QUESTION 1

(a) By the central limit theorem

\[ \frac{\bar{Y} - \mu_Y}{\sigma_Y/\sqrt{n}} \xrightarrow{d} N(0, 1) \text{ as } n \to \infty. \]

It follows that in this case we have

\[ P(\bar{Y} < 101) = P\left( \frac{\bar{Y} - 100}{\sqrt{43/100}} < \frac{101 - 100}{\sqrt{43/100}} \right) \]
\[ \approx P(Z < 1.52) \]
\[ = 0.9357 \]

(b) Similar to part (a), we have

\[ P(101 < \bar{Y} < 103) = P\left( \frac{101 - 100}{\sqrt{43/64}} < \frac{\bar{Y} - 100}{\sqrt{43/64}} < \frac{103 - 100}{\sqrt{43/64}} \right) \]
\[ \approx P(1.22 < Z < 3.66) \]
\[ = P(Z < 3.66) - P(Z \leq 1.22) \]
\[ = 0.9999 - 0.8888 \]
\[ = 0.1111 \]

(c) Finally, by the central limit theorem, we have

\[ P(\bar{Y} > 98) = P\left( \frac{\bar{Y} - 100}{\sqrt{43/165}} > \frac{98 - 100}{\sqrt{43/165}} \right) \]
\[ \approx P(Z > -3.92) \]
\[ = P(Z < 3.92) \text{ by symmetry} \]
\[ = 0.99995 \]
QUESTION 2

(a) First, we compute the t-ratio as follows:

\[ T = \frac{\hat{p} - 0.5}{\sqrt{\hat{p}(1 - \hat{p})/n}} = \frac{0.54 - 0.5}{\sqrt{0.54(0.46)/1055}} = 2.61 \]

Using the central limit theorem and testing against a two-sided alternative at the 5% significance level, our decision rule is

Reject \( H_0 \) if \(|T| > 1.96\)

Since \( T = 2.61 > 1.96 \),

we, thus, reject \( H_0 \).

(b) Note that the (realized) value of the t-statistic here is the same as that computed in part (a). However, now we are testing against a one-sided alternative at the significance level of 5%, and so our decision rule is

Reject \( H_0 \) if \( T > 1.645 \)

Since \( T = 2.61 > 1.645 \),

we of course reject \( H_0 \).

(c) Using the central limit theorem, the approximate 95% confidence interval is given by

\[ \left[ \hat{p} - 1.96\sqrt{\hat{p}(1 - \hat{p})/n}, \hat{p} + 1.96\sqrt{\hat{p}(1 - \hat{p})/n} \right] \]

\[ = \left[ 0.54 - 1.96\sqrt{0.54(0.46)/1055}, 0.54 + 1.96\sqrt{0.54(0.46)/1055} \right] \]

\[ = [0.509, 0.571] \]

QUESTION 3

Note that

\[ P \left( \left| \bar{X} - \mu \right| < 0.1 \sigma \right) \]

\[ = P \left( \frac{\left| \bar{X} - \mu \right|}{\sigma/\sqrt{64}} < \frac{0.1 \sigma}{\sigma/\sqrt{64}} \right) \]

\[ = P \left( \left| Z \right| < 0.8 \right) \]

\[ = P (-0.8 < Z < 0.8) \]

\[ = 2 \left[ P (Z < 0.8) - 0.5 \right] \]

\[ = 0.576. \]
QUESTION 4

(a) The predicted weight is
\[ \hat{\text{Weight}} = -99.41 + 3.94 \times \text{Height} \]
\[ = -99.41 + 3.94 \times 70 \]
\[ = 176.39. \]

(b) The predicted increase in weight is
\[ \Delta \hat{\text{Weight}} = 3.94 \times \Delta \text{Height} \]
\[ = 3.94 \times 1.5 \]
\[ = 5.91. \]

BONUS QUESTION

(a) Note that
\[ E[\overline{Y}] = \frac{1}{n} \sum_{i=1}^{n} E[Y_i] = \frac{1}{n} \sum_{i=1}^{n} \mu_Y = \mu_Y, \]
so \( \overline{Y} \) is unbiased.

(b) Taking expectation, we have
\[ E[\overline{Y}^2] = E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right)^2 \right] \]
\[ = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i Y_j \right] \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} E[Y_i^2] + 2 \frac{1}{n^2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} E[Y_i] E[Y_j] \]
by independence
\[ = \frac{1}{n^2} n (\sigma_Y^2 + \mu_Y^2) + 2 \frac{1}{n^2} \frac{n(n-1)}{2} \mu_Y^2 \]
by identical distribution
\[ = \frac{\sigma_Y^2 + \mu_Y^2}{n} + \mu_Y^2 - \frac{\mu_Y^2}{n} \]
\[ = \frac{\mu_Y^2 + \sigma_Y^2}{n} \neq \mu_Y^2, \]
so \( \overline{Y}^2 \) is not an unbiased estimator of \( \mu_Y^2 \).