Econ 721 Lecture Notes: Models of Conditional Heteroskedasticity[1]

John C. Chao

December 2, 2020

[1] These notes are for instructional purposes only and are not to be distributed outside of the classroom.

Models of Conditional Heteroskedasticity - Motivation

- Consider the $AR(1)$ process

$$Y_t = \beta Y_{t-1} + \varepsilon_t,$$

where $|\beta| < 1$ and $\{\varepsilon_t\} \equiv i.i.d.\left(0, \sigma^2\right)$.

- Note that for this model

$$E\left[Y_{t+1}\right] = 0$$

but

$$E\left[Y_{t+1} | Y_t, Y_{t-1}, ...\right] = E\left[Y_{t+1} | Y_t\right] = \beta Y_t,$$

so that by using information about current and past values of Y_t, this model allows one to improve on ones forecast of the mean-level of Y_{t+1} over that which can be obtained when this information is not used.

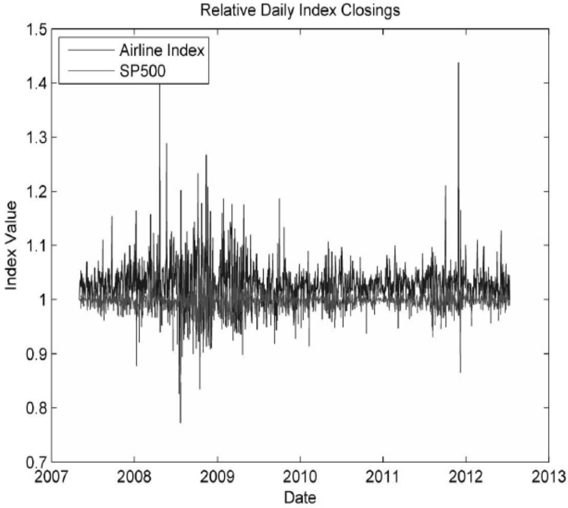
Models of Conditional Heteroskedasticity - Motivation

- **Shortcoming of this model:** The same improvement is not achieved when forecasting the error variance with this model since

$$E\left[\varepsilon_{t+1}^2 \middle| Y_t, Y_{t-1}, ...\right] = E\left[\varepsilon_{t+1}^2\right] = \sigma^2$$

- **Observation:** This model is not rich enough to allow for better prediction of the error variance based on past information. In particular, the independence assumption on the errors precludes any forecast improvement.
- On the other hand, many financial and macroeconomic time series exhibit "volatility clustering." Volatility clustering suggests the possible presence of time dependent variance or time-varying heteroskedasticty that may be forecastable. Interestingly, this can occur even if the time series itself is close to being serially uncorrelated so that the mean-level is difficult to forecast.

Models of Conditional Heteroskedasticity - Empirical Motivation



Why would there be interest in forecasting variance?

- First, in finance, the variance of the return to an asset is a measure of the risk of owning that asset. Hence, investors, particularly those who are risk averse, would naturally be interested in predicting return variances.
- Secondly, the value of some financial derivatives, such as options, depends on the variance of the underlying assets. Thus, an options trader would want to obtain good forecasts of future volatility to help her or him decide on the price at which to buy or sell options.
- Thirdly, being able to forecast variance could allow one to have more accurate forecast intervals that adapt to changing economic conditions.

AutoRegressive Conditional Heteroskedasticity (ARCH) Models

- Here, we will discuss two frequently used models of time-varying heteroskedasticity: the **autoregressive conditional heteroskedasticity (ARCH)** model and its extension, the **generalized ARCH (or GARCH)** model.
- **Regression with ARCH errors:** Consider the following model

$$y_t = x_t'\beta + \varepsilon_t, \text{ for } t = 1, ..., T$$

We make the following assumptions on this model.

- **Assumptions:**

A1 x_t is nonstochastic for all t.

A2 $T^{-1}X'X \to Q_{xx} > 0$ where $\underset{T\times k}{X} = (x_1, ..., x_T)'$.

A3 $\varepsilon_t = u_t \left[\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \cdots + \alpha_p\varepsilon_{t-p}^2\right]^{1/2}$, where $\alpha_i > 0$ for $i = 0, ..., p$ and $\{u_t\} \equiv i.i.d.N(0,1)$ (This assumption specifies an ARCH(p) error process.)

AutoRegressive Conditional Heteroskedasticity (ARCH) Models

- **Remark 1:** We have described here a simple linear regression model with ARCH errors; but, in principle, an ARCH process can be used to model the error variance for any time series regression.
- **Remark 2:** Further conditions on the coefficients $\alpha_1, ..., \alpha_p$ will be given below.
- **Conditional and Unconditional Moments of the ARCH(1) Process:** Let $I_{t-1} = \sigma\left(\varepsilon_{t-1}, \varepsilon_{t-2}, ...\right)$, i.e., the information set generated by $\varepsilon_{t-1}, \varepsilon_{t-2}, ...$ We first consider a special case of the specification given in A3. More specifically, we will consider a first-order ARCH (or ARCH(1)) process for the regression errors, which has the form

$$\varepsilon_t = u_t \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right]^{1/2}$$

where $\alpha_0 > 0$, $0 < \alpha_1 < 1$, and $\{u_t\} \equiv i.i.d.N\left(0, 1\right)$.

AutoRegressive Conditional Heteroskedasticity (ARCH) Models

- **Conditional Moments:**

1. **Conditional Mean**

$$\begin{aligned} E\left[\varepsilon_t | I_{t-1}\right] &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right]^{1/2} E\left[u_t | I_{t-1}\right] \\ &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right]^{1/2} E\left[u_t\right] \\ &= 0 \end{aligned}$$

2. **Conditional Variance**

$$\begin{aligned} E\left[\varepsilon_t^2 | I_{t-1}\right] &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right] E\left[u_t^2 | I_{t-1}\right] \\ &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right] E\left[u_t^2\right] \\ &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right] \\ & \quad \left(\text{since by assumption } E\left[u_t^2\right] = Var\left(u_t\right) = 1\right) \end{aligned}$$

3. **Conditional 4th Moment**

$$\begin{aligned} E\left[\varepsilon_t^4 | I_{t-1}\right] &= \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right]^2 E\left[u_t^4 | I_{t-1}\right] \\ &= \left[\alpha_0^2 + 2\alpha_0\alpha_1\varepsilon_{t-1}^2 + \alpha_1^2\varepsilon_{t-1}^4\right] E\left[u_t^4\right] \\ &= 3\left[\alpha_0^2 + 2\alpha_0\alpha_1\varepsilon_{t-1}^2 + \alpha_1^2\varepsilon_{t-1}^4\right] \\ & \quad \left(\text{since } E\left[u_t^4\right] = 3 \text{ by property of } N(0,1)\right) \end{aligned}$$

ARCH Models

- **Unconditional Moments:**

1. **Unonditional Mean:** By law of iterated expectations

$$E\left[\varepsilon_t\right] = E\left(E\left[\varepsilon_t | I_{t-1}\right]\right) = E\left[0\right] = 0.$$

2. **Autocovariances:** For any integer $j \geq 1$, note that

$$\begin{aligned} E\left[\varepsilon_t \varepsilon_{t-j}\right] &= E\left(\varepsilon_{t-j} E\left[\varepsilon_t | I_{t-1}\right]\right) \\ &\quad \text{(by law of iterated expectations)} \\ &= E\left[\varepsilon_{t-j} \times 0\right] \\ &= 0. \end{aligned}$$

A similar argument can be used to show that, for negative integer j, $E\left[\varepsilon_t \varepsilon_{t-j}\right] = 0$ so that $\{\varepsilon_t\}$ is serially uncorrelated for this model.

- **Remark:** Interestingly, an ARCH process is serially uncorrelated but not independent. These features are important for the modeling of asset returns.

ARCH Models

- **Claim:** The unconditional 4th moment of an $ARCH(1)$ process with $\alpha_0 > 0$ and $\alpha_1 > 0$ exists if and only if $3\alpha_1^2 < 1$. Under these conditions, the (unconditional) 2nd and 4th moments have the explicit form

$$
\begin{aligned}
E\left[\varepsilon_t^2\right] &= \frac{\alpha_0}{1-\alpha_1}, \\
E\left[\varepsilon_t^4\right] &= \left[\frac{3\alpha_0^2}{\left(1-\alpha_1\right)^2}\right]\left[\frac{1-\alpha_1^2}{1-3\alpha_1^2}\right]
\end{aligned}
$$

- **Proof (Sketch):** Define $w_t = \left(\varepsilon_t^4, \varepsilon_t^2\right)'$, and write

$$
\begin{aligned}
E\left[w_t | I_{t-1}\right] &= \begin{pmatrix} E\left[\varepsilon_t^4 | I_{t-1}\right] \\ E\left[\varepsilon_t^2 | I_{t-1}\right] \end{pmatrix} \\
&= \begin{pmatrix} 3\alpha_0^2 \\ \alpha_0 \end{pmatrix} + \begin{pmatrix} 3\alpha_1^2 & 6\alpha_0\alpha_1 \\ 0 & \alpha_1 \end{pmatrix} \begin{pmatrix} \varepsilon_{t-1}^4 \\ \varepsilon_{t-1}^2 \end{pmatrix}
\end{aligned}
$$

ARCH Models

- **Proof (con't):** More succinctly,

$$E\left[w_t \middle| I_{t-1}\right] = b + Aw_{t-1},$$

where

$$b = \begin{pmatrix} 3\alpha_0^2 \\ \alpha_0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 3\alpha_1^2 & 6\alpha_0\alpha_1 \\ 0 & \alpha_1 \end{pmatrix}$$

It follows by the law of iterated expectations that

$$E\left[w_t\right] = b + AE\left[w_{t-1}\right]$$

or, if we let $\gamma_t = E\left[w_t\right]$,

$$\gamma_t = b + A\gamma_{t-1}$$

so that, after rearranging, we can write

$$\left(I_2 - AL\right)\gamma_t = b$$

- **Proof (con't):** If the roots of the determinantal equation

$$\det\left(I_2 - Az\right) = 0$$

are all outside the unit circle (i.e., if all roots have modulus greater than one); then, we can further invert the lag operator and write

$$\gamma_t = \left(I_2 - AL\right)^{-1} b = \left(I_2 - A\right)^{-1} b.$$

Now,

$$\begin{aligned}
\det\left(I_2 - Az\right) &= \det\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3\alpha_1^2 z & 6\alpha_0\alpha_1 z \\ 0 & \alpha_1 z \end{pmatrix}\right\} \\
&= \det\left\{\begin{pmatrix} 1 - 3\alpha_1^2 z & -6\alpha_0\alpha_1 z \\ 0 & 1 - \alpha_1 z \end{pmatrix}\right\} \\
&= \left(1 - 3\alpha_1^2 z\right)\left(1 - \alpha_1 z\right)
\end{aligned}$$

so the roots are

$$z = \frac{1}{3\alpha_1^2} \text{ and } z = \frac{1}{\alpha_1}.$$

- **Proof (con't):** Hence, $|z| > 1$ if and only if $3\alpha_1^2 < 1$. Moreover, by direct calculation using the law of iterated expectations, we obtain

$$
\begin{aligned}
E[w_t] &= \begin{pmatrix} E[\varepsilon_t^4] \\ E[\varepsilon_t^2] \end{pmatrix} \\
&= (I_2 - A)^{-1} b \\
&= \begin{pmatrix} 1 - 3\alpha_1^2 & -6\alpha_0\alpha_1 \\ 0 & 1 - \alpha_1 \end{pmatrix}^{-1} \begin{pmatrix} 3\alpha_0^2 \\ \alpha_0 \end{pmatrix} \\
&= \frac{1}{(1 - 3\alpha_1^2)(1 - \alpha_1)} \begin{pmatrix} 1 - \alpha_1 & 6\alpha_0\alpha_1 \\ 0 & 1 - 3\alpha_1^2 \end{pmatrix} \begin{pmatrix} 3\alpha_0^2 \\ \alpha_0 \end{pmatrix} \\
&= \frac{1}{(1 - 3\alpha_1^2)(1 - \alpha_1)} \begin{pmatrix} (1 - \alpha_1) 3\alpha_0^2 + 6\alpha_0^2\alpha_1 \\ (1 - 3\alpha_1^2)\alpha_0 \end{pmatrix} \\
&= \frac{1}{(1 - 3\alpha_1^2)(1 - \alpha_1)} \begin{pmatrix} 3\alpha_0^2 (1 + \alpha_1) \\ (1 - 3\alpha_1^2)\alpha_0 \end{pmatrix}
\end{aligned}
$$

ARCH Models

- **Proof (con't):** It follows that

$$\begin{aligned} E\left[w_t\right] &= \begin{pmatrix} E\left[\varepsilon_t^4\right] \\ E\left[\varepsilon_t^2\right] \end{pmatrix} \\ &= \begin{pmatrix} \frac{3\alpha_0^2}{(1-\alpha_1)} \frac{1+\alpha_1}{\left(1-3\alpha_1^2\right)} \\ \frac{\alpha_0}{1-\alpha_1} \end{pmatrix} \\ &= \begin{pmatrix} \left[\frac{3\alpha_0^2}{(1-\alpha_1)^2}\right] \left[\frac{1-\alpha_1^2}{\left(1-3\alpha_1^2\right)}\right] \\ \frac{\alpha_0}{1-\alpha_1} \end{pmatrix} \end{aligned}$$

- **Remark (i):** Note that a property of the normal random variable is that if $\eta \sim N\left(0, \sigma_\eta^2\right)$; then, $E\left[\eta^4\right] = 3\sigma_\eta^4$. On the other hand, if $\{\varepsilon_t\}$ follows an ARCH(1) process; then,

$$E\left[\varepsilon_t^4\right] = \left[\frac{3\alpha_0^2}{(1-\alpha_1)^2}\right]\left[\frac{1-\alpha_1^2}{1-3\alpha_1^2}\right] > \frac{3\alpha_0^2}{(1-\alpha_1)^2} = 3\left(E\left[\varepsilon_t^2\right]\right)^2.$$

  since $\left(1-\alpha_1^2\right) / \left(1-3\alpha_1^2\right) > 1$. Hence, the ARCH error process has "fatter-tails" than that implied by the normal distribution.
- **Remark (ii):** Note that the stronger condition $3\alpha_1^2 < 1$ is used to ensure the existence of the (unconditional) fourth moment. If we only wish to specify the existence of the (unconditional) second moment, then we only need to require the weaker condition $\alpha_1 < 1$. In the discussion of higher order ARCH processes below, we shall focus only on cases where we make enough assumptions to ensure the existence of the second moments; hence, the conditions given for higher order ARCH processes will be the analogue of the condition $0 < \alpha_1 < 1$.

ARCH(p) Processes

- The ARCH(1) process described above can be readily generalized to one with an arbitrary finite lag order p

$$\varepsilon_t = u_t \left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2\right]^{1/2}$$

- **Assumptions:**

(i) $\{u_t\} \equiv i.i.d.N(0,1)$.

(ii) $\alpha_i > 0 \quad \text{for } i = 0, 1, ..., p$.

(iii) $\alpha(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p = 0 \Longrightarrow |z| > 1$.

- **Claim:** Assumptions (ii) and (iii) above imply that $\alpha_1 + \cdots + \alpha_p < 1$.
- **Proof of Claim:** To proceed, take derivative of $\alpha(z)$ with respect to z, and we obtain

$$\alpha'(z) = -\left(\alpha_1 + 2\alpha_2 z + \cdots + p\alpha_p z^{p-1}\right) < 0$$

for $z > 0$ given that $\alpha_i > 0 \quad \text{for } i = 0, 1, ..., p$.

ARCH(p) Processes

- **Proof of Claim (con't):** Moreover, since $\alpha(0) = 1$ and $\lim_{z \to \infty} \alpha(z) = -\infty$ and since $\alpha(z)$ is continuous, it follows that the polynomial equation $\alpha(z) = 0$ has only one positive real root, say z^*. Next, note that since all roots of $\alpha(z) = 0$ are outside the unit circle, it must be that $z^* > 1$. Now, by the fact that $\alpha(z)$ is monotonically decreasing for $z \in [0, \infty)$, we have that

$$\alpha(1) = 1 - \alpha_1 - \cdots - \alpha_p > \alpha(z^*) = 0$$

from which it follows immediately that

$$1 > \alpha_1 + \cdots + \alpha_p$$

as required. □

ARCH(p) Processes

- **Some Properties of the ARCH(p) process:** Similar to the ARCH(1) case, it is straightforward to show that if $\{\varepsilon_t\}$ follows an ARCH(p) process, then

1. $E[\varepsilon_t | I_{t-1}] = 0$ and so $E[\varepsilon_t] = 0$.
2. $\{\varepsilon_t\}$ is serially uncorrelated.
3. $E[\varepsilon_t^2 | I_{t-1}] = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2$.

- **Unconditional Variance:** Moreover, note that

$$\begin{aligned}
& E[\varepsilon_t^2] \\
= \; & E\{E[\varepsilon_t^2 | I_{t-1}]\} \\
= \; & E\{[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2] E[u_t^2 | I_{t-1}]\} \\
= \; & E\{[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2] E[u_t^2]\} \\
= \; & \alpha_0 + \alpha_1 E[\varepsilon_{t-1}^2] + \cdots + \alpha_p E[\varepsilon_{t-p}^2]
\end{aligned}$$

ARCH(p) Processes

- **Unconditional Variance (con't):** Now, define $V_{t-j} = E\left[\varepsilon_{t-j}^2\right]$ for $j = 0, ..., p$, and we can write the above relationship in terms of the p^{th} order difference equation

$$V_t = \alpha_0 + \alpha_1 V_{t-1} + \cdots + \alpha_p V_{t-p}$$

or

$$\left(1 - \alpha_1 L - \cdots - \alpha_p L^p\right) V_t = \alpha_0$$

Given the condition $\alpha(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p = 0 \Longrightarrow |z| > 1$, we can invert the lag polynomial to obtain

$$\begin{aligned} V_t &= \left(1 - \alpha_1 L - \cdots - \alpha_p L^p\right)^{-1} \alpha_0 \\ &= \frac{\alpha_0}{1 - \alpha_1 - \cdots - \alpha_p} \\ &> 0 \end{aligned}$$

since $\alpha_0 > 0$ by assumption and $\alpha_1 + \cdots + \alpha_p < 1$, as previously shown.

Effects of ARCH Errors on Estimation of Regression Coefficients

- Consider again the linear regression model

$$y_t = x_t'\beta + \varepsilon_t$$

whose error term ε_t follows an ARCH(p) process. In this case,

$$y_t | I_{t-1} \sim N\left(x_t'\beta, h_t\right)$$

where

$$h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \cdots + \alpha_p\varepsilon_{t-p}^2.$$

Effects of ARCH Errors on Estimation of Regression Coefficients

- The log-likelihood function for this model can be written as

$$l\left(\beta, \alpha\right) = \frac{1}{T}\sum_{t=1}^{T} l_t\left(\beta, \alpha\right)$$

where $\alpha = \left(\alpha_0, \alpha_1, ..., \alpha_p\right)'$ and

$$\begin{aligned} l_t\left(\beta, \alpha\right) &= const - \frac{1}{2}\ln h_t - \frac{1}{2}\varepsilon_t^2 / h_t \\ &= const - \frac{1}{2}\ln h_t - \frac{1}{2}\left(y_t - x_t'\beta\right)^2 / h_t \end{aligned}$$

Effects of ARCH Errors on Estimation of Regression Coefficients

- Under our assumptions, it is easily seen that

$$E[y|X] = X\beta,$$
$$VC(y|X) = \sigma^2 I_T$$

where $y = (y_1, ..., y_T)'$, $X = (x_1, ..., x_T)'$, and $\sigma^2 I_T$. Hence, the conditions of the Gauss-Markov theorem is satisfied and the OLS estimator of β is the best linear unbiased estimator. It is also consistent. However, it is easy to show that the maximum likelihood (ML) estimator of β in this case is different from the OLS estimator and is asymptotically more efficient in the sense that the ML estimator achieves the Cramér-Rao lower bound but the OLS estimator does not.

ARCH-in-Mean Specification

- Asset pricing theory suggests that portfolio with higher perceived risk would have to compensate by yielding higher expected returns. Hence, let

$$r_t = \mu_t + \varepsilon_t,$$

where $\mu_t = E[r_t | I_{t-1}]$. One may want to have a model where μ_t is related to the conditional variance or volatility of the model. One such model, the ARCH-in-mean (or ARCH-M) regression model, was introduced by Engle, Lilien, and Robins (1987). This model takes the form

$$\begin{aligned} y_t &= x_t' \beta + \delta h_t + \varepsilon_t, \\ \varepsilon_t &= \sqrt{h_t} u_t \\ h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2 \end{aligned}$$

for $t = 1, ..., T$. Here, again, we assume that

$$\{u_t\} \equiv i.i.d.N(0,1).$$

Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) Process

- **GARCH(p,q) process:** A useful generalization of the ARCH model is the following GARCH model due to Bollerslev (1986).

$$\varepsilon_t = h_t^{1/2} u_t,$$

where

$$h_t = \alpha_0 + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2.$$

- **Assumptions:**

G1 $\alpha_0 > 0$ and $\alpha_i \geq 0$ (for $i = 1, ..., m$).

G2 $\delta_j \geq 0$ (for $j = 1, ..., r$).

G3 $\{u_t\} \equiv i.i.d.N(0,1)$;

- **Remark:** Note that even a GARCH(1,1) model will allow h_t to depend on ε_t^2 from the distant past. Thus, GARCH provides a clever way of capturing slowly changing variances without having to specify a model that has a lot of parameters to estimate.

GARCH

- **Remark:** Like ARCH, GARCH can also be estimated using the method of maximum likelihood.
- **Example:** GARCH(1,1)
  In this case,

$$h_t = \alpha_0 + \delta_1 h_{t-1} + \alpha_1 \varepsilon_{t-1}^2, \ 0 < \delta_1 < 1 \text{ and } 0 < \alpha_1 < 1$$

  Substituting recursively, we have

$$\begin{aligned} h_t &= \alpha_0 + \delta_1 \left(\alpha_0 + \delta_1 h_{t-2} + \alpha_1 \varepsilon_{t-2}^2 \right) + \alpha_1 \varepsilon_{t-1}^2 \\ &= \delta_1^t h_0 + \alpha_0 \left(1 + \delta_1 + \cdots + \delta_1^{t-1} \right) \\ &\quad + \alpha_1 \left(\varepsilon_{t-1}^2 + \delta_1 \varepsilon_{t-2}^2 + \cdots + \delta_1^{t-1} \varepsilon_0^2 \right) \end{aligned}$$

- **Remark (i):** In practical applications, we need to estimate h_0. Approaches taken in the literature include treating h_0 as an extra parameter of the likelihood function and estimate it jointly with the other parameters. Another approach is to suppose that it is equal to the unconditional variance and estimate it using the formula

$$\widehat{h}_0 = \widehat{\sigma}^2 = \frac{1}{T+1} \sum_{t=1}^{T} \left(y_t - x_t' \widehat{\beta} \right)^{2.}$$

GARCH

- **Remark (ii):** If we envision a situation where the GARCH(1,1) process arises from the infinite past; then, continuing the backward substitution process described previously, we obtain the representation

$$\begin{aligned} h_t &= \lim_{s\to\infty} \delta_1^s h_0 + \frac{\alpha_0}{1-\delta_1} + \alpha_1 \sum_{j=0}^{\infty} \delta_1^j \varepsilon_{t-1-j}^2 \\ &= \frac{\alpha_0}{1-\delta_1} + \alpha_1 \sum_{j=0}^{\infty} \delta_1^j \varepsilon_{t-1-j}^2 \end{aligned}$$

assuming that $\lim_{s\to\infty} \delta_1^s h_0 = 0$. It follows that the GARCH(1,1) model provides a parsimonious way for allowing h_t to depend on an infinite number of lagged ε_t^2's.

Some Properties of the GARCH (r,m) Process

- If $\{\varepsilon_t\}$ follows the GARCH (r,m) process given by

$$\varepsilon_t = h_t^{1/2} u_t,$$

where

$$h_t = \alpha_0 + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2.$$

then, $\{\varepsilon_t\}$ has mean zero and is serially uncorrelated. To see this, note that by direct calculation

$$\begin{aligned} & E\left[\varepsilon_t | I_{t-1}\right] \\ = & \ h_t^{1/2} E\left[u_t | I_{t-1}\right] \\ = & \left[\alpha_0 + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2\right]^{1/2} \\ & \times E\left[u_t | I_{t-1}\right] \\ = & \left[\alpha_0 + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2\right]^{1/2} E\left[u_t\right] \\ = & \ 0. \end{aligned}$$

Some Properties of the GARCH (r,m) Process

- It follows by the law of iterated expectations that

$$E\left[\varepsilon_t\right] = E\left(E\left[\varepsilon_t | I_{t-1}\right]\right) = 0.$$

Moreover, it follows from the law of iterated expectations that for integer $j \geq 1$

$$E\left[\varepsilon_t \varepsilon_{t-j}\right] = E\left(\varepsilon_{t-j} E\left[\varepsilon_t | I_{t-1}\right]\right) = 0.$$

- **A Useful Representation for the GARCH (r,m) Process:** Although it may seem that a GARCH (r,m) process is similar to an ARMA (r,m) process; surprisingly, it is actually more analogous to an ARMA (p, r) process where $p = \max\{r, m\}$. To see this, define

$$v_t = \varepsilon_t^2 - h_t$$

so that

$$\begin{aligned} E\left[v_t | I_{t-1}\right] &= E\left[\varepsilon_t^2 | I_{t-1}\right] - E\left[h_t | I_{t-1}\right] \\ &= h_t - h_t = 0. \end{aligned}$$

Some Properties of the GARCH (r,m) Process

- It follows from the same argument as above that $\{v_t\}$ is a mean zero and serially uncorrelated process. Now, substituting $h_t = \varepsilon_t^2 - v_t$ into

$$h_t = \alpha_0 + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2,$$

we get

$$\begin{aligned} & \varepsilon_t^2 - v_t \\ = \; & \alpha_0 + \delta_1 \left(\varepsilon_{t-1}^2 - v_{t-1} \right) + \cdots + \delta_r \left(\varepsilon_{t-r}^2 - v_{t-r} \right) \\ & + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_m \varepsilon_{t-m}^2 \\ = \; & \alpha_0 + (\delta_1 + \alpha_1) \, \varepsilon_{t-1}^2 + \cdots + (\delta_p + \alpha_p) \, \varepsilon_{t-p}^2 \\ & - \delta_1 v_{t-1} - \cdots \cdot - \delta_r v_{t-r} \end{aligned}$$

Some Properties of the GARCH (r,m) Process

- or

$$\begin{aligned} & \varepsilon_t^2 \\ = \; & \alpha_0 + (\delta_1 + \alpha_1)\, \varepsilon_{t-1}^2 + \cdots + (\delta_p + \alpha_p)\, \varepsilon_{t-p}^2 \\ & + v_t - \delta_1 v_{t-1} - \cdots - \delta_r v_{t-r} \end{aligned}$$

  where in the representation above, we have defined $\delta_j \equiv 0$ for $j > r$ and $\alpha_j \equiv 0$ for $j > m$.

- In light of the ARMA-type representation given above, it seems natural to impose the condition that

$$\psi(z) = 1 - (\delta_1 + \alpha_1)\, z - \cdots - (\delta_p + \alpha_p)\, z^p = 0 \Longrightarrow |z| > 1.$$

Some Properties of the GARCH (r,m) Process

- Under this condition, we can invert the lag polynomial operator

$$\psi(L) = 1 - (\delta_1 + \alpha_1) L - \cdots - (\delta_p + \alpha_p) L^p$$

to obtain a MA-type representation

$$\varepsilon_t^2 = [1 - (\delta_1 + \alpha_1) L - \cdots - (\delta_p + \alpha_p) L^p]^{-1} (\alpha_0 + \eta_t)$$

where

$$\eta_t = v_t - \delta_1 v_{t-1} - \cdots - \delta_r v_{t-r}.$$

It follows that

$$\begin{aligned} E\left[\varepsilon_t^2\right] &= \frac{\alpha_0}{1 - (\delta_1 + \alpha_1) - \cdots - (\delta_p + \alpha_p)} \\ &\quad + E\left\{[1 - (\delta_1 + \alpha_1) L - \cdots - (\delta_p + \alpha_p) L^p]^{-1} \eta_t\right\} \\ &= \frac{\alpha_0}{1 - (\delta_1 + \alpha_1) - \cdots - (\delta_p + \alpha_p)} \end{aligned}$$

since $\{\eta_t\}$ is a mean zero process given that $\{v_t\}$ is mean zero.

Some Properties of the GARCH (r,m) Process

- Suppose we maintain the assumptions

G1 $\alpha_0 > 0$ and $\alpha_i \geq 0$ (for $i = 1, ..., m$).

G2 $\delta_j \geq 0$ (for $j = 1, ..., r$);

Then, a sufficient condition for $0 < E\left[\varepsilon_t^2\right] < \infty$ is that

$$\psi(z) = 1 - (\delta_1 + \alpha_1) z - \cdots - (\delta_p + \alpha_p) z^p = 0 \Longrightarrow |z| > 1.$$

which, based on argument given earlier, implies that

$$(\delta_1 + \alpha_1) + \cdots + (\delta_p + \alpha_p) < 1$$

- Note also that the unconditional variance

$$E\left[\varepsilon_t^2\right] = \frac{\alpha_0}{1 - (\delta_1 + \alpha_1) - \cdots - (\delta_p + \alpha_p)}$$

does not depend on t, so there is no unconditional heteroskedasticity. Coupled with our earlier observations that $E\left[\varepsilon_t\right] = 0$ and $E\left[\varepsilon_t \varepsilon_{t-j}\right] = 0$, we see that $\{\varepsilon_t\}$ is covariance stationary.

Empirical Illustration

- Tsay (2010) provided an empirical illustration using time series data on monthly excess returns of the S&P 500 index. His data set runs from 1926-1991 and has 792 total observations. More specifically, he estimated an AR(3) model with an error process that is GARCH(1,1), i.e.,

$$r_t = \mu + \rho_1 r_{t-1} + \rho_2 r_{t-2} + \rho_3 r_{t-3} + \varepsilon_t$$

where

$$\begin{aligned} \varepsilon_t &= h_t^{1/2} u_t, \\ h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \delta_1 h_{t-1}. \end{aligned}$$

- He obtained the following coefficient estimates

$$\begin{aligned} \widehat{r}_t &= 0.0078 + 0.032 r_{t-1} - 0.029 r_{t-2} - 0.008 r_{t-3} \\ \widehat{h}_t &= 0.000084 + 0.1213 \varepsilon_{t-1}^2 + 0.8523 h_{t-1} \end{aligned}$$