Department of Economics
University of Maryland

John C. Chao
Spring 2013

Economics 422

Final Examination Solution Sheet

QUESTION 1

(a) Note that

$$
\begin{aligned}
& \widehat{\Delta Earn} \\
= & -683.21 + (65.83 \times 26) - \left(1.05 \times 26^2\right) + \left(0.005 \times 26^3\right) \\
& -163.23 \times Female \\
& - \left[-683.21 + (65.83 \times 25) - \left(1.05 \times 25^2\right) + \left(0.005 \times 25^3\right)\right. \\
& \left.-163.23 \times Female\right] \\
= & 65.83 - 1.05\left(26^2 - 25^2\right) + 0.005\left(26^3 - 25^3\right) \\
\approx & 22.04
\end{aligned}
$$

(b) Based on the limited test results presented, it seems that specification (1) should be preferred over specification (2). This is because, for regression (2), the value of the t-statistic on the coefficient of Age^2, Age^3 and Age^4, respectively

$$
\begin{aligned}
t_{\widehat{\beta}_2} &= \frac{-1.69}{1.06} = 1.594, \\
t_{\widehat{\beta}_3} &= \frac{0.015}{0.016} = 0.9375, \\
t_{\widehat{\beta}_3} &= \frac{0.0005}{0.0009} \approx 0.5556
\end{aligned}
$$

so that none of these coefficients are significantly different from zero even at the 10% significance level. On the other hand, for regression (1), the t-statistic on the coefficient of Age, Age^2 and Age^3 are, respectively,

$$
\begin{aligned}
t_{\widehat{\beta}_1} &= \frac{65.83}{9.27} \approx 7.101, \\
t_{\widehat{\beta}_2} &= \frac{-1.05}{0.22} \approx -4.773, \\
t_{\widehat{\beta}_3} &= \frac{0.005}{0.002} = 2.5,
\end{aligned}
$$

all of which are significant even at the 1% significance level. The discrepancy between the two regression results could possibly be explained by the fact that adding Age^4 as a regressor in specification (2) might have caused some multicollinearity. It would be good to use more sophisticated model selection methods to select the order of the polynomial regression, but that is beyond the scope of this course.

QUESTION 2

(a) Taking the natural log transformation on both sides of regression (3), we obtain

$$\begin{aligned} \ln Q_t &= \ln C + \alpha \ln L_t + \beta \ln K_t + u_t \\ &= \gamma + \alpha \ln L_t + \beta \ln K_t + u_t \quad (\text{setting } \gamma = \ln C), \end{aligned}$$

which is a linear regression model since it is linear in coefficients.

(b) Note that we have a log-log specification here. Hence, a 1% change in L_t leads to an α% change in Q_t, so that α is the output elasticity of labor. On the other hand, a 1% change in K_t leads to an β% change in Q_t, so that β is the output elasticity of capital.

(c) The following conditions are sufficient for unbiased and consistent estimation of γ, α, and β.

(i) $E\left[u_t | \ln L_t, \ln K_t\right] = 0$ with probability one;

(ii) $(\ln L_t,\ \ln K_t,\ \ln Q_t)_{t=1}^{T}$ are i.i.d.;

(iii) $E\left(\ln L_t\right)^4 < \infty, E\left(\ln K_t\right)^4 < \infty,\ E\left(\ln Q_t\right)^4 < \infty$, so that large outliers are rare;

(iv) With probability one, there is no perfect multicollinearity.

QUESTION 3

Note first that the TSLS estimator of β for this IV regression model without intercept is

$$\widehat{\beta}^{TSLS} = \frac{\sum_{i=1}^{n} Z_i Y_i}{\sum_{i=1}^{n} Z_i X_i} = \frac{n^{-1}\sum_{i=1}^{n} Z_i Y_i}{n^{-1}\sum_{i=1}^{n} Z_i X_i}$$

assuming that

$$\frac{1}{n}\sum_{i=1}^{n} Z_i X_i \neq 0.$$

Now, estimating the regression

$$\widehat{u}_i = \gamma Z_i + \eta_i,\ i = 1, ..., n.$$

we get by the OLS formula

$$\widehat{\gamma}^{OLS}=\frac{\sum_{i=1}^{n}Z_i\widehat{u}_i}{\sum_{i=1}^{n}Z_i^2}=\frac{n^{-1}\sum_{i=1}^{n}Z_i\widehat{u}_i}{n^{-1}\sum_{i=1}^{n}Z_i^2}$$

Focusing on the numerator of the formula for $\widehat{\gamma}^{OLS}$, we see that

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}Z_i\widehat{u}_i \\
= & \frac{1}{n}\sum_{i=1}^{n}Z_i\left(Y_i-\widehat{\beta}^{TSLS}X_i\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}Z_iY_i-\widehat{\beta}^{TSLS}\frac{1}{n}\sum_{i=1}^{n}Z_iX_i \\
= & \frac{1}{n}\sum_{i=1}^{n}Z_iY_i-\left(\frac{n^{-1}\sum_{i=1}^{n}Z_iY_i}{n^{-1}\sum_{i=1}^{n}Z_iX_i}\times\frac{1}{n}\sum_{i=1}^{n}Z_iX_i\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}Z_iY_i-\frac{1}{n}\sum_{i=1}^{n}Z_iY_i \\
= & 0,
\end{aligned}$$

so that $\widehat{\gamma}^{OLS}=0$, provided of course that

$$\frac{1}{n}\sum_{i=1}^{n}Z_i^2>0,$$

i.e., not all values of Z_i in the sample are zero.

QUESTION 4

,

(a) Given the system of equations

$$\begin{aligned}
Q_i &= \beta_0+\beta_1P_i+\beta_2Z_i+u_i, \\
P_i &= \pi_0+\pi_1Z_i+v_i,
\end{aligned}$$

we solve first for the reduced form of the first equation above as follows

$$\begin{aligned}
Q_i &= \beta_0+\beta_1P_i+\beta_2Z_i+u_i \\
&= \beta_0+\beta_1\left(\pi_0+\pi_1Z_i+v_i\right)+\beta_2Z_i+u_i \\
&= \left(\beta_0+\beta_1\pi_0\right)+\left(\beta_1\pi_1+\beta_2\right)Z_i+\left(u_i+v_i\beta_1\right) \\
&= \varphi_0+\varphi_1Z_i+\eta_i
\end{aligned}$$

Hence, we have the following two equations

$$\begin{aligned}\varphi_0 &= \beta_0 + \beta_1 \pi_0 & \text{(I)}\\ \varphi_1 &= \beta_1 \pi_1 + \beta_2 & \text{(II)}\end{aligned}$$

which link the structural parameters β_0, β_1, and β_2 with the reduced form parameters π_0, π_1, φ_0, and φ_1. Note that although all the reduced form parameters π_0, π_1, φ_0, and φ_1 can be estimated consistently, equation (I) and (II) still leaves with two equations and three unknowns β_0, β_1, and β_2. Hence, none of the structural parameters β_0, β_1, and β_2 are identified; and, in consequence, none of them are consistently estimable.

(b) If $\pi_0 = 0$, then equations (I) becomes

$$\varphi_0 = \beta_0$$

Since φ_0 is a reduced form parameter which can be estimated by OLS consistently from the equation

$$Q_i = \varphi_0 + \varphi_1 Z_i + \eta_i,$$

it follows that, in this case, β_0 is identified and can be estimated consistently. However, equation (II) remains the same in this case, giving one equation for the remaining two structural parameters β_1 and β_2. Hence, neither β_1 nor β_2 is identified and neither can be estimated consistently.

(c) If $\pi_1 = 0$, then equations (II) becomes

$$\varphi_1 = \beta_2$$

Since φ_1 can be estimated by OLS consistently

$$Q_i = \varphi_0 + \varphi_1 Z_i + \eta_i,$$

it follows that β_2 is identified in this case and can be estimated consistently. However, equation (I) remains the same in this case, giving one equation for the two remaining structural parameters β_0 and β_1. Hence, neither β_0 nor β_1 is identified and neither can be estimated consistently.

(d) For the simultaneous equations system

$$\begin{aligned}Q_i &= \beta_0 + \beta_1 P_i + \beta_2 Z_i + u_i,\\ P_i &= \pi_0 + \pi_1 Z_i + \pi_2 W_i + v_i,\end{aligned}$$

we again solve for the reduced form of the first equation as follows

$$\begin{aligned}Q_i &= \beta_0 + \beta_1 P_i + \beta_2 Z_i + u_i\\ &= \beta_0 + \beta_1 (\pi_0 + \pi_1 Z_i + \pi_2 W_i + v_i) + \beta_2 Z_i + u_i\\ &= (\beta_0 + \beta_1 \pi_0) + (\beta_1 \pi_1 + \beta_2) Z_i + \beta_1 \pi_2 W_i + (u_i + v_i \beta_1)\\ &= \varphi_0 + \varphi_1 Z_i + \varphi_2 W_i + \eta_i\end{aligned}$$

This now gives us three equations

$$\begin{aligned} \varphi_0 &= \beta_0 + \beta_1 \pi_0 && \text{(I)} \\ \varphi_1 &= \beta_1 \pi_1 + \beta_2, && \text{(II)} \\ \varphi_2 &= \beta_2 \pi_2, && \text{(III)} \end{aligned}$$

linking the structural parameters β_0, β_1, and β_2 with the reduced form parameters π_0, π_1, φ_0, φ_0, and φ_2. Since all the reduced form parameters π_0, π_1, φ_0, and φ_1 can be estimated consistently, this now gives us three equations with which we can recover the three unknowns β_0, β_1, and β_2. Hence, all three structural parameters β_0, β_1, and β_2 are now identified; and they can all be estimated consistently.

QUESTION 5

Note first that we can rewrite the null hypothesis as

$$H_0 : \beta_2 + 6\beta_3 - 1 = 0.$$

Write

$$\begin{aligned} SALES_i &= \beta_0 + \beta_1 PRICE_i + \beta_2 ADEXP_i + \beta_3 (ADEXP_i)^2 + u_i \\ &= \beta_0 + \beta_1 PRICE_i + (\beta_2 + 6\beta_3 - 1)\, ADEXP_i \\ &\quad + \beta_3 \left[(ADEXP_i)^2 - 6 ADEXP_i \right] + ADEXP_i + u_i \end{aligned}$$

Now, subtracting $ADEXP_i$ from both sides of this equation, we get

$$\begin{aligned} SALES_i - ADEXP_i &= \beta_0 + \beta_1 PRICE_i + (\beta_2 + 6\beta_3 - 1)\, ADEXP_i \\ &\quad + \beta_3 \left[(ADEXP_i)^2 - 6 ADEXP_i \right] + u_i \end{aligned}$$

or

$$Y_i = \beta_0 + \beta_1 PRICE_i + \gamma ADEXP_i + \beta_3 X_i + u_i, \qquad (*)$$

where

$$\begin{aligned} Y_i &= SALES_i - ADEXP_i, \\ \gamma &= \beta_2 + 6\beta_3 - 1, \\ X_i &= (ADEXP_i)^2 - 6 ADEXP_i. \end{aligned}$$

Hence, the null hypothesis

$$H_0 : \beta_2 + 6\beta_3 - 1 = 0$$

is the same as testing the null hypothesis

$$H_0 : \gamma = 0$$

in the regression given by (*). We can then test H_0 by first running the regression (*) and then compute the t-statistic for testing

$$\mathbb{T} = \frac{\widehat{\gamma}}{SE(\widehat{\gamma})}$$

Under regularity conditions, the large sample distribution of $\mathbb{T}$ under H_0 will be $N(0,1)$, so that at the 5% significance level, we will reject H_0 if

$$|\mathbb{T}| > 1.96.$$

Alternatively, we can write

$$\begin{aligned} SALES_i &= \beta_0 + \beta_1 PRICE_i + \beta_2 ADEXP_i + \beta_3 (ADEXP_i)^2 + u_i \\ &= \beta_0 + \beta_1 PRICE_i + (\beta_2 + 6\beta_3 - 1)\frac{1}{6}(ADEXP_i)^2 \\ &\quad + \beta_2\left[ADEXP_i - \frac{1}{6}(ADEXP_i)^2\right] + \frac{1}{6}(ADEXP_i)^2 + u_i \end{aligned}$$

In this case, subtract $\frac{1}{6}(ADEXP_i)^2$ from both sides of this equation, and we get

$$\begin{aligned} SALES_i - \frac{1}{6}(ADEXP_i)^2 &= \beta_0 + \beta_1 PRICE_i + (\beta_2 + 6\beta_3 - 1)\frac{1}{6}(ADEXP_i)^2 \\ &\quad + \beta_2\left[ADEXP_i - \frac{1}{6}(ADEXP_i)^2\right] + u_i \end{aligned}$$

or

$$Y_i = \beta_0 + \beta_1 PRICE_i + \gamma X_{1i} + \beta_2 X_{2i} + u_i, \qquad (**)$$

where

$$\begin{aligned} Y_i &= SALES_i - \frac{1}{6}(ADEXP_i)^2, \\ \gamma &= \beta_2 + 6\beta_3 - 1, \\ X_{1i} &= \frac{1}{6}(ADEXP_i)^2 \\ X_{2i} &= ADEXP_i - \frac{1}{6}(ADEXP_i)^2. \end{aligned}$$

Hence, the null hypothesis

$$H_0 : \beta_2 + 6\beta_3 - 1 = 0$$

is also the same as testing the null hypothesis

$$H_0 : \gamma = 0$$

in the regression given by (**). Hence, we can test H_0 using a t-statistic obtained from estimating γ by OLS from regression (**) in the same way that we have described testing for $\gamma = 0$, based on regression (*).