# Lecture Notes on Autoregression[1]

Econ 624

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[1] These notes are for instructional purposes only and are not to be distributed outside of the classroom.

# First-Order Autoregressive Process (or AR(1) Process)

- Consider the process

$$y_t = \mu + \alpha y_{t-1} + \varepsilon_t, \; t \in \mathbb{Z},$$

where we assume that

$$\begin{aligned} |\alpha| &< 1, \\ \{\varepsilon_t\} &\equiv i.i.d. \left(0, \sigma_\varepsilon^2\right) \text{ with } 0 < \sigma_\varepsilon^2 < \infty. \end{aligned}$$

- To obtain the moving average representation of the $AR(1)$ process, we shall consider here an approach due to Kasparis (2016). To proceed, note that, by using the lag operator notation, we can rewrite this process as

$$\alpha(L)\, y_t = (1 - \alpha L)\, y_t = \mu + \varepsilon_t, \; t \in \mathbb{Z},$$

where

$$\alpha(L) = 1 - \alpha L.$$

# First-Order Autoregressive Process (or AR(1) Process)

- Hence, a moving-average representation can be obtained if we can invert the lag operator $\alpha(L)$ to obtain

$$\begin{aligned} y_t &= \alpha(L)^{-1}(\mu + \varepsilon_t) \\ &= (1 - \alpha L)^{-1}(\mu + \varepsilon_t), \; t \in \mathbb{Z}. \end{aligned}$$

# First-Order Autoregressive Process (or AR(1) Process)

- **Definition:** Let $\mathbb{V}$ be a real (or complex) linear space (vector space). A function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ with the properties

P1

$$\|\varphi\| \geq 0 \quad \text{(positivity)}$$

P2

$$\|\varphi\| = 0 \text{ if and only if } \varphi = 0 \quad \text{(definiteness)}$$

P3

$$\|\alpha\varphi\| = |\alpha| \, \|\varphi\| \quad \text{(homongeneity)}$$

P4

$$\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\| \quad \text{(triangle inequality)}$$

for all $\varphi, \psi \in \mathbb{V}$ and for all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) is called a norm on $\mathbb{V}$. A linear space $\mathbb{V}$ equipped with a norm is called a normed space.

# First-Order Autoregressive Process (or AR(1) Process)

- **Defintion:** A linear operator $A : \mathbb{V} \to \mathbb{W}$ from a normed space $\mathbb{V}$ into a normed space $\mathbb{W}$ is called bounded if there exists a positive constant $C$ such that

$$\|A\varphi\| \leq C \|\varphi\|$$

  for all $\varphi \in \mathbb{V}$. Each number $C$ for which the inequality holds is called a bound for the operator $A$.

- **Remark:** With the aid of linearity of the operator $A$, it is easy to see that there exists a positive constant $C$ such that

$$\|A\varphi\| \leq C \|\varphi\|$$

  if and only if

$$\begin{aligned} \frac{\|A\varphi\|}{\|\varphi\|} &= \left\| A \frac{\varphi}{\|\varphi\|} \right\| \text{ (by homogeneity)} \\ &\leq C \end{aligned}$$

  for all $\varphi \in \mathbb{V}$.

# First-Order Autoregressive Process (or AR(1) Process)

- **Remark (con't):** Moreover, define

$$\varphi^* = \frac{\varphi}{\|\varphi\|}$$

so that

$$\begin{aligned} \|\varphi^*\| &= \left\|\frac{\varphi}{\|\varphi\|}\right\| \text{ (again by homogeneity)} \\ &= \frac{\|\varphi\|}{\|\varphi\|} \\ &= 1 \end{aligned}$$

Note that $\varphi^* \in \mathbb{V}$ since a linear space is closed under scalar multiplication.

# First-Order Autoregressive Process (or AR(1) Process)

- Hence, $A$ is bounded if and only if there exists a positive constant $C$ such that

$$\|A\varphi^*\| \leq C$$

for all $\varphi^* \in \mathbb{V}$ such that $\|\varphi^*\| = 1$. It, thus, follows that $A$ is bounded if and only if

$$\|A\| := \sup_{\|\varphi\|=1} \|A\varphi\| = \sup_{\|\varphi\|\leq 1} \|A\varphi\| < \infty$$

where the number $\|A\|$ is the smallest bound for $A$ and is called the norm of $A$. Thus, a linear operator is bounded if and only if it maps bounded sets in $\mathbb{V}$, i.e., $\{\varphi : \|\varphi\| \leq 1\}$ into bounded sets in $\mathbb{W}$.

# First-Order Autoregressive Process (or AR(1) Process)

- **Definition (Cauchy Sequence):** A sequence $\{\varphi_n\}$ of elements in a normed space $\mathbb{V}$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|\varphi_n - \varphi_m\| < \epsilon$$

for all $n, m \geq N(\epsilon)$; that is, if

$$\lim_{n,m\to\infty} \|\varphi_n - \varphi_m\| = 0.$$

- **Definition (Completeness and Banach Space):** A subset $\mathbb{U}$ of a normed space is called complete if every Cauchy sequence of elements in $\mathbb{U}$ converges to an element in $\mathbb{U}$. A normed space $\mathbb{U}$ is called a Banach space if it is complete.

# First-Order Autoregressive Process (or AR(1) Process)

- **Theorem 1:** Let $\mathbb{B}$ be a Banach space. If $T : \mathbb{B} \to \mathbb{B}$ is bounded linear operator and $\|I - T\| < 1$, where $I : \mathbb{B} \to \mathbb{B}$ denotes the identity operator. Then, $T$ has a bounded inverse operator on $\mathbb{B}$ which is given by the Neumann series

$$T^{-1} = \sum_{k=0}^{\infty} (I - T)^k$$

and which satisfies

$$\|T^{-1}\| \leq \frac{1}{1 - \|I - T\|}.$$

The iterated operators $(I - T)^k$ are defined by $(I - T)^0 := I$ and $(I - T)^k := (I - T)(I - T)^{k-1}$ for $k \in \mathbb{N}$.

# First-Order Autoregressive Process (or AR(1) Process)

- Consider the space $\mathbb{V}$ of sequences $Y(\omega) = \{Y_t(\omega)\}_{t\in\mathbb{Z}}$ on some probability space $(\Omega, \mathfrak{F}, P)$ that satisfies

$$\sup_t E|Y_t| < \infty.$$

  We can take $\mathbb{V}$ to be a normed space, equipped with the (pseudo or semi) norm

$$\|Y\| = \sup_{t\in\mathbb{Z}} E|Y_t|.$$

- **Remark:** Note that any covariance stationary sequence belongs to the space $\mathbb{V}$, but the setup here is more general since covariance stationarity requires a higher (second) moment condition as well as homogeneity of the first two moments with respect to $t$.
- **Lemma 1 (Kasparis, 2016):** The normed space $\mathbb{V}$ is complete and therefore is a Banach space.

- **Some Calculations:** Since

$$\sup_{t\in\mathbb{Z}} E\left|Y_t\right| = \sup_{t\in\mathbb{Z}} E\left|Y_{t-1}\right| = \sup_{t\in\mathbb{Z}} E\left|LY_t\right|,$$

we have

$$\begin{aligned}\left\|L\right\| &= \sup_{\left\|Y\right\|=1} \left\|LY\right\| \\ &= \sup_{\{Y_t\}\in\mathbb{V}:\sup_t E\left|Y_t\right|=1} \left(\sup_{t\in\mathbb{Z}} E\left|Y_{t-1}\right|\right) \\ &= 1.\end{aligned}$$

- **Some Calculations (con't):** Similarly, we have

$$\sup_{t\in\mathbb{Z}} E\left|Y_t\right| = \sup_{t\in\mathbb{Z}} E\left|Y_{t+1}\right| = \sup_{t\in\mathbb{Z}} E\left|L^{-1}Y_t\right|,$$

so that

$$\begin{aligned}\left\|L^{-1}\right\| &= \sup_{\|Y\|=1}\left\|L^{-1}Y\right\| \\ &= \sup_{\{Y_t\}\in\mathbb{V}:\sup_t E|Y_t|=1}\left(\sup_{t\in\mathbb{Z}} E\left|Y_{t+1}\right|\right) \\ &= 1.\end{aligned}$$

where $L^{-1}$ is the inverse of $L$.

# First-Order Autoregressive Process (or AR(1) Process)

- **Some Calculations (con't):**

  Now, given the assumption $|\alpha| < 1$, we have

$$\begin{aligned}\|1-\alpha(L)\| &= \|1-(1-\alpha L)\| \\ &= \|\alpha L\| \\ &= |\alpha|\,\|L\| \\ &= |\alpha| \\ &< 1.\end{aligned}$$

# First-Order Autoregressive Process (or AR(1) Process)

- **Some Calculations (con't):** We can, thus, apply Theorem 1 to the $AR(1)$ case by taking

$$T = \alpha(L) = 1 - \alpha L$$

and note that

$$\begin{aligned} \alpha(L)^{-1} &= \sum_{j=0}^{\infty} (1 - [1 - \alpha L])^j \\ &= \sum_{j=0}^{\infty} \alpha^j L^j. \end{aligned}$$

# First-Order Autoregressive Process (or AR(1) Process)

- **Some Calculations (con't):** It then follows that

$$
\begin{aligned}
y_t &= \alpha(L)^{-1}(\mu + \varepsilon_t) \\
&= \sum_{j=0}^{\infty} \alpha^j L^j (\mu + \varepsilon_t) \\
&= \mu \sum_{j=0}^{\infty} \alpha^j + \sum_{j=0}^{\infty} \alpha^j \varepsilon_{t-j} \\
&= \frac{\mu}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j \varepsilon_{t-j}
\end{aligned}
$$

where the last equality is obtained by applying the summation formula for geometric series

$$
\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}
$$

given that $|\alpha| < 1$.

# First-Order Autoregressive Process (or AR(1) Process)

- **Remarks:**

(i) The representation

$$y_t = \frac{\mu}{1-\alpha} + \sum_{j=0}^{\infty} \alpha^j \varepsilon_{t-j} \tag{1}$$

is often referred to as the stationary solution of the $AR(1)$ process

$$y_t = \mu + \alpha y_{t-1} + \varepsilon_t, \;\; t \in \mathbb{Z}.$$

Moreover, this $AR(1)$ process is said to be causal because it only depends on past innovations.

(ii) From expression (1), we see that $\{y_t\}$ is strictly stationary and ergodic since it is a measurable transformation of $\{\varepsilon_t\}$, which is an $i.i.d.$ sequence and, thus, strictly stationary and ergodic.

- **Remarks (con't):**

(iii) Note, in addition, that in this case

$$\begin{aligned}\sum_{j=0}^{\infty}\left|\psi_j\right| &= \sum_{j=0}^{\infty}\left|\alpha^j\right| \\ &= \sum_{j=0}^{\infty}\left|\alpha\right|^j \\ &= \frac{1}{1-\left|\alpha\right|} \\ &< \infty\end{aligned}$$

so that the moving average coefficients are absolutely summable.

# First-Order Autoregressive Process (or AR(1) Process)

- **Moments:** We can exploit the stationarity of the $AR(1)$ process to calculate its mean, variance, and autocovariances

1. **Mean:**

$$\begin{aligned} E[y_t] &= E[\mu] + E[\alpha y_{t-1}] + E[\varepsilon_t] \\ &= \mu + \alpha E[y_{t-1}] \quad (\text{since } E[\varepsilon_t] = 0 \text{ by assumption}) \end{aligned}$$

This implies that

$$E[y_t] - \alpha E[y_{t-1}] = \mu$$

or

$$(1-\alpha) E[y_t] = \mu \;\; (\text{since by stationarity } E[y_t] = E[y_{t-1}])$$

from which we deduce that

$$E[y_t] = \frac{\mu}{1-\alpha}$$

which is well-defined given that $|\alpha| < 1$.

- **Moments (con't):**

2. **Variance:**

$$\begin{aligned} Var\left(y_t\right) &= Var\left(\mu + \alpha y_{t-1} + \varepsilon_t\right) \\ &= \alpha^2 Var\left(y_{t-1}\right) + Var\left(\varepsilon_t\right) + 2Cov\left(y_{t-1}, \varepsilon_t\right) \\ &= \alpha^2 Var\left(y_{t-1}\right) + \sigma_\varepsilon^2. \end{aligned}$$

It follows that

$$\begin{aligned} \sigma_\varepsilon^2 &= Var\left(y_t\right) - \alpha^2 Var\left(y_{t-1}\right) \\ &= \left(1 - \alpha^2\right) Var\left(y_t\right) \quad \text{(by stationarity)} \end{aligned}$$

or

$$Var\left(y_t\right) = \frac{\sigma_\varepsilon^2}{1 - \alpha^2} < \infty \ \left(\text{given that } 0 < \sigma_\varepsilon^2 < \infty \text{ and } |\alpha| < 1\right)$$

# First-Order Autoregressive Process (or AR(1) Process)

- **Moments (con't):**

3. **Autocovariances:** For positive integer $h$, note that

$$\begin{aligned} Cov\left(y_t, y_{t-h}\right) &= Cov\left(\mu + \alpha y_{t-1} + \varepsilon_t, y_{t-h}\right) \\ &= \alpha Cov\left(y_{t-1}, y_{t-h}\right) + Cov\left(\varepsilon_t, y_{t-h}\right) \\ &= \alpha Cov\left(y_{t-1}, y_{t-h}\right) \end{aligned}$$

Now, for $h = 1$, we have

$$\begin{aligned} Cov\left(y_t, y_{t-1}\right) &= \alpha Cov\left(y_{t-1}, y_{t-1}\right) \\ &= \alpha Var\left(y_t\right) \text{ (by stationarity)} \\ &= \frac{\alpha \sigma_\varepsilon^2}{1-\alpha^2} \\ &= \gamma_y\left(1\right). \end{aligned}$$

# First-Order Autoregressive Process (or AR(1) Process)

3. **Autocovariances (con't):** Iterating backwards, we also see that for $h = 2$

$$
\begin{aligned}
Cov\left(y_t, y_{t-2}\right) &= \alpha Cov\left(y_{t-1}, y_{t-2}\right) \\
&= \alpha Cov\left(y_t, y_{t-1}\right) \quad \text{(by stationarity)} \\
&= \frac{\alpha^2 \sigma_\varepsilon^2}{1-\alpha^2} \\
&= \gamma_y\left(2\right).
\end{aligned}
$$

It is, thus, clear that for any positive integer $h$, we would have

$$
\begin{aligned}
Cov\left(y_t, y_{t-h}\right) &= \frac{\alpha^h \sigma_\varepsilon^2}{1-\alpha^2} \\
&= \gamma_y\left(h\right).
\end{aligned}
$$

3. **Autocovariances (con't):** In addition, for negative integer $h$, we have

$$\begin{aligned} Cov\left(y_t, y_{t-h}\right) &= Cov\left(y_t, y_{t+|h|}\right) \\ &= Cov\left(y_{t-|h|}, y_t\right) \quad \text{(by stationarity)} \\ &= Cov\left(y_t, y_{t-|h|}\right) \\ &= \frac{\alpha^{|h|}\sigma_\varepsilon^2}{1-\alpha^2} \end{aligned}$$

Hence, in general, for any integer $h$, we have

$$\gamma_y\left(h\right) = \frac{\alpha^{|h|}\sigma_\varepsilon^2}{1-\alpha^2}.$$

# p-th Order Autoregressive Process (or AR(p) Process)

- Consider the process

$$y_t = \mu + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + \varepsilon_t,\ t \in \mathbb{Z},$$

where $\{\varepsilon_t\} \equiv i.i.d.\left(0, \sigma_\varepsilon^2\right)$ with $0 < \sigma_\varepsilon^2 < \infty$. Again, using the lag operator notation, we can rewrite this process as

$$\alpha\left(L\right) y_t = \mu + \varepsilon_t,\ t \in \mathbb{Z},$$

where

$$\alpha\left(L\right) = 1 - \alpha_1 L - \cdots - \alpha_p L^p$$

- Now, factor this lag operator polynomial as follows:

$$\begin{aligned} \alpha(L) &= \left(1 - \frac{1}{\rho_1} L\right)\left(1 - \frac{1}{\rho_2} L\right) \times \cdots \times \left(1 - \frac{1}{\rho_p} L\right) \\ &= \alpha_1(L)\, \alpha_2(L) \times \cdots \times \alpha_p(L) \end{aligned}$$

  where $\{\rho_i : i = 1, ..., p\}$ are the roots of the polynomial equation $\alpha(z) = 0$, $z \in \mathbb{C}$, and where

$$\alpha_i(L) = 1 - \frac{1}{\rho_i} L \text{ for } i = 1, ..., p.$$

- We assume that

$$|\rho_i| > 1 \text{ for every } i \in \{1, ..., p\}.$$

- **Lemma 2:** Let $\mathbf{V}$ be a normed space and suppose that the operators $T_i : \mathbf{V} \to \mathbf{V}$, with $i \in \{1, ..., p\}$, commute. Define $T$ as $T_1 T_2 \cdots T_p$. Then, $T$ is invertible if and only if each $T_i$ is invertible.

- Next, let

$$T_i = \alpha_i(L) = 1 - \frac{1}{\rho_i} L$$

and note that

$$\|1 - T_i\| = \left\| 1 - \left(1 - \frac{1}{\rho_i} L\right) \right\| = \left\| \frac{1}{\rho_i} L \right\| = \left| \frac{1}{\rho_i} \right| \|L\| < 1$$

Hence, by Theorem 1 given earlier,

$$\begin{aligned} T_i^{-1} &= \alpha_i(L)^{-1} \\ &= \sum_{j=0}^{\infty} \left(1 - \left[1 - \frac{1}{\rho_i} L\right]\right)^j \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{\rho_i}\right)^j L^j \text{ for every } i \in \{1, ..., p\}. \end{aligned}$$

# p-th Order Autoregressive Process (or AR(p) Process)

- Hence, a moving-average representation for the $AR(p)$ process can be obtained by inverting the lag operator $\alpha(L)$ to obtain

$$
\begin{aligned}
& y_t \\
= \ & \alpha(L)^{-1}(\mu + \varepsilon_t) \\
= \ & \left(1 - \frac{1}{\rho_1}L\right)^{-1}\left(1 - \frac{1}{\rho_2}L\right)^{-1} \times \cdots \times \left(1 - \frac{1}{\rho_p}L\right)^{-1}(\mu + \varepsilon_t)
\end{aligned}
$$

To give a more explicit form of the moving average representation, we look first at the explicit case where $p = 2$.

- **Example ($AR\left(2\right)$ process):** In this case,

$$\alpha\left(L\right)=\left(1-\frac{1}{\rho_1}L\right)\left(1-\frac{1}{\rho_2}L\right)$$

so that

$$\begin{aligned}
&\alpha\left(L\right)^{-1}\\
&= \left(1-\frac{1}{\rho_1}L\right)^{-1}\left(1-\frac{1}{\rho_2}L\right)^{-1}\\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(1-\left[1-\frac{1}{\rho_1}L\right]\right)^{j_1}\left(1-\left[1-\frac{1}{\rho_2}L\right]\right)^{j_2}\\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{j_1}\left(\frac{1}{\rho_2}\right)^{j_2}L^{j_1+j_2}
\end{aligned}$$

- **Example (con't):** $AR(2)$ **Process**
  Let $j = j_1 + j_2$ and $k = j_1$. By rearranging the sums, we can further write

$$\alpha(L)^{-1} = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left(\frac{1}{\rho_1}\right)^k \left(\frac{1}{\rho_2}\right)^{j-k} L^j$$
$$= \sum_{j=0}^{\infty} \psi_j L^j$$

where

$$\psi_j = \sum_{k=0}^{j} \left(\frac{1}{\rho_1}\right)^k \left(\frac{1}{\rho_2}\right)^{j-k}.$$

- $AR(2)$ **Process (con't):** Making use of the representations of $\alpha(L)^{-1}$ given previously, we have

$$
\begin{aligned}
y_t &= \alpha(L)^{-1}(\mu + \varepsilon_t) \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{j_1}\left(\frac{1}{\rho_2}\right)^{j_2} L^{j_1+j_2}\mu \\
&\quad + \sum_{j=0}^{\infty}\sum_{k=0}^{j}\left(\frac{1}{\rho_1}\right)^{k}\left(\frac{1}{\rho_2}\right)^{j-k} L^{j}\varepsilon_t \\
&= \mu\left[\frac{1}{1-(1/\rho_1)}\right]\left[\frac{1}{1-(1/\rho_2)}\right] + \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} \\
&= \mu\left(\prod_{i=1}^{2}\frac{1}{1-(1/\rho_i)}\right) + \psi(L)\varepsilon_t
\end{aligned}
$$

where $\psi(L) = \sum_{j=0}^{\infty}\psi_j L^j$.

- $AR(2)$ **Process (con't):** Moreover, note that

$$\begin{aligned}
\mu\left(\prod_{i=1}^{2}\frac{1}{1-(1/\rho_i)}\right) &= \left(1-\frac{1}{\rho_1}L\right)^{-1}\left(1-\frac{1}{\rho_2}L\right)^{-1}\mu \\
&= \left[\left(1-\frac{1}{\rho_1}L\right)\left(1-\frac{1}{\rho_2}L\right)\right]^{-1}\mu \\
&= \left(1-\alpha_1 L-\alpha_2 L^2\right)^{-1}\mu \\
&= \frac{\mu}{1-\alpha_1-\alpha_2}.
\end{aligned}$$

- $AR\,(2)$ **Process (con't):** Finally, note that the moving-average coefficients in this case are absolutely summable since

$$\begin{aligned}
\sum_{j=0}^{\infty} \left|\psi_j\right| &\leq \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left|\frac{1}{\rho_1}\right|^k \left|\frac{1}{\rho_2}\right|^{j-k} \\
&\leq \sum_{k=0}^{\infty} \left|\frac{1}{\rho_1}\right|^k \sum_{j=0}^{\infty} \left|\frac{1}{\rho_2}\right|^j \\
&= \left(\frac{1}{1-|1/\rho_1|}\right)\left(\frac{1}{1-|1/\rho_2|}\right) \\
&< \infty
\end{aligned}$$

- Returning to the more general $AR(p)$ process, note that, in a similar manner, we have

$$
\begin{aligned}
& y_t \\
= \;& \alpha(L)^{-1}(\mu+\varepsilon_t) \\
= \;& \left(1-\frac{1}{\rho_1}L\right)^{-1}\left(1-\frac{1}{\rho_2}L\right)^{-1}\times\cdots\times\left(1-\frac{1}{\rho_p}L\right)^{-1}(\mu+\varepsilon_t) \\
= \;& \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_p=0}^{\infty}\left\{\left(\frac{1}{\rho_1}\right)^{j_1}\left(\frac{1}{\rho_2}\right)^{j_2}\times\cdots\right. \\
& \left.\cdots\times\left(\frac{1}{\rho_p}\right)^{j_p}L^{j_1+\,\cdots\,+j_p}(\mu+\varepsilon_t)\right\}
\end{aligned}
$$

# p-th Order Autoregressive Process (or AR(p) Process)

- By letting $j = j_1 + \cdots + j_p$ and rearranging the sums, we further obtain

$$
\begin{aligned}
y_t &= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_p=0}^{\infty}\left\{\left(\frac{1}{\rho_1}\right)^{j_1}\left(\frac{1}{\rho_2}\right)^{j_2}\times\cdots\times\left(\frac{1}{\rho_p}\right)^{j_p}\right. \\
&\qquad\qquad \left.\times L^{j_1+\,\cdots\,+j_p}\left(\mu+\varepsilon_t\right)\right\} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_p=0}^{\infty}\left(\frac{1}{\rho_1}\right)^{j_1}\left(\frac{1}{\rho_2}\right)^{j_2}\times\cdots\times\left(\frac{1}{\rho_p}\right)^{j_p}L^{j_1+\,\cdots\,+j_p}\mu \\
&\quad +\sum_{j=0}^{\infty}\sum_{k_1=0}^{j}\sum_{k_2=0}^{j-k_1}\cdots\sum_{k_{p-1}=0}^{j-(k_1+\,\cdots\,+k_{p-2})}\left\{\left(\frac{1}{\rho_1}\right)^{k_1}\left(\frac{1}{\rho_2}\right)^{k_2}\times\cdots\right. \\
&\qquad\qquad \left.\cdots\times\left(\frac{1}{\rho_p}\right)^{j-(k_1+\,\cdots\,+k_{p-1})}L^j\varepsilon_t\right\}.
\end{aligned}
$$

- It follows by argument similar to that given previously for the $AR\left(2\right)$ case that

$$
\begin{aligned}
y_t &= \alpha\left(L\right)^{-1}\left(\mu+\varepsilon_t\right) \\
&= \left(1-\frac{1}{\rho_1}L\right)^{-1}\left(1-\frac{1}{\rho_2}L\right)^{-1}\times\cdots\times\left(1-\frac{1}{\rho_p}L\right)^{-1}\left(\mu+\varepsilon_t\right) \\
&= \mu\left[\frac{1}{1-\left(1/\rho_1\right)}\right]\left[\frac{1}{1-\left(1/\rho_2\right)}\right]\times\cdots\times\left[\frac{1}{1-\left(1/\rho_p\right)}\right] \\
&\quad+\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} \\
&= \mu\left(\prod_{i=1}^{p}\frac{1}{1-\left(1/\rho_i\right)}\right)+\psi\left(L\right)\varepsilon_t
\end{aligned}
$$

# p-th Order Autoregressive Process (or AR(p) Process)

- where

$$\psi_j = \sum_{k_1=0}^{j} \sum_{k_2=0}^{j-k_1} \cdots \sum_{k_{p-1}=0}^{j-(k_1+\cdots+k_{p-2})} \left\{ \left(\frac{1}{\rho_1}\right)^{k_1} \left(\frac{1}{\rho_2}\right)^{k_2} \times \cdots \times \left(\frac{1}{\rho_p}\right)^{j-(k_1+\cdots+k_{p-1})} \right\}.$$

- In addition, note that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left|\psi_j\right| \\
\leq\ & \sum_{j=0}^{\infty}\sum_{k_1=0}^{j}\sum_{k_2=0}^{j-k_1}\cdots\sum_{k_{p-1}=0}^{j-(k_1+\cdots+k_{p-2})}\left\{\left|\frac{1}{\rho_1}\right|^{k_1}\left|\frac{1}{\rho_2}\right|^{k_2}\times\cdots\right. \\
&\left.\cdots\times\left|\frac{1}{\rho_p}\right|^{j-(k_1+\cdots+k_{p-1})}\right\} \\
\leq\ & \sum_{k_1=0}^{\infty}\left|\frac{1}{\rho_1}\right|^{k_1}\sum_{k_2=0}^{\infty}\left|\frac{1}{\rho_2}\right|^{k_2}\times\cdots\times\sum_{j=0}^{\infty}\left|\frac{1}{\rho_p}\right|^{j}
\end{aligned}$$

- so that, applying the summation formula for a convergent geometric series, we obtain

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \left| \psi_j \right| \\
\leq \; & \sum_{k_1=0}^{\infty} \left| \frac{1}{\rho_1} \right|^{k_1} \sum_{k_2=0}^{\infty} \left| \frac{1}{\rho_2} \right|^{k_2} \times \cdots \times \sum_{j=0}^{\infty} \left| \frac{1}{\rho_p} \right|^{j} \\
= \; & \left( \frac{1}{1 - |1/\rho_1|} \right) \left( \frac{1}{1 - |1/\rho_2|} \right) \times \cdots \times \left( \frac{1}{1 - \left|1/\rho_p\right|} \right) \\
< \; & \infty
\end{aligned}
$$

which shows the absolute summability of the moving average coefficients of the $AR\left(p\right)$ process.

# p-th Order Autoregressive Process (or AR(p) Process)

- **Remark:** It follows from the moving average representation

$$y_t = \mu \left( \prod_{i=1}^{p} \frac{1}{1 - (1/\rho_i)} \right) + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

that $\{y_t\}$ is strictly stationary and ergodic since it is measurable transformation of $\{\varepsilon_t\}$, which is an $i.i.d.$ sequence and, thus, strictly stationary and ergodic.

- **Moments:** Exploiting the strictly stationary property, we can obtain the following expressions for the mean, variance, and autocovariances of an $AR\left(p\right)$ process.

1. **Mean:**

$$\begin{aligned} E\left[y_t\right] &= E\left[\mu\right] + E\left[\alpha_1 y_{t-1}\right] + \cdots + E\left[\alpha_p y_{t-p}\right] + E\left[\varepsilon_t\right] \\ &= \mu + \alpha_1 E\left[y_{t-1}\right] + \cdots + \alpha_p E\left[y_{t-p}\right] \\ &\quad \left(\text{since } E\left[\varepsilon_t\right] = 0 \text{ by assumption}\right) \end{aligned}$$

This implies that

$$\begin{aligned} \mu &= E\left[y_t\right] - \alpha_1 E\left[y_{t-1}\right] - \cdots - \alpha_p E\left[y_{t-p}\right] \\ &= \left(1 - \alpha_1 - \cdots - \alpha_p\right) E\left[y_t\right] \\ &\quad \left(\text{since by stationarity } E\left[y_t\right] = E\left[y_{t-h}\right] \text{ for all } h \in \mathbb{Z}\right) \end{aligned}$$

1. **Mean:** From this, we deduce that

$$E\left[y_t\right] = \frac{\mu}{1-\alpha_1-\cdots-\alpha_p} = \mu\left(\prod_{i=1}^{p}\frac{1}{1-\left(1/\rho_i\right)}\right)$$

which is well-defined, since by assumption $z=1$ is not a solution of the polynomial equation $\alpha\left(z\right)=1-\alpha_1 z-\cdots-\alpha_p z^p=0$.

2. **Variance:**

$$\begin{aligned}
Var\left(y_t\right) &= E\left[\left(y_t - E\left[y_t\right]\right)^2\right] \\
&= E\left[\left(y_t - \frac{\mu}{1-\alpha_1-\cdots-\alpha_p}\right)^2\right] \\
&= E\left[\left(\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}\right)^2\right] \\
&= \sum_{j=0}^{\infty}\psi_j^2 E\left[\varepsilon_{t-j}^2\right] \\
&= \sigma_\varepsilon^2\sum_{j=0}^{\infty}\psi_j^2 < \infty
\end{aligned}$$

where the last equality follows from the fact that absolute summability implies square summability and that by assumption $\sigma_\varepsilon^2 < \infty$.

2. **Autocovariances:** Let $h > 0$

$$
\begin{aligned}
& \gamma(h) \\
= \; & Cov(y_t, y_{t-h}) \\
= \; & E\left[(y_t - E[y_t])(y_{t-h} - E[y_t - h])\right] \\
= \; & E\left[\left(y_t - \frac{\mu}{1-\alpha_1-\cdots-\alpha_p}\right)\left(y_{t-h} - \frac{\mu}{1-\alpha_1-\cdots-\alpha_p}\right)\right] \\
= \; & E\left[\left(\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}\right)\left(\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-h-j}\right)\right] \\
= \; & E\left[\left(\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}\right)\left(\sum_{k=h}^{\infty}\psi_{k-h}\varepsilon_{t-k}\right)\right] \quad \text{(by setting } k = j + h\text{)} \\
= \; & \sum_{j=h}^{\infty}\psi_j\psi_{j-h}E\left[\varepsilon_{t-j}^2\right] \quad \text{(given the assumption of independence)}
\end{aligned}
$$

# p-th Order Autoregressive Process (or AR(p) Process)

2. **Autocovariances (con't):** The assumption that $\{\varepsilon_t\}$ is identically distributed then implies that

$$\gamma(h) = \sigma_\varepsilon^2 \sum_{j=h}^{\infty} \psi_j \psi_{j-h} \quad \text{(i.e., } E\left[\varepsilon_{t-j}^2\right] = \sigma_\varepsilon^2 \text{ for all } j \in \mathbb{Z}_+ \cup \{0\}\text{ )}$$

Moreover, since covariance stationarity implies that $\gamma(h) = \gamma(-h)$, we also have for $h < 0$

$$\begin{aligned} \gamma(h) &= \gamma(-h) \\ &= \gamma(|h|) \\ &= \sigma_\varepsilon^2 \sum_{j=|h|}^{\infty} \psi_j \psi_{j-|h|} \end{aligned}$$

2. **Autocovariances (con't):** It follows from these calculations that for all $h \in \mathbb{Z}$,

$$\begin{aligned} \gamma(h) &= Cov(y_t, y_{t-h}) \\ &= \sigma_\varepsilon^2 \sum_{j=|h|}^{\infty} \psi_j \psi_{j-|h|}. \end{aligned}$$