

$$E\left[\frac{z_{\cdot g}' D_P \overline{v}_{\cdot h}}{\mu_n \sqrt{n}}\right]^{2pq} = E\left[\frac{1}{\mu_n \sqrt{n}} \sum_{i=1}^{n} z_{ig} P_{ii} \overline{v}_{ih}\right]^{2pq} = O\left(1/\mu_n^{2pq}\right) \quad \text{for } g = 1, ..., G;\ h = 1, .., G+1.$$

Again, applying the inequality in Lemma B7, we obtain, for $g = 1, ..., G$ and $h = 1, .., G+1$,

$$
\begin{aligned}
& E\left[\frac{1}{\mu_n\sqrt{n}}\sum_{i=1}^{n} z_{ig}P_{ii}\overline{v}_{ih}\right]^{2pq} \\
\leq\ & \frac{1}{\mu_n^{2pq}n^{pq}}\overline{C}^{2pq}\max\left\{(2pq)^{2pq}E\left[\max_i |z_{ig}P_{ii}\overline{v}_{ih}|^{2pq}\right], (2pq)^{pq}\left(\sum_{i=1}^{n} E\left[z_{ig}P_{ii}\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}n^{pq}}\overline{C}^{2pq}\left\{(2pq)^{2pq}E\left[\max_i |z_{ig}P_{ii}\overline{v}_{ih}|^{2pq}\right] + (2pq)^{pq}\left(\sum_{i=1}^{n} E\left[z_{ig}P_{ii}\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}n^{pq}}\overline{C}^{2pq}\left\{(2pq)^{2pq}\sum_{i=1}^{n}|z_{ig}|^{2pq}P_{ii}^{2pq}E|\overline{v}_{ih}|^{2pq} + (2pq)^{pq}\left(\sum_{i=1}^{n} z_{ig}^2P_{ii}^2E\left[\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}}\overline{C}^{2pq}C^*\left\{(2pq)^{2pq}\frac{1}{n^{pq}}\sum_{i=1}^{n}|z_{ig}|^{2pq} + (2pq)^{pq}\left(\frac{1}{n}\sum_{i=1}^{n} z_{ig}^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}}\overline{C}^{2pq}C^*\left\{(2pq)^{2pq}\frac{1}{n^{(pq-1)}}\frac{1}{n}\sum_{i=1}^{n}\|z_i\|^{2pq} + (2pq)^{pq}\left(\frac{1}{n}\sum_{i=1}^{n}\|z_i\|^{2pq}\right)\right\} \\
=\ & O\left(1/\mu_n^{2pq}\right),
\end{aligned}
$$

where the fourth inequality above follows from the fact that $P_{ii} < 1$ by Assumption 1 and by the fact that, under Assumption 8, there exists a constant C^*such that $\left(\sup_{i,h} E\left[\overline{v}_{ih}^2\right]\right)^{pq} \leq \sup_{i,h} E|\overline{v}_{ih}|^{2pq} \leq C^* < \infty$.

For part (c), it suffices to show that

$$
E\left[\frac{z'_{\cdot g}\overline{v}_{\cdot h}}{\mu_n\sqrt{n}}\right]^{2pq} = E\left[\frac{1}{\mu_n\sqrt{n}}\sum_{i=1}^{n} z_{ig}\overline{v}_{ih}\right]^{2pq} = O\left(1/\mu_n^{2pq}\right) \quad \text{for } g = 1, ..., G;\ h = 1, .., G+1.
$$

applying the inequality in Lemma B7, we obtain, for $g = 1, ..., G$ and $h = 1, .., G+1$,

$$
\begin{aligned}
& E\left[\frac{1}{\mu_n\sqrt{n}}\sum_{i=1}^{n} z_{ig}\overline{v}_{ih}\right]^{2pq} \\
\leq\ & \frac{1}{\mu_n^{2pq}n^{pq}}\overline{C}^{2pq}\max\left\{(2pq)^{2pq}E\left[\max_i |z_{ig}\overline{v}_{ih}|^{2pq}\right], (2pq)^{pq}\left(\sum_{i=1}^{n} E\left[z_{ig}\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}n^{pq}}\overline{C}^{2pq}\left\{(2pq)^{2pq}E\left[\max_i |z_{ig}\overline{v}_{ih}|^{2pq}\right] + (2pq)^{pq}\left(\sum_{i=1}^{n} E\left[z_{ig}\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}}\overline{C}^{2pq}\left\{(2pq)^{2pq}\frac{1}{n^{pq}}\sum_{i=1}^{n}|z_{ig}|^{2pq}E|\overline{v}_{ih}|^{2pq} + (2pq)^{pq}\left(\frac{1}{n}\sum_{i=1}^{n} z_{ig}^2E\left[\overline{v}_{ih}\right]^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}}\overline{C}^{2pq}C^*\left\{(2pq)^{2pq}\frac{1}{n^{pq}}\sum_{i=1}^{n}|z_{ig}|^{2pq} + (2pq)^{pq}\left(\frac{1}{n}\sum_{i=1}^{n} z_{ig}^2\right)^{pq}\right\} \\
\leq\ & \frac{1}{\mu_n^{2pq}}\overline{C}^{2pq}C^*\left\{(2pq)^{2pq}\frac{1}{n^{(pq-1)}}\frac{1}{n}\sum_{i=1}^{n}\|z_i\|^{2pq} + (2pq)^{pq}\left(\frac{1}{n}\sum_{i=1}^{n}\|z_i\|^{2pq}\right)\right\} \\
=\ & O\left(1/\mu_n^{2pq}\right).
\end{aligned}
$$

Q.E.D.

Lemma B10: Let $\chi_i = (\xi_i, \zeta_i)'$. Assume that $\{\chi_i\}$ is a sequence of independent 2×1 random vectors such that $E[\chi_i] = 0$ and such that $\sup_i E\|\chi_i\|^\rho \leq C^* < \infty$ for some constant C^* and for some positive integer $\rho > 2$.

Also, let P_{ij} denote the $(i,j)^{th}$ element of the projection matrix $P = Z\left(Z'Z\right)^{-1}Z'$ satisfying Assumptions 1 and 7. Then,

$$E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq\ n}P_{ij}\xi_i\zeta_j\right|^{\rho} = O\left(1\right).$$

Proof :

To proceed, note that by Lemma B8,

$$\begin{aligned}
& E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq\ n}P_{ij}\xi_i\zeta_j\right|^{\rho} \\
\leq\ & 4^{\rho}E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq\ n}P_{ij}\xi_i\widetilde{\zeta}_j\right|^{\rho} \\
\leq\ & \overline{C}^{\rho}\max\left\{\left(\rho\right)^{\rho}\left(\frac{1}{K}\sum_{1\leq\ i\ ,\ j\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2},\left(\rho\right)^{3\rho/2}E^{\xi}\max_i\left(\frac{1}{K}\sum_{1\leq\ j\ \leq\ n}P_{ij}^2\xi_i^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2},\right. \\
& \left.\left(\rho\right)^{3\rho/2}E^{\widetilde{\zeta}}\max_j\left(\frac{1}{K}\sum_{1\leq\ i\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\widetilde{\zeta}_j^2\right)^{\rho/2},\left(\rho\right)^{2\rho}\frac{1}{K^{\rho/2}}E\left[\max_{i,j}\left(\left|P_{ij}\right|^{\rho}\left|\xi_i\right|^{\rho}\left|\widetilde{\zeta}_j\right|^{\rho}\right)\right]\right\} \\
\leq\ & \overline{C}^{\rho}\left\{\left(\rho\right)^{\rho}\left(\frac{1}{K}\sum_{1\leq\ i\ ,\ j\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2}+\left(\rho\right)^{3\rho/2}E^{\xi}\max_i\left(\frac{1}{K}\sum_{1\leq\ j\ \leq\ n}P_{ij}^2\xi_i^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2}\right. \\
& \left.+\left(\rho\right)^{3\rho/2}E^{\widetilde{\zeta}}\max_j\left(\frac{1}{K}\sum_{1\leq\ i\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\widetilde{\zeta}_j^2\right)^{\rho/2}+\left(\rho\right)^{2\rho}\frac{1}{K^{\rho/2}}E\left[\max_{i,j}\left(\left|P_{ij}\right|^{\rho}\left|\xi_i\right|^{\rho}\left|\widetilde{\zeta}_j\right|^{\rho}\right)\right]\right\}
\end{aligned}$$

Next, observe that

$$\begin{aligned}
\frac{1}{K}\sum_{1\leq\ i\ ,\ j\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\sigma_{\widetilde{\zeta},j}^2 &\leq \left(C^*\right)^{4/\rho}\frac{1}{K}\sum_{i=1}^{n}\sum_{j=1}^{n}P_{ij}^2\ \ \text{[by Liapunov's inequality]} \\
&= \left(C^*\right)^{4/\rho}<\infty
\end{aligned}$$

so that

$$\left(\frac{1}{K}\sum_{1\leq\ i\ ,\ j\ \leq\ n}P_{ij}^2\sigma_{\xi,i}^2\sigma_{\widetilde{\zeta},j}^2\right)^{\rho/2}\leq\left(C^*\right)^2<\infty \tag{21}$$

Also,

$$
\begin{aligned}
& E^{\xi} \max_{i} \left(\frac{1}{K} \sum_{1 \leq\ j\ \leq\ n} P_{ij}^{2} \xi_{i}^{2} \sigma_{\widetilde{\zeta},j}^{2} \right)^{\rho/2} \\
\leq\ & \frac{1}{K^{\rho/2}} \sum_{i=1}^{n} E^{\xi} \left(\sum_{1 \leq\ j\ \leq\ n} P_{ij}^{2} \xi_{i}^{2} \sigma_{\widetilde{\zeta},j}^{2} \right)^{\rho/2} \\
=\ & \frac{1}{K^{\rho/2}} \sum_{i=1}^{n} E\left|\xi_{i}\right|^{\rho} \left(\sum_{1 \leq\ j\ \leq\ n} P_{ij}^{2} \sigma_{\widetilde{\zeta},j}^{2} \right)^{\rho/2} \\
\leq\ & \frac{1}{K^{\rho/2}} \sum_{i=1}^{n} \left(C^{*}\right)^{2} \left(\sum_{1 \leq\ j\ \leq\ n} P_{ij}^{2} \right)^{\rho/2} \\
\leq\ & \left(C^{*}\right)^{2} \frac{1}{K^{\rho/2}} \sum_{i=1}^{n} P_{ii}^{\rho/2} = O\left(n^{-(\rho-2)/2}\right) \quad \text{for } \rho > 2,
\end{aligned} \tag{22}
$$

where the order of magnitude given above follows from the fact that given Assumption 7, we have

$$
\sum_{i=1}^{n} P_{ii}^{\rho/2} \leq C_{P}^{\rho/2} \sum_{i=1}^{n} \left(\frac{K}{n}\right)^{\rho/2} = C_{P}^{\rho/2} \left(\frac{K^{\rho/2}}{n^{(\rho-2)/2}}\right).
$$

Similarly,

$$
E^{\zeta} \max_{j} \left(\frac{1}{K} \sum_{1 \leq\ i\ \leq\ n,\ i \neq j} P_{ij}^{2} \sigma_{\xi,i}^{2} \widetilde{\zeta}_{j}^{2} \right)^{\rho/2} \leq \left(C^{*}\right)^{2} \frac{1}{K^{\rho/2}} \sum_{j=1}^{n} P_{jj}^{\rho/2} = O\left(n^{-(\rho-2)/2}\right) = o\left(1\right) \tag{23}
$$

for $\rho > 2$. Moreover,

$$
\begin{aligned}
& \frac{1}{K^{\rho/2}} E\left[\max_{i,j} \left(\left|P_{ij}\right|^{\rho} \left|\xi_{i}\right|^{\rho} \left|\widetilde{\zeta}_{j}\right|^{\rho}\right)\right] \\
\leq\ & \frac{1}{K^{\rho/2}} E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \left|P_{ij}\right|^{\rho} \left|\xi_{i}\right|^{\rho} \left|\widetilde{\zeta}_{j}\right|^{\rho}\right] = \frac{1}{K^{\rho/2}} \sum_{i=1}^{n}\sum_{j=1}^{n} \left|P_{ij}\right|^{\rho} E\left[\left|\xi_{i}\right|^{\rho}\right] E\left[\left|\widetilde{\zeta}_{j}\right|^{\rho}\right] \\
\leq\ & \left(C^{*}\right)^{2} \frac{1}{K^{\rho/2}} \sum_{i=1}^{n}\sum_{j=1}^{n} \left|P_{ij}\right|^{\rho} \\
\leq\ & \left(C^{*}\right)^{2} C_{P}^{(\rho-2)} \left(\frac{K}{n}\right)^{(\rho-2)} \frac{1}{K^{\rho/2}} \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij}^{2} \\
=\ & \left(C^{*}\right)^{2} C_{P}^{(\rho-2)} \left(\frac{K}{n}\right)^{(\rho-2)} \frac{1}{K^{\rho/2}} \sum_{i=1}^{n} P_{ii} \\
=\ & O\left(\frac{K^{(\rho-2)/2}}{n^{(\rho-2)}}\right) \\
=\ & O\left(n^{(a/2-1)(\rho-2)}\right) = o\left(1\right) \text{ for } \rho > 2, \text{ [since } a/2 < 1 \text{ by Assumption 7]}
\end{aligned} \tag{24}
$$

where the third inequality above follows from the fact $\left|P_{ij}\right|^{(\rho-2)} \leq P_{ii}^{(\rho-2)/2} P_{jj}^{(\rho-2)/2} \leq C_{P}^{(\rho-2)} \left(K/n\right)^{(\rho-2)}$. It follows from (21)-(24) that

$$E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq\ n} P_{ij}\xi_i\zeta_j\right|^{\rho}=O\left(1\right),$$

as desired. *Q.E.D.*

Remark: Note that there is nothing in the argument above which requires that $K\to\infty$ as $n\to\infty$. Hence, the conclusion of Lemma B10 holds also for the case where K is fixed. Indeed, the inequalities given in Lemmas B7 and B8 hold both for the case where $K\to\infty$ and for the case where K is fixed, and the moment calculations that we will make using these inequalities will also hold for both cases as well.

Lemma B11: Under Assumptions 1-4, 7, and 8; the following results hold:

(a)

$$E\left[\left\|\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right]=O\left(\frac{1}{\mu_n^{2pq}}\right);$$

(b)

$$E\left[\left\|\frac{z'\left(M+D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right]=O\left(\frac{1}{\mu_n^{2pq}}\right);$$

(c)

$$E\left[\left\|\frac{z'M\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right]=O\left(\frac{1}{\mu_n^{2pq}}\right);$$

where p and q are as given in Assumption 8.

Proof:

For part (a), note that by T, Loève's c_r inequality, and parts (a) and (b) of Lemma B9 above, we have

$$\begin{aligned}
E\left[\left\|\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right] &\leq E\left[\left\|\frac{z'P\overline{V}}{\mu_n\sqrt{n}}\right\|+\left\|\frac{z'D_P\overline{V}}{\mu_n\sqrt{n}}\right\|\right]^{2pq} \\
&\leq 2^{2pq-1}\left\{E\left\|\frac{z'P\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}+E\left\|\frac{z'D_P\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right\} \\
&= O\left(1/\mu_n^{2pq}\right),
\end{aligned}$$

as desired.

To show part (b), note that by T, Loève's c_r inequality, part (c) of Lemma B9, and part (a) of this lemma, we have

$$\begin{aligned}
E\left[\left\|\frac{z'\left(M+D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right] &= E\left[\left\|\frac{z'\overline{V}}{\mu_n\sqrt{n}}-\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|\right]^{2pq} \\
&\leq E\left[\left\|\frac{z'\overline{V}}{\mu_n\sqrt{n}}\right\|+\left\|\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|\right]^{2pq} \\
&\leq 2^{2pq-1}\left\{E\left\|\frac{z'\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}+E\left\|\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq}\right\} \\
&= O\left(1/\mu_n^{2pq}\right),
\end{aligned}$$

as desired.

Part (c) can be shown similar to (b) by applying T, Loève's c_r inequality, and parts (a) and (c) of Lemma B9. For brevity, we do not give the full argument. *Q.E.D.*

Lemma B12: Under Assumptions 1-3, 7, and 8; the following results hold

(a)
$$E\left\|\frac{\overline{V}'\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}=O\left(\frac{1}{n^{pq/2}}\right)\;;$$

(b)
$$E\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|^{2pq}=O\left(\frac{K^{pq}}{\mu_n^{4pq}}\right);$$

(c)
$$E\left[\left\|\frac{\overline{V}'\left(M+D_P\right)\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}\right]=O\left(\frac{1}{n^{pq/2}}\right),$$

where p and q are as given in Assumption 8.

Proof: For part (a), it suffices to show that

$$\begin{aligned}&E\left[\frac{\overline{v}_{\cdot g}'\overline{v}_{\cdot h}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_{gh,i}\right]^{pq}\\ =\;&E\left[\frac{1}{n}\sum_{i=1}^{n}\left(\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right)\right]^{pq}=O\left(1/n^{pq/2}\right)\quad\text{for } g=1,...,G;\;h=1,..,G+1;\end{aligned}$$

where $\Xi_{gh,i}$ denotes the $(g,h)^{th}$ element of $\Xi_i = Var(\varepsilon_i, V_i')$. Applying the inequality in Lemma B7, we obtain

$$
\begin{aligned}
& E\left[\frac{1}{n}\sum_{i=1}^{n}\left(\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right)\right]^{pq} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{pq}\max\left\{(pq)^{pq}E\left[\max_i\left|\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right|^{pq}\right],(pq)^{pq/2}\left(\sum_{i=1}^{n}E\left[\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right]^2\right)^{pq/2}\right\} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{pq}\left\{(pq)^{pq}E\left[\max_i\left|\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right|^{pq}\right]+(pq)^{pq/2}\left(\sum_{i=1}^{n}E\left[\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right]^2\right)^{pq/2}\right\} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{pq}\left\{(pq)^{pq}\sum_{i=1}^{n}E\left|\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right|^{pq}+(pq)^{pq/2}\left(\sum_{i=1}^{n}E\left[\overline{v}_{ig}^2\overline{v}_{ih}^2\right]-\sum_{i=1}^{n}\Xi_{gh,i}^2\right)^{pq/2}\right\} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{pq}\left\{(pq)^{pq}\sum_{i=1}^{n}E\left|\overline{v}_{ig}\overline{v}_{ih}-\Xi_{gh,i}\right|^{pq}+(pq)^{pq/2}\left(\sum_{i=1}^{n}E\left[\overline{v}_{ig}^2\overline{v}_{ih}^2\right]+\sum_{i=1}^{n}\Xi_{gh,i}^2\right)^{pq/2}\right\} \\
\leq\ & \frac{1}{n^{pq}}\overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\left(\sum_{i=1}^{n}E\left|\overline{v}_{ig}\overline{v}_{ih}\right|^{pq}+\sum_{i=1}^{n}\left|\Xi_{gh,i}\right|^{pq}\right)+(pq)^{pq/2}\left(\sum_{i=1}^{n}E\left[\overline{v}_{ig}^2\overline{v}_{ih}^2\right]+\sum_{i=1}^{n}\Xi_{gh,i}^2\right)^{pq/2}\right\} \\
& \text{[by Loève's } c_r \text{ inequality]} \\
\leq\ & \overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\frac{1}{n^{pq-1}}\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{E\left|\overline{v}_{ig}\right|^{2pq}}\sqrt{E\left|\overline{v}_{ih}\right|^{2pq}}+\frac{1}{n}\sum_{i=1}^{n}\left|\Xi_{gh,i}\right|^{pq}\right)\right. \\
& \qquad\left.+(pq)^{pq/2}\frac{1}{n^{pq/2}}\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{E\left(\overline{v}_{ig}^4\right)}\sqrt{E\left(\overline{v}_{ih}^4\right)}+\frac{1}{n}\sum_{i=1}^{n}\Xi_{gh,i}^2\right)^{pq/2}\right\}\text{ [by CS]} \\
\leq\ & \overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\frac{1}{n^{pq-1}}\left(\frac{1}{n}\sum_{i=1}^{n}E\left\|\overline{V}_i\right\|^{2pq}+\frac{1}{n}\sum_{i=1}^{n}\left(E\left\|\overline{V}_i\right\|^2\right)^{pq}\right)\right. \\
& \qquad\left.+(pq)^{pq/2}\frac{1}{n^{pq/2}}\left(\frac{1}{n}\sum_{i=1}^{n}E\left\|\overline{V}_i\right\|^4+\frac{1}{n}\sum_{i=1}^{n}\left(E\left\|\overline{V}_i\right\|^2\right)^2\right)^{pq/2}\right\} \\
\leq\ & \overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\frac{1}{n^{pq-1}}\left(\frac{2}{n}\sum_{i=1}^{n}E\left\|\overline{V}_i\right\|^{2pq}\right)+(pq)^{pq/2}\frac{1}{n^{pq/2}}\left(\frac{2}{n}\sum_{i=1}^{n}E\left\|\overline{V}_i\right\|^4\right)^{pq/2}\right\} \\
& \text{[by Liapunov's inequality]} \\
\leq\ & \overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\frac{1}{n^{pq-1}}\left(\frac{2}{n}\sum_{i=1}^{n}E\left\|\overline{V}_i\right\|^{2pq}\right)+(pq)^{pq/2}\frac{1}{n^{pq/2}}\left(\frac{2}{n}\sum_{i=1}^{n}\left(E\left\|\overline{V}_i\right\|^{2pq}\right)^{2/(pq)}\right)^{pq/2}\right\} \\
\leq\ & \overline{C}^{pq}\left\{(pq)^{pq}2^{pq-1}\frac{1}{n^{pq-1}}2\widetilde{C}+(pq)^{pq/2}\frac{1}{n^{pq/2}}2^{pq/2}\widetilde{C}\right\}\quad\text{[by Assumption 8]} \\
=\ & O\left(1/n^{pq/2}\right),
\end{aligned}
$$

For part (b), let $\overline{v}_{\cdot g}$ denote the g^{th} column of $\overline{V}$, and note that it suffices to show that

$$
\begin{aligned}
& E\left|\frac{\overline{v}_{\cdot g}'\left[P-D_P\right]\overline{v}_{\cdot h}}{\mu_n^2}\right|^{2pq} \\
= & \left(\frac{K^{pq}}{\mu_n^{4pq}}\right) E\left|\frac{1}{\sqrt{K}}\sum_{1\leq\ i\neq j\ \leq\ n} P_{ij}\overline{v}_{ig}\overline{v}_{jh}\right|^{2pq} = O\left(\frac{K^{pq}}{\mu_n^{4pq}}\right),\ \text{for } g=1,...,G+1;\ h=1,..,G+1;
\end{aligned}
$$

but this follows immediately from Lemma B10 by taking $\xi_i=\overline{v}_{ig}$ and $\eta_j=\overline{v}_{jh}$ for $g=1,...,G+1$; $h=1,..,G+1$. Hence, the desired conclusion follows.

Finally, to show part (c), note that

$$
\begin{aligned}
& E\left[\left\|\frac{\overline{V}'\left(M+D_P\right)\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}\right] \\
= & E\left[\left\|\left(\frac{\overline{V}'\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right)-\frac{\overline{V}'\left(P-D_P\right)\overline{V}}{n}\right\|\right]^{pq} \\
\leq & E\left[\left\|\frac{\overline{V}'\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|+\left\|\frac{\overline{V}'\left(P-D_P\right)\overline{V}}{n}\right\|\right]^{pq} \quad \text{[by T]} \\
\leq & 2^{pq-1}\left\{E\left\|\frac{\overline{V}'\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}+\left(\frac{\mu_n^2}{n}\right)^{pq}E\left\|\frac{\overline{V}'\left(P-D_P\right)\overline{V}}{\mu_n^2}\right\|^{pq}\right\} \text{[by Loève's } c_r \text{ inequality]} \\
\leq & 2^{pq-1}\left\{E\left\|\frac{\overline{V}'\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}+\left(\frac{\mu_n^2}{n}\right)^{pq}\left(E\left\|\frac{\overline{V}'\left(P-D_P\right)\overline{V}}{\mu_n^2}\right\|^{2pq}\right)^{1/2}\right\} \\
& \text{[by Liapunov's inequality]} \\
= & O\left(\max\left\{\frac{1}{n^{pq/2}},\left(\frac{\sqrt{K}}{n}\right)^{pq}\right\}\right) \text{ [by parts (a) and (b) of this lemma]} \\
= & O\left(1/n^{pq/2}\right).
\end{aligned}
$$

Q.E.D.

Lemma B13: If Assumption 9 is satisfied, then the following results hold for each n such that $n-K\geq L=G+1$:

(a) $\overline{X}'M\overline{X}/n$ is postive definite with probability one;

(b) $\overline{X}'\overline{X}/n$ is postive definite with probability one;

where $\overline{X}=\left[\begin{array}{cc} y & X\end{array}\right]$.

Proof: To prove (a), note that, under Assumption 9, $X_*'MX_*/n$ is positive definite with probability one for n sufficiently large. Now, since

$$\frac{\overline{X}'M\overline{X}}{n}=\frac{D'^{-1}X_*'MX_*D^{-1}}{n},$$

where

$$D^{-1}=\left(\begin{array}{cc} 1 & 0 \\ -\delta_0 & I_G\end{array}\right);$$

we deduce that $\overline{X}'M\overline{X}/n$ is positive definite with probability one as well for n sufficiently large.

To show (b), write

$$\frac{\overline{X}'\overline{X}}{n} = \frac{\overline{X}'M\overline{X}}{n} + \frac{\overline{X}'P\overline{X}}{n}.$$

Since the matrix $\overline{X}'P\overline{X}/n$ is positive semidefinite with probability one, it follows from the eigenvalue inequality given in part (a) of Exercise 12.40 of Abadir and Magnus (2005) and from the result given in part (a) of this lemma that

$$\lambda_{\min}\left(\frac{\overline{X}'\overline{X}}{n}\right) \geq \lambda_{\min}\left(\frac{\overline{X}'M\overline{X}}{n}\right) + \lambda_{\min}\left(\frac{\overline{X}'P\overline{X}}{n}\right) > 0,$$

so that $\overline{X}'\overline{X}/n$ is postive definite with probability one for n such that $n - K \geq L$. *Q.E.D.*

Appendix C

Lemma C1: If Assumptions 1 and 4 are satisfied, then the following results hold.

(a)

$$\left\|\frac{z'Mz}{n}\right\| \leq \frac{1}{n}\sum_{i=1}^{n}\|z_i - \pi_{Kn}Z_i\|^2 = o(1)$$

(b)

$$\lambda_{\max}\left(\frac{z'Mz}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{n}\|z_i - \pi_{Kn}Z_i\|^2 = o(1)$$

Proof: To show part (a), write

$$\begin{aligned}
\left\|\frac{z'Mz}{n}\right\| &= \left\|\frac{z'[I_n - P]z}{n}\right\| = \left\|\frac{(z - Z\pi'_{Kn})'[I_n - P](z - Z\pi'_{Kn})}{n}\right\| \\
&\leq \left\|\frac{(z - Z\pi'_{Kn})'(z - Z\pi'_{Kn})}{n}\right\| \quad \text{[using part (b) of Lemma B4]} \\
&= \sqrt{tr\left\{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(z_i - \pi_{Kn}Z_i)(z_i - \pi_{Kn}Z_i)'(z_j - \pi_{Kn}Z_j)(z_j - \pi_{Kn}Z_j)'\right\}} \\
&\leq \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}(z_i - \pi_{Kn}Z_i)'(z_i - \pi_{Kn}Z_i)\right)^2} \\
&= \frac{1}{n}\sum_{i=1}^{n}(z_i - \pi_{Kn}Z_i)'(z_i - \pi_{Kn}Z_i) \\
&= \frac{1}{n}\sum_{i=1}^{n}\|z_i - \pi_{Kn}Z_i\|^2 \to 0 \text{ as } n \to \infty \quad \text{[by Assumption 4]},
\end{aligned}$$

where the second inequality above follows from using the fact that $tr[AB] = tr[BA]$ and then applying CS.

Part (b) follows immediately from part (a) by noting that

$$\lambda_{\max}\left(\frac{z'Mz}{n}\right) \leq \sqrt{\sum_{g=1}^{G}\lambda_g^2\left(\frac{z'Mz}{n}\right)} = \left\|\frac{z'Mz}{n}\right\|.$$

Q.E.D.

Lemma C2: Suppose that Assumptions 1-4 and 7 hold. In addition, if Assumptions 8 and 9 are satisfied; then, for n sufficiently large, there exists a constant $C^* > 0$ such that

$$\left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A}} \leq \frac{2\sqrt{G}}{C_5} \mathbb{I}_{\mathcal{A}},$$

where G is the number of endogenous regressors in the IV regression.

Proof: To begin, write

$$\begin{aligned}
& \left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A}} \\
= & \left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \\
& + \left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \qquad (25)
\end{aligned}$$

We will consider each term on the right-hand side of (25) in turn.

To proceed, note first that by part (c) of Lemma B5, we have

$$\begin{aligned}
& \left\| \left(S_n^{-1} X' (P - D_P) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' (M + D_P) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
\leq & \frac{\sqrt{G}}{\min\limits_{1 \leq g \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' (P - D_P) X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' (M + D_P) X \right] S_n'^{-1} \right) \right|} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}}.
\end{aligned}$$

To analyze $\min\limits_{1 \leq g \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' (P - D_P) X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' (M + D_P) X \right] S_n'^{-1} \right) \right|$, we make use of the fact that $X = \Upsilon + U = z S_n' / \sqrt{n} + U$ and write

$$\begin{aligned}
& S_n^{-1} \left(X' [P - D_P] X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' [M + D_P] X \right) S_n'^{-1} \\
= & S_n^{-1} \left(z S_n' / \sqrt{n} + U \right)' [P - D_P] \left(z S_n' / \sqrt{n} + U \right) S_n'^{-1} \\
& - \widetilde{\kappa}_{HLIM} S_n^{-1} \left(z S_n' / \sqrt{n} + U \right)' [M + D_P] \left(z S_n' / \sqrt{n} + U \right) S_n'^{-1} \\
& + \frac{C}{n} S_n^{-1} \left(z S_n' / \sqrt{n} + U \right)' [M + D_P] \left(z S_n' / \sqrt{n} + U \right) S_n'^{-1} \\
= & \frac{z' [I_n - D_P] z}{n} - \frac{z' M z}{n} + \frac{\left(z' [P - D_P] U S_n'^{-1} + S_n^{-1} U' [P - D_P] z \right)}{\sqrt{n}} + S_n^{-1} U' [P - D_P] U S_n'^{-1} \\
& - \widetilde{\kappa}_{HLIM} \frac{z' [M + D_P] z}{n} - \widetilde{\kappa}_{HLIM} \frac{\left(z' [M + D_P] U S_n'^{-1} + S_n^{-1} U' [M + D_P] z \right)}{\sqrt{n}} \\
& - \widetilde{\kappa}_{HLIM} S_n^{-1} U' [M + D_P] U S_n'^{-1} + \frac{C}{n} S_n^{-1} U' [M + D_P] U S_n'^{-1} + \frac{C}{n} \frac{z' [M + D_P] z}{n} \\
& + \frac{C}{n} \frac{\left(z' [M + D_P] U S_n'^{-1} + S_n^{-1} U' [M + D_P] z \right)}{\sqrt{n}}
\end{aligned}$$

Next, note that conditional on the event $\mathcal{A} \cap \mathcal{B}$, we have

$$
\begin{aligned}
& \min_{1 \leq\, g\, \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' \left(P - D_P \right) X - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} X' \left(M + D_P \right) X \right] S_n'^{-1} \right) \right| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} + \lambda_{\min} \left(- \frac{\left(z - Z\pi'_{Kn} \right)' M \left(z - Z\pi'_{Kn} \right)}{n} \right) \\
& + \lambda_{\min} \left(\frac{\left(z' \left[P - D_P \right] U S_n'^{-1} + S_n^{-1} U' \left[P - D_P \right] z \right)}{\sqrt{n}} \right) + \lambda_{\min} \left(S_n^{-1} U' \left[P - D_P \right] U S_n'^{-1} \right) \\
& + \widetilde{\kappa}_{HLIM} \lambda_{\min} \left(- \frac{z' \left[M + D_P \right] z}{n} \right) + \widetilde{\kappa}_{HLIM} \lambda_{\min} \left(- \frac{\left(z' \left[M + D_P \right] U S_n'^{-1} + S_n^{-1} U' \left[M + D_P \right] z \right)}{\sqrt{n}} \right) \\
& + \widetilde{\kappa}_{HLIM} \lambda_{\min} \left(- S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right) + \frac{C}{n} \lambda_{\min} \left(S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right) + \frac{C}{n} \lambda_{\min} \left(\frac{z' \left[M + D_P \right] z}{n} \right) \\
& + \frac{C}{n} \lambda_{\min} \left(\frac{\left(z' \left[M + D_P \right] U S_n'^{-1} + S_n^{-1} U' \left[M + + D_P \right] z \right)}{\sqrt{n}} \right) \\
& \text{[obtained by applying inductively the result of Exercise 12.40 part (a) in Abadir and Magnus (2005)]} \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \lambda_{\max} \left\{ \frac{\left(z - Z\pi'_{Kn} \right)' \left(z - Z\pi'_{Kn} \right)}{n} \right\} - 2 \left\| \frac{z' \left[P - D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
& - \left\| S_n^{-1} U' \left[P - D_P \right] U S_n'^{-1} \right\| - \left| \widehat{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| \\
& - 2 \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \left| \widetilde{\kappa}_{HLIM} \right| \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| \\
& - \frac{C}{n} \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| - \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| - 2 \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2 \left\| \frac{z' \left[P - D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \left\| S_n^{-1} U' \left[P - D_P \right] U S_n'^{-1} \right\| \\
& - \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| - 2 \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \left| \widetilde{\kappa}_{HLIM} \right| \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| \\
& - \frac{C}{n} \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| - \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| - 2 \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
& \text{[using the same argument as the proof of Lemma C1 part (a)],} \qquad (26)
\end{aligned}
$$

where $H_n = z' \left[I_n - D_P \right] z / n$.

Now, there exists a constant C_1 such that

$$
\begin{aligned}
& \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| \\
\leq \quad & \left\| \frac{z' M z}{n} \right\| + \left\| \frac{z' D_P z}{n} \right\| \quad \text{[by T]} \\
\leq \quad & \left\| \frac{z' M z}{n} \right\| + C_P \left(\frac{K}{n} \right) \left\| \frac{z' z}{n} \right\| \quad \text{by Assumption 7} \\
\leq \quad & C_1 < \infty \quad \text{[by Assumptions 1, 2, and 7 and Lemma C1 part (a)]} \qquad (27)
\end{aligned}
$$

for all n sufficiently large. Also, let $D_\mu = diag\left(\mu_{1n}, .., \mu_{Gn}\right)$, and note that

$$\begin{aligned}
& \left\| \frac{z'\left[P - D_P\right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
\leq\ & \left\| S_n'^{-1} \right\| \left\| \frac{z'\left[P - D_P\right] U}{\sqrt{n}} \right\| \quad = \sqrt{tr\left\{ D_\mu^{-1} \left(\widetilde{S}_n' \widetilde{S}_n \right)^{-1} D_\mu^{-1} \right\}} \frac{1}{\sqrt{n}} \sqrt{tr\left\{ U'\left[P - D_P\right] zz' \left[P - D_P\right] U \right\}} \\
\leq\ & \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}' \widetilde{S} \right)}} \frac{1}{\mu_n \sqrt{n}} \sqrt{tr\left\{ U'\left[P - D_P\right] zz' \left[P - D_P\right] U \right\}} \\
\leq\ & \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}' \widetilde{S} \right)}} \frac{1}{\mu_n \sqrt{n}} \sqrt{v'\left[P - D_P\right] zz' \left[P - D_P\right] v + tr\left\{ U'\left[P - D_P\right] zz' \left[P - D_P\right] U \right\}} \\
=\ & \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}' \widetilde{S} \right)}} \frac{1}{\mu_n \sqrt{n}} \sqrt{tr\left\{ \begin{pmatrix} v' \\ U' \end{pmatrix} \left[P - D_P\right] zz' \left[P - D_P\right] \begin{pmatrix} v & U \end{pmatrix} \right\}} \\
=\ & \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}' \widetilde{S} \right)}} \left\| \frac{z'\left[P - D_P\right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
\leq\ & \sqrt{C_2} \left\| \frac{z'\left[P - D_P\right] \overline{V}}{\mu_n \sqrt{n}} \right\| \qquad (28)
\end{aligned}$$

for constant C_2 such that $G/\lambda_{\min}\left(\widetilde{S}' \widetilde{S} \right) \leq C_2 < \infty$; note that such constant exists in light of Assumption 2. Moreover, by similar argument, we also have

$$\left\| S_n^{-1} U' \left[P - D_P\right] U S_n'^{-1} \right\| \leq C_2 \left\| \frac{\overline{V}'\left[P - D_P\right] \overline{V}}{\mu_n^2} \right\| \qquad (29)$$

$$\left\| \frac{z'\left[M + D_P\right] U S_n'^{-1}}{\sqrt{n}} \right\| \leq \sqrt{C_2} \left\| \frac{z'\left[M + D_P\right] \overline{V}}{\mu_n \sqrt{n}} \right\| \qquad (30)$$

$$\begin{aligned}
& \left\| S_n^{-1} U' \left[M + D_P\right] U S_n'^{-1} \right\| \\
\leq\ & \left(\frac{n}{\mu_n^2} \right) \left\| \frac{\mu_n^2 S_n^{-1} \left(U'\left[M + D_P\right] U - E\left\{ U'\left[M + D_P\right] U \right\} \right) S_n'^{-1}}{n} \right\| \\
& + \left(\frac{n}{\mu_n^2} \right) \left\| \frac{\mu_n^2 S_n^{-1} E\left(U'\left[M + D_P\right] U \right) S_n'^{-1}}{n} \right\| \quad \text{[by T]} \\
\leq\ & C_2 \left(\frac{n}{\mu_n^2} \right) \left\| \frac{\left(\overline{V}'\left[M + D_P\right] \overline{V} - E\left[\overline{V}'\left[M + D_P\right] \overline{V} \right] \right)}{n} \right\| + C_2 \left(\frac{n}{\mu_n^2} \right) \left\| \frac{E\left\{ \overline{V}'\left[M + D_P\right] \overline{V} \right\}}{n} \right\| \\
=\ & C_2 \left(\frac{n}{\mu_n^2} \right) \left\| \frac{\overline{V}'\left[M + D_P\right] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| + C_2 \left(\frac{n}{\mu_n^2} \right) \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\|, \qquad (31)
\end{aligned}$$

where $\Xi_i = Var\left(\varepsilon_i, V_i'\right)$ and where the last equality follows in light of the fact that by direct calculation

$$\begin{aligned}
E\left[e_g' \overline{V}' \left[M + D_P\right] \overline{V} e_h \right] &= tr\left\{ \left[M + D_P\right] E\left(\overline{v}_{\cdot h} \overline{v}_{\cdot g}' \right) \right\} = tr\left\{ \left[M + D_P\right] diag\left(\Xi_{1,gh}, ..., \Xi_{n,gh} \right) \right\} \\
&= \sum_{i=1}^n \left(1 - P_{ii}\right) \Xi_{i,gh} + \sum_{i=1}^n P_{ii} \Xi_{i,gh} = \sum_{i=1}^n \Xi_{i,gh} \quad \text{for } g, h = 1, .., G+1;
\end{aligned}$$

from which we deduce that

$$\frac{E\left\{\overline{V}'\left[M+D_P\right]\overline{V}\right\}}{n}=\frac{1}{n}\sum_{i=1}^{n}\Xi_i.$$

Moreover, note that

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\| &\leq \frac{1}{n}\sum_{i=1}^{n}\left\|\Xi_i\right\|=\sqrt{\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left(E\left[\overline{v}_{ig}\overline{v}_{ih}\right]\right)^2} \\
&\leq \sqrt{\left(\sum_{g=1}^{G+1}E\left[\overline{v}_{ig}^2\right]\right)^2} \quad \text{[by CS]} \\
&= \left(E\left[\sum_{g=1}^{G+1}\overline{v}_{ig}^2\right]^{pq}\right)^{1/pq} \quad \text{[by Liapunov's inequality]} \\
&= E\left\|\overline{V}_i\right\|^{2pq} \\
&\leq (\widetilde{C})^{1/pq}:=C_3<\infty \quad \text{[by Assumption 8].} \qquad (32)
\end{aligned}$$

It follows from (27)-(32) that, conditioning on the event $\mathcal{A}\cap\mathcal{B}$, we can further bound (26) by

$$\begin{aligned}
&\min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n^{\prime-1}\right)\right| \\
\geq\ & \lambda_{\min}\left\{H_n\right\}-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2-2\sqrt{C_2}\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|-C_2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\| \\
&-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)C_1-2\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)\sqrt{C_2}\left\|\frac{z'\left[M+D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\| \\
&-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)C_2\left(\frac{n}{\mu_n^2}\right)\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)\left(\frac{n}{\mu_n^2}\right)C_2C_3 \\
\geq\ & \lambda_{\min}\left\{H_n\right\}-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2-2\eta_1\sqrt{C_2}-C_2\eta_2-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)C_1 \\
&-2\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)\eta_4\sqrt{C_2}-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)C_2\left(\frac{n}{\mu_n^2}\right)\eta_3-\left(\left|\widetilde{\kappa}_{HLIM}\right|+\frac{C}{n}\right)\left(\frac{n}{\mu_n^2}\right)C_2C_3, \qquad (33)
\end{aligned}$$

which holds for n sufficiently large.

Next, we derive an upper bound for $\left|\widetilde{\kappa}_{HLIM}\right|$ conditional on $\mathcal{A}\cap\mathcal{B}$. To proceed, let $\widetilde{\delta}_\Delta=\left[\begin{array}{cc}1 & -\widetilde{\delta}'_{HLIM}\end{array}\right]'$, and note that

$$\begin{aligned}
\widetilde{\kappa}_{HLIM} &= \frac{\widetilde{\delta}'_\Delta\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta\overline{X}'\left[M+D_P\right]\overline{X}\widetilde{\delta}_\Delta} \\
&= \left(\frac{\mu_n^2}{n}\right)\frac{\widetilde{\delta}'_\Delta\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_\Delta/\mu_n^2}{\widetilde{\delta}'_\Delta\overline{X}'\left[M+D_P\right]\overline{X}\widetilde{\delta}_\Delta/n} \qquad (34)
\end{aligned}$$

Focusing first on the numerator of (34), we have

$$
\begin{aligned}
&\frac{\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_{\Delta}}{\mu_n^2} \\
=\ &\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\left(\frac{z'\left[P-D_P\right]z}{n}\right)\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}+\frac{\widetilde{\delta}_{\Delta}'\overline{V}'\left[P-D_P\right]z}{\mu_n\sqrt{n}}\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n} \\
&+\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\frac{z'\left[P-D_P\right]\overline{V}\widetilde{\delta}_{\Delta}}{\mu_n\sqrt{n}}+\frac{\widetilde{\delta}_{\Delta}'\overline{V}'\left[P-D_P\right]\overline{V}\widetilde{\delta}_{\Delta}}{\mu_n^2} \\
=\ &\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\left(\frac{z'\left[I_n-D_P\right]z}{n}\right)\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n} \\
&-\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\left(\frac{z'\left[I_n-P\right]z}{n}\right)\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n} \\
&+\frac{\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)'\overline{V}'\left[P-D_P\right]z}{\mu_n\sqrt{n}}\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}+\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\frac{z'\left[P-D_P\right]\overline{V}\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)}{\mu_n\sqrt{n}} \\
&+\frac{\delta_{\Delta,0}'\overline{V}'\left[P-D_P\right]z}{\mu_n\sqrt{n}}\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}+\frac{\left(\delta_0-\widetilde{\delta}_{HLIM}\right)'\widetilde{S}_n D_\mu}{\mu_n}\frac{z'\left[P-D_P\right]\overline{V}\delta_{\Delta,0}}{\mu_n\sqrt{n}} \\
&+\frac{\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)'\overline{V}'\left[P-D_P\right]\overline{V}\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)}{\mu_n^2}+\frac{\delta_{\Delta,0}'\overline{V}'\left[P-D_P\right]\overline{V}\delta_{\Delta,0}}{\mu_n^2} \\
&+\frac{\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)'\overline{V}'\left[P-D_P\right]\overline{V}\delta_{\Delta,0}}{\mu_n^2}+\frac{\delta_{\Delta,0}'\overline{V}'\left[P-D_P\right]\overline{V}\left(\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right)}{\mu_n^2} \\
\geq\ &\left(\lambda_{\min}\left(H_n\right)-\lambda_{\max}\left(\frac{z'Mz}{n}\right)\right)\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2 \\
&-2\left\|\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right\|\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\| \\
&-2\left\|\delta_{\Delta,0}\right\|\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\| \\
&-\left\|\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right\|^2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|-\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|\left\|\delta_{\Delta,0}\right\|^2 \\
&-2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|\left\|\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right\|\left\|\delta_{\Delta,0}\right\|
\end{aligned}
$$

where the inequality above follows from using the result of Exercise 12.40 part (a) in Abadir and Magnus (2005) and from applying T, parts (a) and (b) of Lemma B5, and the submultiplicativity of the Euclidean norm. Next, note that

$$\begin{aligned}
\left\|\widetilde{\delta}_{\Delta}-\delta_{\Delta,0}\right\| &= \left\|\widetilde{\delta}_{HLIM}-\delta_{0}\right\| \\
&\leq \left\|\widetilde{S}_{n}^{-1}\right\| \left\|\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)\right\| \\
&\leq \sqrt{\frac{G}{\lambda_{\min}\left(\widetilde{S}_{n}'\widetilde{S}_{n}\right)}} \left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\| \\
&\leq \sqrt{C_{2}} \left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\|
\end{aligned}$$

where the second inequality follows from the fact that $\mu_n \leq \mu_{jn}$ for $j = 1, ..., G$. Applying this bound and also the inequality in part (b) of Lemma C1, we obtain

$$\begin{aligned}
&\frac{\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_{P}\right]\overline{X}\widetilde{\delta}_{\Delta}}{\mu_{n}^{2}} \\
\geq\ & \left(\lambda_{\min}\left(H_{n}\right)-\frac{1}{n}\sum_{i=1}^{n}\left\|z_{i}-\pi_{Kn}Z_{i}\right\|^{2}\right)\left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\|^{2} \\
&-2\left\|\frac{z'\left[P-D_{P}\right]\overline{V}}{\mu_{n}\sqrt{n}}\right\|\sqrt{C_{2}}\left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\|^{2}-2\left\|\delta_{\Delta,0}\right\|\left\|\frac{z'\left[P-D_{P}\right]\overline{V}}{\mu_{n}\sqrt{n}}\right\|\left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\| \\
&-C_{2}\left\|\frac{\overline{V}'\left[P-D_{P}\right]\overline{V}}{\mu_{n}^{2}}\right\|\left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\|^{2}-2\left\|\frac{\overline{V}'\left[P-D_{P}\right]\overline{V}}{\mu_{n}^{2}}\right\|\left\|\delta_{\Delta,0}\right\|\sqrt{C_{2}}\left\|\frac{D_{\mu}\widetilde{S}_{n}'\left(\delta_{0}-\widetilde{\delta}_{HLIM}\right)}{\mu_{n}}\right\| \\
&-\left\|\frac{\overline{V}'\left[P-D_{P}\right]\overline{V}}{\mu_{n}^{2}}\right\|\left\|\delta_{\Delta,0}\right\|^{2}.
\end{aligned}$$

Now, Assumption 7 implies that there exists a positive constant C_4 such that $\left\|\delta_{\Delta,0}\right\|^2 \leq G + \left\|\delta_0\right\|^2 \leq C_4 < \infty$. Moreover, by Assumption 2, there exists a constant C_* such that $\lambda_{\min}\left(z'z/n\right) \geq C_* > 0$ for n sufficiently large. Hence, together with Assumption 7, this implies that, for n sufficiently large,

$$\begin{aligned}
\lambda_{\min}\left(H_{n}\right) &= \lambda_{\min}\left(\frac{z'\left[I_{n}-D_{P}\right]z}{n}\right) \\
&\geq \left(1-C_{P}\frac{K}{n}\right)\lambda_{\min}\left(\frac{z'z}{n}\right) \\
&\geq \left(1-C_{P}\frac{K}{n}\right)C_{*} \ := C_{5}>0 \qquad (35)
\end{aligned}$$

In addition, Assumption 4 implies that for n sufficiently large

$$\frac{1}{n}\sum_{i=1}^{n}\left\|z_{i}-\pi_{Kn}Z_{i}\right\|^{2} \leq \frac{C_{5}}{6}. \qquad (36)$$

It follows that

$$
\begin{aligned}
&\frac{\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_{\Delta}}{\mu_n^2}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
\geq\ & \Bigg\{\left(\lambda_{\min}\left(H_n\right)-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2\right)\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2 \\
&-2\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|\sqrt{C_2}\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2-2\left\|\delta_{\Delta,0}\right\|\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\| \\
&-C_2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2-2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|\left\|\delta_{\Delta,0}\right\|\sqrt{C_2}\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\| \\
&-\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|\left\|\delta_{\Delta,0}\right\|^2\Bigg\}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
\geq\ & \left(\lambda_{\min}\left(H_n\right)-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2-2\eta_1\left[\sqrt{C_2}+\sqrt{C_4}\right]-\eta_2\left[\sqrt{C_2}+\sqrt{C_4}\right]^2\right) \\
&\quad\times\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
\geq\ & \left(C_5-\frac{C_5}{6}-\frac{2C_5\left[\sqrt{C_2}+\sqrt{C_4}\right]}{24\left[\sqrt{C_2}+\sqrt{C_4}\right]}-\frac{C_5\left[\sqrt{C_2}+\sqrt{C_4}\right]^2}{6\left[\sqrt{C_2}+\sqrt{C_4}\right]^2}\right)\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
&\text{[using expressions (2)-(5) and (35)-(36) above]} \\
\geq\ & \left(C_5-\frac{C_5}{6}-\frac{2C_5}{12}-\frac{C_5}{6}\right)\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
=\ & \frac{C_5}{2}\left\|\frac{D_\mu\widetilde{S}_n'\left(\delta_0-\widetilde{\delta}_{HLIM}\right)}{\mu_n}\right\|^2\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
\geq\ & 0,
\end{aligned}
$$

so that $\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}$ is non-negative for sufficiently large n. Combined with the fact that $\widetilde{\delta}_{\Delta}'\overline{X}'\left[M+D_P\right]\overline{X}\widetilde{\delta}_{\Delta}$ is positive with probability one for sufficiently large n in light of Lemma B13 part (a), we deduce that $\widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}$ is non-negative, from which it follows that

$$
\begin{aligned}
0 &\leq \widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
&= \left(\frac{\mu_n^2}{n}\right)\frac{\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_{\Delta}/\mu_n^2}{\widetilde{\delta}_{\Delta}'\overline{X}'\left[M+D_P\right]\overline{X}\widetilde{\delta}_{\Delta}/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
&\leq \left(\frac{\mu_n^2}{n}\right)\frac{\delta_{\Delta,0}'\overline{X}'\left[P-D_P\right]\overline{X}\delta_{\Delta,0}/\mu_n^2}{\delta_{\Delta,0}'\overline{X}'\left[M+D_P\right]\overline{X}\delta_{\Delta,0}/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \\
&= \left(\frac{\mu_n^2}{n}\right)\frac{\varepsilon'\left[P-D_P\right]\varepsilon/\mu_n^2}{\varepsilon'\left[M+D_P\right]\varepsilon/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}.
\end{aligned}
$$

To derive an upper bound for $\widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}$, we first note that

$$\begin{aligned}\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\mu_n^2}\right| &= \left|\frac{\delta_{\Delta,0}'\overline{V}'\left[P-D_P\right]\overline{V}\delta_{\Delta,0}}{\mu_n^2}\right| \\ &\leq \left\|\delta_{\Delta,0}\right\|^2\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\| \\ &\leq C_4\eta_2\end{aligned}$$

and that, by Assumption 8, there exists a constant C_6 such that

$$\begin{aligned}\left|\frac{\varepsilon'\left[M+D_P\right]\varepsilon}{n}\right| &\geq \left|\frac{E\left\{\varepsilon'\left[M+D_P\right]\varepsilon\right\}}{n}\right| - \left|\frac{\varepsilon'\left[M+D_P\right]\varepsilon - E\left\{\varepsilon'\left[M+D_P\right]\varepsilon\right\}}{n}\right| \\ &= \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 - \left|\frac{\varepsilon'\left[M+D_P\right]\varepsilon}{n} - \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right| \\ &= \frac{1}{n}\sum_{i=1}^{n}\delta_{\Delta,0}'\Xi_i\delta_{\Delta,0} - \left|\frac{\delta_{\Delta,0}'\overline{V}'\left[M+D_P\right]\overline{V}\delta_{\Delta,0}}{n} - \frac{1}{n}\sum_{i=1}^{n}\delta_{\Delta,0}'\Xi_i\delta_{\Delta,0}\right| \\ &\geq \left\|\delta_{\Delta,0}\right\|^2\frac{1}{n}\sum_{i=1}^{n}\lambda_{\min}\left(\Xi_i\right) - \left\|\delta_{\Delta,0}\right\|^2\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n} - \frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\| \\ &\geq C_6 - C_4\eta_3,\end{aligned}$$

for all n sufficiently large, from which it follows that

$$0 \leq \widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \leq \left(\frac{\mu_n^2}{n}\right)\frac{\varepsilon'\left[P-D_P\right]\varepsilon/\mu_n^2}{\varepsilon'\left[M+D_P\right]\varepsilon/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \leq \left(\frac{\mu_n^2}{n}\right)\frac{C_4\eta_2}{C_6-C_4\eta_3}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \leq \left(\frac{\mu_n^2}{n}\right)\frac{2C_4\eta_2}{C_6}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}} \tag{37}$$

Further analysis of the lower bound given by (33) using (37) yields

$$\begin{aligned}&\min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X - \left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right| \\ \geq\ & \lambda_{\min}\left\{H_n\right\} - \frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2 - 2\eta_1\sqrt{C_2} - C_2\eta_2 - \left(\frac{\mu_n^2}{n}\frac{2C_4\eta_2}{C_6}+\frac{C}{n}\right)C_1 \\ & -2\left(\frac{\mu_n^2}{n}\frac{2C_4\eta_2}{C_6}+\frac{C}{n}\right)\eta_4\sqrt{C_2} - \left(\frac{\mu_n^2}{n}\frac{2C_4\eta_2}{C_6}+\frac{C}{n}\right)C_2\left(\frac{n}{\mu_n^2}\right)\eta_3 - \left(\frac{\mu_n^2}{n}\frac{2C_4\eta_2}{C_6}+\frac{C}{n}\right)\left(\frac{n}{\mu_n^2}\right)C_2C_3 \\ \geq\ & \lambda_{\min}\left\{H_n\right\} - \frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2 - 2\eta_1\sqrt{C_2} - 4\frac{\sqrt{C_2}C_4}{C_6}\eta_4\eta_2 \\ & -\left(C_2+\frac{2C_1C_4}{C_6}+\frac{2C_2C_4}{C_6}\eta_3+\frac{2C_2C_3C_4}{C_6}\right)\eta_2 - C\left(\frac{C_1}{n}+\frac{2\sqrt{C_2}}{n}\eta_4+\frac{C_2}{\mu_n^2}\eta_3+\frac{C_2C_3}{\mu_n^2}\right) \\ \geq\ & C_5 - \frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2 - \frac{2C_5\sqrt{C_2}}{24\left(\sqrt{C_2}+\sqrt{C_4}\right)} - 4\frac{\sqrt{C_2}C_4}{C_6}\frac{C_6}{8\sqrt{GC_4C_7}}\frac{C_5\sqrt{GC_7}}{12\sqrt{C_2C_4}} \\ & -\left(C_2+\frac{2C_1C_4}{C_6}+\frac{2C_2C_4}{C_6}\frac{C_6}{2C_4}+\frac{2C_2C_3C_4}{C_6}\right)\frac{C_5C_6}{48\left[C_2\left(C_6+C_3C_4\right)+C_1C_4\right]} \\ & -C\left(\frac{C_1}{n}+\frac{2\sqrt{C_2}}{n}\frac{C_6}{8\sqrt{GC_4C_7}}+\frac{C_2}{\mu_n^2}\frac{C_6}{2C_4}+\frac{C_2C_3}{\mu_n^2}\right) \\ \geq\ & C_5 - \frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2 - \frac{C_5}{12} - \frac{C_5}{24} - \frac{C_5}{24} - \frac{C}{n}\left(C_1+\frac{C_6\sqrt{C_2}}{4\sqrt{GC_4C_7}}\right) - \frac{C}{\mu_n^2}\left(\frac{C_2C_6+2C_2C_3C_4}{2C_4}\right)\end{aligned} \tag{38}$$

where the third inequality above follows from making use of (2)-(5) and the lower bound given by (35). Now, let n be sufficiently large so that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\|z_i-\pi_{Kn}Z_i\|^2 &\leq \frac{C_5}{6},\ \frac{C}{n}\left(C_1+\frac{C_6\sqrt{C_2}}{4\sqrt{GC_4C_7}}\right)\leq\frac{C_5}{12},\\ \frac{C}{\mu_n^2}\left(\frac{C_2C_6+2C_2C_3C_4}{2C_4}\right) &= \frac{C}{n^b}\left(\frac{C_2C_6+2C_2C_3C_4}{2C_4}\right)\leq\frac{C_5}{12},\end{aligned}\tag{39}$$

and we have

$$\begin{aligned}&\min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right|\\ \geq\ & C_5-\frac{C_5}{6}-\frac{C_5}{12}-\frac{C_5}{24}-\frac{C_5}{24}-\frac{C_5}{12}-\frac{C_5}{12}=\frac{C_5}{2},\end{aligned}\tag{40}$$

from which it follows by part (c) of Lemma B5 that

$$\begin{aligned}&\left\|\left(S_n^{-1}X'\left(P-D_P\right)XS_n'^{-1}-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}S_n^{-1}X'\left(M+D_P\right)XS_n'^{-1}\right)^{-1}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}\\ \leq\ & \frac{\sqrt{G}}{\min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right|}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}\\ \leq\ & \frac{2\sqrt{G}}{C_5}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}.\end{aligned}\tag{41}$$

Next, we consider the second term on the right-hand side of (25). In this case, since $\widetilde{\kappa}_{HLIM}\geq 0$ under event $\mathcal{C}$, we have that

$$\begin{aligned}0 &\leq \widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\\ &= \left(\frac{\mu_n^2}{n}\right)\frac{\widetilde{\delta}_{\Delta}'\overline{X}'\left[P-D_P\right]\overline{X}\widetilde{\delta}_{\Delta}/\mu_n^2}{\widetilde{\delta}_{\Delta}'\overline{X}'\left[M+D_P\right]\overline{X}\widetilde{\delta}_{\Delta}/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\\ &\leq \left(\frac{\mu_n^2}{n}\right)\frac{\delta_{\Delta,0}'\overline{X}'\left[P-D_P\right]\overline{X}\delta_{\Delta,0}/\mu_n^2}{\delta_{\Delta,0}'\overline{X}'\left[M+D_P\right]\overline{X}\delta_{\Delta,0}/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\\ &= \left(\frac{\mu_n^2}{n}\right)\frac{\varepsilon'\left[P-D_P\right]\varepsilon/\mu_n^2}{\varepsilon'\left[M+D_P\right]\varepsilon/n}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}.\end{aligned}$$

It follows by the same argument as (37) that

$$\widetilde{\kappa}_{HLIM}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\leq\left(\frac{\mu_n^2}{n}\right)\frac{2C_4\eta_2}{C_6}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\tag{42}$$

Moreover, following the same argument as that used to derive (33), (38), and (40) above, we have that conditional on $\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}$

$$
\begin{aligned}
& \min_{1 \leq \ g \ \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' \left(P - D_P \right) X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' \left(M + D_P \right) X \right] S_n'^{-1} \right) \right| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2\sqrt{C_2} \left\| \frac{z' \left[P - D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| - C_2 \left\| \frac{\overline{V}' \left[P - D_P \right] \overline{V}}{\mu_n^2} \right\| \\
& - \left(\widetilde{\kappa}_{HLIM} + \frac{C}{n} \right) C_1 - 2 \left(\widetilde{\kappa}_{HLIM} + \frac{C}{n} \right) \sqrt{C_2} \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
& - \left(\widetilde{\kappa}_{HLIM} + \frac{C}{n} \right) C_2 \left(\frac{n}{\mu_n^2} \right) \left\| \frac{\overline{V}' \left[M + D_P \right] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| - \left(\widetilde{\kappa}_{HLIM} + \frac{C}{n} \right) \left(\frac{n}{\mu_n^2} \right) C_2 C_3 \\
\geq \quad & C_5 - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - \frac{C_5}{12} - \frac{C_5}{24} - \frac{C_5}{24} - \frac{C}{n} \left(C_1 + \frac{C_6 \sqrt{C_2}}{4\sqrt{G C_4 C_7}} \right) - \frac{C}{\mu_n^2} \left(\frac{C_2 C_6 + 2 C_2 C_3 C_4}{2 C_4} \right) \\
\geq \quad & \frac{C_5}{2}
\end{aligned}
$$

for n sufficiently large so that (39) holds. As before, it follows from applying part (c) of Lemma B5 that

$$
\begin{aligned}
& \left\| \left(S_n^{-1} X' \left(P - D_P \right) X S_n'^{-1} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} S_n^{-1} X' \left(M + D_P \right) X S_n'^{-1} \right)^{-1} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \\
\leq \quad & \frac{\sqrt{G}}{\min_{1 \leq \ g \ \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' \left(P - D_P \right) X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' \left(M + D_P \right) X \right] S_n'^{-1} \right) \right|} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \\
\leq \quad & \frac{2\sqrt{G}}{C_5} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \qquad (43)
\end{aligned}
$$

Finally, we consider the third term on the right-hand side of (25).We start by deriving an upper bound for $\left| \widetilde{\kappa}_{HLIM} \right| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C}$. Since $\widetilde{\kappa}_{HLIM} < 0$ under event $\mathcal{C}^C$ and since the denominator term $\widehat{\delta}_{\Delta}' \overline{X}' \left[M + D_P \right] \overline{X} \widehat{\delta}_{\Delta}$ is positive almost surely for all n sufficiently large, we must have

$$\begin{aligned}
0 &\geq \frac{\widetilde{\delta}_\Delta' \overline{X}' \left[P - D_P\right] \overline{X} \widehat{\delta}_\Delta}{\mu_n^2} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
&\geq \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \left(\left\{ \lambda_{\min}\left(H_n\right) - \frac{1}{n} \sum_{i=1}^n \left\| z_i - \pi_{Kn} Z_i \right\|^2 \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \right. \\
&\quad -2 \left\| \frac{z' \left[P - D_P\right] \overline{V}}{\mu_n \sqrt{n}} \right\| \left\| \widetilde{S}_n^{-1} \right\| \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 - 2 \left\| \delta_{\Delta,0} \right\| \left\| \frac{z' \left[P - D_P\right] \overline{V}}{\mu_n \sqrt{n}} \right\| \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\| \\
&\quad - \left\| \frac{\overline{V}' \left[P - D_P\right] \overline{V}}{\mu_n^2} \right\| \left\| \widetilde{S}_n^{-1} \right\|^2 \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \\
&\quad \left. -2 \left\| \frac{\overline{V}' \left[P - D_P\right] \overline{V}}{\mu_n^2} \right\| \left\| \delta_{\Delta,0} \right\| \left\| \widetilde{S}_n^{-1} \right\| \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\| - \left\| \frac{\overline{V}' \left[P - D_P\right] \overline{V}}{\mu_n^2} \right\| \left\| \delta_{\Delta,0} \right\|^2 \right) \\
&\geq \left\{ \lambda_{\min}\left(H_n\right) - \frac{1}{n} \sum_{i=1}^n \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2\eta_1 \sqrt{C_2} - \eta_2 C_2 \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
&\quad -2\sqrt{C_4} \left\{ \eta_1 + \sqrt{C_2} \eta_2 \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} - C_4 \eta_2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C}
\end{aligned}$$

Making use of (2), (3), and (35) and letting n be sufficiently large so that $n^{-1} \sum_{i=1}^n \left\| z_i - \pi_{Kn} Z_i \right\|^2 \leq C_5/6$, we obtain

$$\begin{aligned}
&\left\{ \lambda_{\min}\left(H_n\right) - \frac{1}{n} \sum_{i=1}^n \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2\eta_1 \sqrt{C_2} - \eta_2 C_2 \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\geq &\left\{ C_5 - \frac{C_5}{6} - \frac{2C_5\sqrt{C_2}}{24\left(\sqrt{C_2} + \sqrt{C_4}\right)} - \frac{C_5 C_2}{6\left[\sqrt{C_2} + \sqrt{C_4}\right]^2} \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\geq &\left\{ C_5 - \frac{C_5}{6} - \frac{C_5}{12} - \frac{C_5}{6} \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\geq &\frac{C_5}{2} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\|^2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \geq 0
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &\geq \frac{\widetilde{\delta}_\Delta' \overline{X}' \left[P - D_P\right] \overline{X} \widetilde{\delta}_\Delta}{\mu_n^2} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
&\geq -2\sqrt{C_4} \left\{ \eta_1 + \sqrt{C_2} \eta_2 \right\} \left\| \frac{D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right)}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} - C_4 \eta_2 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
&\geq -\left\{ 2\sqrt{C_4}\eta_1 + \left(2\sqrt{C_2 C_4} + C_4 \right) \eta_2 \right\} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C}, \qquad (44)
\end{aligned}$$

where the last inequality follows from the fact that, under event $\mathcal{B}^C$, $\left\| D_\mu \widetilde{S}_n' \left(\delta_0 - \widetilde{\delta}_{HLIM} \right) / \mu_n \right\| < 1$. Now, (44) implies that

$$\left| \frac{\widetilde{\delta}_\Delta' \overline{X}' \left[P - D_P \right] \overline{X} \widetilde{\delta}_\Delta}{\mu_n^2} \right| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \leq \left\{ 2\sqrt{C_4}\eta_1 + \left(2\sqrt{C_2 C_4} + C_4 \right) \eta_2 \right\} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \tag{45}$$

Turning our attention now to the denominator term of $\widetilde{\kappa}_{HLIM}$, let $\Delta = \left[\begin{array}{cc} \delta_0 & I_G \end{array} \right]$, we can write

$$\begin{aligned}
& \frac{\overline{X}' \left[M + D_P \right] \overline{X}}{n} \\
= \quad & \frac{\Delta' \Upsilon' \left[M + D_P \right] \Upsilon \Delta}{n} + \frac{\Delta' \Upsilon' \left[M + D_P \right] \overline{V}}{n} + \frac{\overline{V}' \left[M + D_P \right] \Upsilon \Delta}{n} + \frac{\overline{V}' \left[M + D_P \right] \overline{V}}{n} \\
= \quad & \frac{\Delta' S_n z' \left[M + D_P \right] z S_n' \Delta}{n^2} + \frac{\Delta' S_n z' \left[M + D_P \right] \overline{V}}{n^{3/2}} + \frac{\overline{V}' \left[M + D_P \right] z S_n' \Delta}{n^{3/2}} \\
& + \left(\frac{\overline{V}' \left[M + D_P \right] \overline{V} - E\left[\overline{V}' \left[M + D_P \right] \overline{V} \right]}{n} \right) + \frac{E\left[\overline{V}' \left[M + D_P \right] \overline{V} \right]}{n}
\end{aligned}$$

so that

$$\begin{aligned}
& \lambda_{\min} \left(\frac{\overline{X}' \left[M + D_P \right] \overline{X}}{n} \right) \\
\geq \quad & \lambda_{\min} \left(\frac{E\left[\overline{V}' \left[M + D_P \right] \overline{V} \right]}{n} + \frac{\Delta' S_n z' \left[M + D_P \right] z S_n' \Delta}{n^2} \right) \\
& - \left\| \frac{\left(\overline{V}' \left[M + D_P \right] \overline{V} - E\left[\overline{V}' \left[M + D_P \right] \overline{V} \right] \right)}{n} \right\| - 2 \left\| \frac{\Delta' S_n z' \left[M + D_P \right] \overline{V}}{n^{3/2}} \right\|
\end{aligned}$$

Next, note that, by Assumption 8,

$$\begin{aligned}
& \lambda_{\min} \left(\frac{E\left[\overline{V}' \left[M + D_P \right] \overline{V} \right]}{n} + \frac{\Delta' S_n z' \left[M + D_P \right] z S_n' \Delta}{n^2} \right) \\
\geq \quad & \lambda_{\min} \left(\frac{1}{n} \sum_{i=1}^{n} \Xi_i \right) + \lambda_{\min} \left(\frac{\Delta' S_n z' \left[M + D_P \right] z S_n' \Delta}{n^2} \right) \\
= \quad & \lambda_{\min} \left(\frac{1}{n} \sum_{i=1}^{n} \Xi_i \right) \geq C_6 > 0
\end{aligned} \tag{46}$$

for all n sufficiently large. Moreover, by Assumption 7 part (e), there exists a constant C_7 such that $\lambda_{\max} \left(\widetilde{S}_n' \widetilde{S}_n \right) \leq$

C_7 for all n sufficiently large, so that

$$\begin{aligned}
& \left\| \frac{\Delta' S_n z' \left[M + D_P \right] \overline{V}}{n^{3/2}} \right\| \\
\leq \quad & \left\| \frac{\mu_n \Delta' \widetilde{S}_n D_\mu}{n} \right\| \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
= \quad & \sqrt{tr \left\{ \frac{\mu_n^2 D_\mu \widetilde{S}_n' \Delta \Delta' \widetilde{S}_n D_\mu}{n^2} \right\}} \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
\leq \quad & \left(\frac{\mu_n}{\sqrt{n}} \right) \sqrt{tr \left(\Delta \Delta' \right) \lambda_{\max} \left(\widetilde{S}_n' \widetilde{S}_n \right) tr \left(D_\mu' D_\mu / n \right)} \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
= \quad & \left(\frac{\mu_n}{\sqrt{n}} \right) \left\| n^{-1/2} D_\mu \right\| \sqrt{\left(G + \left\| \delta_0 \right\|^2 \right) \lambda_{\max} \left(\widetilde{S}_n' \widetilde{S}_n \right)} \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
\leq \quad & \sqrt{G C_4 C_7} \eta_4, \qquad (47)
\end{aligned}$$

where last inequality follows in part from the fact that $\left\| n^{-1/2} D_\mu \right\| \leq \sqrt{G}$ since $\mu_{jn} \leq \sqrt{n}$ $(j = 1, .., G)$ for all n sufficiently large. Conditioning on $\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C$, it follows from (46) and (47) that

$$\lambda_{\min} \left(\frac{\overline{X}' \left[M + D_P \right] \overline{X}}{n} \right) \geq C_6 - \eta_3 - 2\sqrt{G C_4 C_7} \eta_4 \qquad (48)$$

for all n sufficiently large. (45) and (48) together imply that

$$\begin{aligned}
\left| \widetilde{\kappa}_{HLIM} \right| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \quad = \quad & \left(\frac{\mu_n^2}{n} \right) \frac{\left| \widetilde{\delta}_\Delta' \overline{X}' \left[P - D_P \right] \overline{X} \widetilde{\delta}_\Delta / \mu_n^2 \right|}{\widetilde{\delta}_\Delta' \overline{X}' \left[M + D_P \right] \overline{X} \widetilde{\delta}_\Delta / n} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq \quad & \left(\frac{\mu_n^2}{n} \right) \frac{\left| \widetilde{\delta}_\Delta' \overline{X}' \left[P - D_P \right] \overline{X} \widetilde{\delta}_\Delta / \mu_n^2 \right|}{\lambda_{\min} \left(\overline{X}' \left[M + D_P \right] \overline{X} / n \right)} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq \quad & \left(\frac{\mu_n^2}{n} \right) \frac{2\sqrt{C_4} \eta_1 + \left(2\sqrt{C_2 C_4} + C_4 \right) \eta_2}{C_6 - \eta_3 - 2\sqrt{G C_4 C_7} \eta_4} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq \quad & \left(\frac{\mu_n^2}{n} \right) \frac{2 \left[2\sqrt{C_4} \eta_1 + \left(2\sqrt{C_2 C_4} + C_4 \right) \eta_2 \right]}{C_6} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C}, \qquad (49)
\end{aligned}$$

where the last inequality follows from (4) and (5) which imply that

$$\eta_3 \leq \frac{C_6}{4} \text{ and } \eta_4 \leq \frac{C_6}{8\sqrt{G C_4 C_7}}.$$

Using the result given by (49), we have, conditioning on $\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C$,

$$
\begin{aligned}
& \min_{1 \leq\ g\ \leq G} \left| \lambda_g \left(S_n^{-1} \left[X' \left(P - D_P \right) X - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} X' \left(M + D_P \right) X \right] S_n'^{-1} \right) \right| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \widetilde{\kappa}_{HLIM} \lambda_{\min} \left(\frac{z' \left[M + D_P \right] z}{n} \right) - \widetilde{\kappa}_{HLIM} \lambda_{\min} \left(S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right) - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 \\
& -2 \left\| \frac{z' \left[P - D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \left\| S_n^{-1} U' \left[P - D_P \right] U S_n'^{-1} \right\| - 2 \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
& - \frac{C}{n} \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| - \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| - 2 \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2 \left\| \frac{z' \left[P - D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \left\| S_n^{-1} U' \left[P - D_P \right] U S_n'^{-1} \right\| \\
& -2 \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| - \frac{C}{n} \left\| S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right\| - \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| \\
& -2 \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] U S_n'^{-1}}{\sqrt{n}} \right\| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2 \sqrt{C_2} \left\| \frac{z' \left[P - D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| - C_2 \left\| \frac{\overline{V}' \left[P - D_P \right] \overline{V}}{\mu_n^2} \right\| \\
& -2 \sqrt{C_2} \left| \widetilde{\kappa}_{HLIM} \right| \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| - C_2 \left(\frac{C}{n} \right) \left(\frac{n}{\mu_n} \right) \left\| \frac{\overline{V}' \left[M + D_P \right] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \\
& -C_2 \left(\frac{C}{n} \right) \left(\frac{n}{\mu_n} \right) \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| - \frac{C}{n} \left\| \frac{z' \left[M + D_P \right] z}{n} \right\| - 2 \sqrt{C_2} \left(\frac{C}{n} \right) \left\| \frac{z' \left[M + D_P \right] \overline{V}}{\mu_n \sqrt{n}} \right\| \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2 \eta_1 \sqrt{C_2} - C_2 \eta_2 - \left(\frac{\mu_n^2}{n} \right) \frac{4 \sqrt{C_2} \left[2 \sqrt{C_4} \eta_1 + \left(2 \sqrt{C_2 C_4} + C_4 \right) \eta_2 \right]}{C_6} \eta_4 \\
& -C_2 \left(\frac{C}{\mu_n} \right) \eta_3 - \left(\frac{C}{\mu_n} \right) C_2 C_3 - \frac{C}{n} C_1 - 2 \sqrt{C_2} \left(\frac{C}{n} \right) \eta_4 \\
\geq \quad & \lambda_{\min} \left\{ H_n \right\} - \frac{1}{n} \sum_{i=1}^{n} \left\| z_i - \pi_{Kn} Z_i \right\|^2 - 2 \sqrt{C_2} \left(1 + \frac{4 \sqrt{C_4}}{C_6} \eta_4 \right) \eta_1 \\
& - \left(C_2 + \frac{4 \sqrt{C_2} \left(2 \sqrt{C_2 C_4} + C_4 \right)}{C_6} \eta_4 \right) \eta_2 - C_2 \left(\frac{C}{\mu_n} \right) \eta_3 - \left(\frac{C}{\mu_n} \right) C_2 C_3 - \frac{C}{n} C_1 - 2 \sqrt{C_2} \left(\frac{C}{n} \right) \eta_4 \qquad (50)
\end{aligned}
$$

where the second inequality holds because both $-\widetilde{\kappa}_{HLIM} \lambda_{\min} \left(z' \left[M + D_P \right] z / n \right)$ and $-\widetilde{\kappa}_{HLIM} \lambda_{\min} \left(S_n^{-1} U' \left[M + D_P \right] U S_n'^{-1} \right)$ are non-negative conditional on the event $\left\{ \widetilde{\kappa}_{HLIM} < 0 \right\}$. Making use of

(2)-(5), and (35), we can further bound expression (50) by

$$
\begin{aligned}
& \min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right| \\
\geq\quad & C_5-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2-2\sqrt{C_2}\left(1+\frac{4\sqrt{C_4}}{C_6}\frac{C_6}{8\sqrt{GC_4C_7}}\right)\frac{C_5\sqrt{GC_7}}{12\sqrt{C_2}\left(2\sqrt{GC_7}+1\right)} \\
& -\left(C_2+\frac{4\sqrt{C_2}\left(2\sqrt{C_2C_4}+C_4\right)}{C_6}\frac{C_6}{8\sqrt{GC_4C_7}}\right)\frac{C_5\sqrt{GC_7}}{12C_2\sqrt{GC_7}+12C_2+6\sqrt{C_2C_4}} \\
& -C_2\left(\frac{C}{\mu_n}\right)\frac{C_6}{2C_4}-\left(\frac{C}{\mu_n}\right)C_2C_3-\frac{C}{n}C_1-2\sqrt{C_2}\left(\frac{C}{n}\right)\frac{C_6}{8\sqrt{GC_4C_7}} \\
=\quad & C_5-\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2-\frac{C_5}{12}-\frac{C_5}{12}-\frac{C}{n}\left(C_1+\frac{C_6\sqrt{C_2}}{4\sqrt{GC_4C_7}}\right) \\
& -\frac{C}{\mu_n}\left(\frac{C_2C_6+2C_2C_3C_4}{2C_4}\right)
\end{aligned}
$$

Next, let n be sufficiently large so that (39) holds, and we have

$$
\begin{aligned}
& \min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right| \\
\geq\quad & C_5-\frac{C_5}{6}-\frac{C_5}{12}-\frac{C_5}{12}-\frac{C_5}{12}-\frac{C_5}{12}=\frac{C_5}{2},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
& \left\|\left(S_n^{-1}X'\left(P-D_P\right)XS_n'^{-1}-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}S_n^{-1}X'\left(M+D_P\right)XS_n'^{-1}\right)^{-1}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C} \\
\leq\quad & \frac{\sqrt{G}}{\min_{1\leq\ g\ \leq G}\left|\lambda_g\left(S_n^{-1}\left[X'\left(P-D_P\right)X-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left(M+D_P\right)X\right]S_n'^{-1}\right)\right|}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C} \\
\leq\quad & \frac{2\sqrt{G}}{C_5}\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}. \qquad (51)
\end{aligned}
$$

(41), (43), and (51) immediately imply the desired conclusion. *Q.E.D.*

Lemma C3: Suppose that Assumptions 1-4 and 7 hold. In addition, if Assumptions 8 and 9 are satisfied; then, for n sufficiently large, there exists a positive constant C_8 such that

$$
\left\|\frac{S_n^{-1}X'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widehat{\kappa}_{HLIM}-\frac{C}{n}\right\}\frac{S_n^{-1}X'\left(M+D_P\right)\varepsilon}{\mu_n}\right\|\mathbb{I}_{\mathcal{A}}\leq C_8\mathbb{I}_{\mathcal{A}}.
$$

Proof: To proceed, first write

$$
\begin{aligned}
& \frac{S_n^{-1}X'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}\frac{S_n^{-1}X'\left(M+D_P\right)\varepsilon}{\mu_n} \\
=\quad & \frac{z'\left(P-D_P\right)\varepsilon}{\sqrt{n}\mu_n}+\frac{S_n^{-1}U'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}\left(\frac{z'\left(M+D_P\right)\varepsilon}{\sqrt{n}\mu_n}+\frac{S_n^{-1}U'\left(M+D_P\right)\varepsilon}{\mu_n}\right) \\
=\quad & \frac{z'\left(P-D_P\right)\overline{V}\delta_{\Delta,0}}{\sqrt{n}\mu_n}+\frac{S_n^{-1}F_2'\overline{V}'\left(P-D_P\right)\overline{V}\delta_{\Delta,0}}{\mu_n} \\
& -\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}\left(\frac{z'\left(M+D_P\right)\overline{V}\delta_{\Delta,0}}{\sqrt{n}\mu_n}+\frac{S_n^{-1}F_2'\overline{V}'\left(M+D_P\right)\overline{V}\delta_{\Delta,0}}{\mu_n}\right),
\end{aligned}
$$

so that by T and the submultiplicativity of the Euclidean norm

$$
\begin{aligned}
& \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \\
\leq \quad & \|\delta_{\Delta,0}\| \left\| \frac{z' (P - D_P) \overline{V}}{\mu_n \sqrt{n}} \right\| + \|\delta_{\Delta,0}\| \left\| D_\mu^{-1} \right\| \left\| \widetilde{S}_n^{-1} \right\| \|F_2\| \left\| \frac{\overline{V}' (P - D_P) \overline{V}}{\mu_n} \right\| \\
& + |\widetilde{\kappa}_{HLIM}| \|\delta_{\Delta,0}\| \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| + |\widetilde{\kappa}_{HLIM}| \|\delta_{\Delta,0}\| \left\| D_\mu^{-1} \right\| \left\| \widetilde{S}_n^{-1} \right\| \|F_2\| \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n} \right\| \\
& + \frac{C}{n} \|\delta_{\Delta,0}\| \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| + \frac{C}{n} \|\delta_{\Delta,0}\| \left\| D_\mu^{-1} \right\| \left\| \widetilde{S}_n^{-1} \right\| \|F_2\| \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n} \right\| \\
\leq \quad & \sqrt{C_4} \left\| \frac{z' (P - D_P) \overline{V}}{\mu_n \sqrt{n}} \right\| + \sqrt{G C_2 C_4} \left\| \mu_n D_\mu^{-1} \right\| \left\| \frac{\overline{V}' (P - D_P) \overline{V}}{\mu_n^2} \right\| + |\widetilde{\kappa}_{HLIM}| \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \\
& + |\widetilde{\kappa}_{HLIM}| \left\| \mu_n D_\mu^{-1} \right\| \sqrt{G C_2 C_4} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n^2} \right\| + \frac{C}{n} \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \\
& + \left\| \mu_n D_\mu^{-1} \right\| \sqrt{G C_2 C_4} \frac{C}{n} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n^2} \right\| \\
\leq \quad & \sqrt{C_4} \left\| \frac{z' (P - D_P) \overline{V}}{\mu_n \sqrt{n}} \right\| + G \sqrt{C_2 C_4} \left\| \frac{\overline{V}' (P - D_P) \overline{V}}{\mu_n^2} \right\| + |\widetilde{\kappa}_{HLIM}| \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \\
& + |\widetilde{\kappa}_{HLIM}| G \sqrt{C_2 C_4} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n^2} \right\| + \frac{C}{n} \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \\
& + G \sqrt{C_2 C_4} \frac{C}{n} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n^2} \right\|, \qquad (52)
\end{aligned}
$$

where the last inequality above follows from the fact that $\left\| \mu_n D_\mu^{-1} \right\| \leq \sqrt{G}$ given that $\mu_n \leq \mu_{jn}$ $(j = 1, .., G)$ by Assumption 2. Next, write

$$
\begin{aligned}
& \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A}} \\
= \quad & \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \\
& + \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C}
\end{aligned}
$$

Using (37) and (52), we obtain

$$
\begin{aligned}
& \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
\leq \; & \sqrt{C_4} \left\| \frac{z' (P - D_P) \overline{V}}{\mu_n \sqrt{n}} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + G \sqrt{C_2 C_4} \left\| \frac{\overline{V}' (P - D_P) \overline{V}}{\mu_n^2} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \left(\frac{\mu_n^2}{n} \right) \frac{2 C_4^{3/2}}{C_6} \eta_2 \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + \left(\frac{\mu_n^2}{n} \right) \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \eta_2 \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{\mu_n^2} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \frac{C}{n} \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
\leq \; & \sqrt{C_4} \left\| \frac{z' (P - D_P) \overline{V}}{\mu_n \sqrt{n}} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + G \sqrt{C_2 C_4} \left\| \frac{\overline{V}' (P - D_P) \overline{V}}{\mu_n^2} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \frac{2 C_4^{3/2}}{C_6} \eta_2 \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \eta_2 \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \eta_2 \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + \frac{C}{n} \sqrt{C_4} \left\| \frac{z' (M + D_P) \overline{V}}{\sqrt{n} \mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \left\| \frac{\overline{V}' (M + D_P) \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
\leq \; & \Bigg\{ \sqrt{C_4} \eta_1 + G \sqrt{C_2 C_4} \eta_2 + \frac{2 C_4^{3/2}}{C_6} \eta_2 \eta_4 + \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \eta_2 \eta_3 + \frac{2 G \sqrt{C_2} C_3 C_4^{3/2}}{C_6} \eta_2 \\
& + \frac{C}{n} \sqrt{C_4} \eta_4 + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \eta_3 + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} C_3 \Bigg\} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
= \; & \Bigg\{ \sqrt{C_4} \eta_1 + \left(G \sqrt{C_2 C_4} + \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \eta_3 + \frac{2 G \sqrt{C_2} C_3 C_4^{3/2}}{C_6} \right) \eta_2 + \frac{2 C_4^{3/2}}{C_6} \eta_4 \eta_2 \\
& + \frac{C}{n} \sqrt{C_4} \eta_4 + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \eta_3 + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} C_3 . \Bigg\} \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \qquad (53)
\end{aligned}
$$

Making use of (2)-(5), we can further bound expression (53) as follows

$$
\begin{aligned}
& \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widetilde{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
\leq \; & \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \Bigg\{ \frac{C_8 \sqrt{C_4}}{4 \sqrt{C_4}} + \left(G \sqrt{C_2 C_4} + \frac{2 G \sqrt{C_2} C_4^{3/2}}{C_6} \frac{C_6}{2 C_4} + \frac{2 G \sqrt{C_2} C_3 C_4^{3/2}}{C_6} \right) \frac{C_8 C_6}{16 G \sqrt{C_2 C_4} \left[C_6 + C_3 C_4 \right]} \\
& + \frac{2 C_4^{3/2}}{C_6} \frac{C_6}{8 \sqrt{G C_4 C_7}} \left(\frac{C_8 \sqrt{G C_7}}{2 C_4} \right) + \frac{C}{n} \sqrt{C_4} \frac{C_6}{8 \sqrt{G C_4 C_7}} + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} \frac{C_6}{2 C_4} + G \sqrt{C_2 C_4} \frac{C}{\mu_n^2} C_3 \Bigg\} \\
= \; & \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \left\{ \frac{C_8}{4} + \frac{C_8}{8} + \frac{C_8}{8} + \frac{C}{n} \frac{C_6}{8 \sqrt{G C_7}} + \frac{C}{\mu_n^2} G \sqrt{C_2 C_4} \left(\frac{C_6}{2 C_4} + C_3 \right) \right\} .
\end{aligned}
$$

Now, let n be sufficiently large so that

$$
\frac{C}{n} \frac{C_6}{8 \sqrt{G C_7}} \leq \frac{C_8}{4}, \; \frac{C}{\mu_n^2} G \sqrt{C_2 C_4} \left(\frac{C_6}{2 C_4} + C_3 \right) \leq \frac{C_8}{4}, \qquad (54)
$$

then it follows that

$$\begin{aligned}
&\left\|\frac{S_n^{-1}X'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}\frac{S_n^{-1}X'\left(M+D_P\right)\varepsilon}{\mu_n}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}\\
\leq\;&\left(\frac{C_8}{4}+\frac{C_8}{8}+\frac{C_8}{8}+\frac{C_8}{4}+\frac{C_8}{4}\right)\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}=C_8\mathbb{I}_{\mathcal{A}\cap\mathcal{B}}<\infty
\end{aligned}\tag{55}$$

By similar argument, we also have under (54)

$$\begin{aligned}
&\left\|\frac{S_n^{-1}X'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widehat{\kappa}_{HLIM}-\frac{C}{n}\right\}\frac{S_n^{-1}X'\left(M+D_P\right)\varepsilon}{\mu_n}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\\
\leq\;&\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}\left\{\sqrt{C_4}\eta_1+\left(G\sqrt{C_2C_4}+\frac{2G\sqrt{C_2}C_4^{3/2}}{C_6}\eta_3+\frac{2G\sqrt{C_2}C_3C_4^{3/2}}{C_6}\right)\eta_2\right.\\
&\left.+\frac{2C_4^{3/2}}{C_6}\eta_4\eta_2+\frac{C}{n}\sqrt{C_4}\eta_4+G\sqrt{C_2C_4}\frac{C}{\mu_n^2}\eta_3+G\sqrt{C_2C_4}\frac{C}{\mu_n^2}C_3\right\}\\
\leq\;&C_8\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}}<\infty.
\end{aligned}\tag{56}$$

Moreover, using (49)and (52), we have

$$\begin{aligned}
&\left\|\frac{S_n^{-1}X'\left(P-D_P\right)\varepsilon}{\mu_n}-\left\{\widehat{\kappa}_{HLIM}-\frac{C}{n}\right\}\frac{S_n^{-1}X'\left(M+D_P\right)\varepsilon}{\mu_n}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
\leq\;&\sqrt{C_4}\left\|\frac{z'\left(P-D_P\right)\overline{V}}{\mu_n\sqrt{n}}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}+G\sqrt{C_2C_4}\left\|\frac{\overline{V}'\left(P-D_P\right)\overline{V}}{\mu_n^2}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
&+\left(\frac{\mu_n^2}{n}\right)\frac{2\left[2\sqrt{C_4}\eta_1+\left(2\sqrt{C_2C_4}+C_4\right)\eta_2\right]}{C_6}\sqrt{C_4}\left\|\frac{z'\left(M+D_P\right)\overline{V}}{\sqrt{n}\mu_n}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
&+\frac{2\left[2\sqrt{C_4}\eta_1+\left(2\sqrt{C_2C_4}+C_4\right)\eta_2\right]}{C_6}G\sqrt{C_2C_4}\left\|\frac{\overline{V}'\left(M+D_P\right)\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^n\Xi_i\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
&+\frac{2\left[2\sqrt{C_4}\eta_1+\left(2\sqrt{C_2C_4}+C_4\right)\eta_2\right]}{C_6}G\sqrt{C_2C_4}\left\|\frac{1}{n}\sum_{i=1}^n\Xi_i\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
&+\frac{C}{n}\sqrt{C_4}\left\|\frac{z'\left(M+D_P\right)\overline{V}}{\sqrt{n}\mu_n}\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}+\frac{C}{\mu_n^2}G\sqrt{C_2C_4}\left\|\frac{\overline{V}'\left(M+D_P\right)\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^n\Xi_i\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
&+\frac{C}{\mu_n^2}G\sqrt{C_2C_4}\left\|\frac{1}{n}\sum_{i=1}^n\Xi_i\right\|\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\\
\leq\;&\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\left\{\sqrt{C_4}\eta_1+G\sqrt{C_2C_4}\eta_2+\frac{2\sqrt{C_4}\left[2\sqrt{C_4}\eta_1+\left(2\sqrt{C_2C_4}+C_4\right)\eta_2\right]}{C_6}\eta_4\right.\\
&+\frac{2G\sqrt{C_2C_4}\left[2\sqrt{C_4}\eta_1+\left(2\sqrt{C_2C_4}+C_4\right)\eta_2\right]}{C_6}\left(\eta_3+C_3\right)\\
&\left.+\frac{C}{n}\sqrt{C_4}\eta_4+\frac{C}{\mu_n^2}G\sqrt{C_2C_4}\left(\eta_3+C_3\right)\right\}\\
=\;&\mathbb{I}_{\mathcal{A}\cap\mathcal{B}^C\cap\mathcal{C}^C}\left\{\left(\sqrt{C_4}+\frac{4C_4}{C_6}\eta_4+\frac{4GC_4\sqrt{C_2}}{C_6}\eta_3+\frac{4GC_3C_4\sqrt{C_2}}{C_6}\right)\eta_1\right.\\
&+\left(G\sqrt{C_2C_4}+\frac{2C_4\left[2\sqrt{C_2}+\sqrt{C_4}\right]}{C_6}\eta_4+\frac{2GC_4\sqrt{C_2}\left[2\sqrt{C_2}+\sqrt{C_4}\right]}{C_6}\eta_3\right.\\
&\left.\left.+\frac{2GC_3C_4\sqrt{C_2}\left[2\sqrt{C_2}+\sqrt{C_4}\right]}{C_6}\right)\eta_2+\frac{C}{n}\sqrt{C_4}\eta_4+\frac{C}{\mu_n^2}G\sqrt{C_2C_4}\left(\eta_3+C_3\right)\right\}
\end{aligned}\tag{57}$$

Making use of (2)-(5), we can further bound expression (57) as follows

$$\begin{aligned}
&\left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq\ & \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \left\{ \left(\sqrt{C_4} + \frac{4GC_4\sqrt{C_2}}{C_6}\eta_3 + \frac{4GC_3C_4\sqrt{C_2}}{C_6} \right) \eta_1 + \frac{4C_4}{C_6}\eta_4\eta_1 + \frac{2C_4\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6}\eta_4\eta_2 \right. \\
& + \left(G\sqrt{C_2C_4} + \frac{2GC_4\sqrt{C_2}\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6}\eta_3 + \frac{2GC_3C_4\sqrt{C_2}\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6} \right) \eta_2 \\
& \left. + \frac{C}{n}\sqrt{C_4}\eta_4 + \frac{C}{\mu_n^2} G\sqrt{C_2C_4}\left(\eta_3 + C_3\right) \right\} \\
\leq\ & \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \left\{ \left(\sqrt{C_4} + \frac{4GC_4\sqrt{C_2}}{C_6}\frac{C_6}{2C_4} + \frac{4GC_3C_4\sqrt{C_2}}{C_6} \right) \frac{C_8C_6}{8\left(C_6\sqrt{C_4} + 2G\sqrt{C_2}\left[C_6 + 2C_3C_4\right]\right)} \right. \\
& + \frac{4C_4}{C_6}\frac{C_6}{8\sqrt{GC_4C_7}}\frac{C_8\sqrt{GC_7}}{4\sqrt{C_4}} + \frac{2C_4\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6}\frac{C_6}{8\sqrt{GC_4C_7}}\frac{C_8\sqrt{GC_7}}{2\left[2\sqrt{C_2C_4} + C_4\right]} \\
& + \left(G\sqrt{C_2C_4} + \frac{2GC_4\sqrt{C_2}\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6}\frac{C_6}{2C_4} + \frac{2GC_3C_4\sqrt{C_2}\left[2\sqrt{C_2} + \sqrt{C_4}\right]}{C_6} \right) \\
& \times \frac{C_8C_6}{8G\sqrt{C_2}\left[C_6\sqrt{C_4} + \left(2\sqrt{C_2} + \sqrt{C_4}\right)\left(C_6 + 2C_3C_4\right)\right]} \\
& \left. + \frac{C}{n}\sqrt{C_4}\frac{C_6}{8\sqrt{GC_4C_7}} + \frac{C}{\mu_n^2} G\sqrt{C_2C_4}\left(\frac{C_6}{2C_4} + C_3\right) \right\} \\
\leq\ & \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \left\{ \frac{C_8}{8} + \frac{C_8}{8} + \frac{C_8}{8} + \frac{C_8}{8} + \frac{C}{n}\frac{C_6}{8\sqrt{GC_7}} + \frac{C}{\mu_n^2} G\sqrt{C_2C_4}\left(\frac{C_6}{2C_4} + C_3\right) \right\}
\end{aligned}$$

Now, let n be sufficiently large so that (54) holds, and we have

$$\begin{aligned}
&\left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq\ & \left(\frac{C_8}{8} + \frac{C_8}{8} + \frac{C_8}{8} + \frac{C_8}{8} + \frac{C_8}{4} + \frac{C_8}{4} \right) \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} = C_8 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} < \infty
\end{aligned} \tag{58}$$

It follows immediately by (55), (56), and (58) that

$$\begin{aligned}
&\left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A}} \\
=\ & \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} \\
& + \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} \\
& + \left\| \frac{S_n^{-1} X' (P - D_P) \varepsilon}{\mu_n} - \left\{ \widehat{\kappa}_{HLIM} - \frac{C}{n} \right\} \frac{S_n^{-1} X' (M + D_P) \varepsilon}{\mu_n} \right\| \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
\leq\ & C_8 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}} + C_8 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}} + C_8 \mathbb{I}_{\mathcal{A} \cap \mathcal{B}^C \cap \mathcal{C}^C} \\
=\ & C_8 \mathbb{I}_{\mathcal{A}},
\end{aligned}$$

which establishes the desired conclusion. *Q.E.D.*

Lemma C4: If Assumptions 1-4 and 7-9 are satisfied, then the following results hold.

(a) $E\left\|adj\left(X_*'MX_*/n\right)\right\|^{pq/G}=O\left(1\right);$

(b) $E\left\|adj\left(X'MX/n\right)\right\|^{pq/(G-1)}=O\left(1\right)$ for $G\geq 2$;

(c) $E\left\|X'\left[M+D_P\right]\varepsilon/n\right\|^{pq}=O\left(1\right);$

(d) $E\left\|X_*'X_*/n\right\|^{pq}=O\left(1\right),$

(e) $E\left[\det\left(X_*'MX_*/n\right)/\det\left(X'MX/n\right)\right]^{pq^*}=O\left(1\right),$

where p and q are as given in Assumption 8.

Proof: To prove part (a), first write

$$E\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^{pq/G}=E\left(\sqrt{tr\left\{\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]'\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]\right\}}\right)^{pq/G}$$

Consider the spectral decomposition

$$\frac{X_*'MX_*}{n}=HLH',$$

where $H\in O\left(G+1\right)$ and $L=diag\left(l_1,..,l_{G+1}\right)$. Without loss of generality, let $l_1\geq\cdots\geq l_{G+1}>0$ $a.s.$, where the almost sure positivity of the eigenvalues holds for all n sufficiently large, since

$$\frac{X_*'MX_*}{n}=\frac{D'^{-1}\overline{X}'M\overline{X}D^{-1}}{n},$$

and so Assumptions 9 implies that $X_*'MX_*/n$ is positive definite with probability one for all n sufficiently large. It follows by using this spectral decomposition that

$$\begin{aligned}adj\left(\frac{X_*'MX_*}{n}\right) &= \left[\det\left(HLH'\right)\right]\left(HLH'\right)^{-1}\\ &= \det\left(H'H\right)\det\left(L\right)HL^{-1}H'\\ &= \left(\prod_{g=1}^{G+1}l_g\right)HL^{-1}H' \quad \text{[by } \det\left(H'H\right)=\det\left(I_{G+1}\right)=1\text{]}\end{aligned}$$

Next, note that

$$
\begin{aligned}
& \sqrt{tr\left\{\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]'\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]\right\}} \\
= & \sqrt{tr\left\{\left(\prod_{g=1}^{G+1} l_g\right)^2 HL^{-1}H'HL^{-1}H'\right\}} = \sqrt{tr\left\{\left(\prod_{g=1}^{G+1} l_g\right)^2 L^{-2}\right\}} = \sqrt{\left(\prod_{g=1}^{G+1} l_g^2\right)\sum_{h=1}^{G+1}\frac{1}{l_h^2}} = \sqrt{\sum_{h=1}^{G+1}\left(\prod_{g\neq h}^{G+1} l_g^2\right)} \\
\leq & \sqrt{(G+1)\prod_{g=1}^{G} l_g^2} = \sqrt{G+1}\prod_{g=1}^{G} l_g \\
\leq & \frac{\sqrt{G+1}}{G^G}\left(\sum_{g=1}^{G} l_g\right)^G \leq \frac{(G+1)^{G+1/2}}{G^G}\left(\frac{1}{G+1}tr\left\{\frac{X_*'MX_*}{n}\right\}\right)^G \\
\leq & \frac{(G+1)^{2G+1/2}}{G^G}\left(\frac{1}{(G+1)^2}\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left|\frac{x_{*,\cdot g}'Mx_{*,\cdot h}}{n}\right|\right)^G \quad [x_{*,\cdot g} \text{ and } x_{*,\cdot h} \text{ denote the } g^{th} \text{ and } h^{th} \text{ columns of } X_*] \\
\leq & \frac{(G+1)^{2G+1/2}}{G^G}\left(\sqrt{\frac{1}{(G+1)^2}\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left(\frac{x_{*,\cdot g}'Mx_{*,\cdot h}}{n}\right)^2}\right)^G \quad \text{[by Liapunov's inequality]} \\
= & \frac{(G+1)^{G+1/2}}{G^G}\left(\sqrt{\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left(\frac{x_{*,\cdot g}'Mx_{*,\cdot h}}{n}\right)^2}\right)^G \\
= & \frac{(G+1)^{G+1/2}}{G^G}\left(\left\|\frac{X_*'MX_*}{n}\right\|\right)^G
\end{aligned}
$$

where the second inequality above follows as a result of the arithmetic-geometric mean inequality, i.e.,

$$
\left(\prod_{g=1}^{G} l_g\right)^{1/G} \leq \frac{1}{G}\sum_{g=1}^{G} l_g \Longrightarrow \prod_{g=1}^{G} l_g \leq \left(\frac{1}{G}\sum_{g=1}^{G} l_g\right)^G.
$$

Hence, applying Theorem 15.2 part (iv) of Billingsley (1986), we get

$$
\begin{aligned}
& E\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^{pq/G} \\
= & E\left(\sqrt{tr\left\{\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]'\left[adj\left(\frac{X_*'MX_*}{n}\right)\right]\right\}}\right)^{pq/G} \\
\leq & \left(\frac{(G+1)^{G+1/2}}{G^G}\right)^{pq/G} E\left(\left\|\frac{X_*'MX_*}{n}\right\|\right)^{pq} \qquad (59)
\end{aligned}
$$

Next, we want to obtain an upper bound for the expectation $E\left(\|X_*'MX_*/n\|\right)^{pq}$. To proceed, let $D_\mu = diag\left(\mu_{1n},...,\mu_{Gn}\right)$ as before, and note that since

$$
X_* = \left[\begin{array}{cc} \varepsilon & X \end{array}\right] = \Upsilon F_2' + \left[\begin{array}{cc} \varepsilon & U \end{array}\right] = \frac{1}{\sqrt{n}}zD_\mu\widetilde{S}_n'F_2' + \overline{U},
$$

with $F_2 = \left[\begin{array}{cc} 0 & I_G \end{array}\right]'$, we obtain

$$\begin{aligned}
&\left\| \frac{X_*' M X_*}{n} \right\| \\
\leq \; & \|F_2\|^2 \left\| \widetilde{S}_n \right\|^2 \left\| n^{-1/2} D_\mu \right\|^2 \left\| \frac{z' M z}{n} \right\| + 2 \|F_2\| \left\| \widetilde{S}_n \right\| \left\| n^{-1/2} D_\mu \right\| \left\| \frac{z' M \overline{U}}{n} \right\| \\
& + \left\| \frac{\overline{U}' [M + D_P] \overline{U}}{n} \right\| \quad \text{[by T and submultiplicativity of Frobenius norm]}
\end{aligned}$$

Moreover, note that using the fact that $\overline{U} = \overline{V} D^{-1}$, we have

$$\begin{aligned}
\left\| \frac{z' M \overline{U}}{n} \right\| &\leq \left\| \frac{z' M \overline{V}}{n} \right\| \left\| D^{-1} \right\| \quad \text{[by submultiplicativity of Frobenius norm]} \\
&= \sqrt{G + 1 + \|\delta_0\|^2} \left\| \frac{z' M \overline{V}}{n} \right\| \\
&\leq \overline{C} \left\| \frac{z' M \overline{V}}{n} \right\| \qquad \text{[by Assumption 7]} \\
&= \overline{C} \left(\frac{\mu_n}{\sqrt{n}} \right) \left\| \frac{z' M \overline{V}}{\mu_n \sqrt{n}} \right\|,
\end{aligned}$$

from which it follows by Theorem 15.2 part (iv) of Billingsley (1986), Liapunov's inequality, and part (c) of Lemma B11 that

$$E \left\| \frac{z' M \overline{U}}{n} \right\|^{pq} \leq \overline{C}^{pq} \left(\frac{\mu_n}{\sqrt{n}} \right)^{pq} E \left\| \frac{z' M \overline{V}}{\mu_n \sqrt{n}} \right\|^{pq} \leq \overline{C}^{pq} \left(\frac{\mu_n}{\sqrt{n}} \right)^{pq} \left(E \left\| \frac{z' M \overline{V}}{\mu_n \sqrt{n}} \right\|^{2pq} \right)^{1/2} = O\left(\frac{1}{n^{pq/2}} \right). \tag{60}$$

In addition, by direct calculation, it is easy to see that

$$E \left[\frac{\overline{U}' [M + D_P] \overline{U}}{n} \right] = \frac{1}{n} \sum_{i=1}^{n} \Omega_i$$

so that

$$\begin{aligned}
\left\| \frac{\overline{U}' [M + D_P] \overline{U}}{n} \right\| &\leq \left\| \frac{\overline{U}' [M + D_P] \overline{U}}{n} - E \left[\frac{\overline{U}' [M + D_P] \overline{U}}{n} \right] \right\| + \left\| E \left[\frac{\overline{U}' [M + D_P] \overline{U}}{n} \right] \right\| \\
& \quad \text{[by T]} \\
&= \left\| \frac{\overline{U}' [M + D_P] \overline{U}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Omega_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Omega_i \right\| \\
&= \left\| \frac{D'^{-1} \overline{V}' [M + D_P] \overline{V} D^{-1}}{n} - \frac{1}{n} \sum_{i=1}^{n} D'^{-1} \Xi_i D^{-1} \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} D'^{-1} \Xi_i D^{-1} \right\| \\
& \quad \text{[by (\textbf{??}) and (\textbf{??})]} \\
&\leq \left\| D^{-1} \right\|^2 \left[\left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right] \\
& \quad \text{[by submultiplicativity of the Frobenius norm]} \\
&= \left(G + 1 + \|\delta_0\|^2 \right) \left[\left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right] \\
&\leq \overline{C} \left[\left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right] \quad \text{[by Assumption 7]}
\end{aligned}$$

It follows from applying Theorem 15.2 part (iv) of Billingsley (1986) to that

$$
\begin{aligned}
E\left\|\frac{\overline{U}'\left[M+D_P\right]\overline{U}}{n}\right\|^{pq} &\leq \overline{C}^{pq}E\left[\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|\right]^{pq} \\
&\leq 2^{pq-1}\left\{E\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}\right\} \\
&\quad \text{[by Loève's } c_r \text{ inequality]} \\
&\leq 2^{pq-1}\left\{E\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq}+\widetilde{C}\right\} \\
&\quad \text{[using (32) above]} \\
&= O\left(\frac{1}{n^{pq/2}}\right)+O\left(1\right)=O\left(1\right) \quad \text{[by part (c) of Lemma B12]} \qquad (61)
\end{aligned}
$$

Given (60), (61), and part (a) of Lemma C1; a further application of Theorem 15.2 part (iv) of Billingsley (1986) and Loève's c_r inequality yields

$$
\begin{aligned}
& E\left\|\frac{X_*'MX_*}{n}\right\|^{pq} \\
\leq\; & 3^{pq-1}\left\{\left\|F_2\right\|^{2pq}\left\|\widetilde{S}_n\right\|^{2pq}\left\|n^{-1/2}D_\mu\right\|^{2pq}\left\|\frac{z'Mz}{n}\right\|^{pq}\right. \\
& \left.+2^{pq}\left\|F_2\right\|^{pq}\left\|\widetilde{S}_n\right\|^{pq}\left\|n^{-1/2}D_\mu\right\|^{pq}E\left\|\frac{z'M\overline{U}}{n}\right\|^{pq}+E\left\|\frac{\overline{U}'\left[M+D_P\right]\overline{U}}{n}\right\|^{pq}\right\} \\
=\; & o\left(1\right)+O\left(n^{-pq/2}\right)+O\left(1\right)=O\left(1\right), \qquad (62)
\end{aligned}
$$

since $\left\|F_2\right\|^2=G$, and $\left\|\widetilde{S}_n\right\|$ is bounded and $\left\|n^{-1/2}D_\mu\right\|=O\left(1\right)$ under Assumption 2. We deduce from (59) and (62) that

$$
E\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^{pq/G}=O\left(1\right),
$$

which is the desired conclusion.

Part (b) can be shown in a manner very similar to part (a) above. Hence, for brevity, we omit the proof.

To show part (c), first write

$$
\frac{X'\left[M+D_P\right]\varepsilon}{n}=\frac{n^{-1/2}\widetilde{S}_nD_\mu z'\left[M+D_P\right]\varepsilon}{n}+\frac{U'\left[M+D_P\right]\varepsilon}{n}
$$

from which it follows by T and the submultiplicativity of the Euclidean norm that

$$\begin{aligned}
& \left\| \frac{X' [M + D_P] \varepsilon}{n} \right\| \\
\leq \; & \left\| \widetilde{S}_n \right\| \left\| n^{-1/2} D_\mu \right\| \left\| \frac{z' [M + D_P] \varepsilon}{n} \right\| + \left\| \frac{U' [M + D_P] \varepsilon}{n} \right\| \\
\leq \; & \left\| \widetilde{S}_n \right\| \left\| n^{-1/2} D_\mu \right\| \left\| \frac{z' [M + D_P] \overline{V}}{n} \right\| \|\delta_{\Delta,0}\| + \|F_2\| \left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} \right\| \|\delta_{\Delta,0}\| \\
& \text{[by (??) and the submultiplicativity of the Frobenius norm]} \\
\leq \; & \left\| \widetilde{S}_n \right\| \left\| n^{-1/2} D_\mu \right\| \|\delta_{\Delta,0}\| \left\| \frac{z' [M + D_P] \overline{V}}{n} \right\| + \|F_2\| \|\delta_{\Delta,0}\| \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \\
& + \|F_2\| \|\delta_{\Delta,0}\| \left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \quad \text{[by T]} \\
\leq \; & \overline{C} \left\{ \left\| \frac{z' [M + D_P] \overline{V}}{n} \right\| + \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| + \left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\| \right\}
\end{aligned}$$

where the last inequality holds for n sufficiently large in light of the fact that$\left\| n^{-1/2} D_\mu \right\| = O(1)$; $\left\| \widetilde{S} \right\|$ is bounded and$\|\delta_{\Delta,0}\|$ is bounded under Assumption 7; and $\|F_2\| = \sqrt{G}$. Part (iv) of Theorem 15.2 of Billingsley (1986) and Loève's c_r inequality then imply that

$$\begin{aligned}
& E \left\| \frac{X' [M + D_P] \varepsilon}{n} \right\|^{pq} \\
\leq \; & 3^{pq-1} \overline{C}^{pq} \left\{ E \left\| \frac{z' [M + D_P] \overline{V}}{n} \right\|^{pq} + \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\|^{pq} + E \left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\|^{pq} \right\} \\
\leq \; & 3^{pq-1} \overline{C}^{pq} \left\{ \left(E \left\| \frac{z' [M + D_P] \overline{V}}{n} \right\|^{2pq} \right)^{1/2} + \left\| \frac{1}{n} \sum_{i=1}^n \Xi_i \right\|^{pq} + E \left\| \frac{\overline{V}' [M + D_P] \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right\|^{pq} \right\} \\
= \; & O\left(\mu_n^{-pq} \right) + O(1) + O\left(n^{-pq/2} \right) = O(1),
\end{aligned}$$

where the orders of magnitude are obtained from making use of part (b) of Lemma B11, part (c) of Lemma B12, and expression (32).

Finally, to show part (d), write

$$\begin{aligned}
& \frac{X_*' X_*}{n} \\
= \; & F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z'z}{n} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + \frac{\overline{U}' z}{n} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z' \overline{U}}{n} + \frac{\overline{U}' \overline{U}}{n} \\
= \; & F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z'z}{n} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + \left(\frac{1}{\sqrt{n}} \right) D^{-1} \frac{\overline{V}' z}{\sqrt{n}} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + \left(\frac{1}{\sqrt{n}} \right) F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z' \overline{V}}{\sqrt{n}} D^{-1} + D^{-1\prime} \frac{\overline{V}' \overline{V}}{n} D^{-1} \\
= \; & F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z'z}{n} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + \left(\frac{1}{\sqrt{n}} \right) D^{-1} \frac{\overline{V}' z}{\sqrt{n}} \frac{D_\mu}{\sqrt{n}} \widetilde{S}_n' F_2' + \left(\frac{1}{\sqrt{n}} \right) F_2 \widetilde{S}_n \frac{D_\mu}{\sqrt{n}} \frac{z' \overline{V}}{\sqrt{n}} D^{-1} \\
& + D^{-1\prime} \left[\frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^n \Xi_i \right] D^{-1} + \frac{1}{n} \sum_{i=1}^n D^{-1\prime} \Xi_i D^{-1}.
\end{aligned}$$

It follows by T and the submultiplicativity of the Euclidean norm that

$$
\begin{aligned}
& \left\| \frac{X_*' X_*}{n} \right\| \\
\leq \; & \|F_2\|^2 \left\| \widetilde{S}_n \right\|^2 \left\| n^{-1/2} D_\mu \right\|^2 \left\| \frac{z'z}{n} \right\| + 2 \left(\frac{1}{\sqrt{n}} \right) \|F_2\| \left\| D^{-1} \right\| \left\| \widetilde{S}_n \right\| \left\| n^{-1/2} D_\mu \right\| \left\| \frac{z' \overline{V}}{\sqrt{n}} \right\| \\
& + \left\| D^{-1} \right\|^2 \left\{ \left\| \frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right\} \\
= \; & \|F_2\|^2 \left\| \widetilde{S}_n \right\|^2 \left\| n^{-1/2} D_\mu \right\|^2 \left\| \frac{z'z}{n} \right\| + 2 \left(\frac{1}{\sqrt{n}} \right) \|F_2\| \sqrt{G + 1 + \|\delta_{\Delta,0}\|^2} \left\| \widetilde{S}_n \right\|^2 \left\| n^{-1/2} D_\mu \right\| \left\| \frac{z' \overline{V}}{\sqrt{n}} \right\| \\
& + \left(G + 1 + \|\delta_{\Delta,0}\|^2 \right) \left\{ \left\| \frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right\} \\
\leq \; & \overline{C} \left\{ \left\| \frac{z'z}{n} \right\| + \left(\frac{1}{\sqrt{n}} \right) \left\| \frac{z' \overline{V}}{\sqrt{n}} \right\| + \left\| \frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\| \right\}
\end{aligned}
$$

where the last inequality holds for n sufficiently large in light of the fact that $\left\| n^{-1/2} D_\mu \right\| = O(1)$; $\left\| \widetilde{S}_n \right\|$ is bounded and $\|\delta_{\Delta,0}\|$ is bounded under Assumption 7; and $\|F_2\| = \sqrt{G}$. Part (iv) of Theorem 15.2 of Billingsley (1986) and Loève's c_r inequality then imply that

$$
\begin{aligned}
& E \left\| \frac{X_*' X_*}{n} \right\|^{pq} \\
\leq \; & 4^{pq-1} \overline{C}^{pq} \left\{ \left\| \frac{z'z}{n} \right\|^{pq} + \left(\frac{1}{\sqrt{n}} \right)^{pq} E \left\| \frac{z' \overline{V}}{\sqrt{n}} \right\|^{pq} + E \left\| \frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\|^{pq} + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\|^{pq} \right\} \\
\leq \; & 4^{pq-1} \overline{C}^{pq} \left\{ \left\| \frac{z'z}{n} \right\|^{pq} + \left(\frac{1}{\sqrt{n}} \right)^{pq} \left(E \left\| \frac{z' \overline{V}}{\sqrt{n}} \right\|^{2pq} \right)^{1/2} + E \left\| \frac{\overline{V}' \overline{V}}{n} - \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\|^{pq} + \left\| \frac{1}{n} \sum_{i=1}^{n} \Xi_i \right\|^{pq} \right\}. \quad (63)
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
\left\| \frac{z'z}{n} \right\|^{pq} & = \left[\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(z_i' z_j \right)^2 \right]^{pq/2} \quad [z_i' \text{ denote the } i^{th} \text{ row of } z] \\
& \leq \left[\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \|z_i\|^2 \|z_j\|^2 \right]^{pq/2} \quad [\text{by CS}] \\
& = \left[\frac{1}{n} \sum_{i=1}^{n} \|z_i\|^2 \right]^{pq} \\
& \leq \frac{1}{n} \sum_{i=1}^{n} \|z_i\|^{2pq} \quad [\text{by Liapunov's inequality}] \\
& = O(1) \;\; [\text{by Assumption 8}] \quad (64)
\end{aligned}
$$

Applying (64), part (c) of Lemma B9, and part (a) of Lemma B12 to the upper bound in expression (63) above; we deduce that

$$
\begin{aligned}
E \left\| \frac{X_*' X_*}{n} \right\|^{pq} & = O(1) + O\left(n^{-pq/2} \right) O(1) + O\left(n^{-pq/2} \right) + O(1) \\
& = O(1),
\end{aligned}
$$

which is the conclusion desired.

To prove part (e), let $\lambda_1^2 > \lambda_2^2 > \cdots > \lambda_{G+1}^2 > 0$ be the ordered eigenvalues of the matrix $X_*'MX_*/n$. Note that, by the interlacing eigenvalues theorem for bordered matrices (Theorem 4.3.8 of Horn and Johnson, 1985), we have that

$$\prod_{k=1}^{G} \lambda_{k+1}^2 \leq \det\left(\frac{X'MX}{n}\right).$$

It follows that

$$\begin{aligned}
& E\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^{pq^*} \\
\leq\ & E\left[\frac{\prod_{j=1}^{G+1} \lambda_j^2}{\prod_{k=1}^{G} \lambda_{k+1}^2}\right]^{pq^*} = E\left[\lambda_1^{2pq^*}\right] \\
\leq\ & E\left(tr\left\{\frac{X_*'MX_*}{n}\right\}\right)^{pq^*} \\
=\ & E\left(\sum_{g=1}^{G+1} \frac{x_{*,\cdot g}'Mx_{*,\cdot g}}{n}\right)^{pq^*} \\
\leq\ & (G+1)^{2pq^*} E\left(\frac{1}{(G+1)^2}\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left|\frac{x_{*,\cdot g}'Mx_{*,\cdot h}}{n}\right|\right)^{pq^*} \\
\leq\ & (G+1)^{2pq^*} E\left(\sqrt{\frac{1}{(G+1)^2}\sum_{g=1}^{G+1}\sum_{h=1}^{G+1}\left(\frac{x_{*,\cdot g}'Mx_{*,\cdot h}}{n}\right)^2}\right)^{pq^*} \\
=\ & (G+1)^{pq^*} E\left(\left\|\frac{X_*'MX_*}{n}\right\|\right)^{pq^*}
\end{aligned}$$

where the third inequality above follows from Loève's c_r inequality. Next, using the upper bound

$$\begin{aligned}
& E\left(\left\|\frac{X_*'MX_*}{n}\right\|\right)^{pq^*} \\
\leq\ & 4^{pq^*-1}\Bigg[G^{pq^*}\left\|\widetilde{S}_n\right\|^{2pq^*}\left\|n^{-1/2}D_\mu\right\|^{2pq^*}\left(\frac{1}{n}\sum_{i=1}^{n}\|z_i-\pi_{Kn}Z_i\|^2\right)^{pq^*} \\
& +2^{3pq^*/2-1}G^{pq^*/2}\left\|\widetilde{S}_n\right\|^{pq^*}\left\|n^{-1/2}D_\mu\right\|^{pq^*}\left(\frac{1}{n}\sum_{i=1}^{n}\|z_i-\pi_{Kn}Z_i\|^2\right)^{pq^*/2} \\
& \quad\times\left(\|\delta_0\|^2+G+1\right)^{pq^*/2}\left\{E\left(\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|\right)^{pq^*/2}+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq^*/2}\right\} \\
& +\left(\|\delta_0\|^2+G+1\right)^{pq^*}\left\{E\left(\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|\right)^{pq^*}+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq^*}\right\}\Bigg],
\end{aligned}$$

we obtain

$$
\begin{aligned}
& E\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^{pq^*} \\
\leq\ & (G+1)^{pq^*} 4^{pq^*-1}\Bigg[G^{pq^*}\left\|\widetilde{S}_n\right\|^{2pq^*}\left\|n^{-1/2}D_\mu\right\|^{2pq^*}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2\right)^{pq^*} \\
& +2^{3pq^*/2-1}G^{pq^*/2}\left\|\widetilde{S}_n\right\|^{pq^*}\left\|n^{-1/2}D_\mu\right\|^{pq^*}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|z_i-\pi_{Kn}Z_i\right\|^2\right)^{pq^*/2} \\
& \times\left(\left\|\delta_0\right\|^2+G+1\right)^{pq^*/2}\left\{E\left(\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|\right)^{pq^*/2}+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq^*/2}\right\} \\
& +\left(\left\|\delta_0\right\|^2+G+1\right)^{pq^*}\left\{E\left(\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n}-\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|\right)^{pq^*}+\left\|\frac{1}{n}\sum_{i=1}^{n}\Xi_i\right\|^{pq^*}\right\}\Bigg]. \\
=\ & O(1)
\end{aligned}
$$

given part (c) of Lemma B12 and and expression (32) and given that $\left\|n^{-1/2}D_\mu\right\| = O(1)$; $\left\|\widetilde{S}_n\right\|$ is bounded; and $\left\|\delta_{\Delta,0}\right\|$ is bounded under Assumption 7. *Q.E.D.*

Lemma C5: If Assumptions 1-4 and 7-9 are satisfied, then the following statements hold.

(a) $E\left\|X'\left[P-D_P\right]\varepsilon/\sqrt{n}\right\|^{2pq} = O(1)$;

(b) $E\left\|\varepsilon'\left[P-D_P\right]\varepsilon/\mu_n^2\right\|^{pq} = O\left(K^{pq/2}/\mu_n^{2pq}\right)$,

(c) $E\left\|X'\left[M+D_P\right]\varepsilon/n\right\|^{pd_1} = O(1)$;

(d) $E\left\|adj\left(X'MX/n\right)\right\|^{pd_1/G} = O(1)$;

where $d_1 = (G+1)(1+\eta)$ and where p, q, and η are as defined in Assumption 8.

Proof:

For part (a), first write

$$
\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}} = \widetilde{S}_n\left(\frac{D_\mu}{\sqrt{n}}\right)\mu_n\frac{z'\left[P-D_P\right]\varepsilon}{\sqrt{n}\mu_n}+\left(\sqrt{\frac{K}{n}}\right)\frac{U'\left[P-D_P\right]\varepsilon}{\sqrt{K}}.
$$

It follows by T and the submultiplicativity of the Euclidean norm that

$$
\begin{aligned}
& \left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\| \\
\leq\ & \left\|\widetilde{S}_n\right\|\left\|\frac{D_\mu}{\sqrt{n}}\right\|\mu_n\left\|\frac{z'\left[P-D_P\right]\varepsilon}{\sqrt{n}\mu_n}\right\|+\left(\sqrt{\frac{K}{n}}\right)\left\|\frac{U'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right\| \\
=\ & \left\|\widetilde{S}_n\right\|\left\|\frac{D_\mu}{\sqrt{n}}\right\|\mu_n\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\sqrt{n}\mu_n}\right\|\left\|\delta_{\Delta,0}\right\|+\left(\sqrt{\frac{K}{n}}\right)\left\|F_2\right\|\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\sqrt{K}}\right\|\left\|\delta_{\Delta,0}\right\| \\
& \text{[by (5.4) of the paper and the submultiplicativity of the Frobenius norm]} \\
\leq\ & \overline{C}\left\{\mu_n\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\sqrt{n}\mu_n}\right\|+\left(\sqrt{\frac{K}{n}}\right)\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\sqrt{K}}\right\|\right\}
\end{aligned}
$$

where the last inequality holds for n sufficiently large in light of the fact that $\|F_2\| = \sqrt{G}$; $\left\|\widetilde{S}_n\right\|$ is bounded; $\sqrt{K/n} = O(1)$ and $\left\|n^{-1/2}D_\mu\right\| = O(1)$ under Assumptions 2 and 7; and $\|\delta_{\Delta,0}\|$ is bounded under Assumption 7. Part (iv) of Theorem 15.2 of Billingsley (1986) and Loève's c_r inequality then imply that

$$\begin{aligned}
& E\left\|\frac{X'[P-D_P]\varepsilon}{\sqrt{n}}\right\|^{2pq} \\
\leq\; & 2^{2pq-1}\overline{C}^{2pq}\left\{\mu_n^{2pq}E\left\|\frac{z'[P-D_P]\overline{V}}{\sqrt{n}\mu_n}\right\|^{2pq}+\left(\frac{K}{n}\right)^{pq}E\left\|\frac{\overline{V}'[P-D_P]\overline{V}}{\sqrt{K}}\right\|^{2pq}\right\} \\
=\; & O\left(\mu_n^{2pq}\right)O\left(\mu_n^{-2pq}\right)+O\left(K^{pq}/n^{pq}\right) \\
=\; & O(1)
\end{aligned}$$

where the orders of magnitude are obtained from part (a) of Lemma B11 and (b) of Lemma B12.

To show part (b), write

$$\frac{\varepsilon'[P-D_P]\varepsilon}{\sqrt{K}} = \frac{\delta_{\Delta,0}'\overline{V}'[P-D_P]\ \overline{V}\delta_{\Delta,0}}{\sqrt{K}}$$

so that by the submultiplicativity of the Euclidean norm and by Theorem 15.2 part(iv) of Billingsley (1986), we have

$$\begin{aligned}
E\left\|\frac{\varepsilon'[P-D_P]\varepsilon}{\mu_n^2}\right\|^{pq} &\leq \|\delta_{\Delta,0}\|^2 E\left\|\frac{\overline{V}'[P-D_P]\ \overline{V}}{\mu_n^2}\right\|^{pq} \\
&\leq \overline{C}E\left\|\frac{\overline{V}'[P-D_P]\ \overline{V}}{\mu_n^2}\right\|^{pq} \text{ [by Assumption 7]} \\
&\leq \overline{C}\sqrt{E\left\|\frac{\overline{V}'[P-D_P]\ \overline{V}}{\mu_n^2}\right\|^{2pq}} \\
&= O\left(K^{pq/2}/\mu_n^{2pq}\right) \text{ [by part (b) of Lemma B12]}
\end{aligned}$$

This establishes the desired result.

To show part (c), note first that

$$\begin{aligned}
d_1 &= (1+\eta)(G+1) \\
&< (1+\eta)[2(G+1)+\varphi(a,b)] \\
&= q
\end{aligned} \tag{65}$$

since $\eta > 0$ and $\varphi(a,b) \geq 0$ for all a, b. It follows by applying Liapunov's inequality that

$$\begin{aligned}
E\left\|\frac{X'[M+D_P]\varepsilon}{n}\right\|^{pd_1} &\leq \left(E\left\|\frac{X'[M+D_P]\varepsilon}{n}\right\|^{pq}\right)^{d_1/q} \\
&= O(1) \text{ [by part (c) of Lemma C4].}
\end{aligned}$$

To show part (d), note that for $G = 1$, we set

$$adj\left(\frac{X'MX}{n}\right) := 1$$

so that $E\|adj(X'MX/n)\|^{pd_1/G}$ is trivially bounded. For $G \geq 2$, we make use of (65) and note that

$$E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{pd_1/G} \leq \left(E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{pq/(G-1)}\right)^{(G-1)d_1/(Gq)} \quad \text{[by Liapunov's inequality]}$$
$$= O(1) \quad \text{[by part (b) of Lemma C4]},$$

which establishes the desired conclusion. *Q.E.D.*

Lemma C6: Suppose that Assumptions 1-4 and 8 are satisfied, and suppose that Assumption 7 hold but with $0 < a \leq 1$. Then, the following results hold.

(a) Let

$$r_1 = (1+\eta)\left[\frac{2(G+1)+\varphi(a,b)}{\varphi(a,b)}\right];$$

then,

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O(1)$$

(b) Let

$$\rho_1 = (1+\eta)\left[\frac{2(G+1)+\varphi(a,b)}{G+1+\varphi(a,b)}\right];$$

then,

$$n^{p/2} \left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1} = O(1)$$

(c) Let $s_1 = \rho_1$; then,

$$K^{p/2} \left(P\left(\mathcal{A}_n^C\right)\right)^{1/s_1} = O(1)$$

Proof: To show part (a), note that by (1), M, parts (a)-(b) of Lemma B11 and parts (b)-(c) of Lemma B12, we have

$$\begin{aligned}
P\left(\mathcal{A}_n^C\right) &= P\left(\mathcal{A}_{1,n}^C \cup \mathcal{A}_{2,n}^C \cup \mathcal{A}_{3,n}^C \cup \mathcal{A}_{4,n}^C\right) \\
&\leq P\left(\mathcal{A}_{1,n}^C\right) + P\left(\mathcal{A}_{2,n}^C\right) + P\left(\mathcal{A}_{3,n}^C\right) + P\left(\mathcal{A}_{4,n}^C\right) \\
&= P\left\{\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\| \geq \eta_1\right\} + P\left\{\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\| \geq \eta_2\right\} \\
&\quad + P\left\{\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n} - \frac{1}{n}\sum_{i=1}^n \Xi_i\right\| \geq \eta_3\right\} + P\left\{\left\|\frac{z'\left[M+D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\| \geq \eta_4\right\} \\
&\leq \eta_1^{-2pq} E\left\|\frac{z'\left[P-D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} + \eta_2^{-2pr_2} E\left\|\frac{\overline{V}'\left[P-D_P\right]\overline{V}}{\mu_n^2}\right\|^{2pq} \\
&\quad + \eta_3^{-pq} E\left\|\frac{\overline{V}'\left[M+D_P\right]\overline{V}}{n} - \frac{1}{n}\sum_{i=1}^n \Xi_i\right\|^{pq} + \eta_4^{-2pr_2} E\left\|\frac{z'\left[M+D_P\right]\overline{V}}{\mu_n\sqrt{n}}\right\|^{2pq} \\
&= O\left(\frac{1}{\mu_n^{2pq}}\right) + O\left(\frac{K^{pq}}{\mu_n^{4pq}}\right) + O\left(\frac{1}{n^{pq/2}}\right) + O\left(\frac{1}{\mu_n^{2pq}}\right) = O\left(\max\left\{\frac{1}{\mu_n^{2pq}}, \frac{K^{pq}}{\mu_n^{4pq}}, \frac{1}{n^{pq/2}}\right\}\right), \quad (66)
\end{aligned}$$

where

$$q = (1+\eta)\left[2(G+1)+\varphi(a,b)\right]$$

Note that all expectations in (66) above exist in light of Assumption 8. By Assumption 7, we must have $a/2 < b \leq 1$, and note that

$$\left\{\frac{a}{2} < b \leq 1\right\} = \left\{\frac{a}{2} < b \leq \frac{1}{2}\right\} \cup \left\{\frac{1}{2} < b \leq 1\right\}$$

First, consider the case $a/2 < b \leq 1/2$. In this case, we have

$$\frac{1}{n^{pq/2}} = O\left(\frac{1}{n^{bpq}}\right) = O\left(\frac{1}{\mu_n^{2pq}}\right)$$

so that in this case

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{\mu_n^{2pq/r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right)$$

Next, note that for $a/2 < b \leq 1/2$

$$\frac{q}{r_1} = \varphi(a, b) = \begin{cases} \frac{a \vee (1-4(2b-a))/2}{2b-a} & \text{for } 2b - a < b \\ \frac{a \vee (1-4b)/2}{b} & \text{for } 2b - a \geq b \end{cases}$$

Consider first the case where $\{a/2 < b \leq 1/2\} \cap \{2b - a < b\}$. In this case, it is easily seen that under Assumptions 2 and 7

$$\begin{aligned} \frac{K^p}{\mu_n^{2pq/r_1}} &= O\left(n^{ap-bpq/r_1}\right), \\ K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1} &= O\left(n^{ap-(2b-a)pq/r_1}\right), \end{aligned}$$

so that, given $2b - a < b$, we have

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{\mu_n^{2pq/r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O\left(n^{ap-(2b-a)pq/r_1}\right).$$

Now, since

$$\begin{aligned} & ap - (2b-a)\, pq/r_1 \\ = \;& p\left[a - (2b-a)\left(\frac{a \vee (1 - 4(2b-a))/2}{2b-a}\right)\right] \\ = \;& p\left[a - \{a \vee (1 - 4(2b-a))/2\}\right] \\ \leq \;& p\left[a - a\right] = 0. \end{aligned}$$

It follows that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} \mathbb{I}\{a/2 < b \leq 1/2 \cap 2b - a < b\} = O\left(\max\left\{\frac{K^p}{\mu_n^{2pq/r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O(1). \tag{67}$$

Now, consider the case $\{a/2 < b \leq 1/2\} \cap \{2b - a \geq b\}$. Under Assumptions 2 and 7, we have

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{\mu_n^{2pq/r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O\left(n^{ap-bpq/r_1}\right),$$

given that $2b - a \geq b$. Next, note that

$$\begin{aligned} ap - bpq/r_1 &= p\left[a - b\left(\frac{a \vee (1-4b)/2}{b}\right)\right] \\ &= p\left[a - \{a \vee (1-4b)/2\}\right] \\ &\leq p\left[a - a\right] = 0 \end{aligned}$$

from which it follows that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} \mathbb{I}\{a/2 < b \leq 1/2 \cap 2b - a \geq b\} = O\left(\max\left\{\frac{K^p}{\mu_n^{2pq/r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O(1). \quad (68)$$

Turning our attention now to the case $1/2 < b \leq 1$, we note that in this case

$$\frac{1}{\mu_n^{2pq}} = \frac{1}{n^{bpq}} = o\left(\frac{1}{n^{pq/2}}\right)$$

so that here

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{n^{pq/2r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right)$$

Moreover, for $1/2 < b \leq 1$

$$\frac{q}{r_1} = \varphi(a,b) = \begin{cases} \frac{a}{2b-a} & \text{for } 2b - a < \frac{1}{2} \\ 2a & \text{for } 2b - a \geq \frac{1}{2} \end{cases}$$

Taking the first case where $\{1/2 < b \leq 1\} \cap \{2b - a < 1/2\}$, it is easily seen that under Assumptions 2 and 7

$$\begin{aligned} \frac{K^p}{n^{pq/2r_1}} &= O\left(n^{ap - \frac{1}{2}pq/r_1}\right), \\ K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1} &= O\left(n^{ap-(2b-a)pq/r_1}\right), \end{aligned}$$

so that, given $2b - a < 1/2$, we have that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{n^{pq/2r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O\left(n^{ap-(2b-a)pq/r_1}\right).$$

Now, note that

$$\begin{aligned} ap - (2b-a)\, pq/r_1 &= p\left\{a - (2b-a)\left(\frac{a \vee (1 - 4(2b-a))/2}{2b-a}\right)\right\} \\ &= p\{a - [a \vee (1 - 4(2b-a))/2]\} \\ &\leq p[a - a] = 0. \end{aligned}$$

It follows that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} \mathbb{I}\{1/2 < b \leq 1 \cap 2b - a < 1/2\} = O\left(\max\left\{\frac{K^p}{n^{pq/2r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O(1). \quad (69)$$

Next, consider the case $\{1/2 < b \leq 1\} \cap \{2b - a \geq 1/2\}$. In this case, under Assumptions 2 and 7, we have that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} = O\left(\max\left\{\frac{K^p}{n^{pq/2r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O\left(n^{ap-\frac{1}{2}pq/r_1}\right).$$

Now,

$$ap - \frac{1}{2}pq/r_1 = ap - \frac{1}{2}p(2a) = 0$$

from which it follows that

$$K^p \left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} \mathbb{I}\{1/2 < b \leq 1 \cap 2b - a \geq 1/2\} = O\left(\max\left\{\frac{K^p}{n^{pq/2r_1}}, K^p\left(\frac{K}{\mu_n^4}\right)^{pq/r_1}\right\}\right) = O(1). \quad (70)$$

From (67)-(70), we deduce immediately that

$$\begin{aligned}
& K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1} \\
= \; & K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{a/2<b\leq 1/2\right\}+K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{1/2<b\leq 1\right\} \\
= \; & K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a<b\right\} \\
& +K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a\geq b\right\} \\
& +K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a<1/2\right\} \\
& +K^p\left(P\left(\mathcal{A}_n^C\right)\right)^{1/r_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a\geq 1/2\right\} \\
= \; & O\left(1\right),
\end{aligned}$$

which is the result to be proved for part (a).

To show part (b), again we first consider the case $a/2<b\leq 1/2$ where

$$\frac{1}{n^{pq/2}}=O\left(\frac{1}{n^{bpq}}\right)=O\left(\frac{1}{\mu_n^{2pq}}\right)$$

so that, in this case, we have

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)$$

Next, note that for $a/2<b\leq 1/2$

$$\begin{aligned}
\frac{q}{\rho_1} &= G+1+\varphi\left(a,b\right) \\
&= \left\{\begin{array}{ll} G+1+\frac{a\vee(1-4[2b-a])/2}{2b-a} & \text{for } 2b-a<b \\ G+1+\frac{a\vee(1-4b)/2}{b} & \text{for } 2b-a\geq b \end{array}\right. .
\end{aligned}$$

Taking the first case where $\{a/2<b\leq 1/2\}\cap\{2b-a<b\}$, it is easily seen that under Assumptions 2 and 7

$$\begin{aligned}
\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}} &= O\left(n^{p/2-bpq/\rho_1}\right), \\
n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1} &= O\left(n^{p/2-(2b-a)pq/\rho_1}\right)
\end{aligned}$$

so that, given that $2b-a<b$, we have

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(n^{p/2-(2b-a)pq/\rho_1}\right).$$

Moreover,

$$\begin{aligned}
& \frac{p}{2}-\left(2b-a\right)p\frac{q}{\rho_1} \\
= \; & p\left\{\frac{1}{2}-\left(2b-a\right)\left(G+1+\frac{a\vee\left(1-4\left[2b-a\right]\right)/2}{2b-a}\right)\right\} \\
\leq \; & p\left\{\frac{1}{2}-\left(2b-a\right)\left(G+1\right)-\left[\frac{1}{2}-2\left(2b-a\right)\right]\right\} \\
= \; & -p\left(2b-a\right)\left(G-1\right) \\
\leq \; & 0
\end{aligned}$$

It follows that

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a<b\right\}=O\left(\max\left\{\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(1\right). \quad (72)$$

Now, consider the case $\{a/2<b\leq 1/2\}\cap\{2b-a\geq b\}$. In this case, under Assumptions 2 and 7, we have that

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(n^{p/2-bpq/\rho_1}\right).$$

Moreover,

$$\begin{aligned}
&\frac{p}{2}-bp\frac{q}{\rho_1}\\
=\ & p\left\{\frac{1}{2}-b\left[G+1+\frac{a\vee(1-4b)/2}{b}\right]\right\}\\
\leq\ & p\left\{\frac{1}{2}-b\left(G+1\right)-\left[\frac{1}{2}-2b\right]\right\}\\
=\ & -pb\left(G-1\right)\\
\leq\ & 0
\end{aligned}$$

from which it follows that

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a\geq b\right\}=O\left(\max\left\{\frac{n^{p/2}}{\mu_n^{2pq/\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(1\right). \quad (73)$$

Turning our attention now to the case $1/2<b\leq 1$, we note that in this case

$$\frac{1}{\mu_n^{2pq}}=\frac{1}{n^{bpq}}=o\left(\frac{1}{n^{pq/2}}\right)$$

so that here

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{n^{pq/2\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)$$

Moreover, for $1/2<b\leq 1$

$$\begin{aligned}
\frac{q}{\rho_1} &= G+1+\varphi\left(a,b\right)\\
&= \begin{cases} G+1+\frac{a}{2b-a} & \text{for } 2b-a<1/2\\ G+1+2a & \text{for } 2b-a\geq 1/2\end{cases}..
\end{aligned}$$

Consider first the case where $\{1/2<b\leq 1\}\cap\{2b-a<1/2\}$. In this case, it is easily seen that under Assumptions 2 and 7

$$\begin{aligned}
\frac{n^{p/2}}{n^{pq/2\rho_1}} &= O\left(n^{\frac{1}{2}p(1-q/\rho_1)}\right),\\
n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1} &= O\left(n^{p/2-(2b-a)pq/\rho_1}\right)
\end{aligned}$$

so that, given that $2b-a<1/2$, we have that

$$n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{n^{pq/2\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(n^{p/2-(2b-a)pq/\rho_1}\right).$$

Moreover, note that

$$
\begin{aligned}
& \frac{p}{2}-(2b-a)\,p\frac{q}{\rho_1} \\
= & \; p\left\{\frac{1}{2}-(2b-a)\left(G+1+\frac{a}{2b-a}\right)\right\} \\
= & \; p\left\{\frac{1}{2}-(2b-a)\,(G+1)-a\right\} \\
= & \; p\left\{\frac{1}{2}-2b\,(G+1)+aG\right\} \\
< & \; p\left\{\frac{1}{2}-(G+1)+aG\right\} \quad \text{[since } b>1/2 \text{ in this case]} \\
= & \; -p\left(\frac{1}{2}+[1-a]\,G\right) \\
\leq & \; 0 \quad \text{[since } a\leq 1 \text{ by Assumption 7]}.
\end{aligned}
$$

It follows that

$$
n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a<1/2\right\}=O\left(\max\left\{\frac{n^{p/2}}{n^{pq/2\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(1\right). \tag{74}
$$

Now, consider the case $\{1/2<b\leq 1\}\cap\{2b-a\geq 1/2\}$. Here, under Assumptions 2 and 7 and given that $2b-a\geq 1/2$, we have

$$
n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}=O\left(\max\left\{\frac{n^{p/2}}{n^{pq/2\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=O\left(n^{\frac{1}{2}p(1-q/\rho_1)}\right).
$$

Since

$$
\frac{1}{2}p\left(1-\frac{q}{\rho_1}\right)=\frac{1}{2}p\left(1-[G+1+2a]\right)=-\frac{1}{2}p\left(G+2a\right)<0
$$

from which it follows that

$$
n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a\geq 1/2\right\}=O\left(\max\left\{\frac{n^{p/2}}{n^{pq/2\rho_1}},n^{p/2}\left(\frac{K}{\mu_n^4}\right)^{pq/\rho_1}\right\}\right)=o\left(1\right). \tag{75}
$$

From (72)-(75), we deduce immediately that

$$
\begin{aligned}
& n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1} \\
= & \; n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{a/2<b\leq 1/2\right\}+n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{1/2<b\leq 1\right\} \\
= & \; n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a<b\right\} \\
& +n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{a/2<b\leq 1/2\cap 2b-a\geq b\right\} \\
& +n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a<1/2\right\} \\
& +n^{p/2}\left(P\left(\mathcal{A}_n^C\right)\right)^{1/\rho_1}\mathbb{I}\left\{1/2<b\leq 1\cap 2b-a\geq 1/2\right\} \\
= & \; O\left(1\right),
\end{aligned}
$$

which is the result to be proved for part (b).

The proof of part (c) follows immediately from that of part (b) by noting that $K<n$ and $s_1=\rho_1$. *Q.E.D.*

Lemma C7: Suppose that Assumptions 1, 7, and 9 are satisfied; then, there exists a positive constant $\overline{C}$ such that for all n sufficiently large

$$E\left(\det\left[X_*'MX_*/n\right]\right)^{-p(1+\eta)/\eta} \leq \overline{C} < \infty;$$

where $X_* = \begin{bmatrix} \varepsilon & X \end{bmatrix}$.

Proof: To proceed, note that by Liapunov's inequality

$$E\left(\det\left[X_*'MX_*/n\right]\right)^{-p(1+\eta)/\eta} \leq \left(E\left[\left(\det\left[X_*'MX_*/n\right]\right)^{-2p(1+\eta)/\eta}\right]\right)^{1/2},$$

so that the desired result follows immediately from the boundedness of $E\left[\left(\det\left[X_*'MX_*/n\right]\right)^{-2p(1+\eta)/\eta}\right]$ as given in Assumption 9. *Q.E.D.*

Lemma C7: If Assumptions 1-4, 7, 8, and 9 are satisfied, then

$$E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] = O(1)$$

Proof: To proceed, write

$$\begin{aligned}
\left\|\widehat{\delta}_{HFUL}\right\| &= \left\|\delta_0 + \left(X'\left[P - D_P\right]X - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]X\right)^{-1}\right. \\
&\qquad \left.\times\left(X'\left[P - D_P\right]\varepsilon - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]\varepsilon\right)\right\| \\
&\leq \left\|\delta_0\right\| + \left\{\left\|\left(X'\left[P - D_P\right]X - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]X\right)^{-1}\right\|\right. \\
&\qquad \left.\times\left\|X'\left[P - D_P\right]\varepsilon - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]\varepsilon\right\|\right\},
\end{aligned}$$

where the inequality above follows from T and the submultiplicativity of the Euclidean norm. Note that, since the matrix $X'\left[P_Z - D_P\right]X - \widetilde{\kappa}_{HLIM}X'\left[M + D_P\right]X + (C/n)\,X'D_PX$ is positive semidefinite by Lemma B3, we can apply part (a) of Lemma B4 to obtain

$$\begin{aligned}
&\left\|\left(X'\left[P - D_P\right]X - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]X\right)^{-1}\right\| \\
\leq\ &\left\|\left(\frac{C}{n}X'MX\right)^{-1}\right\| = \left[\det\left(C\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|.
\end{aligned}$$

Using this inequality, it follows that

$$\begin{aligned}
&\left\|\widehat{\delta}_{HFUL}\right\| \\
\leq\ &\left\|\delta_0\right\| + \left\|\left(\frac{C}{n}X'MX\right)^{-1}\right\|\left\|X'\left[P - D_P\right]\varepsilon - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]\varepsilon\right\| \\
=\ &\left\|\delta_0\right\| + C^{-G}\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\| \\
&\qquad \times\left\|X'\left[P - D_P\right]\varepsilon - \left\{\widetilde{\kappa}_{HLIM} - \frac{C}{n}\right\}X'\left[M + D_P\right]\varepsilon\right\|. \qquad (76)
\end{aligned}$$

Next, let $\widetilde{\delta}_\Delta = \left(1, -\widetilde{\delta}'_{HLIM}\right)'$, $\delta_{\Delta,0} = \left(1, -\delta'_0\right)'$, $X_* = \left[\begin{array}{cc} \varepsilon & X \end{array}\right]$ and $\overline{X} = \left[\begin{array}{cc} y & X \end{array}\right]$, and note that

$$\widetilde{\kappa}_{HLIM} = \frac{\widetilde{\delta}'_\Delta \overline{X}' \left[P - D_P\right] \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta} = \min_{\delta_\Delta : \delta_{\Delta,1} = 1} \frac{\delta'_\Delta \overline{X}' \left[P - D_P\right] \overline{X} \delta_\Delta}{\delta'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \delta_\Delta},$$

from which it follows that

$$-\frac{\widetilde{\delta}'_\Delta \overline{X}' D_P \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta} \leq \frac{\widetilde{\delta}'_\Delta \overline{X}' \left[P - D_P\right] \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta} \leq \frac{\delta'_{\Delta,0} \overline{X}' \left[P - D_P\right] \overline{X} \delta_{\Delta,0}}{\delta'_{\Delta,0} \overline{X}' \left[M + D_P\right] \overline{X} \delta_{\Delta,0}} = \frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}$$

so that

$$\begin{aligned}
\left|\widetilde{\kappa}_{HLIM}\right| &\leq \max\left\{\frac{\widetilde{\delta}'_\Delta \overline{X}' D_P \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta}, \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right|\right\} \\
&\leq \frac{\widetilde{\delta}'_\Delta \overline{X}' D_P \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta} + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \\
&\leq C_P \left(\frac{K}{n}\right) \frac{\widetilde{\delta}'_\Delta \overline{X}' \overline{X} \widetilde{\delta}_\Delta}{\widetilde{\delta}'_\Delta \overline{X}' \left[M + D_P\right] \overline{X} \widetilde{\delta}_\Delta} + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \quad \text{[by Assumption 7]} \\
&\leq C_P \left(\frac{K}{n}\right) \left\{\max_{\gamma : \|\gamma\| = 1} \gamma' \left[\left(\frac{\overline{X}' \left[M + D_P\right] \overline{X}}{n}\right)^{-1/2} \frac{\overline{X}' \overline{X}}{n} \left(\frac{\overline{X}' \left[M + D_P\right] \overline{X}}{n}\right)^{-1/2}\right] \gamma\right\} + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \\
&\leq C_P \left(\frac{K}{n}\right) \left\|\left(\frac{\overline{X}' \left[M + D_P\right] \overline{X}}{n}\right)^{-1/2} \frac{\overline{X}' \overline{X}}{n} \left(\frac{\overline{X}' \left[M + D_P\right] \overline{X}}{n}\right)^{-1/2}\right\| + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \\
&= C_P \left(\frac{K}{n}\right) \left\|\frac{D\left(\delta_0\right)'^{-1} \overline{X}' \overline{X} D\left(\delta_0\right)^{-1}}{n} \left(\frac{D\left(\delta_0\right)'^{-1} \overline{X}' \left[M + D_P\right] \overline{X} D\left(\delta_0\right)^{-1}}{n}\right)^{-1}\right\| + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \\
&= C_P \left(\frac{K}{n}\right) \left\|\frac{X'_* X_*}{n} \left(\frac{X'_* \left[M + D_P\right] X_*}{n}\right)^{-1}\right\| + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' \left[M + D_P\right] \varepsilon}\right| \\
&\leq C_P \left(\frac{K}{n}\right) \left\|\frac{X'_* X_*}{n}\right\| \left\|\left(\frac{X'_* M X_*}{n}\right)^{-1}\right\| + \left|\frac{\varepsilon' \left[P - D_P\right] \varepsilon}{\varepsilon' M \varepsilon}\right| \qquad (77)
\end{aligned}$$

where $D\left(\delta_0\right) = \left(\begin{array}{cc} 1 & 0 \\ \delta_0 & I_G \end{array}\right)$ as defined in Lemma A3 of the paper, so that $D\left(\delta_0\right)^{-1} = \left(\begin{array}{cc} 1 & 0 \\ -\delta_0 & I_G \end{array}\right)$ and $X_* = \left[\begin{array}{cc} \varepsilon & X \end{array}\right]$. Applying the upper bound (77) to (76), we obtain, via T and the submultiplicativity of the

Frobenius norm

$$
\begin{aligned}
& \left\|\widehat{\delta}_{HFUL}\right\| \\
\leq\ & \|\delta_0\| + C^{-G}\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\| \\
& \times\left\|X'\left[P-D_P\right]\varepsilon-\left\{\widetilde{\kappa}_{HLIM}-\frac{C}{n}\right\}X'\left[M+D_P\right]\varepsilon\right\| \\
\leq\ & \|\delta_0\| + C^{-G}\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\sqrt{n}\left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\| \\
& +C^{-G}\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\left[\det\left(\frac{X_*'MX_*}{n}\right)\right]^{-1} \\
& \times\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|C_PK\left\|\frac{X_*'X_*}{n}\right\|\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \\
& +C^{-G}\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\sqrt{K}\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right| \\
& \times\left|\left(\frac{\varepsilon'M\varepsilon}{n}\right)^{-1}\right|\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \\
& +C^{-(G-1)}C\left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \\
\leq\ & \|\delta_0\| + \left(C\left\{\frac{n-K}{n}\right\}\right)^{-G}\left[\det\left(\frac{X'MX}{n-K}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\sqrt{n}\left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\| \\
& +\left(C\left\{\frac{n-K}{n}\right\}\right)^{-G}\left[\det\left(\frac{X'MX}{n-K}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\left(\frac{n}{n-K}\right)^{L}\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-1} \\
& \quad\times\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|C_PK\left\|\frac{X_*'X_*}{n}\right\|\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \\
& +\left(C\left\{\frac{n-K}{n}\right\}\right)^{-G}\left(\frac{n}{n-K}\right)\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\sqrt{K}\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right| \\
& \quad\times\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \\
& +\left(C\left\{\frac{n-K}{n}\right\}\right)^{-G}C\left[\det\left(\frac{X'MX}{n-K}\right)\right]^{-1}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\| \qquad (78)
\end{aligned}
$$

Let G be the number of endogenous regressors in the IV regression and a be as defined in Assumption 7, and we shall consider four different cases: (i) $G \geq 2$ and $0 < a \leq 1$; (ii) $G \geq 2$ and $a = 0$; (iii) $G = 1$ and $0 < a \leq 1$; (iv) $G = 1$ and $a = 0$.We start with the case $G \geq 2$ and $0 < a \leq 1$. It follows from (78) and Loève's c_r inequality that we can obtain the upper bound

$$
\begin{aligned}
& E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] \\
\leq\ & 5^{p-1}\left\|\delta_0\right\|^p + 5^{p-1}\Bigg\{C^{-Gp} n^{p/2} E\left(\left[\det\left(\frac{X'MX}{n}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-Gp} C_P^p K^p E\left(\left[\det\left(\frac{X'MX}{n}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left[\det\left(\frac{X_*'MX_*}{n}\right)\right]^{-p}\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^p\right. \\
& \qquad\qquad \left. \times\left\|\frac{X_*'X_*}{n}\right\|^p\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-Gp} K^{p/2} E\left(\left[\det\left(\frac{X'MX}{n}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right|^p\left|\frac{\varepsilon'M\varepsilon}{n}\right|^{-p}\right. \\
& \qquad\qquad \left. \times\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-(G-1)p} E\left(\left[\det\left(\frac{X'MX}{n}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right)\Bigg\} \\
& \text{[by Loève's } c_r \text{ inequality]} \\
\leq\ & 5^{p-1}\left\|\delta_0\right\|^p \\
& + 5^{p-1}\left(\frac{n}{n-K}\right)^{pL}\Bigg\{C^{-Gp} n^{p/2} E\left(\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^p\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\right. \\
& \qquad\qquad \left. \times\left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-Gp} C_P^p\left(\frac{n}{n-K}\right)^{pL} K^p E\left(\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^p\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-2p}\right. \\
& \qquad\qquad \left. \times\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^p\left\|\frac{X_*'X_*}{n}\right\|^p\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-Gp} K^{p/2} E\left(\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right|^p\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right) \\
& + C^{-(G-1)p} E\left(\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^p\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-p}\left\|adj\left(C\frac{X'MX}{n}\right)\right\|^p\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right)\Bigg\}
\end{aligned}
$$

Applying the generalized Hölder inequality, we further obtain

$$
\begin{aligned}
& E\left[\left\|\widehat{\delta}_{HFUL}\right\|^{p}\mathbb{I}_{\mathcal{A}^C}\right] \\
\leq\ & \overline{C}+5^{p-1}\left(\frac{n}{n-K}\right)^{pL}\left\{C^{-p}n^{p/2}\left[P\left(\mathcal{A}^C\right)\right]^{1/\rho_1}\left(E\left\|\frac{X'\left[P-D_P\right]\varepsilon}{\sqrt{n}}\right\|^{p\rho_2}\right)^{1/\rho_2}\right. \\
& \qquad\times\left(E\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-p\rho_3}\right)^{1/\rho_3}\left(E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{p\rho_4}\right)^{1/\rho_4} \\
& \qquad\times\left(E\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^{p\rho_5}\right)^{1/\rho_5} \\
& +C^{-p}C_P^p\left(\frac{n}{n-K}\right)^{pL}K^p\left[P\left(\mathcal{A}^C\right)\right]^{1/r_1}\left(E\left\|\frac{X_*'X_*}{n}\right\|^{pr_2}\right)^{1/r_2}\left(E\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^{pr_3}\right)^{1/r_3} \\
& \times\left(E\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-2pr_4}\right)^{1/r_4}\left(E\left\|adj\left(\frac{X_*'MX_*}{n}\right)\right\|^{pr_5}\right)^{1/r_5} \\
& \times\left(E\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^{pr_6}\right)^{1/r_6}\left(E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{pr_7}\right)^{1/r_7} \\
& +C^{-p}K^{p/2}\left[P\left(\mathcal{A}^C\right)\right]^{1/s_1}\left(E\left|\frac{\varepsilon'\left[P-D_P\right]\varepsilon}{\sqrt{K}}\right|^{ps_2}\right)^{1/s_2}\left(E\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^{ps_3}\right)^{1/s_3} \\
& \times\left(E\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-ps_4}\right)^{1/s_4}\left(E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{ps_5}\right)^{1/s_5} \\
& +\left(E\left\|\frac{X'\left[M+D_P\right]\varepsilon}{n}\right\|^{pd_1}\right)^{1/d_1}\left(E\left[\det\left(\frac{X_*'MX_*}{n-K}\right)\right]^{-pd_2}\right)^{1/d_2}\left(E\left\|adj\left(\frac{X'MX}{n}\right)\right\|^{pd_3}\right)^{1/d_3} \\
& \left.\times\left(E\left[\frac{\det\left(X_*'MX_*/n\right)}{\det\left(X'MX/n\right)}\right]^{pd_4}\right)^{1/d_4}\right\} \qquad (79)
\end{aligned}
$$

Next, we choose the exponents $\rho_1,..,\rho_5$; $r_1,...,r_7$; $s_1,..,s_5$; and $d_1,..,d_4$ as follows

$$
\begin{aligned}
\rho_1 &= (1+\eta)\left[\frac{2\left(G+1\right)+\varphi\left(a,b\right)}{G+1+\varphi\left(a,b\right)}\right], \\
\rho_2 &= \rho_5=q=(1+\eta)\left[2\left(G+1\right)+\varphi\left(a,b\right)\right],\ \rho_3=\left(\frac{1+\eta}{\eta}\right), \\
\rho_4 &= \frac{q}{G-1}\leq q \text{ for } G\geq 2; \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= (1+\eta)\left[\frac{2\left(G+1\right)+\varphi\left(a,b\right)}{\varphi\left(a,b\right)}\right], \\
r_2 &= r_3=r_6=q=(1+\eta)\left[2\left(G+1\right)+\varphi\left(a,b\right)\right], \\
r_4 &= \left(\frac{1+\eta}{\eta}\right),\ r_5=\frac{q}{G},\ r_7=\frac{q}{G-1}; \qquad (81)
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= (1+\eta)\left[\frac{2\left(G+1\right)+\varphi\left(a,b\right)}{G+1+\varphi\left(a,b\right)}\right],\ s_2=s_3=q, \\
s_4 &= \left(\frac{1+\eta}{\eta}\right),\ s_5=\frac{q}{G-1}\leq q \text{ for } G\geq 2; \qquad (82)
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= d_4 = (1+\eta)(G+1) < q \\
d_2 &= \left(\frac{1+\eta}{\eta}\right), \quad d_3 = \frac{d_1}{G-1},
\end{aligned} \tag{83}
$$

It is easily verified that

$$
1 = \sum_{j=1}^{4} \frac{1}{\rho_j} = \sum_{j=1}^{7} \frac{1}{r_j} = \sum_{j=1}^{5} \frac{1}{s_j} = \sum_{j=1}^{4} \frac{1}{d_j}
$$

as required. Note also that these choices of exponents satisfy the moment condition of Assumption 8. It follows, thus, by applying Assumption 9 and Lemmas C4-C7 that

$$
E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] = o(1) + O(1) + o(1) + O(1) = O(1)
$$

for the case where $G \geq 2$ and $0 < a \leq 1$.

Next, consider the case where $G \geq 2$ and $a = 0$. When $a = 0$, we may have $\varphi(0, b) = 0$; but the generalized Hölder inequality still applies if we take $r_1 = \infty$ with all other exponents $\rho_1, .., \rho_4$; $r_2, ..., r_7$; $s_1, .., s_5$; and $d_1, ..., d_4$ chosen as in (80)-(83) above (but with $\varphi(a, b) = 0$). Hence, an upper bound of the form given by (79) still holds but with the L_{r_1} norm $\left(E\mathbb{I}_{\mathcal{A}^C}^{r_1}\right)^{1/r_1} = \left[P\left(\mathcal{A}^C\right)\right]^{1/r_1}$ (for finite r_1) replaced by the L_∞ norm

$$
\|\mathbb{I}_{\mathcal{A}^C}\|_\infty = ess \sup |\mathbb{I}_{\mathcal{A}^C}| = \inf\{x : P(|\mathbb{I}_{\mathcal{A}^C}| > x) = 0\} = 1.
$$

Since K is fixed in this case, the term $K^p \|\mathbb{I}_{\mathcal{A}^C}\|_\infty$ is trivially bounded. Applying Assumption 9 and Lemmas C4-C5, C7, and parts (b) and (c) of Lemma C6 then allows us to deduce that $E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] = O(1)$ for this case.

We now turn our attention to the case where $G = 1$ and $0 < a \leq 1$. Note that, when $G = 1$, $X'MX/n$ is 1×1, so that we have

$$
\begin{aligned}
\left(\frac{X'MX}{n}\right)^{-1} &= \left[\det\left(\frac{X'MX}{n}\right)\right]^{-1}, \\
adj\left(\frac{X'MX}{n}\right) &= 1.
\end{aligned} \tag{84}
$$

Again, we can still obtain an upper bound of the form (79) by applying the generalized Hölder inequality, this time with $\rho_4 = r_7 = s_5 = d_3 = \infty$ and with all other exponents $\rho_1, ., \rho_3$; $r_1, ..., r_6$; $s_1, .., s_4$; and d_1, d_2, d_4 chosen as in (80)-(83) above (setting $G = 1$). The L_∞ norm of $[adj(X'MX/n)]^p$ is clearly bounded in light of (84); so that using parts (a), (c), and (d) of Lemma C4; Assumption 9; and Lemmas C5-C7; we deduce that $E\left[\left\|\widehat{\delta}_{HFUL}\right\|^p \mathbb{I}_{\mathcal{A}^C}\right] = O(1)$ for this case as well.

Finally, the case where both $G = 1$ and $a = 0$ can be handled in an obvious way by combining the arguments above. *Q.E.D.*

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