Lecture Notes on Linear Time Series Models[1]

Econ 624

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[1] These notes are for instructional purposes only and are not to be distributed outside of the classroom.

Moving Average Processes

- $MA(1)$ **Process:** The first-order moving average process, or $MA(1)$ process is given by

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

  where $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$. This model is called a "moving average" process because y_t is a weighted average of the shocks ε_t and ε_{t-1}.
- **Remark:** Recall that a white noise process is not necessarily $i.i.d.$ Rather, $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$ if it is mean zero, is serially uncorrelated, and is covariance stationary with homoskedastic variance σ_ε^2.
- **Moments:** It is straightforward to calculate the first two moments of a $MA(1)$ process. Note that

$$E[y_t] = E[\mu] + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}] = \mu,$$

Moving Average Processes

- **Moments (con't):**

$$\begin{aligned} Var\left(y_t\right) &= Var\left(\mu + \varepsilon_t + \theta\varepsilon_{t-1}\right) \\ &= Var\left(\varepsilon_t\right) + \theta^2 Var\left(\varepsilon_{t-1}\right) + 2\theta Cov\left(\varepsilon_t, \varepsilon_{t-1}\right) \\ &= \left(1+\theta^2\right)\sigma_\varepsilon^2 \\ &= \gamma\left(0\right). \end{aligned}$$

and

$$\begin{aligned} Cov\left(y_{t+1}, y_t\right) &= Cov\left(\mu + \varepsilon_{t+1} + \theta\varepsilon_t, \mu + \varepsilon_t + \theta\varepsilon_{t-1}\right) \\ &= \theta Var\left(\varepsilon_t\right) \\ &= \theta\sigma_\varepsilon^2 \\ &= \gamma\left(1\right). \end{aligned}$$

- **Autocorrelation:**

$$
\begin{aligned}
Corr\left(y_{t+1}, y_t\right) &= \frac{Cov\left(y_{t+1}, y_t\right)}{\sqrt{Var\left(y_{t+1}\right)}\sqrt{Var\left(y_t\right)}} \\
&= \frac{\gamma\left(1\right)}{\gamma\left(0\right)} \\
&= \frac{\theta\sigma_\varepsilon^2}{\left(1+\theta^2\right)\sigma_\varepsilon^2} \\
&= \frac{\theta}{\left(1+\theta^2\right)} \\
&= \rho\left(1\right).
\end{aligned}
$$

- **Higher-Order Autocovariance and Autocorrelation:** For $h \geq 2$

$$\begin{aligned} Cov\left(y_{t+h}, y_t\right) &= Cov\left(\mu + \varepsilon_{t+h} + \theta\varepsilon_{t+h-1}, \mu + \varepsilon_t + \theta\varepsilon_{t-1}\right) \\ &= 0 \\ &= \gamma\left(h\right) \end{aligned}$$

  It follows that

$$\rho\left(h\right) = Corr\left(y_{t+h}, y_t\right) = \frac{\gamma\left(h\right)}{\gamma\left(0\right)} = 0 \text{ for } h \geq 2.$$

- **Remark:** Note that the $MA\left(1\right)$ process with $\theta \neq 0$ has a non-zero first-order autocorrelation, but all higher-order autocorrelations are zero. If $\theta > 0$; then, y_{t+1} and y_t are positively correlated while, if $\theta < 0$, y_{t+1} and y_t are negatively correlated. Under the assumption that $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$, it is covariance stationary, as the calculations above have shown, but it is not necessarily strictly stationary.

Moving Average Processes

- **Remark (con't):** On the other hand, if we assume that $\{\varepsilon_t\} \equiv i.i.d.\left(0, \sigma_\varepsilon^2\right)$; then, $\{y_t\}$ will be strictly stationary (and ergodic). That this is true is given in the following two theorems.
- **Theorem 1:** Suppose that

(i) $\{X_t\}$ is strictly stationary;
(ii) $\varphi(\cdot) : \mathbb{R}^\infty \to \mathbb{R}$ is measurable;
(iii) $Y_t = \varphi(...., X_{t-1}, X_t, X_{t+1}, ...)$

Then, $\{Y_t\}$ is strictly stationary.

- **Theorem 2:** Suppose that

(i) $\{X_t\}$ is strictly stationary and ergodic;
(ii) $\varphi(\cdot) : \mathbb{R}^\infty \to \mathbb{R}$ is measurable;
(iii) $Y_t = \varphi(...., X_{t-1}, X_t, X_{t+1}, ...)$

Then, $\{Y_t\}$ is strictly stationary and ergodic.

MA(q) Process

- The q^{th}- order moving average process, or $MA(q)$ process is given by

$$\begin{aligned} y_t &= \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ &= \mu + \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \text{ (letting } \theta_0 = 1) \\ &= \mu + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \end{aligned}$$

wherre $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$.

MA(q) Process

- **Example:** $MA(3)$ process

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}$$

It is straightforward to calculate the following autocovariances

1. $h = 1$

$$\begin{aligned}
& Cov(y_{t+1}, y_t) \\
= \; & Cov(\mu + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} + \theta_3 \varepsilon_{t-2}, \\
& \qquad \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}) \\
= \; & \theta_1 \sigma_\varepsilon^2 + \theta_1 \theta_2 \sigma_\varepsilon^2 + \theta_2 \theta_3 \sigma_\varepsilon^2 \\
= \; & \sigma_\varepsilon^2 [\theta_0 \theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3] \quad (\text{letting } \theta_0 = 1) \\
= \; & \sigma_\varepsilon^2 \sum_{j=0}^{2} \theta_j \theta_{j+1} \\
= \; & \sigma_\varepsilon^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} \quad (\text{given that } q = 3 \text{ and } h = 1)
\end{aligned}$$

- **Example (con't):** $MA(3)$ process

2. $h = -1$

$$
\begin{aligned}
& Cov\left(y_{t-1}, y_t\right) \\
= \;& Cov\left(\mu + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \theta_2 \varepsilon_{t-3} + \theta_3 \varepsilon_{t-4}, \right. \\
& \qquad \left. \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3}\right) \\
= \;& \theta_1 \sigma_\varepsilon^2 + \theta_1 \theta_2 \sigma_\varepsilon^2 + \theta_2 \theta_3 \sigma_\varepsilon^2 \\
= \;& \sigma_\varepsilon^2 \left[\theta_0 \theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3\right] \quad (\text{letting } \theta_0 = 1) \\
= \;& \sigma_\varepsilon^2 \sum_{j=0}^{2} \theta_j \theta_{j+1} \\
= \;& \sigma_\varepsilon^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} \quad (\text{given that } q = 3 \text{ and } h = -1)
\end{aligned}
$$

- **Example (con't):** $MA(3)$ process

3. $h = 2$

$$
\begin{aligned}
& Cov\left(y_{t+2}, y_t\right) \\
= \quad & Cov\left(\mu + \varepsilon_{t+2} + \theta_1\varepsilon_{t+1} + \theta_2\varepsilon_t + \theta_3\varepsilon_{t-1},\right. \\
& \qquad \left. \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3}\right) \\
= \quad & \theta_2\sigma_\varepsilon^2 + \theta_1\theta_3\sigma_\varepsilon^2 \\
= \quad & \sigma_\varepsilon^2\left[\theta_0\theta_2 + \theta_1\theta_3\right] \quad (\text{letting } \theta_0 = 1) \\
= \quad & \sigma_\varepsilon^2 \sum_{j=0}^{1} \theta_j\theta_{j+2} \\
= \quad & \sigma_\varepsilon^2 \sum_{j=0}^{q-|h|} \theta_j\theta_{j+|h|} \quad (\text{given that } q = 3 \text{ and } h = 2)
\end{aligned}
$$

MA(q) Process

- **Example (con't):** $MA(3)$ process

4. $h = -2$

$$
\begin{aligned}
& Cov\left(y_{t-2}, y_t\right) \\
= \; & Cov\left(\mu + \varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4} + \theta_3\varepsilon_{t-5},\right. \\
& \qquad \left.\mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \theta_3\varepsilon_{t-3}\right) \\
= \; & \theta_2\sigma_\varepsilon^2 + \theta_1\theta_3\sigma_\varepsilon^2 \\
= \; & \sigma_\varepsilon^2\left[\theta_0\theta_2 + \theta_1\theta_3\right] \quad \text{(letting } \theta_0 = 1\text{)} \\
= \; & \sigma_\varepsilon^2 \sum_{j=0}^{1} \theta_j\theta_{j+2} \\
= \; & \sigma_\varepsilon^2 \sum_{j=0}^{q-|h|} \theta_j\theta_{j+|h|} \quad \text{(given that } q = 3 \text{ and } h = -2\text{)}
\end{aligned}
$$

MA(q) Process

- **Moments of a $MA(q)$ process:** More generally, we have

$$
\begin{aligned}
y_t &= E[\mu] + \sum_{j=0}^{q} \theta_j E[\varepsilon_{t-j}] = \mu, \\
Var(y_t) &= \sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j^2 \ \left(\text{again letting } \theta_0 = 1\right), \\
Cov(y_{t+h}, y_t) &= \gamma(h) = \begin{cases} \sigma_\varepsilon^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & \text{if } |h| \leq q \\ 0 & \text{if } |h| > q \end{cases}, \\
Corr(y_{t+h}, y_t) &= \rho(h) \\
&= \begin{cases} \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} / \sum_{j=0}^{q} \theta_j^2 & \text{if } |h| \leq q \\ 0 & \text{if } |h| > q \end{cases}
\end{aligned}
$$

MA(q) Process

- **Remarks:**

(i) The above calculations show that the $MA(q)$ process is covariance stationary and will be strictly stationary (and ergodic) if we assume that $\{\varepsilon_t\} \equiv i.i.d.\left(0, \sigma_\varepsilon^2\right)$. Moreover, a $MA(q)$ process has q non-zero autocorrelations, while autocorrelation of order higher than q are all zero.

(ii) A $MA(q)$ process with moderate size q can still have considerably more complicated dependence than the $MA(1)$ process. One specific pattern which can be induced by a MA process is smoothing. Suppose, for example, that the coefficients $\theta_j = 1$ for all $j = 0, 1, ..., q$; then, y_t is a smoothed version of the shocks ε_t.

Infinite-Order Moving Average Process

- An infinite-order moving average process, denoted $MA(\infty)$, also known as a linear process is given by

$$y_t = \mu + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}$$

where we assume absolute summability of the moving average coefficients, i.e.,

$$\sum_{j=0}^{\infty} |\theta_j| < \infty$$

and where $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$.

Infinite-Order Moving Average Process

- Note that the absolute summability condition given above is stronger than the square summability condition

$$\sum_{j=0}^{\infty} \theta_j^2 < \infty.$$

To see that this is true, note first that

$$\sum_{j=0}^{\infty} |\theta_j| < \infty \Longrightarrow \sup_{j \in \mathbb{Z}_+ \cup \{0\}} |\theta_j| < \infty$$

Next, observe that

$$\sum_{j=0}^{\infty} \theta_j^2 = \sum_{j=0}^{\infty} |\theta_j|^2 \leq \sup_{j \in \mathbb{Z}_+ \cup \{0\}} |\theta_j| \sum_{j=0}^{\infty} |\theta_j| < \infty$$

so that absolute summability implies square summability.

Infinite-Order Moving Average Process

- On the other hand, the converse does not hold. To give a counterexample, consider $MA(\infty)$ process where $\theta_0 = 1$ and $\theta_j = 1/j$ for $j \geq 1$. Note that, in this case,

$$\sum_{j=0}^{\infty} \theta_j^2 = 1 + \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

but

$$\sum_{j=0}^{\infty} |\theta_j| = 1 + \sum_{j=1}^{\infty} \frac{1}{j} = \infty$$

so that square summability does not necessarily imply absolute summability.

Moments of an Infinite-Order Moving Average Process

- It is straightforward to calculate the mean, variance, autocovariances, and autocorrelations of a linear process

$$y_t = \mu + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}$$

under the assumptions that

(i) $\sum_{j=0}^{\infty} |\theta_j| < \infty$
(ii) $\{\varepsilon_t\} \sim WN\left(0, \sigma_\varepsilon^2\right)$ with $0 < \sigma_\varepsilon^2 < \infty$.

In particular, the mean and variance are given by

$$\begin{aligned} E\left[y_t\right] &= \mu, \\ Var\left(y_t\right) &= \gamma(0) = \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \theta_j^2 < \infty \end{aligned}$$

(since absolute summability implies square summability)

Moments of an Infinite-Order Moving Average Process

- In addition, the autocovariances and autocorrelations are given by

$$
\begin{aligned}
Cov\left(y_{t+h}, y_t\right) &= \gamma\left(h\right) \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+|h|} \\
Corr\left(y_{t+h}, y_t\right) &= \frac{\gamma\left(h\right)}{\gamma\left(0\right)} \\
&= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \theta_j \theta_{j+|h|} / \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \theta_j^2 \\
&= \sum_{j=0}^{\infty} \theta_j \theta_{j+|h|} / \sum_{j=0}^{\infty} \theta_j^2
\end{aligned}
$$

Lag Operator

- A lag operator, denoted by the symbol L, maps a sequence $\{X_t\}_{t=-\infty}^{\infty}$ into a sequence $\{Y_t\}_{t=-\infty}^{\infty}$ such that

$$Y_t = LX_t = X_{t-1} \text{ for all } t \in \mathbb{Z}$$

- Repeated application leads to

$$L^k X_t = X_{t-k} \text{ for all } t \in \mathbb{Z} \text{ and } k \in \mathbb{N}$$

- Note also that L is a linear operator in the sense that for two sequences $\{X_t\}_{t=-\infty}^{\infty}$ and $\{Z_t\}_{t=-\infty}^{\infty}$ and two scalars $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$,

$$\begin{aligned} L(\alpha X_t + \beta Z_t) &= \alpha L X_t + \beta L Z_t \\ &= \alpha X_{t-1} + \beta Z_{t-1} \end{aligned}$$

for all $t \in \mathbb{Z}$

Linear Operator

- Using the lag operator, we can, for example, give the following, alternative representation for the $MA(q)$ process

$$\begin{aligned} y_t &= \mu + \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ &= \mu + \theta_0 \varepsilon_t + \theta_1 L \varepsilon_t + \cdots + \theta_q L^q \varepsilon_t \\ &= \mu + (\theta_0 + \theta_1 L + \cdots + \theta_q L^q) \varepsilon_t \\ &= \mu + \theta(L) \varepsilon_t, \end{aligned}$$

where

$$\theta(L) = \theta_0 + \theta_1 L + \cdots + \theta_q L^q$$

is a q^{th} degree polynomial in the lag operator L.