Quadratic Voting *

Steven P. Lalley[†] E. Glen Weyl[‡]

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Abstract

We argue that quadratic pricing of votes on collective decisions is analogous to linear pricing of private goods and thus solves the tyranny of majority created by the one-person-one-vote rule. To do so we propose a solution concept for costly voting models where individuals take the price of influence in units of votes as given. Under this concept, quadratic pricing of votes is the only rule that is always efficient. We then show that all type-symmetric Bayes-Nash equilibria of an independent private values Quadratic Voting game converge to this efficient price-taking outcome as the population size grows large, with inefficiency generically decaying as $1/N$. We discuss the robustness of these conclusions and their implications for market and mechanism design.

Keywords: social choice, collective decisions, large markets, costly voting, vote trading

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†Department of Statistics, University of Chicago: 5734 S. University Avenue, Chicago, IL 60637; lalley@galton.uchicago.edu, http://www.stat.uchicago.edu/~lalley/.

‡Microsoft Research New England, One Memorial Drive, Cambridge, MA 02142 and Department of Economics, University of Chicago: 1126 E. 59th Street, Chicago, IL 60637; glenweyl@microsoft.com, http://www.glenweyl.com.

> (T)he "one man one vote" rule gives everyone minimum share in public decision-making, but it also sets...a maximum...it does not permit the citizens to register the widely different intensities with which they hold their respective political convictions and opinions.

– Albert O. Hirschman, *Shifting Involvements: Private Interest and Public Action*

> (T)he will of those whose qualifications, when both sides are added up, are the greatest, should prevail.

– Aristotle, *The Politics*, Book VI, Part III

1 Introduction

Prohibitions on gay marriage seem destined to be remembered as classic examples of the "tyranny of the majority" that has plagued democracy since the ancient world. While in many countries a(n increasingly narrow) majority of voters oppose the practice, the value it brings to those directly affected seems likely to be an order of magnitude larger than the costs accruing to those opposed.[1] However, one-person-one-vote (1p1v) offers no opportunity to express intensity of preference, allowing such inefficient policies to persist. While most developed countries have institutions, such as independent judiciaries and log-rolling, that help protect minorities, these are often costly and insufficient (Posner and Weyl, Forthcoming). In this paper we argue for a simple solution: allow individuals to cast as many (continuous) votes as they wish, but charge them the square of the votes they cast.[2]

The basic problem we seek to address is that 1p1v rations rather than prices votes, resulting in externalities across individuals. This contrast with the market mechanisms for allocating private goods where individuals pay the opportunity cost of their consumption, leading to social efficiency (Smith, 1776). We therefore, in Section 2, consider a simple class of *costly voting* rules under which individuals purchase any continuous number of votes they wish using a quasi-linear numeraire. To study such rules, we propose a "price-taking" model where individuals take as given the *vote-price of influence*, the number of votes it takes to have a unit of influence on the outcome. This *price* for short plays the same role in coordinating behavior that prices do in a market for private goods. Optimization given price-taking, together with a fixed total supply of influence, inspired by general statistical limits on influence as derived by Al-Najjar and Smorodinsky (2000) define a *price-taking equilibrium*.

[1]A rough calibration to the California Proposition 8 referendum by Weyl (2015a) suggests a potential Pareto improvement of nearly $1000 per California voter was possible over the prohibition that was enacted through 1p1v.

[2]To our knowledge this mechanism was first proposed, in its present form, by Weyl in an earlier version of this paper (circulated in February 2012) as "Quadratic Vote Buying".

We show that for any convex vote costs a unique equilibrium exists. Limiting cases yield familiar predictions: in nearly linear vote buying equilibrium approaches the dictatorship of the single individual with the most intense preference typically derived from linear vote buying models and as the cost becomes extremely convex 1p1v results. We then use this concept in Subsection 2.5 to extend to discrete decisions Hylland and Zeckhauser (1980)'s argument that efficiency occurs in equilibrium for all value configurations if and only the pricing rule is quadratic.[3] This uniqueness contrasts with the complete information, game theoretic environment studied by Groves and Ledyard (1977) where many rules are efficient (Maskin, 1999).

Individuals in this model have an assumed-linear value of acquiring "influence", a concept without clear micro-foundation in this non-stochastic price-theoretic environment.[4] In Section 3 we therefore study a canonical, quasi-linear independent bounded-support private values model with a small aggregate noise that smooths payoffs.

We prove that in any type-symmetric Bayes-Nash equilibrium, at least one of which exists, any social waste associated with equilibrium is eliminated as the population grows large. For the generic case when the mean of the value distribution is non-zero, equilibrium takes a surprising form, where a vanishingly small tail of "extremists", from the side of the distribution opposite to its mean, purchase enough votes to win the election with high probability. Their existence, despite occurring only with probability $1/N$, is sufficient to provide others the incentive to buy sufficient votes to deter extremists from being more active. Approximate calculations of constants on this decay rate and numerical simulations by Weyl (2015a) suggest resulting inefficiency is very small for reasonable parameter values in moderately large populations (with thousands of individuals).

While we focus here on establishing these motivating properties of Quadratic Voting (QV), in companion work we and others have investigated its robustness to a variety of changes in the model and practical considerations. In Section 4 we briefly discuss our tentative conclusion, based on this work, that QV is a more practically viable approximately efficient alternative to 1p1v than others previously proposed by economists. Finally in Section 5 we briefly describe on-going work related on QV.

2 Price-Taking Equilibrium of Costly Voting

As we discuss in Section 4, the mechanisms economists have previously proposed for collective decision-making strike most as complex and fitted tightly to the formal modeling environments that motivated them. Furthermore they have proved fragile in other analyses and have thus

[3] Also, our quasi-linearity assumption allows us to extend their result to a single decision.

[4] Readers who are interested primarily in the core economic intuitions behind and applications of QV may wish to skip Section 3, which is quite technical. On the other hand, readers primarily interested in micro-founded results may wish to skip Section 2.

been widely dismissed as impractical. Like over-fitted statistical models, they appear to perform poorly out of sample.

We therefore take an alternative approach inspired by the literature on over-fitting (Vapnik and Chervonenkis, 1971): we consider a simple class of mechanisms that are as analogous as possible to the linear-pricing market mechanism studied by Smith (1776) and analyze them using a price-taking approximation. This uniquely identifies a simple mechanism that we hope, consequently, will be more robust than those previously considered by economists. While placing such arbitrary restraints on the class of mechanisms considered may seem restrictive, statistical learning theory (Blumer et al., 1987) suggests it increases the reliability of extrapolation based upon such an analysis. Similarly while our price-taking concept is somewhat ad-hoc, Section 3 shows that it is the limit of any type-symmetric equilibrium of a canonical game theoretic micro-foundation and Subsection 4.1 discussed the robustness of this conclusion.

2.1 Model

Consider a finite collection N of individuals making a binary collective decision about whether to maintain a shared status quo or adopt an alternative A. Each individual $i = 1, \ldots, N$ is characterized by a *value* $u_i \in \mathbb{R}$ describing her willingness to pay, out of a quasi-linear numeraire, to see the alternative adopted over the status quo; negative values represent willingness to pay to maintain the status quo. For normalization we assume that an individual gains u_i if the alternative is adopted, but loses it if the status quo is maintained, so that her net value for changing the outcome is $2u_i$.

We study a class of *costly voting mechanisms*. Whether the alternative is implemented is determined by a vote in which each individual selects a scalar $v_i \in \mathbb{R}$ and the alternative is implemented if and only if $\sum_i v_i \geq 0$. Each individual pays a cost $c(v_i)$ where c is differentiable, convex, even and strictly monotone increasing in $|v_i|$ and receives a refund $r_i(v_{-i})$ such that $\sum_i c(v_i) = \sum_i r_i(v_{-i})$, where v_{-i} is the vector of votes by other individuals.[5]

2.2 Definition of equilibrium

We begin by defining our equilibrium concept formally and then motivate it.

Definition 1. A *collective decision problem* is a triple $\{N, S, u\}$ of a *number of individuals* $N \in \mathbb{Z}_{++}$, a *supply of influence* $S \in \mathbb{R}_{++}$ and an N-dimensional *value vector* $u \in \mathbb{R}^N$.

Definition 2. A *price-taking equilibrium* of a collective decision problem $\{N, S, u\}$ under costly voting rule c a triple $\{I^\star, p^\star, 1_A^\star\}$ of an influence vector $I^\star \in \mathbb{R}^N$, *price* $p^\star \in \mathbb{R}_{++}$ and an action

[5]We do not specify precisely which refund rule is used as it is irrelevant to the analysis that follows in this and the next two sections (given that it in no way depends on the individual's own choice), but a simple one obeying this budget balance condition is that each individual receives $\frac{\sum_{j\neq i} c(v_j)}{N-1}$.

$1_A^\star \in \{0, 1\}$ such that

1. *Price-taking*: for each $i \in 1, \ldots, N$, $I_i^\star$ maximizes $2u_i I_i - c\left(pI_i\right)$ over all choices of $I_i \in \mathbb{R}$.
2. *Market clearing*: $\sum_{i=1}^N |I_i^\star| = S$.
3. *Majority rules*: $1_A^\star = 1_{V^\star \geq 0}$ where $V^\star \equiv \sum_i p^\star I_i^\star$.

The *equilibrium votes* corresponding to an equilibrium $\{I^\star, p^\star, 1_A^\star\}$ is the vector $v^\star$ whose ith entry is $v_i^\star = p^\star I_i^\star$.

Under our solution concept, individuals choose an amount of influence to exert over the decision, taking as given the linear price of this influence in units of votes. The market clears if the total absolute value of influence acquired equals an exogenous supply S. The most natural micro-foundation for this concept is the chance that individuals perceive of their changing the outcome in their preferred direction. Section 3 develops this interpretation rigorously (but narrowly).

This chance of any vote influencing the outcome of the aggregate decision is likely to be small in large elections for reasons familiar to economists. As Mailath and Postelwaite (1990) show, only in very special, complete information environments is it possible to make a large number of individuals pivotal for a single binary decision each with a large probability. In fact, Al-Najjar and Smorodinsky (2000) prove strong and robust upper bounds on the total influence exerted on average on the outcome. Extensive empirical evidence confirms these predictions (Mulligan and Hunter, 2003; Gelman et al., 2010). While the precise amount of influence "available" depends on the particular form and parameters of the information environment, the basic principles of it being in limited supply and thus it being small for almost all individuals is robust across all such environments (Gelman et al., 2002). This motivates our assumption that there is a limited total supply of influence.

Clearly this influence arises (only) from the possibility that the decision may be tied. Thus the price of influence in units of votes is the inverse of the chance (density with which) a tie occurs. We refer to this price as the *vote-price of influence* or simply *price* for short. Mueller (1973, 1977) and Laine (1977) argue that, in a somewhat different context, this price is insensitive to number of votes an individual purchases. This seems a reasonable extension of the previous intuition limiting the total size of influence, because it appears impossible for an individual with very little influence over the final decision to significantly impact the chance a tie occurs; after all an election's chance of being tied only changes by the chance of one side's victory rising or falling. But by Al-Najjar and Smorodinsky's arguments, it is impossible for a large number of individuals to have significant influence. This is the intuition behind our price-taking assumption.

2.3 Existence and uniqueness

As a prelude to our main analysis, we now consider two results, one technical and one substantive, that illustrate attractive properties of price-taking equilibrium.

Lemma 1. For any collective decision problem $\{N, S, u\}$ and any vote cost $c(\cdot)$ there exists a unique price-taking equilibrium.

Proof. By differentiability and convexity, price-taking is equivalent to

$$2u_i = pc'(pI_i).$$

By the same properties, c' is invertible and its inverse is continuous; denote this inverse $\gamma(\cdot)$. Then, if we let $I(p;u)$ be the unique vector of price-taking influences voters acquire when the price is p,

$$I_i(p;u) = \frac{\gamma\left(\frac{2u_i}{p}\right)}{p}. \tag{1}$$

γ is strictly monotone increasing by strict convexity of c, has the same sign as I_i by the fact that c is even and increasing in the absolute value of its argument and is continuous by differentiability of c. $|I_i(p;u)|$, and thus $|I(p;u)|$, is thus strictly monotone decreasing and continuous in p. Thus $D(p) \equiv \sum_{i=1}^{N} |I_i(p;u)|$ is strictly monotone decreasing in p and continuous. By definition, any equilibrium must have $D(p) = S$; thus there may be at most one equilibrium.

Furthermore note that as $p \to \infty$, $I_i(p;u) \to 0$ for all i because $\gamma\left(\frac{u_i}{p}\right)$ is strictly declining in p. By the same logic, $I_i(p;u) \to \pm\infty$ as $p \to 0$. Thus $\lim_{p\to 0} D(p) = \infty$ and $\lim_{p\to\infty} D(p) = 0$. Thus by continuity and the intermediate value theorem there is some p for which $D(p) = S$. This p and $I(p;u)$ constitute an equilibrium as equation (1) fully characterizes the price-taking condition by our arguments above. Given this, the outcome $1_A^\star$ immediately follows uniquely. $\square$

Note the role that each of convexity and quasi-linearity play here. Absent convexity, equilibrium could easily fail to exist because it might be that individuals would, at each value of p, choose either a very large number of votes or a very small number of votes, potentially violating market clearing. Absent quasi-linearity, income effects of changes in p could result in multiple equilibria just as in classical general equilibrium theory.

2.4 Limit cases as validation

Does the concept yield substantively reasonable conclusions? Consider the special case of convex power costs $c(v) = k|v|^x$, where $x > 1$. Then price taking requires that

$$\frac{u_i}{p} = \frac{kx}{2}\text{sign}(v_i^\star)|v_i^\star|^{x-1} \iff v_i^\star = \left(\frac{2}{kxp}\right)^{\frac{1}{x-1}} \text{sign}(u_i)|u_i|^{\frac{1}{x-1}}.$$

Thus, in this class of mechanisms, the decision always favors whichever side has a greater value of $\sum_i |u_i|^{\frac{1}{x-1}}$. Two extreme cases yield particularly simple and familiar results.

For any fixed vector u and for all i, $\lim_{x\to\infty} |u_i|^{\frac{1}{x-1}} \to 1$. Thus, as the cost of voting becomes arbitrarily convex the decision is determined by which side has more individuals, that is 1p1v majority rule. A case yielding a less obvious conclusion is the limit as $x \to 1$ of linear voting costs. Let $i^\star$ be the index of the (generically unique) individual with the largest $|u_i|$. Note that for any $j \neq i^\star$

$$\lim_{x\to 1} \frac{|u_{i^\star}|^{\frac{1}{x-1}}}{|u_j|^{\frac{1}{x-1}}} = \infty \implies \lim_{x\to 1} \frac{|u_{i^\star}|^{\frac{1}{x-1}}}{\sum_{j\neq i^\star} |u_j|^{\frac{1}{x-1}}} \to \infty.$$

Thus as $x \to 1$ the outcome is generically the dictatorship of the single individual with the most intense preference.

This result is predicted by several recent studies of equilibria in linear vote buying models with new solution concepts or set-ups that resolve classic problems nonexistence in linear vote-buying models, arising from lack of convexity.[6] Casella et al. (2012) propose a notion of *ex-ante* equilibrium under which they, and Casella and Turban (2014), show that in every case they can study the single individual with the most intense preference wins with high probability, regardless of all other parameters. Dekel et al. (2008, 2009) and Dekel and Wolinsky (2012) find similar and in many cases identical results in a range of game theoretic vote buying models.

2.5 The uniquely robust efficiency of Quadratic Voting

We now characterize the class of voting costs yielding robustly efficient outcomes, as now formally defined, under this tractable and apparently reasonable solution concept.

Definition 3. A voting rule $c(\cdot)$ is *robustly efficient* if, for all collective decision problems $\{N, S, u\}$, in the unique equilibrium $1_A^\star = 1_{U\geq 0}$ where $U \equiv \sum_i u_i$.

Theorem 1. $c(\cdot)$ is robustly efficient if and only if $c(v) = kv^2$ for some $k > 0$.

Proof. First consider the "if" direction. By price-taking and our analysis in Subsection 2.4, at any price-taking equilibrium we must have

$$v_i^\star = \frac{1}{kp^\star}\text{sign}\,(u_i)\,|u_i| = \frac{u_i}{kp^\star}.$$

Thus

$$\text{sign}\left(\sum_i v_i^\star\right) = \text{sign}\left(\sum_i \frac{u_i}{kp^\star}\right) = \text{sign}\left(\sum_i u_i\right),$$

[6] It is also consistent with a long tradition of informal argument about the impact of (linear) vote buying and lobbying on political outcomes (Olson, 1965) and formal results on the private provision of public goods with linear technology (Bergstrom et al., 1986).

because $k, p^\star > 0$.

For the "only if" direction, equation 1 implies

$$v_i^\star = \gamma\left(\frac{2u_i}{p}\right). \tag{2}$$

By the proof of Lemma 1 (and by the argument in the text above), for any number of individuals N and value vector u, adjusting S can lead to any desired value of p. Thus we can, without loss of generality with respect to considering robust efficiency, assume that $p = 2$ as long as we do not adjust S in the rest of the proof. Thus equation (2) becomes $v_i^\star = \gamma(u_i)$. The only homogeneous of degree one functions of a single variable are linear, so either γ is linear or it is not homogeneous of degree one. In the first case, inversion and integration yields that c takes the desired form, given that γ is also even by the evenness of c. In the second case, there must exist some values $u' > 0, \kappa > 1$ (again by evenness) such that $\gamma(\kappa u') \neq \kappa\gamma(u')$. Suppose, without loss of generality, that $\gamma(\kappa u') > \kappa\gamma(u')$ and let $\Delta \equiv \frac{\gamma(\kappa u')}{\kappa\gamma(u')} - 1$. Let $N^\star$ be the least integer strictly greater than $\frac{2\kappa(1+\Delta)}{\Delta}$ and let $N^{\star\star}$ be the greatest integer strictly less than $\frac{N^\star}{k}$.

Consider a collective decision problem where $N^{\star\star}$ individuals have value $-\kappa u'$ and $N^\star$ individuals have value u' and there are no other individuals in the economy. Then note that

$$\sum_i u_i = N^\star u' - N^{\star\star}\kappa u' > N^\star u' - \frac{N^\star}{\kappa}\kappa u' = 0$$

so the sign of $\sum_i u_i > 0$. However,

$$\sum_i v_i^\star = N^\star\gamma(u') - N^{\star\star}\gamma(\kappa u') = \gamma(u')\left[N^\star - N^{\star\star}\kappa(1+\Delta)\right] <$$

$$\gamma(u')\left[N^\star - (N^\star - \kappa)(1+\Delta)\right] = \kappa\gamma(u')\left[1 + \Delta - \Delta\frac{N^\star}{\kappa}\right] < \kappa\gamma(u')\left[1 + \Delta - 2(1+\Delta)\right] < 0,$$

using the fact that $k, \gamma > 0$ by the monotonicity of γ. Thus unless $\gamma(\cdot)$ is homogeneous of degree 1, $c(\cdot)$ cannot be robustly efficient. □

The proof formalizes the intuition that quadratic functions are the only ones with linear derivatives and thus the only ones where individuals equating their marginal utility to the *marginal cost* of a vote will buy votes in proportion to their utility. As Smith (1776) observed about the linear pricing of private goods, quadratic pricing leads a voter who intends only her own gain to be led by an invisible hand to promote an end that is no part of her intention.

Our result is most closely related to three from the literature. Groves and Ledyard (1977) show that quadratic pricing can be used to achieve optimality in the provision of continuous public goods under complete information. However, under complete information, many other pricing schemes (Greenberg et al., 1977), many of them (Maskin, 1999) far more fragile than the quadratic mechanism, achieve optimality.[7]

A closer result, therefore, is that in unpublished and publicly unavailable work by Hylland and Zeckhauser (1980).[8] They show, in a Walrasian model analogous to ours where individuals take the price of influence as constant, that quadratic pricing of continuous public goods using artificial currency is the unique pricing rule that achieves an analog to the First and Second Fundamental Welfare Theorems. We extend their analysis to the discrete decisions through our notion of price-taking equilibrium and consider a case with only a single choice byu introducing a private numeraire good. We also extend their analysis by providing explicit non-cooperative foundations for Walrasian equilibrium in the next section.[9]

Finally, Goeree and Zhang (2013), in work that was circulated after the first draft of this paper, consider the large population limit of the Expected Externality mechanism of Arrow (1979) and d'Aspremont and Gérard-Varet (1979) for a binary, quasilinear collective decision like ours. In the case when the mean of the distribution of values is 0, this limit is payments that are approximately quadratic, so that a quadratic pricing mechanism (with a particular value of k) gives approximate incentives for truthful value revelation. While much of our analysis below focuses on the generic case when the mean of the value distribution is not 0, when the limit of the Expected Externality mechanism is nothing like quadrtic, it was also this connection that led us to consider the quadratic form.[10]

3 Convergence from Independent Private Values

We now follow Cournot (1838) and Satterthwaite and Williams (1989) who provided coherent non-cooperative game theoretic models that converge to market equilibrium by building an analogous model of convergence towards our price-taking equilibrium for collective decisions. Rigorous proofs of all results in this section appear in an appendix following the main text of the paper.

[7] This led much of the literature to consider such schemes generally unattractive (Bailey, 1994).

[8] This was recently revived by Benjamin et al. (2013) and Chung and Duggan (2014) who, like us until after the first version of this paper was published, were unaware of Hylland and Zeckhauser's previous work.

[9] See our conclusion for a discussion of our on-going work with Hylland and Zeckhauser to update and publish their original work.

[10] To gain some intuition for this, consider the classic problem of choosing the level of consumption of a good causing a negative externality. Each individual reports her schedule of harms and the optimal level of the externality is determined by equating demand to the vertical sum of the harm schedules. As Tideman (1983) observes, the Vickrey payment is the externality on other individuals of a given individual's report, which is the area between private demand and the cost curve for all other individuals between the quantity that would prevail absent an individual's report and the quantity that prevails with this report. Note that this is a deadweight loss triangle and, as such, grows quadratically in the change in quantity induced by the individual's report.

3.1 Model

We consider an environment of symmetric, independent private values, analogous to the most canonical models of double auctions (Satterthwaite and Williams, 1989). There are N voters $i = 1, \ldots, N$. Each voter is characterized by a value, u_i; these are drawn independently and identically from a continuous probability distribution F supported by a finite interval $[\underline{u}, \overline{u}]$, with associated density f and $\underline{u} < 0 < \overline{u}$.[11] For normalization, we assume that the numeraire has been scaled so that $\min(|\underline{u}|, \overline{u}) \geq 1$. We denote by μ, σ^2 and μ_3 respectively the mean, variance and raw third moment of u under F, and we assume that f is smooth and bounded away from 0 on $[\underline{u}, \overline{u}]$. Each individual knows her own value, but knows nothing about those of the other agents except that they were obtained by random sampling from F.

Each individual buys v_i votes, where $v_i \in \mathbb{R}$, and earns utility

$$u_i \Psi(V) - v_i^2 \tag{3}$$

where $V \equiv \sum_{i=1}^{N} v_i$.[12] Voters are expected wealth maximizers; thus, voter i chooses v_i to maximize

$$E\left[u_i \Psi\left(V_{-i} + v_i\right)\right] - v_i^2, \tag{4}$$

where $V_{-i} \equiv \sum_{j \neq i} v_i$ is the sum of all votes cast. The *payoff function* $\Psi : \mathbb{R} \to [-1, 1]$ is a smoothed proxy for the Heaviside function (viz. a discontinuous jump from -1 to 1 at 0): we assume that it is an odd, nondecreasing, C^∞ function such that for some $\delta < 1/\sqrt{2}$,

$$\Psi(x) = \text{sgn}(x) \quad \text{for all} \ \ |x| \geq \delta$$

and such that Ψ is strictly increasing, with positive derivative, on the interval $(-\delta, \delta)$. Thus, the derivative $\psi = \Psi'$ is twice an even probability density with support $[-\delta, \delta]$ that is strictly positive on $(-\delta, \delta)$. The assumption that $\delta < \sqrt{2}$ ensures that a voter with value $v_i \leq -1$ would find it worthwhile to purchase 2δ votes to sway the election in the event that the sum V_{-i} of the other votes were exactly $+\delta$. This in turn implies that there must exist pairs of values (α, w) with $-\delta < w < \delta$ and $\alpha \geq \delta$ such that $(1 - \Psi(w))\, |\underline{u}| - (\delta - w)^2$ is positive, and so there will exist extrema for the difference. We shall assume throughout that the following *steepness hypothesis* (roughly stating that there is a unique extremal at which the second derivative is negative) holds.

[11] Weyl (2015a) use heuristic arguments to conjecturally extend our analysis to the case of unbounded distributions and find that the decay of inefficiency is slower in this case if the value distribution has fat tails and $\mu \neq 0$, occurring at rate $N^{a - 1/a+1}$, where a is the rate of decay of the Pareto tail of the distribution of sign opposite to that of μ. For $a = 3$, a reasonable approximation to the US income distribution, this leads to a much slower decay of inefficiency as $1/\sqrt{N}$ and EI of several percentage points even in populations of many thousands. However, once we reach population sizes in the millions EI is again negligible.

[12] Again we do not focus on the the refunding of revenue raised as it has no impact on incentives under QV so long as, e.g. each individual receives back the same share of revenue or only receives revenues raised from individuals other than herself. However, note that QV may be budget balanced in a variety of ways without impacting any of the analysis that follows.

Assumption 1. There exists $w \in [-\delta, \delta]$ such that

$$(1 - \Psi(w)) |\underline{u}| > (\delta - w)^2.$$

Furthermore, there exist a unique pair (α, w) of real numbers such that $-\delta \leq w < \delta < \alpha$ and

$$\begin{aligned} (1 - \Psi(w)) |\underline{u}| &= (\alpha - w)^2 \quad \text{and} \\ (1 - \Psi(w')) |\underline{u}| &\leq (\alpha - w')^2 \quad \text{for all } w' \neq w, \end{aligned} \tag{5}$$

and that

$$2 + \psi'(w) |\underline{u}| > 0.$$

For $\delta < 1/\sqrt{2}$ this hypothesis holds generically among the class of C^∞ payoff functions Ψ.

While we use Ψ to determine payoffs in lieu of the Heaviside function primarily for technical convenience, the function Ψ can also be interpreted as representing some exogenous uncertainty around close elections, arising from judicially supervised recount procedures such as those during the 2000 United States Presidential election.[13] We conjecture our results also hold if the Heaviside function is used, but we have not been able to prove them formally without the exogenous smoothness provided by Ψ.[14]

We define the *expected inefficiency* resulting as $EI \equiv \frac{1}{2} - \frac{E[U\Psi(V)]}{2E[|U|]} \in [0, 1]$. This is the unique negative monotone linear transformation of aggregate utility realized $U\Psi(V)$ that is normalized to have range of the unit interval.

3.2 Existence of Equilibria

Lemma 2. For any $N > 1$ there exists a type-symmetric Bayes-Nash Equilibrium v that is monotone increasing.

This result follows directly from Reny (2011)'s Theorem 4.5 for symmetric games.[15]

Now consider the optimal behavior of an individual in such an equilibrium, given that the other individuals use the Bayes-Nash strategy $v(\cdot)$.[16] The expected utility of an individual with value u who buys votes v is $E[u\Psi(v + V_{N-1})] - v^2$, where V_{N-1} is the sum of $N - 1$ independent draws of $v(u)$, where u has distribution F. By smoothness of Ψ, maximization of expected utility

[13] Ψ may also be interpreted as representing the possibility of only-partial victories in sufficiently tight elections.

[14] All the constant calculations and numerical results quoted from Weyl (2015a) use the Heaviside function.

[15] All of Reny's conditions can easily be checked, so we highlight only the less obvious ones. Continuity of payoffs in actions follows from the smoothed payoffs imposed through Ψ. Type-conditional utility is only bounded from above, not below, but boundedness from below can easily be restored by simply deleting for each value type u votes of magnitude greater $\sqrt{2|u|}$. The existence of a monotone best-response follows from the clear super-modularity of payoffs.

[16] We conjecture that these results are true of asymmetric equilibria as well, given that the "smallness" of each agent rules out significant asymmetries. However, we have not proved this.

implies the necessary condition

$$uE\left[\psi\left(v+V_{N-1}\right)\right]=2v \implies v=\frac{1}{p(v)}u,$$

where $p(x) \equiv \frac{2}{E[\psi(x+V_{N-1})]}$ is the price perceived by an individual buying votes x. This price now has the rigorous interpretation of the inverse of the chance of an individual being pivotal in changing the outcome.

3.3 Efficiency

The following theorem follows directly from the explicit characterizations of Bayes-Nash equilibria discussed in the following sections.

Theorem 2. For a given sampling distribution F and payoff function Ψ satisfying the hypotheses specified above, there exist constants $\alpha_N > 0$ satisfying $\lim \alpha_N = 0$ such that for any type-symmetric Bayes-Nash equilibrium, EI is bounded above by α_N.

We conjecture that $\alpha_N = w/(N-1)$ for some constant w, and we give explicit expressions for w in terms of F in the next two subsections. These depend on whether $\mu = 0$ or $\mu \neq 0$ and we can prove this rate of decay holds for the latter case.

3.4 Characterization of equilibrium in the zero mean case

The reason for this dependence is that the structure of Bayes-Nash equilibrium differs radically depending on whether $\mu = 0$ or $\mu \neq 0$. Although non-generic, the case $\mu = 0$ corresponds most closely to the simplest intuition for why p is approximately constant in the limit and may arise frequently in the equilibrium of a broader political game where candidates or candidate initiatives converge toward efficiency (Ledyard, 1984).

Theorem 3. For any sampling distribution F with mean $\mu = 0$ that satisfies the hypotheses of Subsection 3.1 there exist constants $\epsilon_N \to 0$ such that in any type-symmetric Bayes-Nash equilibrium, $v(u)$ is C^∞ on $[\underline{u}, \overline{u}]$ and satisfies the following approximate proportionality rule:

$$\left|\frac{v(u)}{u}-\frac{1}{p_N}\right| \leq \frac{\epsilon_N}{p_N} \quad \text{where} \quad p_N=\frac{\sqrt{\sigma}\sqrt[4]{\pi(N-1)}}{\sqrt[4]{2}}. \tag{6}$$

Furthermore, there exist constants $\alpha_N, \beta_N \to 0$ such that in any equilibrium the vote total $V = V_N$ and expected inefficiency satisfy

$$EI < \alpha_N \quad \text{and} \quad |E[V]| \leq \beta_N \sqrt{\text{var}(V)}. \tag{7}$$

Thus, in any equilibrium agents buy votes approximately in proportion to their values u_i, which corresponds to their behavior under price-taking, as described in the previous section. This *approximate proportionality rule*, as the proof makes clear, holds because in any equilibrium each voter perceives approximately the same *pivot probability*, that is, the probability that the vote total V will be in the range $[-\delta, \delta]$ where a small increment to one's vote would move Ψ and thus the same price. Given that this is true (though not nearly as easy as it sounds to prove), it is not difficult to understand why the number of votes bought by a typical voter should decay as $1/\sqrt[4]{N}$. For if the vote function $v(u)$ in a Bayes-Nash equilibrium follows a proportionality rule $v(u) \approx \beta u$ then the constant β must be the consensus pivot probability; on the other hand, by the local limit theorem of probability (Feller, 1971) if $\beta = CN$ for some constants $a, C > 0$ then the chance that $V \in [-\delta, \delta]$ would be of order $N^{\alpha - \frac{1}{2}}$, and so a must be $1/4$.

Weyl (2015a) use heuristics to argue that the inefficiency of QV decays like $\mu_3^2/16\sigma^6(N-1)$. They consider an example calibrated to California's 2008 gay marriage referendum with uniform value distributions and the fraction of non-gays supporting gay marriage adjusted to ensure a 0 mean in aggregate. In this example, the value of the constant is approximately 4.5. Consequently, in a community of 101 individuals, inefficiency is inefficiency is 4.5%, all resulting from gay marriage being too frequently defeated as it would be with near-certainty in democracy. In a city of 100, 001 it is a negligible .0045%. They also find that these limiting constants typically overstate EI by 2-3 times in more precise numerical calculations for fixed and moderate population sizes.

3.5 Characterization of equilibrium in the non-zero mean case

When μ is not zero the nature of equilibrium is quite different: for sufficiently large N, any type-symmetric Bayes-Nash equilibrium has a massive jump discontinuity in the extreme tail of the value distribution.[17]

Theorem 4. Assume that the sampling distribution F has mean $\mu > 0$ and satisfies the hypotheses of Subsection 3.1, and that the payoff function satisfies Assumption 1. Then for any $\epsilon > 0$, if N is sufficiently large then for any Bayes-Nash equilibrium $v(u)$,

(i) $v(u)$ has a single discontinuity at u_*, where $|u_* + |\underline{u}| - \gamma(N-1)^{-2}| < \epsilon(N-1)^{-2}$;
(ii) $|v(u) + \sqrt{2|\underline{u}|}| < \epsilon$ for $u \in [\underline{u}, u_*)$;
(iii) $|v(u) - Cu/N-1| < \varepsilon_N/N-1$ where $C = f(\underline{u})\psi(w)/\mu$ and $\varepsilon_N \to 0$ as $N \to \infty$; and
(iv) $P\{|V - \alpha| > \epsilon\} < \epsilon$.

Here $\gamma > 0$ is a constant that depends only on the distribution F.

Thus, an agent with value u will buy approximately $CN^{-1}u$ votes, where C is a constant $C > 0$ depending on the sampling distribution F, *unless* u is in the extreme lower tail of F, in

[17] Without loss of generality we focus on the case when $\mu > 0$.

which case the agent will buy approximately $\alpha - w \approx -\sqrt{2|\underline{u}|}$ votes, enough to single-handedly win the election. Agents of the first type will be called *moderates*, and agents of the second kind *extremists*. Because the tail region in which extremists reside has $F-$probability on the order N^{-2}, the sample of agents will contain an extremist with probability only on the order N^{-1}, and will contain two or more extremists with probability on the order N^{-2}. Given that the sample contains no extremists, the conditional probability that $|V - \alpha| > \epsilon$ is $O(e^{-\varrho n})$ for some $\varrho > 0$, by standard large deviations estimates, and so the event that $V < 0$ essentially coincides with the event that the sample contains an extremist. Thus, we have the following corollary.

Corollary 2. Under the hypotheses of Theorem 4, EI is of order $1/N$.

Weyl (2015a) use heuristic arguments to derive a constant of $|\underline{u}|/\mu$ on this rate and calibrate it in the gay marriage example (without adjusting the proportion of gay marriage backers among the straight population) to a bit more than twice its value in the $\mu = 0$ case of the previous subsection. Thus inefficiency remains very small in reasonable populations (say more than a few thousand) in this case, and more precise numerical simulations suggest that these calculations overstate EI by 2-3 times for moderate population sizes.

Why does equilibrium take this somewhat counter-intuitive form in this case? For an agent i with value u_i in the "bulk" of the value distribution F, there is very little information about the vote total V in the agent's value u_i, and so for most such agents their perceived price $p\left(v\left(u_i\right)\right)$ will be approximately $p \equiv 2/E[\psi(V)]$; that is the price will be approximately its average value. Thus, in the bulk of the distribution the function $v(u)$ will be approximately linear in u. Therefore, by the law of large numbers, the vote total will, with high probability, be near μ/p. Since $\mu > 0$, agents with negative values will, with high probability, be on the losing side of the election.

However, if the price p were too large – large enough that $\mu N/p$ is not much greater than δ – then agents with values near $\underline{u}$ would be able to swing the whole election at cost $4\delta^2 < 2\left|\underline{u}\right|$, which they would clearly find it in their interest to do. They cannot "steal" the election in this way with non-vanishing probability in equilibrium, however, as it would raise the pivotal chance greatly, thereby reducing p, given that the extremist must also satisfy her necessary optimization condition and the price she perceives can only be smaller than that perceived by moderates by the probability of her existence on which she conditions. On the other hand, p cannot be so small that it would no longer be worthwhile for any extremist to steal the election, because then the probability of the status quo being maintained would be exponentially small implying that the pivotal event would be exponentially small and thus p would be exponentially large. The unique limiting equilibrium described above is the only point balancing these two considerations.

4 Discussion

While the preceding results motivate an interest in QV as a potentially useful mechanism, economists have proposed several other mechanisms with similar or even superior efficiency properties in the simple game theoretic environment we studied in the previous section. As we discuss below, these mechanism are widely viewed, even by economists, as being of little practical value. In this section we briefly discuss some companion work that has led us to the tentative conclusion that QV has a better chance of being practically relevant.

4.1 Robustness and comparison to other mechanisms

Weyl (2015b) explores the performance of QV under a range of modeling environments beyond the canonical one we study in the previous section where other mechanisms proposed by economists have proved very fragile. We now very briefly summarize leading conclusions of this work.[18]

4.1.1 Collusion and fraud

The best-known efficient mechanism for collective decision-making in the environment of the previous section is that proposed by Vickrey (1961), Clarke (1971) and Groves (1973) (VCG), which is fully efficient even in finite populations. However this mechanism is now almost universally rejected by market designs as practically useful for collective decision-making (Ausubel and Milgrom, 2005; Rothkopf, 2007). Perhaps the leading concern is its sensitivity to collusion; typically any two colluding individuals or any one individuals who can fraudulently represent herself as two can obtain whatever outcome they/she desire(s) at no cost to themselves and in (at least approximate) equilibrium.

Collusion and fraud are therefore an important concerns with QV; they typically benefits those perpetrating them and may harm the efficiency of QV. However, three forces that limit the harm created by collusion and fraud in market economies also do so in QV: the collusive group or fraudulent misrepresentation must be large to significantly dent efficiency, collusion creates incentives for unilateral deviation and both cause reactions by other agents that may make these activities self-defeating. Weyl shows that the first two forces put significant limits on the plausibility of collusion and fraud in the $\mu = 0$ case and the third force puts significant limits in the $\mu \neq 0$ case. Even in the worst case, collusion in a population of a million will require on the order of thousands of participants to be effective against these limits. When combined with the sorts of legal prohibitions against collusion used in market economies, these seem likely to make collusion and fraud significant, but not devastating, challenges for QV.

[18] These results are all based on approximate calculations and thus are conjectural to greater and lesser degrees, as discussed in Weyl.

4.1.2 Aggregate uncertainty

Another class of well-known mechanisms (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979; Ledyard and Palfrey, 1994, 2002; Krishna and Morgan, 2001; Goeree and Zhang, 2013) are only defined if the value distribution is common knowledge, as we assumed above, but again are typically more efficient than QV under this assumption.[19] A natural question is then how QV performs when there is some aggregate uncertainty about the distribution of values, leading potentially to different individuals differing in their estimates of the price.

Weyl does not derive any general results, but explores several examples. In most cases QV is not fully efficient under aggregate uncertainty even in arbitrarily large populations. However, its inefficiency is quite small, never greater than 10% and almost always much smaller than this. It nearly always greatly outperforms 1p1v, except in a few cases when the value distribution, conditional on the aggregate uncertainty, is known to be symmetric. Weyl also explores an example that additionally incorporates common values, but violates Feddersen and Pesendorfer (1997)'s conditions for 1p1v to aggregate information, in which QV still manages to aggregate information; this conclusion seems less robust, however, given the complexity of the common values environment.

4.1.3 Voter behavior

Ledyard (1984), and Myerson (2000) and Krishna and Morgan (Forthcoming) after him, argued that, under some very special conditions 1p1v could be efficient if the costs of voting deterred nearly all (but the most intense or lowest cost) individuals from voting. In practice we see large turnouts that suggest voters are either not as rational as this model requires or are motivated by factors other than changing the outcome of the election (Blais, 2000).

While such behavior destroys the efficiency of costly 1p1v (Ledyard, 1984), Weyl shows that it can sometimes actually enhance the efficiency of QV, almost never leads it to perform worse than 1p1v and usually only slightly reduces its efficiency. He studies two models, one where voters are motivated to express their preferences through their votes and one where they misestimate their chance of being pivotal. In both cases these motives help deter extremists, accelerating convergence towards efficiency, but they may introduce noise into the process which can cause some limiting inefficiency if and only if $\mu = 0$. In experiments on QV in the laboratory (Goeree and Zhang, 2013) and the field (Cárdenas et al., 2014) this logic appears to play out: QV is highly efficient because individuals buy votes close to in proportion to their values, but not because the standard game theoretic equilibrium developed in the previous section is played. Instead, individuals buy more votes on average than they "should", but do so in proportion to their values on average, maintainin efficiency.

[19] Many of these further rely on this common knowledge extends to the mechanism designer who can condition the mechanism on it, which clearly our analysis above does not.

4.1.4 Small populations

Weyl (2015a)'s bounds on inefficiency that we cited above are essentially vacuous for reasonable parameter values in population sizes below 50 or so. This contrasts sharply with mechanisms like VCG which, in their focal equilibrium, are perfectly efficient in any population size. However, Weyl also solves computationally for the equilibria of QV for a variety of distributions in finite population and find that our bounds are vastly conservative in all cases they consider. They do not have any examples with greater than 4% EI and usually it is much less, while 1p1v is often highly inefficient. This suggests that while our analytic results apply to large populations, QV may perform quite well even in fairly small groups.

4.2 Applications

The robustness of QV to these different environments and in experiments (Goeree and Zhang, 2013; Cárdenas et al., 2014) has led us to believe has a real chance of being a useful paradigm for practical collective decision-making. As a result we are pursuing a variety practical implementations of it, ranging from the near-term, commercial and small-scale to longer-term but higher impact aspirations.

In this first category, we have created a commercial venture, Collective Decision Engines, with Eric Posner and Kevin Slavin to commercialize implementations and variations on QV, initially for applications to market research but eventually in other domains. In the second category, we have written a series of articles jointly with Eric Posner exploring philosophical, practical and legal issues related to using QV for bankruptcy restructuring (Posner and Weyl, 2013), corporate governance (Posner and Weyl, 2014) and large-scale public decisions (Posner and Weyl, Forthcoming). Our hope is that lessons learned and validation gained from the first category of applications will eventually allow the second to become more practically plausible as QV moves from being a simple mechanism to a richly articulated governance paradigm.

5 Conclusion

Economists have typically been skeptical of the possibility of public decisions being taken as efficiently as private goods are allocated, as reflected in the formal results of Arrow (1951), Samuelson (1954), Gibbard (1973) and Satterthwaite (1975) and in informal attitudes in work such as Friedman (1962). In this paper we have argued that this attitude may be an artifact of particular institutions. Public goods do not appear to pose a fundamentally harder mechanism design problem than that posed by private goods. We highlighted a number of symmetries between QV and market mechanisms for the allocation of private goods, such as the double auction.

In addition to results explicitly described above, a variety of on-going work, by us and others, articulates these ideas further and builds on the work here. Building off of Hylland and Zeckhauser (1980), we plan a collaboration with Hylland and Zeckhauser to show that the welfare theorems apply to an economy where public goods are allocated by QV and that if strategy-proofness is relaxed in precisely the same way that makes the double auction to be "approximately strategy-proof" in large markets the welfare theorems can be approximated using QV in finite populations. We also plan to explore variations on QV that expand the range of cases where it can be applied.

In collaboration with Jerry Green and Scott Duke Kominers we are proposing an alternative to eminent domain as a procedure for the assembly of complements and studying its fairness and efficiency properties. *Public Choice* plans a special issue for 2017, the fortieth anniversary of its special issue on the use of the VCG mechanism for public choice, on QV for which (roughly twelve) papers representing a variety of disciplines and perspectives (e.g. law, history of economic thought, etc.) have been commissioned.

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Appendix

In this appendix we prove Theorems 3 and 4. We take as given the model assumptions laid out in Subsection 3.1 of the text, though we adopt some slightly altered notation for convenience: in particular, we assume that the sampling distribution $F = F_U$ is supported by a finite interval $[\underline{u}, \overline{u}]$, that its mean $\mu = \mu_U$ is nonnegative, and that its density is smooth and strictly positive on this interval; and we assume that the payoff function Ψ satisfies Assumption 1. For notational convenience we shall write $n = N + 1$ for the number of voters minus one and S_n for the vote total of the first n voters. For brevity, we shall refer to *type-symmetric Bayes-Nash equilibria* as *Nash equilibria* and we drop the brackets following the expectation operator E in cases where this does not cause confusion. Except when explicitly noted otherwise, all references to sections and subsections in this appendix are internal.

The proofs of both Theorems 3 and 4 will rely on a number of auxiliary properties of Nash equilibria that will be established in Sections B and C. In Section B we will prove that for every mixed-strategy Nash equilibrium there is an equivalent pure-strategy Nash equilibrium $u \mapsto v(u)$, and we will show that this must satisfy the fundamental necessary condition

$$E\psi(S_n + v(u))u = 2v(u) \tag{8}$$

where $S_n = \sum_{i=1}^{n} v(U_i)$ denotes the vote total for $n = N-1$ independent voters, all employing the strategy $v(u)$, whose values U_i are independently drawn from F_U. We will then proceed to use this necessary condition to deduce a series of properties that any Nash equilibrium must have, at least when n is sufficiently large. The reader should take note that *a priori* we know nothing about the function $v(u)$, and so even though we have made strong assumptions about the distribution of the utility values U_i we cannot appeal to the classical laws of probability governing sums of independent, identically distributed random variables to deduce anything about the sums $S_n = \sum_{i=1}^{n} v(U_i)$ until we know more about the function $v(u)$. Thus, in establishing the needed properties of Nash equilibria we shall be forced to bootstrap our way.

Following is a brief synopsis of the main steps.

1. *Strict monotonicity*: First we shall establish (Lemma 4) the relatively straightforward result that any type-symmetric equilibrium strategies must be *strictly* monotone. This then implies that any equilibrium strategy is continuous except at perhaps countably many points, which in turn implies that any type-symmetric Bayes-Nash equilibrium almost surely coincides with a pure-strategy Bayes-Nash equilibrium (Lemma 3).

2. *Weak consensus*: We will then consider voters in the "bulk" of the value distribution. We will prove (Lemma 6) that for any such voter the conditional distribution (given the voter's value u) of the vote total nearly coincides with the unconditional distribution, and use this to deduce from the necessary condition (8) that the ratio $v(u)/u$ is bounded above and below by positive constants independent of n and of the particular equilibrium.

3. *Concentration*: The equality (8) implies that a voter with value u will purchase a large number $|v(u)|$ of votes only if the distribution of S_n is highly concentrated in the interval $[-\delta, \delta]$

where ψ is non-zero. Using the weak consensus estimates on $v(u)/u$, we will show that concentration inequalities for sums of i.i.d. random variables imply (Lemma 7) that when n is large either $|v(u)|$ is bounded above by $C/\sqrt{n}$ in the bulk of the distribution, for some small C, or no voter (including extremists) will buy more than a vanishingly small number of votes. We will then deduce from this that the F_U-probability of the extremist regions cannot be larger than $n^{-3/2}$ for large n.

4. *Discontinuity size and smoothness*: Next we will show (Lemma 11), using the necessary condition (8) and another concentration argument, that the size of a discontinuity in the equilibrium vote function $v(u)$ must be bounded below. In view of the concentration results discussed above, this will imply that discontinuities can only occur at values u within distance $O(n^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$. By differentiating in the necessary condition (using the smoothness of $\psi = \Psi'$) we will then obtain a first-order differential equation for $v(u)$ that has no singularities in the region where v is continuous; this will imply that v is smooth up to within distance $O(n^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$.

5. *Approximate proportionality*: The various results on the size of the vote function $v(u)$ in the bulk of the distribution, the size and location of discontinuities, and concentration of the vote total will then be brought to bear on the necessary condition (8) to prove the *approximate proportionality* rule for Bayes-Nash equilibria (Lemma 6): in the bulk of the value distribution (up to within distance $O(n^{-3/2})$ of one of the endpoints), the ratio $v(u)/u$ must be almost constant, with relative error converging to zero uniformly as $n \to \infty$. Furthermore, if v has no discontinuities then the approximate proportionality rule extends all the way to the endpoints $\underline{u}, \overline{u}$. Given the approximate proportionality of the vote function $v(u)$, concentration inequalities and uniform versions of the Central Limit Theorem (the Berry-Esseen Theorem) can then be deduced for the vote total.

Once these properties of Nash equilibria have been established, we will then prove Theorems 3 and 4 from Subsections 3.4 and 3.5 of the main text, respectively.

A Terminology

A *pure strategy* is a Borel measurable function $v : [\underline{u}, \overline{u}] \to [-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$; when a pure strategy v is adopted, each agent buys $v(u)$ votes, where u is the agent's utility. A *mixed strategy* is a Borel measurable[20] function $\pi_V : [\underline{u}, \overline{u}] \to \Pi$, where Π is the collection of Borel probability measures on $[-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$; when a mixed strategy π_V is adopted, each agent i will buy a random number V_i of votes, where $V_1, V_2, \dots$ are conditionally independent given the utilities $U_1, U_2, \dots$ and V_i has conditional distribution $\pi_V(U_i)$. Clearly, the set of mixed strategies contains the pure strategies.

A *best response* for an agent with utility u to a strategy (either pure or mixed) is a value v such that

$$E\Psi(v + S_n)u - v^2 = \sup_{\tilde{v}} E\Psi(\tilde{v} + S_n)u - \tilde{v}^2, \tag{9}$$

where S_n is the sum of the votes of the other n agents when these agents all play the specified strategy and E denotes expectation. (Thus, under E, the random variables V_i of the n other

[20]The space of Borel probability measures on $[-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$ is given the topology of weak convergence; Borel measurability of a function with range Π is relative to the Borel field induced by this topology. Proposition 3 below implies that in the Quadratic Voting game only pure strategies are relevant, so measurability issues will play no role in this paper.

voters are distributed in accordance with the strategy and the sampling rule for utility values U_i described above.) Since Ψ is continuous and bounded, the equation (9) and the dominated convergence theorem imply that for each u the set of best responses is closed, and hence has well-defined maximal and minimal elements $v_+(u), v_-(u)$.

A mixed strategy π_V is a *Nash equilibrium* if for every $u \in [\underline{u}, \overline{u}]$ the measure $\pi_V(u)$ is supported by the set of best responses to π_V for an agent with utility u.

B Necessary Conditions for Nash Equilibrium

Let π_V be a mixed-strategy Nash equilibrium, and let S_n be the sum of the votes of n agents with utilities U_i gotten by random sampling from F_u, all acting in accordance with the strategy π_V. For an agent with utility u, a best response v must satisfy equation (9), and so in particular for every $\Delta > 0$, if $u > 0$ then

$$\begin{aligned} &E\left\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\right\} u \leq 2\Delta v + \Delta^2 \quad \text{and} \\ &E\left\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\right\} u \leq -2\Delta v + \Delta^2 \end{aligned} \tag{10}$$

Similarly, if $u < 0$ and $\Delta > 0$ then

$$\begin{aligned} &E\left\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\right\} u \leq -2\Delta v + \Delta^2 \quad \text{and} \\ &E\left\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\right\} u \leq 2\Delta v + \Delta^2 \end{aligned} \tag{11}$$

Since Ψ is C^∞ and its derivative ψ has compact support, differentiation under the expectation if permissible. Thus, we have the following necessary condition.

Lemma 3. If π_V is a mixed-strategy Nash equilibrium then for every u a best response v must satisfy

$$E\psi(S_n + v)u = 2v. \tag{12}$$

Consequently, every pure-strategy Nash equilibrium $v(u)$ must satisfy the functional equation

$$E\psi(S_n + v(u))u = 2v(u). \tag{13}$$

□

Lemma 4. Let π_V be a mixed-strategy Nash equilibrium, and let $v, \tilde{v}$ be best responses for agents with utilities $u, \tilde{u}$, respectively. If $u = 0$ then $v = 0$, and if $u < \tilde{u}$, then $v \leq \tilde{v}$. Consequently, any pure-strategy Nash equilibrium $v(u)$ is a nondecreasing function of u, and therefore has at most countably many discontinuities and is differentiable almost everywhere.

Proof. It is obvious that the only best response for an agent with $u = 0$ is $v = 0$, and the monotonicity of the payoff function Ψ implies that a best response v for an agent with utility u must be of the same sign as u. If $v, \tilde{v}$ are best responses for agents with utilities $0 \leq u < \tilde{u}$, then by definition

$$\begin{aligned} &E\Psi(\tilde{v} + S_n)\tilde{u} - \tilde{v}^2 \geq E\Psi(v + S_n)\tilde{u} - v^2 \quad \text{and} \\ &E\Psi(v + S_n)u - v^2 \geq E\Psi(\tilde{v} + S_n)u - \tilde{v}^2, \end{aligned}$$

and so, after re-arrangement of terms,

$$(E\Psi(\tilde{v}+S_n)-E\Psi(v+S_n))\tilde{u}\geq \tilde{v}^2-v^2 \quad \text{and}$$
$$(E\Psi(\tilde{v}+S_n)-E\Psi(v+S_n))u\leq \tilde{v}^2-v^2.$$

Hence,

$$(E\Psi(\tilde{v}+S_n)-E\Psi(v+S_n))(\tilde{u}-u)\geq 0.$$

The monotonicity of Ψ implies that if $0\leq\tilde{v}<v$ then $E\Psi(\tilde{v}+S_n)\leq E\Psi(v+S_n)$, and so it follows that the two expectations must be equal, since $\tilde{u}-u>0$. But if the two expectations were equal then v could not possibly be a best response at u, because an agent with utility u could obtain the same expected payoff $E\Psi(v+S_n)u$ at a lower vote cost by purchasing $\tilde{v}$ votes. This proves that if $0\leq u<\tilde{u}$ then best responses $v,\tilde{v}$ for agents with utilities $u,\tilde{u}$ must satisfy $0\leq v\leq\tilde{v}$. A similar argument shows that if $u<\tilde{u}\leq 0$ then best responses $v,\tilde{v}$ for agents with utilities $u,\tilde{u}$ must satisfy $v\leq\tilde{v}\leq 0$.

□

Proposition 3. If a mixed strategy π_V is a Nash equilibrium, then the set of utility values $u\in[\underline{u},\overline{u}]$ for which there is more than one best response (and hence the set of values u such that $\pi_V(u)$ is not supported by just a single point $v(u)$) is at most countable.

Proof of Proposition 3. For each u denote by $v_-(u)$ and $v_+(u)$ the minimal and maximal best responses at u. Proposition 4 implies that if $u<\tilde{u}$ then $v_+(u)\leq v_-(\tilde{u})$. Consequently, for any $\epsilon>0$ the set of utilities values u at which $v_+(u)-v_-(u)\geq\epsilon$ must be finite, because otherwise $v_+(u)\to\infty$ as $u\to\overline{u}$, which is impossible since best responses must take values between $-\sqrt{2\,|\underline{u}|}$ and $\sqrt{2\overline{u}}$. □

Since by hypothesis the values U_i are sampled from a distribution F that is absolutely continuous with respect to Lebesgue measure, there is zero probability that one of the votes i will have utility value U_i equal to one of the countably many values where there is more than one best response. Consequently, for every Nash equilibrium there is an equivalent pure-strategy Nash equilibrium $v(u)$. Henceforth, we shall consider only pure-strategy Nash equilibria; whenever we refer to a *Nash equilibrium* we will mean a pure-strategy Nash equilibrium.

Corollary 4. Every pure-strategy Nash equilibrium $v(u)$ is a strictly increasing function of $u\in[\underline{u},\overline{u}]$, and therefore is continuous except at possibly countably many points.

Lemma 5. If $v(u)$ is a Nash equilibrium, then $v(u)\neq 0$ for all $u\neq 0$.

Proof. If $v(u)=0$ for some $u>0$ then by Proposition 4 $v(u')=0$ for all $u'\in(0,u)$. Since the utility density $f_U(u)$ is strictly positive on $[\underline{u},\overline{u}]$, it follows that there is positive probability p that every agent in the sample casts vote $V_i=0$. But then an agent with utility u could improve her expectation by buying $\varepsilon>0$ votes, where $\varepsilon\ll u\psi(0)p$, because then the expected utility gain would be at least

$$u\Psi(\varepsilon)p\sim u\psi(0)p\varepsilon$$

at a cost of ε^2. Since by hypothesis $\psi(0)>0$, the expected utility gain would overwhelm the increased vote cost for small $\varepsilon>0$. □

Corollary 5. Any Nash equilibrium $v(u)$ is *strictly* monotone on $[\underline{u},\overline{u}]$.

Proof. Propositions 3 and 5 imply that $E\psi(S_n + v(u)) > 0$ for every $u \neq 0$. Now differentiation of the necessary condition (13) gives

$$E\psi(S_n + v(u)) = (2 - E\psi'(S_n + v(u)))v'(u)$$

at every u where $v(u)$ is differentiable. Since such points are dense in $[\underline{u}, \overline{u}]$, and since ψ and ψ' are C^∞ functions with compact support, it follows that $v'(u) \neq 0$ on a dense set. But $v'(u) \geq 0$ at every point where the derivative exists, so it follows that $v'(u) > 0$ almost everywhere, and this implies that v is strictly monotone. □

C Continuity and Smoothness

C.1 Weak consensus bounds

According to Lemma 3, in any Nash equilibrium the number of votes $v(u)$ purchased by an agent with utility u must satisfy the necessary condition (13). It is natural to expect that when the sample size $n+1$ is large the effect of adding a single vote v to the aggregate total S_n should be small, and so the function $v(u)$ should satisfy the approximate proportionality rule

$$2v(u) \approx E\psi(S_n)u.$$

As we will show later, this naive approximation can fail badly for utility values u in the extreme tails of the distribution F_U, and even in the bulk of the distribution the relative error in the approximation can be significant. Nevertheless, the idea of approximate population consensus on the expectations $E\psi(v(u) + S_n)$ can be used to obtain weak bounds that we will find useful. The following lemma states, roughly, that if it is optimal for some agent in the bulk of the population to buy a moderately large number of votes, then *most* agents will be forced to buy a moderately large number of votes.

Lemma 6. For every $\epsilon > 0$ there exist constants $\alpha, \beta > 0$ such that for all sufficiently large n and any Nash equilibrium $v(u)$

$$\frac{v(u)}{u} \geq \alpha \max(-v(\underline{u} + \epsilon), v(\overline{u} - \epsilon)) - e^{-\beta n} \quad \text{for all } |u| > 2\epsilon. \tag{14}$$

Proof. It suffices to establish the lower bound $\alpha v(\overline{u} - \epsilon) - e^{-\beta n}$, as the other half of (14) can be proved in virtually the same way. The main idea is that, for an agent with utility u not in the tails of the distribution F_U, the joint distribution of the sample $U_1, U_2, \ldots, U_{n+1}$ conditional on the agent's value u is not appreciably different than the *unconditional* distribution; that is, the agent gets very little information from knowing her own utility value u.

Set $u_\epsilon = \overline{u} - \epsilon$ and $p_\epsilon = 1 - F_U(u_\epsilon)$ where F_U is the cumulative distribution function of the utility distribution. Let $N = N_\epsilon$ be the number of points in the sample $U_1, U_2, \ldots, U_n$ that fall in the interval $[\overline{u}-\epsilon, \overline{u}]$, and let $U = U_{n+1}$ be independent of $U_1, U_2, \ldots, U_n$. Then

$$E(\psi(v(U)+S_n) \,|\, U < u_\epsilon) = \sum_{m \geq 0} \binom{n}{m} p_\epsilon^m (1-p_\epsilon)^{n-m} E(\psi(v(U)+S_n) \,|\, U < u_\epsilon, N = m),$$
$$E(\psi(v(U)+S_n) \,|\, U \geq u_\epsilon) = \sum_{m \geq 0} \binom{n}{m} p_\epsilon^m (1-p_\epsilon)^{n-m} E(\psi(v(U)+S_n) \,|\, U \geq u_\epsilon, N = m)$$

Now conditional on $N = m$, the sample $U_1, U_2, \ldots, U_n$ is obtained by choosing m points at random according to the conditional distribution of U given $U \geq u_\epsilon$ and $n - m$ according to the conditional distribution of U given $U < u_\epsilon$. Consequently, for each $m \geq 0$,

$$E(\psi(v(U)+S_n) \,|\, U \geq u_\epsilon, N = m) = E(\psi(v(U)+S_n) \,|\, U < u_\epsilon, N = m+1).$$

Furthermore, for any small $\epsilon' > 0$ and for m in the range $[np_\epsilon - n\epsilon', np_\epsilon + n\epsilon']$, the ratio

$$\binom{n}{m} p_\epsilon^m (1-p_\epsilon)^{n-m} \Big/ \binom{n}{m+1} p_\epsilon^{m+1} (1-p_\epsilon)^{n-m-1}$$

is between $1/2$ and 2. Since the binomial-(n, p_ϵ) distribution puts only an exponentially small (in n) mass outside the interval $[np_\epsilon - n\epsilon', np_\epsilon + n\epsilon']$, it follows that for some constants α', β' depending on ϵ but not n,

$$E(\psi(v(U)+S_n) \,|\, U < u_\epsilon) \geq \alpha' E(\psi(v(U)+S_n) \,|\, U \geq u_\epsilon) - e^{-\beta' n} \tag{15}$$

for all sufficiently large n.

A similar argument proves that for suitable constants $\alpha'', \beta'' > 0$, and for any interval $J \subset [\underline{u}, \overline{u}]$ of length ϵ not overlapping $[\overline{u}-\epsilon, \overline{u}]$,

$$E(\psi(v(U)+S_n) \,|\, U \in J) \geq \alpha'' E(\psi(v(U)+S_n) \,|\, U \geq u_\epsilon) - e^{-\beta'' n}. \tag{16}$$

To see this, let N be the number of points in the sample $U_1, U_2, \ldots, U_n$ that fall in the interval $[\overline{u}-\epsilon, \overline{u}]$, and let N' be the number of points in the sample that fall in J. Decompose the conditional expectations $E(\psi(v(U)+S_n) \,|\, U \in J)$ and $E(\psi(v(U)+S_n) \,|\, U > u_\epsilon)$ according to the values of N and N', and use the identity

$$E(\psi(v(U)+S_n) \,|\, U \in J, N = m+1, N' = m') = E(\psi(v(U)+S_n) \,|\, U > u_\epsilon, N = m, N' = m'+1).$$

As in the proof of (15), the ratio

$$\frac{P\{N = m+1, N' = m'\}}{P\{N = m, N' = m'+1\}}$$

is near one for all pairs (m, m') except those in the tails of the joint distribution, and the tails are exponentially small, by standard estimates for the multinomial distribution.

Now recall that any Nash equilibrium $v(u)$ is monotone, and satisfies the necessary condition $2v(u) = E\psi(v(u)+S_n)u$. Since any $u \in [\epsilon, u_\epsilon - \epsilon]$ is the right endpoint of an interval $J = [u-\epsilon, u]$

of length ϵ that does not intersect $[u_\epsilon, \overline{u}]$, it follows that for any such u,

$$\begin{aligned} E\psi(v(u) + S_n) &\geq E(\psi(v(U) + S_n) \,|\, U \in [u - \epsilon, u]) \\ &\geq \alpha'' E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon) - e^{-\beta'' n} \\ &\geq \alpha'' E\psi(v(u_\epsilon + S_n)) - e^{-\beta'' n} \\ &= \alpha'' v(u_\epsilon)/\overline{u} - e^{-\beta'' n} \end{aligned}$$

and similarly for any $u \in [\underline{u} + 2\epsilon, -\epsilon]$. The assertion (14) now follows from another application of the necessary condition (13). □

C.2 Concentration and size constraints

Since the vote total S_n is the sum of independent, identically distributed random variables $v(U_i)$ (albeit with unknown distribution), its distribution is subject to concentration restrictions, such as those imposed by the following lemma.

Lemma 7. For any $\epsilon > 0$ there exists a constant $\gamma = \gamma(\epsilon) < \infty$ such that for all sufficiently large values of n and any Nash equilibrium $v(u)$, if

$$\max(v(\overline{u} - \epsilon), -v(\underline{u} + \epsilon) \geq \gamma/\sqrt{n}, \tag{17}$$

then

$$P\{|S_n + v| \leq \delta\} < \epsilon \quad \text{for all } v \in \mathbb{R} \tag{18}$$

and therefore

$$\frac{|2v(u)|}{|u|} \leq \epsilon \|\psi\|_\infty \quad \text{for all } u \in [\underline{u}, \overline{u}]\,. \tag{19}$$

We will deduce Lemma 7 from the following general fact about sums of independent, identically distributed random variables.

Lemma 8. Fix $\delta > 0$. For any $\epsilon > 0$ and any $C < \infty$ there exists $C' = C'(\epsilon, C) > 0$ and $n' = n'(\epsilon, C) < \infty$ such that the following statement is true: if $n \geq n'$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables such that

$$E|Y_1 - EY_1|^3 \leq C \text{var}(Y_1)^{3/2} \quad \text{and} \quad \text{var}(Y_1) \geq C'/n \tag{20}$$

then for every interval $J \subset \mathbb{R}$ of length δ or greater, the sum $S_n = \sum_{i=1}^n Y_i$ satisfies

$$P\{S_n \in J\} \leq \epsilon |J|/\delta. \tag{21}$$

The proof of the proposition, a routine exercise in the use of Fourier methods, is relegated to Section F, at the end of this appendix.

Proof of Lemma 7. Inequality (19) follows from (18), by the necessary condition (13) for Nash equilibria. Hence, it suffices to show that (17) implies (18).

Lemma 6 implies that there are constants $\alpha, \beta > 0$ such that for every $u \in [\underline{u}, \overline{u}] \setminus [-2\epsilon, 2\epsilon]$ the ratio $v(u)/u$ is at least $\alpha v(u_\epsilon) - e^{-\beta n}$, where $u_\epsilon = \overline{u} - \epsilon$. Since the utility density f_U is bounded below, it follows that for suitable constants $0 < C < \infty$ and $p > 0$, for every sufficiently large n

and every Nash equilibrium $v(u)$ there is an interval $[u', u''] \subset [\overline{u}/2, \overline{u})$ of probability p such that $u' < u_\epsilon < u''$ and

$$v(u_\epsilon) \leq Cv(u'') \leq C^2 v(u'). \tag{22}$$

Similarly, there exists an interval $[u_*, u_{**}] \subset [\underline{u}, 0]$ of probability p such that

$$|v(u_*)| \leq C|v(u_{**})|. \tag{23}$$

Let $N^\star$ be the number of points U_i in the sample $U_1, U_2, \dots, U_n$ that fall in $[u_*, u_{**}] \cup [u', u'']$, and let S_n^* be the sum of the votes $v(U_i)$ for those agents whose utility values fall in this range. Observe that N has the binomial-$(n, 2p)$ distribution, and that conditional on the event $N^\star = m$ and $S_n - S_n^* = w$, the random variable S_n^* is the sum of m independent random variables Y_i whose variance is at least $v(u')^2/4$ and whose third moment obeys the restriction (20) (this follows from the inequalities (22)–(23)). Consequently, by Lemma 8, if $v(u_\epsilon)\sqrt{n}$ is sufficiently large then the conditional probability, given $N^\star = n \geq np$ and $S_n - S_n^* = w$, that S_n^* lies in any interval of length δ is bounded above by $\epsilon/2$. Since $P\{N \leq np\}$ is, for large n, much less than $\epsilon/2$, the inequality (18) follows.

□

Lemma 7 implies that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium $v(u)$ the absolute value $|v(u)|$ can assume large values only at utility values u within distance ϵ of one of the endpoints $\underline{u}, \overline{u}$. The following proposition improves this to the extreme tails of the distribution.

Lemma 9. For any $0 < C < \infty$ there exists $C' > 0$ such that for all sufficiently large n and any Nash equilibrium $v(u)$ satisfies the inequality

$$|v(u)| \leq C \quad \text{for all } \; u \in \left[\underline{u} + C'n^{-3/2}, \overline{u} - C'n^{-3/2}\right]. \tag{24}$$

Proof. Fix $C > 0$, and suppose that $2v(u_*) \geq C$ for some $u_* > 0$. Since any Nash equilibrium v is monotone, we must have $2v(u) \geq C$ for all $u \geq u_*$, and by the necessary condition (13) it follows that

$$E\psi(v(u) + S_n)u \geq C \quad \Longrightarrow \quad E\psi(v(u) + S_n) \geq C/\overline{u} \quad \forall\, u \geq u_*. \tag{25}$$

Consequently, the distribution of S_n is concentrated: since the function ψ has support $[-\delta, \delta]$, the probability that $S_n + v(u) \in [-\delta, \delta]$ must be at least $C/\overline{u}\|\psi'\|_\infty$. Thus, Lemma 7 implies that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ (depending on both ϵ and C, but not on n) such that

$$\max\left(-v\left(\underline{u} + \epsilon\right), v(\overline{u} - \epsilon)\right) \leq \gamma_\epsilon/\sqrt{n}. \tag{26}$$

In particular, for all sufficiently large n,

$$\begin{aligned}
v(\overline{u}/2) \leq \frac{\gamma_{\overline{u}/2}}{\sqrt{n}} \quad &\Longrightarrow \quad E\psi(v(\overline{u}/2) + S_n) \leq \frac{2\gamma_{\overline{u}/2}}{\overline{u}\sqrt{n}} \\
&\Longrightarrow \quad E\psi(S_n) \leq \frac{2\gamma_{\overline{u}/2}}{\overline{u}\sqrt{n}} + \|\psi'\|_\infty v(\overline{u}/2) \\
&\Longrightarrow \quad E\psi(S_n) \leq \frac{C_{\overline{u}/2}}{\sqrt{n}}
\end{aligned} \tag{27}$$

for a constant $C_{\overline{u}/2} < \infty$ that may depend on $\overline{u}/2$ and C but not on either n or the particular Nash equilibrium.

Fix C' large, and suppose that $2v(u_*) \geq C$ for $u_* = \overline{u} - C'n^{-3/2}$. Let N_* be the number of points U_i in the sample $U_1, U_2, \ldots, U_n$ that fall in the interval $[u_*, \overline{u}]$; by our assumptions concerning the sampling procedure, the random variable N_* has the binomial distribution with mean

$$EN_* = n \int_{u_*}^{\overline{u}} f_U(u)\, du = C'C_f n^{-1/2}$$

where C_f is the mean value of f_U on the interval $[u_*, \overline{u}]$ (which for large n will be close to $f_U(\overline{u}) > 0$). Since EN_* is vanishingly small for large n, the assumption $v(u_*) \geq C$ implies that

$$E\psi(v(u) + S_n)\mathbf{1}\{N_* = 0\} \geq C/2\overline{u} \quad \text{for all } u \geq u_*. \tag{28}$$

This expectation can be decomposed by partitioning the probability space into the event $G = \{U_n \in [\underline{u} + \epsilon, \overline{u} - \epsilon]\}$ and its complement. On the event G, the contribution of $v(U_n)$ to the vote total S_n is at most $\gamma_\epsilon/\sqrt{n}$ in absolute value, by (26). On the complementary event G^c the integrand is bounded above by $\|\psi\|_\infty$. Therefore,

$$\begin{aligned} E\psi(v(u) + S_n)\mathbf{1}\{N_* = 0\} &\leq P(G^c)\|\psi\|_\infty + E\psi(v(u) + S_n)\mathbf{1}\{N_* = 0\}\mathbf{1}_G \\ &\leq P(G^c)\|\psi\|_\infty + E\psi(v(u) + S_{n-1})\mathbf{1}\{N_* = 0\} + \|\psi'\|_\infty(\gamma_\epsilon/\sqrt{n}) \\ &\leq \epsilon' + E\psi(v(u) + S_{n-1})\mathbf{1}\{N_* = 0\} \end{aligned}$$

where $\epsilon' > 0$ can be made arbitrarily small by choosing $\epsilon > 0$ small and n large. This together with inequality (28) implies that for large n,

$$E\psi(v(u) + S_{n-1})\mathbf{1}\{N_* = 0\} \geq C/4\overline{u} \quad \text{for all } u \geq u_*. \tag{29}$$

Now consider the conditional distribution of S_n given that $N_* = 1$: this can be simulated by generating S_{n-1} from the conditional distribution of S_{n-1} given that $N_* = 0$ and then adding an independent $v(U)$ where $U = U_n$ is drawn from the conditional distribution of U given that $U \geq u_*$. Consequently, by inequality (29),

$$E(\psi(S_n) \,|\, N_* = 1) = E(\psi(S_{n-1} + v(U)) \,|\, N_* = 0) \geq C/4\overline{u}.$$

But this implies that

$$E(\psi(S_n)) \geq (C/2\overline{u})P\{N_* \geq 1\} \approx CC'C_f/(2\overline{u}\sqrt{n}).$$

For large C' this is incompatible with inequality (27) when n is sufficiently large. □

C.3 Discontinuities

Since any Nash equilibrium $v(u)$ is monotone in the utility u, it can have at most countably many discontinuities. Moreover, since any Nash equilibrium is bounded in absolute value by $\sqrt{2\max(|\underline{u}|, \overline{u})}$ (as no agent will pay more for votes than she could gain in expected utility) the sum of the jumps is bounded by $\sqrt{2\max(|\underline{u}|, \overline{u})}$. We will now show that there is a *lower* bound on the size of $|v|$ at a discontinuity.

Lemma 10. Let $v(u)$ be a Nash equilibrium. If v is discontinuous at $u \in (\underline{u}, \overline{u})$ then

$$E\psi'(\tilde{v} + S_n)u = 2 \tag{30}$$

for some $\tilde{v} \in [v_-, v_+]$, where v_- and v_+ are the left and right limits of $v(u')$ as $u' \to u$.

Proof. The necessary condition (13) holds at all u' in a neighborhood of u, so by monotonicity of v and continuity of ψ, the Equation (13) must hold when $v(u)$ is replaced by either of $v_\pm$, that is,

$$\begin{aligned} 2v_+ &= E\psi(v_+ + S_n)u \quad \text{and} \\ 2v_- &= E\psi(v_- + S_n)u. \end{aligned}$$

Subtracting one equation from the other and using the differentiability of ψ we obtain

$$2v_+ - 2v_- = uE\int_{v_-}^{v_+} \psi'(t + S_n)\, dt = u\int_{v_-}^{v_+} E\psi'(t + S_n)\, dt.$$

The result then follows from the mean value theorem of calculus. □

Lemma 11. There is a constant $\Delta > 0$ such that for all sufficiently large n, at any point u of discontinuity of a Nash equilibrium,

$$\begin{aligned} v(u_+) &\geq \Delta \quad \text{if } u \geq 0 \quad \text{and} \\ v(u_-) &\leq -\Delta \quad \text{if } u \leq 0. \end{aligned} \tag{31}$$

Consequently, there is a constant $\beta < \infty$ not depending on the sample size n such that for all sufficiently large n no Nash equilibrium $v(u)$ has a discontinuity at a point u at distance greater than $\beta n^{-3/2}$ from one of the endpoints $\underline{u}, \overline{u}$.

Proof. Since the function ψ has support contained in the interval $[-\delta, \delta]$, equation (30) implies that v can have a discontinuity only if the distribution of S_n is highly concentrated: specifically,

$$P\{S_n + \tilde{v} \in [-\delta, \delta]\} \geq \frac{2}{\|\psi'\| \max\left(|\underline{u}|, \overline{u}\right)}. \tag{32}$$

In fact, since ψ' vanishes at the endpoints of $[-\delta, \delta]$, there exists $0 < \delta' < \delta$ such that

$$P\{S_n + \tilde{v} \in [-\delta', \delta']\} \geq \frac{1}{\|\psi'\| \max\left(|\underline{u}|, \overline{u}\right)}. \tag{33}$$

Lemma 7 asserts that strong concentration of the distribution of S_n can occur only if $|v(u)|$ is vanishingly small in the interior of the interval $[\underline{u}, \overline{u}]$. In particular, if $\epsilon < \left(\|\psi'\| \max\left(|\underline{u}|, \overline{u}\right)\right)^{-1}$ and n is sufficiently large then $|v(u)| < \gamma_\epsilon/\sqrt{n}$ for all $u \in [\underline{+}\epsilon, \overline{u} - \epsilon]$. But $v(u)$ must satisfy the necessary condition (13) at all such u, so

$$E\psi(v(u) + S_n)|u| \leq 2\gamma_\epsilon/\sqrt{n}$$

for all $u \in [\underline{u} + \epsilon, \overline{u} - \epsilon]$. Since the function ψ is positive and bounded away from 0 in any interval $[-\delta'', \delta'']$ where $0 < \delta'' < \delta$, it follows from (33) that for sufficiently large n,

$$|\tilde{v}| \geq (\delta - \delta')/3 := \Delta.$$

Thus, by the monotonicity of Nash equilibria, at every point u of discontinuity we must have (31). Lemma 9 now implies that any such discontinuities can occur only within a distance $\beta n^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. $\square$

C.4 Smoothness

Since Nash equilibria are monotone, by Lemma 4, they are necessarily differentiable almost everywhere. We will show that in fact differentiability must hold at *every* u, except near the endpoints $\underline{u}, \overline{u}$.

Lemma 12. If $v(u)$ is a Nash equilibrium then at every u where v is differentiable,

$$E\psi(S_n + v(u)) + E\psi'(S_n + v(u))uv'(u) = 2v'(u). \tag{34}$$

Proof. This is a routine consequence of the necessary condition (13) and the smoothness of the function ψ. $\square$

Equation (34) can be rewritten as a first-order differential equation:

$$v'(u) = \frac{E\psi(S_n + v(u))}{2 - E\psi'(S_n + v(u))u}. \tag{35}$$

This differential equation becomes singular at any point where the denominator approaches 0, but is regular in any interval where $E\psi'(S_n + v(u))u \leq 1$. The following lemma implies that this will be the case on any interval where $|v(u)|$ remains sufficiently small.

Lemma 13. For any $\alpha > 0$ there exists a constant $\beta = \beta_\alpha > 0$ such that for any strategy $v(u)$, any $\tilde{v} \in \mathbb{R}$, any $u \in [\underline{u}, \overline{u}]$, and all n,

$$\begin{aligned} E|\psi'(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta \;\text{ and} \\ E|\psi''(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta. \end{aligned} \tag{36}$$

Proof. Recall that $\psi/2$ is a C^∞ probability density with support $[-\delta, \delta]$ and such that ψ is *strictly* positive in the open interval $(-\delta, \delta)$. Consequently, on any interval $J \subset (-\delta, \delta)$ where $|\psi'|$ (or $|\psi''|$) is bounded below by a positive number, so is ψ.

Fix $\epsilon > 0$ so small that $\epsilon \max(\underline{u}, \overline{u}) < \alpha/2$. In order that $E|\psi'(\tilde{v} + S_n)u| \geq \alpha$, it must be the case that the event $\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\}$ contributes at least $\alpha/2$ to the expectation; hence,

$$P\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\} \geq \frac{\alpha}{2\|\psi'\|_\infty \max(\underline{u}, \overline{u})}.$$

But on this event the random variable $\psi(\tilde{v} + S_n)$ is bounded below by a positive number $\eta = \eta_\epsilon$, so it follows that

$$E\psi(\tilde{v} + S_n)|u| \geq \frac{\eta\alpha}{2\|\psi'\|_\infty \max(\underline{u}, \overline{u})}.$$

A similar argument proves the corresponding result for ψ''. $\square$

Lemma 14. There exist constants $C, \alpha > 0$ such that for all sufficiently large n, any Nash equilibrium $v(u)$ is continuously differentiable on any interval where $|v(u)| \leq C$ (and therefore, by

Proposition 9, on $\left(\underline{u}+C'n^{-3/2}, \overline{u}-C'n^{-3/2}\right))$, and the derivative satisfies

$$\alpha \leq \frac{v'(u)}{E\psi(v(u)+S_n)} \leq \alpha^{-1}. \tag{37}$$

Proof. The function $v(u)$ is differentiable almost everywhere, by Lemma 4, and at every point u where $v(u)$ is differentiable the differential equation (35) holds. By Lemma 11, the sizes of discontinuities are bounded below, and so if $C>0$ is sufficiently small then a Nash equilibrium $v(u)$ can have no discontinuities on any interval where $|v(u)| \leq C$. Furthermore, if $C>0$ is sufficiently small then by Lemma 13 and the necessary condition (13), we must have $E\psi'(v(u)+S_n) \leq 1$ on any interval where $|v(u)| \leq C$. Since the functions $v \mapsto E\psi(S_n+v)$ and $v \mapsto E\psi'(S_n+v)$ are continuous (by dominated convergence), it now follows from Equation (35) that if $C>0$ is sufficiently small then on any interval where $|v(u)| \leq C$ the function $v'(u)$ extends to a continuous function. Finally, since the denominator in equation (35) is at least 1 and no larger than $2+\|\psi'\|_\infty$, the inequalities (37) follow. □

Similar arguments show that Nash equilibria have derivatives of higher orders provided the sample size is sufficiently large. The proof of Theorem 3 in Section E below will require information about the second derivative $v''(u)$. This can be obtained by differentiating under the expectations in (35):

$$v''(u) = \frac{E\psi'(v(u)+S_n)v'(u)}{2-E\psi'(S_n+v(u))u} + \frac{E\psi(v(u)+S_n)(E\psi''(v(u)+S_n)v'(u)u+E\psi'(v(u)+S_n)}{(2-E\psi'(S_n+v(u)u))^2}. \tag{38}$$

A repetition of the proof of Lemma 14 now shows that for suitable constants $C, \beta > 0$ and all sufficiently large n, any Nash equilibrium $v(u)$ is twice continuously differentiable on any interval where $|v(u)| \leq C$ and satisfies the inequalities

$$\beta \leq \frac{v''(u)}{E\psi(v(u)+S_n)} \leq \beta^{-1}. \tag{39}$$

C.5 Approximate proportionality

The information that we now have about the form of Nash equilibria can be used to sharpen the heuristic argument given in Subsection C.1 to support the "approximate proportionality rule". Recall that in a Nash equilibrium the number of votes $v(u)$ purchased by an agent with utility u must satisfy the equation $2v(u) = E\psi(v(u)+S_n)u$. We have shown in Proposition 9 that for any Nash equilibrium, $v(u)$ must be small except in the extreme tails of the distribution (in particular, for all u at distance much more than $n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$). Since ψ is uniformly continuous, it follows that the expectation $E\psi(v(u)+S_n)$ cannot differ by very much from $E\psi(S_n)$.

Unfortunately, this argument only shows that the approximation $2v(u) \approx E\psi(S_n)u$ is valid up to an error of size $\epsilon_n|u|$ where $\epsilon_n \to 0$ as $n \to \infty$. However, as $n \to \infty$ the expectation $E\psi(S_n) \to 0$, and so the error in the approximation above might be considerably larger than the approximation itself. Proposition 6 makes the stronger assertion that when n is large the relative error in the approximate proportionality rule is small.

Proposition 6. For any $\epsilon > 0$ there exist constants $n_\epsilon < \infty$ and $C < \infty$ such that if $n \geq n_\epsilon$ then for any Nash equilibrium $v(u)$ and for all $u \in \left[\underline{u} + Cn^{-3/2}, \overline{u} - Cn^{-3/2}\right]$,

$$(1-\epsilon)E\psi(S_n)|u| \leq |2v(u)| \leq (1+\epsilon)E\psi(S_n)|u|. \tag{40}$$

Furthermore, for all sufficiently large n any Nash equilibrium $v(u)$ with no discontinuities must satisfy (40) for all $u \in [\underline{u}, \overline{u}]$.

Proof of Proposition 6. Since ψ has compact support, it and all of its derivatives are uniformly continuous and uniformly bounded, and so the function $v \mapsto E\psi(v + S_n)$ is differentiable with derivative $E\psi'(v + S_n)$. Consequently, by Taylor's theorem, for every u there exists $\tilde{v}(u)$ intermediate between 0 and $v(u)$ such that

$$2v(u) = E\psi(v(u) + S_n)u = E\psi(S_n)u + E\psi'(\tilde{v}(u) + S_n)v(u)u. \tag{41}$$

We will argue that for all $C > 0$ sufficiently small, if $|v(u)| \leq C$ then the expectation $E\psi'(\tilde{v}(u) + S_n)$ remains below ϵ in absolute value, provided n is sufficiently large. Proposition 9 will then imply that there exists $C' < \infty$ such that (40) holds for all $u \in (\underline{u}, \overline{u})$ at distance greater than $C'n^{-3/2}$ from the endpoints $\underline{u}, \overline{u}$.

If $|2v(u)| \leq C$ then $|E\psi(v(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$, by the necessary condition (13). By Lemma 11, if $C < \Delta$, where Δ is the discontinuity threshold, then $v(u)$ is continuous on any interval $[0, u_C]$ where $|v(u)| \leq C$, and so for each u in this interval there is a $u' \in [0, u]$ such that $\tilde{v}(u) = v(u')$. Consequently, $|E\psi(\tilde{v}(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$. But Lemma 13 implies that for any $\epsilon > 0$, if $C > 0$ is sufficiently small then for all n and any Nash equilibrium $v(u)$,

$$|E\psi'(\tilde{v}(u) + S_n)| < \epsilon$$

on any interval $[0, u_C]$ where $|v(u)| \leq C$. Thus, the error in the approximation (41) will be small when n is large and $|v(u)| < C$, for $u > 0$. A similar argument applies for $u \leq 0$.

Finally, suppose that $v(u)$ is a Nash equilibrium with no discontinuities. By Lemma 9, for any $C > 0$ there exists $C' < \infty$ such that $|v(u)| \leq C/2$ except at arguments u within distance $C'/n^{3/2}$ of one of the endpoints. Moreover, Lemma 14 implies that if C is sufficiently small then on any interval where $|v(u)| \leq C$ the function v is differentiable, with derivative $v'(u) \leq C''$ for some constant $C'' < \infty$ not depending on n or on the particular Nash equilibrium. It then follows that

$$v(\overline{u}) \leq C/2 + C'C''n^{-3/2} \leq C$$

provided n is large. Since $C > 0$ can be chosen arbitrarily small, it follows that $v(u)$ must satisfy the proportionality relations (40) on $[0, \overline{u}]$. A similar argument applies to the interval $[\underline{u}, 0]$. □

C.6 Consequences of Proposition 6

Proposition 6 puts strong constraints on the distribution of the vote total S_n in a Nash equilibrium. According to this proposition, the approximate proportionality rule (40) holds for all $u \in [\underline{u}, \overline{u}]$ except those values u within distance $Cn^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Call such values *extremists*, and denote by G the event that the sample $U_1, U_2, \ldots, U_n$ contains no extremists. By Proposition 6, on the event G the approximate proportionality rule (40) will apply for each agent; furthermore, for Nash equilibria with no discontinuities, (40) holds for all $u \in [\underline{u}, \overline{u}]$. Thus, *conditional* on the event G (or, for continuous Nash equilibria, *unconditionally*) the random variables $v(U_i)$ are (at least for sufficiently large n) bounded above and below by $E\psi(S_n)\overline{u}$ and $E\psi(S_n)\underline{u}$, and so Hoeffding's inequality applies.

Corollary 7. Let G be the event that the sample U_i contains no extremists. Then for all sufficiently large n and any Nash equilibrium $v(u)$,

$$P(|S_n - ES_n| \geq tE\psi(S_n) \,|\, G) \leq \exp\{-2t^2/n \max\left(|\underline{u}|^2, \overline{u}^2\right)\}; \tag{42}$$

and for any Nash equilibrium with no discontinuities,

$$P(|S_n - ES_n| \geq tE\psi(S_n)) \leq \exp\{-2t^2/n \max\left(|\underline{u}|^2, \overline{u}^2\right)\}. \tag{43}$$

Proposition 6 also implies uniformity in the normal approximation to the distribution of S_n, because the proportionality rule (40) guarantees that the ratio of the third moment to the $3/2$ power of the variance of $v(U_i)$ is uniformly bounded. Hence, by the Berry-Esseen theorem, we have the following corollary.

Corollary 8. There exists $\kappa < \infty$ such that for all sufficiently large n and any Nash equilibrium $v(u)$, the vote total S_n satisfies

$$\sup |P((S_n - ES_n) \leq t\sqrt{\mathrm{var}(S_n)} \,|\, G) - \Phi(t)| \leq \kappa n^{-1/2}; \tag{44}$$

and for any Nash equilibrium with no discontinuities,

$$\sup |P((S_n - ES_n) \leq t\sqrt{\mathrm{var}(S_n)}) - \Phi(t)| \leq \kappa n^{-1/2}. \tag{45}$$

Here Φ denotes the standard normal cumulative distribution function.

D Unbalanced Populations: Proof of Theorem 4

D.1 Concentration of the vote total

Lemma 15. If $\mu > 0$ then for all large n no Nash equilibrium $v(u)$ has a discontinuity at a non-negative value of u. Moreover, if $\mu > 0$ then for any $\epsilon > 0$, if n is sufficiently large then in any Nash equilibrium the vote total S_n must satisfy
(i) $ES_n \in [\delta - \epsilon, \delta + \epsilon + \sqrt{2\,|\underline{u}|}]$ and
(ii) $P\{|S_n - ES_n| > \epsilon\} < \epsilon$.
Furthermore, there is a constant $\gamma > 0$ such that for any $\epsilon > 0$, if n is sufficiently large and $v(u)$ is a Nash equilibrium with no discontinuities, then
(iii) $P\{|S_n - ES_n| > \epsilon\} < e^{-\gamma n}$.

Proof. By Lemma 11, a Nash equilibrium $v(u)$ can have no discontinuities at distance greater than $Cn^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Agents with such utilities are designated *extremists*; the event G that the sample $U_1, U_2, \ldots, U_n$ contains no extremists has probability $1 - O(n^{-1/2})$.

By Proposition 6, any Nash equilibrium $v(u)$ obeys the approximate proportionality Rule (40) except in the extremist regime. The contribution of extremists to ES_n is vanishingly small for large n, since $P(G^c) = O(n^{-1/2})$ and $|v| \leq \max(\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}})$. Consequently, (40) implies that for any $\epsilon > 0$, if n is large then

$$E\psi(S_n)\mu_U(1-\epsilon) \leq ES_n/n \leq E\psi(S_n)\mu_U(1+\epsilon). \tag{46}$$

Since $\mu_U > 0$, this implies that $ES_n \geq 0$ for all sufficiently large n.

Suppose now that $ES_n < \delta - 2\epsilon'$ for some small $\epsilon' > 0$. If $\epsilon > 0$ is sufficiently small relative to ϵ' then (46) implies that $nE\psi(S_n)\mu_U \leq \delta - \epsilon'/2$. But then Hoeffding's inequality (42) (for this a weaker Chebyshev bound would suffice), together with the fact that $P(G^c) \leq Kn^{-1/2}$, implies that

$$P\{S_n \in [-\delta/2, \delta - \epsilon'/4]\} \geq 1 - \epsilon$$

for large n. This is impossible, though, because we would then have

$$E\psi(S_n) \geq (1-\epsilon) \min_{v \in [-\delta/2, \delta - \epsilon'/4]} \psi(v),$$

and since ψ is bounded away from 0 on any compact sub-interval of $(-\delta, \delta)$ this contradicts the fact that $nE\psi(S_n) < \delta - \epsilon'/2$. This proves that for all large n and all Nash equilibria, $ES_n \geq \delta - 2\epsilon'$.

Next suppose that $ES_n > \delta + \sqrt{2\,|\underline{u}|} + 2\epsilon'$, where $\epsilon' > 0$. The proportionality rule (40) (applied with some $\epsilon > 0$ small relative to ϵ') then implies that $nE\psi(S_n) > \delta + \sqrt{2\,|\underline{u}|} + \epsilon'$. Hence, by the Hoeffding inequality (42), there exists $\gamma = \gamma(\epsilon') > 0$ such that

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|} \,|\, G) \leq e^{-\gamma n},$$

because on the event $S_n \leq \delta + \sqrt{2\,|\underline{u}|}$ the sum S_n must deviate from its expectation by more than $nE\psi(S_n)\epsilon'$. Hence, for all $v \in [-\sqrt{2\,|\underline{u}|}, 0] \leq 0$,

$$E\psi(v + S_n) \leq e^{-\gamma n}\|\psi\|_\infty + P(G^c)\|\psi\|_\infty.$$

Thus, $|v\,(\underline{u})\,|$ must be vanishingly small, and so by Lemma 11 there can be no discontinuities in $[\underline{u}, 0]$. But this implies that the proportionality rule (40) holds for all $u \in \left[\underline{u}, \overline{u} - Cn^{-3/2}\right]$, and so another application of Hoeffding's inequality (coupled with the observation that $v(u)/u \geq (1-\epsilon)E\psi(S_n)$ holds for all $u \in [\underline{u}, \overline{u}]$ if v has no discontinuities at negative values of u) implies that

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|}) \leq e^{-\gamma n} \quad \Longrightarrow \quad E\psi(S_n) \leq e^{-\gamma n}\|\psi\|_\infty,$$

which is a contradiction. This proves assertion (i).

Since ES_n is now bounded away from 0 and ∞, it follows as before that $nE\psi(S_n)$ is bounded away from 0 and ∞, and so the proportionality rule (40) implies that the conditional variance of S_n given the event G is $O(n^{-1})$. The assertion (ii) therefore follows from Chebyshev's inequality and the bound $P(G^c) = O(n^{-1/2})$. Given (i) and (ii), we can now conclude that there can be no discontinuities at nonnegative values of u, because in view of Proposition 11, the monotonicity of Nash equilibria, and the necessary condition (13), this would entail that

$$E\psi(v(\overline{u}) + S_n)\overline{u} \geq 2\Delta,$$

which is incompatible with (i) and (ii).

Finally, if v is a Nash equilibrium with no discontinuities then Corollary 7 implies the exponential bound (iii). $\square$

D.2 Proof of Theorem 4

Lemma 15 implies that for large n the distribution of S_n must be highly concentrated near ES_n in any Nash equilibrium, and for any $\epsilon > 0$ there exists $\gamma > 0$ such that for any Nash equilibrium with no discontinuities,

$$P\{|S_n - ES_n| \geq \epsilon\} \leq e^{-\gamma n}.$$

Hence, if $ES_n > \delta + \epsilon$ then $E\psi(S_n) < e^{-\gamma n}$. But Proposition 6 asserts that if a Nash equilibrium $v(u)$ has no discontinuities then the proportionality rule (40) holds for all $u \in [\underline{u}, \overline{u}]$, and so

$$ES_n \leq (1+\epsilon')nE\psi(S_n) \leq (1+\epsilon')ne^{-\gamma n},$$

contradicting the fact that $ES_n \geq \delta - \epsilon$. This proves that for large n, any Nash equilibrium $v(u)$ with no discontinuities must satisfy $ES_n < \delta + \epsilon$.

Suppose that $ES_n < \alpha - 2\epsilon$ for some $\epsilon > 0$. If $\epsilon > 0$ is sufficiently small, then for some $\epsilon' > 0$ depending on ϵ,

$$(1 - \Psi(w+\epsilon)))(|\underline{u}| - \epsilon') > (\alpha - \epsilon - w)^2.$$

Consequently, if $ES_n \leq \alpha - 2\epsilon$, then an agent with utility $u \in [\underline{u}, \underline{u} + \epsilon']$ purchasing $\alpha + \epsilon - w$ votes would have expected payoff at least

$$-\Psi(w+\epsilon))\,(|\underline{u}| - \epsilon')\,P\{S_n \leq \alpha - \epsilon\} - (\alpha - \epsilon - w)^2.$$

This strictly dominates the expected payoff $\approx \underline{u}P\{S_n \geq \alpha - 3\epsilon\}$ for buying votes in accordance with the approximate proportionality rule (40). But any Nash equilibrium must satisfy the rule (40) except in the extremist regime, so we have a contradiction. This proves that for all sufficiently large n, in any Nash equilibrium we must have $ES_n > \alpha - 2\epsilon$. It follows that for all sufficiently large n, every Nash equilibrium has a discontinuity. The discontinuity must be located within distance $Cn^{-3/2}$ of the endpoint $\underline{u}$, by Lemma 11.

Now suppose that $ES_n > \alpha + 3\epsilon$. Then, by Hoeffding's inequality, $P(S_n \leq \alpha + 2\epsilon \,|\, G)$ is exponentially small for large n. Furthermore, since (α, w) is the unique pair satisfying (5),

$$(1 - \Psi(w' + \epsilon)))\,|\underline{u}| + 2\,|\underline{u}|\,e^{-\gamma n} < (\alpha + 2\epsilon - w')^2 \quad \text{for all } w' \in [-\delta, \delta]$$

and so it would be suboptimal for an agent with utility value $\underline{u}$ to buy more than $\alpha + 2\epsilon - \delta$ votes. Clearly it would also be suboptimal to buy more than Δ but no more than $\alpha + 2\epsilon - \delta$ votes, where Δ is the discontinuity threshold (cf. Lemma 11), because this would leave the expected utility payoff below $\underline{u}(1 - e^{-\gamma n})$. Consequently, if $ES_n > \alpha + 3\epsilon$ then for large n no Nash equilibrium would have a discontinuity; since we have shown that for large n every Nash equilibrium has a discontinuity it follows that ES_n cannot exceed $\alpha + 3\epsilon$ for large n. We have therefore proved that for any $\epsilon > 0$, if n is sufficiently large then (a) every Nash equilibrium has a discontinuity in the extremist regime near $\underline{u}$; and (b) $|ES_n - \alpha| < \epsilon$. Assertion (iv) of the theorem follows, by Proposition 15.

Let $v(u)$ be a Nash equilibrium, and let u_* be the rightmost point u_* of discontinuity of v. Consider the strategy $v(u)$ for an agent with utility value $u < u_*$: since v is monotone, $v(u) \leq -\Delta$. Moreover, the expected payoff for an agent with utility u must exceed the expected payoff under the alternative strategy of buying no votes. The latter expectation is approximately $\underline{u}$, because S_n is highly concentrated near $ES_n > \alpha - \epsilon$ and so $E\Psi(S_n) \approx 1$. On the other hand, the expected payoff at u for an agent playing the Nash strategy v is approximately

$$\Psi(\alpha - v(u))\,(\underline{u}) - v(u)^2.$$

Consequently, since (α, w) is the unique pair such that relations (5) hold, we must have

$$|v(u)| \approx \alpha - w.$$

This proves assertion (ii).

That v has only a single point of discontinuity u_* follows from the hypothesis (5). Recall (cf. Lemma 10) that if v is discontinuous at u then $E\psi'(\tilde{v} + S_n) = 2$ for some $\tilde{v}$ intermediate between the right and left limits $v(u+)$ and $v(u-)$. But any discontinuity u must occur within distance $\beta n^{-3/2}$ of $\underline{u}$, and if $u < u_*$ then $v(u) \approx -\alpha + w$. Hence, since the distribution of S_n is concentrated in a neighborhood of α,

$$E\psi'(v(u\pm) + S_n) \approx \psi'(w),$$

and so by Assumption 1, for $u \in [\underline{u}, u_*)$ there cannot be a value $\tilde{v} \in [v(u-), v(u+)]$ satisfying the necessary condition $E\psi'(\tilde{v} + S_n) = 2$ for a discontinuity.

Finally, since u_* must be within distance $Cn^{-3/2}$ of $\underline{u}$, the conditional probability that there are at least two extremists in the sample $U_1, U_2, \ldots, U_n$ given that there is at least one is of order $O(n^{-1/2})$. Consequently,

$$E\psi(S_n) = \psi(w) f_U(\underline{u})(u_* + |\underline{u}|) + O(n^{-1/2}(u_* + |\underline{u}|)).$$

On the other hand, since $ES_n \approx \alpha$, the proportionality rule (40) implies that $nE\psi(S_n) \approx \alpha$. Therefore,

$$u_* + \underline{u} \sim \gamma n^{-2}$$

where γ is the unique solution of the equation $\alpha = \gamma\psi(w) f_U(\underline{u})$. This proves assertions (i) and (iii).

□

E Balanced Populations: Proof of Theorem 3

E.1 Continuity of Nash equilibria

Proposition 9. If $\mu = 0$, then for all sufficiently large values of the sample size n no Nash equilibrium $v(u)$ has a discontinuity in $[\underline{u}, \overline{u}]$. Moreover, for any $\epsilon > 0$, if n is sufficiently large then every Nash equilibrium $v(u)$ satisfies

$$\|v\|_\infty \leq \epsilon. \tag{47}$$

Proof. The size of any discontinuity is bounded below by a positive constant Δ, by Proposition 11, so it suffices to prove the assertion (47). By Proposition 7, for any $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon)$ such that if n is sufficiently large then any Nash equilibrium $v(u)$ satisfying $\|v\|_\infty > \epsilon$ must also satisfy $|v(u)| \leq \gamma/\sqrt{n}$ for all u not within distance ϵ of one of the endpoints $\underline{u}, \overline{u}$. Hence, the approximate proportionality relation (40) implies that

$$E\psi(S_n) \leq \frac{C}{\sqrt{n}} \tag{48}$$

for a suitable $C = C(\gamma)$. Since $v(u)/u$ is within a factor $(1+\epsilon)^{\pm 1}$ of $E\psi(S_n)$ for all u not within distance $C'_\epsilon n^{-3/2}$ of $\underline{u}$ or $\overline{u}$, it follows from Chebyshev's inequality that for any $\alpha > 0$ there exists

$\beta = \beta(\alpha)$ such that

$$P\{|S_n - ES_n| \geq \beta\} \leq \alpha.$$

On the other hand, if $\|v\|_\infty \geq \epsilon$, then by the necessary condition (13), there is some u such that

$$P\{S_n + v(u) \in [-\delta, \delta]\} \geq \frac{\epsilon}{\|\psi\|_\infty \max(|\underline{u}|, \overline{u})}.$$

Since S_n is concentrated around ES_n, it follows that ES_n must be at bounded distance from $v(u)$, and so the Berry–Esseen bound (44) implies that $P\{S_n \in [-\delta/2, \delta/2]\}$ is bounded below. But this in turn implies that $E\psi(S_n)$ is bounded below, which for large n is impossible in view of (48). Thus, if n is sufficiently large then no Nash equilibrium $v(u)$ can have $\|v\|_\infty \geq \epsilon$.

□

Since $\|v\|_\infty$ is small for any Nash equilibrium v, the distribution of the vote total S_n cannot be too highly concentrated. This in turn implies that the proportionality constant $E\psi(S_n)$ in (40) cannot be too small.

Lemma 16. For any $C < \infty$ there exists $n_C < \infty$ such that for all $n \geq n_C$ and every Nash equilibrium,

$$nE\psi(S_n) \geq C. \tag{49}$$

Proof. By the approximate proportionality rule (40) and the necessary condition (13), for any $\epsilon > 0$ and all sufficiently large n,

$$|ES_n| \leq n\epsilon E\psi(S_n) E|U|.$$

Thus, by Hoeffding's inequality (Corollary 7), if $nE\psi(S_n) < C$ then the distribution of S_n must be highly concentrated in a neighborhood of 0. But if this were so we would have, for all large n,

$$E\psi(S_n) \approx \psi(0) > 0,$$

which is a contradiction. □

E.2 Edgeworth expansions

For the analysis of the case $\mu_U = 0$ refined estimates of the errors in the approximate proportionality rule (40) will be necessary. These we will derive from the Edgeworth expansion for the density of a sum of independent, identically distributed random variables (cf. Feller (1971), Ch. XVI, sec. 2, Th. 2). The relevant summands here are the random variables $v(U_i)$, and since the function $v(u)$ depends on the particular Nash equilibrium (and hence also on n), it will be necessary to have a version of the Edgeworth expansion in which the error is precisely quantified. The following variant of Feller's Theorem 2 (which can be proved in the same manner as in Feller) will suffice for our purposes.

Proposition 10. Let $Y_1, Y_2, \ldots, Y_n$ be independent, identically distributed random variables with mean $EY_1 = 0$, variance $EY_1^2 = 1$, and finite $2r$th moment $E|Y_1|^{2r} = \mu_{2r} \leq m_{2r}$. Assume that the distribution of Y_1 has a density $f_1(y)$ whose Fourier transform $\hat{f}_1$ satisfies $|\hat{f}_1(\theta)| \leq g(\theta)$, where g is a C^{2r} function such that $g \in L^\nu$ for some $\nu \geq 1$ and such that for every $\epsilon > 0$,

$$\sup_{|\theta| \geq \epsilon} g(\theta) < 1. \tag{50}$$

Then there is a sequence $\epsilon_n \to 0$ depending only on m_{2r} and on the function g such that the density $f_n(y)$ of $\sum_{i=1}^n Y_i/\sqrt{n}$ satisfies

$$\left| f_n(x) - \frac{e^{-x^2/2}}{\sqrt{2\pi n}} \left(1 + \sum_{k=3}^{2r} n^{-(k-2)/2} P_k(x) \right) \right| \leq \frac{\epsilon_n}{n^{-r+1}} \tag{51}$$

for all $x \in \mathbb{R}$, where $P_k(x) = C_k H_k(x)$ is a multiple of the kth Hermite polynomial $H_k(x)$, and C_k is a continuous function of the moments $\mu_3, \mu_4, \ldots, \mu_k$ of Y_1.

The following lemma ensures that in any Nash equilibrium the sums $S_n = \sum_{i=1}^n v(U_i)$, after suitable renormalization, meet the requirements of Proposition 10.

Lemma 17. There exist constants $0 < \sigma_1 < \sigma_2 < m_{2r} < \infty$ and a function $g(\theta)$ satisfying the hypotheses of Proposition 10 (with $r = 4$) such that for all sufficiently large n and any Nash equilibrium $v(u)$ the following statement holds. If $w(u) = 2v(u)/E\psi(S_n)$ then
  (a) $\sigma_1^2 < \mathrm{var}(w(U_i)) < \sigma_2^2$;
  (b) $E|w(U_i) - Ew(U_i)|^{2r} \leq m_{2r}$; and
  (c) the random variables $w(U_i)$ have density $f_W(w)$ whose Fourier transform is bounded in absolute value by g.

Proof. These statements are consequences of the proportionality relations (40) and the smoothness of Nash equilibria. By Proposition 9, Nash equilibria are continuous on $[\underline{u}, \overline{u}]$ and for large n satisfy $\|v\|_\infty < \epsilon$, where $\epsilon > 0$ is any small constant. Consequently, by Proposition 6, the proportionality relations (40) hold on the entire interval $[\underline{u}, \overline{u}]$. Since $EU_1 = 0$, it follows that for any $\epsilon > 0$, if n is sufficiently large then $|Ew(U_i)| < \epsilon$, and so assertions (a)–(b) follow routinely from (40).

The existence of the density $f_W(w)$ follows from the smoothness of Nash equilibria, which was established in Subsection C.4. In particular, by Proposition 14, inequalities (39), and the proportionality relations (40), if the sample size n is sufficiently large and v is any continuous Nash equilibrium then v is twice continuously differentiable on $[\underline{u}, \overline{u}]$, and there are constants $\alpha, \beta > 0$ not depending on n or on the particular Nash equilibrium such that the derivatives satisfy

$$\alpha \leq \frac{v'(u)}{E\psi(S_n)} \leq \alpha^{-1} \quad \text{and} \quad \beta \leq \frac{v''(u)}{E\psi(S_n)} \leq \beta^{-1} \tag{52}$$

for all $u \in [\underline{u}, \overline{u}]$. Consequently, if U is a random variable with density $f_U(u)$ then the random variable $W := 2v(U)/E\psi(S_n)$ has density

$$f_W(w) = f_U(u)E\psi(S_n)/(2v'(u)) \quad \text{where } w = 2v(u)/E\psi(S_n). \tag{53}$$

Furthermore, the density $f_W(w)$ is continuously differentiable, and its derivative

$$f'_W(w) = \frac{f'_U(u)(E\psi(S_n))^2}{4v'(u)^2} - \frac{f_U(u)(E\psi(S_n))^2 v''(u)}{4v'(u)^3}$$

satisfies

$$|f'_W(w)| \leq \kappa \tag{54}$$

where $\kappa < \infty$ is a constant that does not depend on either n or on the choice of Nash equilibrium.

It remains to prove the existence of a dominating function $g(\theta)$ for the Fourier transform of f_W. This will be done in three pieces: (i) for values $|\theta| \leq \gamma$, where $\gamma > 0$ is a small fixed constant;

(ii) for values $|\theta| \geq K$, where K is a large but fixed constant; and (iii) for $\gamma < |\theta| < K$. Region (i) is easily dealt with, in view of the bounds (a)–(b) on the second and third moments and the estimate $|Ew(U)| < \epsilon'$, as these together with Taylor's theorem imply that for all $|\theta| < 1$,

$$|\hat{f}_W(\theta) - (1 + i\theta Ew(U) - \theta^2 \mathrm{var}(w(U))/2| \leq m_3|\theta|^3.$$

Next consider region (ii), where $|\theta|$ is large. Integration by parts shows that

$$\hat{f}_W(\theta) = \int_{w\underline{u}}^{w(\overline{u})} f_W(w)e^{i\theta w}\,dw = -\int_{w\underline{u}}^{w(\overline{u})} \frac{e^{i\theta w}}{i\theta} f_W'(w)\,dw + \frac{e^{i\theta w}}{i\theta} f_W(w)\Big|_{w\underline{u}}^{w(\overline{u})};$$

since $f_W(w)$ is uniformly bounded at $w\underline{u}$ and $w(\overline{u})$, by (52) and (53), and since $|f_W'(w)| \leq \kappa$, by (54), it follows that there is a constant $C < \infty$ such that for all sufficiently large n and all Nash equilibria,

$$|\hat{f}_W(\theta)| \leq C/|\theta| \quad \forall\, \theta \neq 0.$$

Thus, setting $g(\theta) = C/|\theta|$ for all $|\theta| \geq 2C$, we have a uniform bound for the Fourier transforms $\hat{f}_W(\theta)$ in the region (ii).

Finally, to bound $|\hat{f}_W(\theta)|$ in the region (iii) of intermediate $\theta-$values, we use the proportionality rule once again to deduce that $|w(u) - u| < \epsilon$. Therefore,

$$\begin{aligned}
\hat{f}_W(\theta) &= \int_{\underline{u}}^{\overline{u}} e^{i\theta w(u)} f_U(u)\,du \\
&= \int_{\underline{u}}^{\overline{u}} e^{i\theta u} f_U(u)\,du + \int_{\underline{u}}^{\overline{u}} (e^{i\theta w(u)} - e^{i\theta u}) f_U(u)\,du \\
&= \hat{f}_U(\theta) + R(\theta)
\end{aligned}$$

where $|R(\theta)| < \epsilon'$ uniformly for $|\theta| \leq C$ and $\epsilon' \to 0$ as $\epsilon \to 0$. Since $\hat{f}_U$ is the Fourier transform of an absolutely continuous probability density, its absolute value is bounded away from 1 on the complement of $[-\gamma, \gamma]$, for any $\gamma > 0$. Since $\epsilon > 0$ can be made arbitrarily small (cf. Proposition 6), it follows that there is a continuous, positive function $g(\theta)$ that is bounded away from 1 on $|\theta| \in [\gamma, C]$ such that $|\hat{f}_W)\theta| \leq g(\theta)$ for all $|\theta| \in [\gamma, C]$. The extension of g to the whole real line can now be done by smoothly interpolating at the boundaries of regions (i), (ii), and (iii). □

E.3 Proof of Theorem 3

Since the function ψ is smooth and has compact support, differentiation under the expectation in the necessary condition $2v(u) = E\psi(v(u) + S_n)u$ is permissible, and so for every $u \in [-\underline{u}, \overline{u}]$ there exists $\tilde{v}(u)$ intermediate between 0 and $v(u)$ such that

$$2v(u) = E\psi(S_n)u + E\psi'(S_n + \tilde{v}(u))v(u)u. \tag{55}$$

The proof of Theorem 3 will hinge on the use of the Edgeworth expansion (Proposition 10) to approximate each of the two expectations in (55) precisely.

As in Lemma 17, let $w(u) = 2v(u)/E\psi(S_n)$. We have already observed, in the proof of Lemma 17, that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium, $|Ew(U)| < \epsilon$.

It therefore follows from the proportionality rule that

$$\left|\frac{4\,\mathrm{var}(v(U))}{(E\psi(S_n))^2\sigma_U^2} - 1\right| \leq \epsilon \quad \text{and} \quad \left|\frac{E|v(u) - Ev(u)|^k}{(E\psi(S_n))^k E|U|^k}\right| < \epsilon \;\; \forall\, k \leq 8. \tag{56}$$

Moreover, Lemma 17 and Proposition 10 imply that the distribution of S_n has a density with an Edgeworth expansion, and so for any continuous function $\varphi : [-\delta, \delta] \to \mathbb{R}$,

$$E\varphi(S_n) = \int_{-\delta}^{\delta} \varphi(x) \frac{e^{-y^2/2}}{\sqrt{2\pi n}\sigma_V} \left(1 + \sum_{k=3}^{m} n^{-(k-2)/2} P_k(y)\right) dx + r_n(\varphi) \tag{57}$$

where

$$\begin{aligned} \sigma_V^2 &:= \mathrm{var}(v(U)), \\ y = y(x) &= (x - ES_n)/\sqrt{\mathrm{var}(S_n)}, \end{aligned}$$

and $P_k(y) = C_k H_3(y)$ is a multiple of the kth Hermite polynomial. The constants C_k depend only on the first k moments of $w(U)$, and consequently are uniformly bounded by constants C_k' not depending on n or on the choice of Nash equilibrium. The error term $r_n(\varphi)$ satisfies

$$|r_n(\varphi)| \leq \frac{\epsilon_n}{n^{(m-2)/2}} \int_{-\delta}^{\delta} \frac{|\varphi(x)|}{\sqrt{2\pi \mathrm{var}(S_n)}}\, dx. \tag{58}$$

In the special case $\varphi = \psi$, (57) and the remainder estimate (58) (with $m = 4$) imply that

$$E\psi(S_n) \leq \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\, dx + o(n^{-1}\sigma_V^{-1}).$$

Since $4\,\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ for large n, this implies that for a suitable constant $\kappa < \infty$,

$$E\psi(S_n) \leq \frac{\kappa}{\sqrt[4]{n}}. \tag{59}$$

Claim 11. There exist constants $\alpha_n \to \infty$ such that in every Nash equilibrium,

$$|ES_n| \leq \alpha_n^{-1}\sqrt{\mathrm{var}(S_n)} \quad \text{and} \tag{60}$$

$$\mathrm{var}(S_n) \geq \alpha_n^2. \tag{61}$$

Proof of Theorem 3. Before we begin the proof of the claim, we indicate how it will imply Theorem 3. If (60) and (61) hold, then for every $x \in [-\delta, \delta]$,

$$|y(x)| \leq (1 + 2\delta)/\alpha_n \to 0.$$

Consequently, the dominant term in the Edgeworth expansion (57) for $\varphi = \psi$ (with $m = 4$), is the first, and so for any $\epsilon > 0$, if n is sufficiently large,

$$E\psi(S_n) = \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\, dx(1 \pm \epsilon).$$

(Here the notation $(1 \pm \epsilon)$ means that the ratio of the two sides is bounded above and below by $(1 \pm \epsilon)$.) Since $4\,\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ this will imply that

$$\sqrt{\pi n/2}\sigma_U(E\psi(S_n))^2 = \int_{-\delta}^{\delta} \psi(x)\,dx(1 \pm \epsilon) = 2 \pm 2\epsilon,$$

proving the assertion (7). □

Proof of Claim 11. First we deal with the remainder term $r_n(\varphi)$ in the Edgeworth expansion (57). By Lemma 16, the expectation $E\psi(S_n)$ is at least C/n for large n, and so by (56) the variance of S_n must be at least C'/n. Consequently, by (58), the remainder term $r_n(\varphi)$ in (57) satisfies

$$|r_n(\varphi)| \leq C'' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-2)/2}\sqrt{\text{var}(S_n)}} \leq C''' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-3)/2}}.$$

Suitable choice of m will make this bound small compared to any desired monomial n^{-A}, and so we may ignore the remainder term in the arguments to follow.

Suppose that there were a constant $C < \infty$ such that along some sequence $n \to \infty$ there were Nash equilibria satisfying var$(S_n) \leq C$. By (56), this would force $C/n \leq E\psi(S_n) \leq C'/\sqrt{n}$. This in turn would force

$$C''\text{var}(S_n)\log n \geq |ES_n|^2 \geq C'''\text{var}(S_n)\log n, \tag{62}$$

because otherwise the dominant term in the Edgeworth series for $E\psi(S_n)$ would be either too large or too small asymptotically (along the sequence $n \to \infty$) to match the requirement that $C/n \leq E\psi(S_n) \leq C'/\sqrt{n}$. (Observe that since the ratio $|ES_n|^2/\text{var}(S_n)$ is bounded above by $C'' \log n$, the terms $e^{-y^2/2}P_k(y)$ in the integral (57) are of size at most $(\log n)^A$ for some A depending on m, and so the first term in the Edgeworth series is dominant.) We will show that (62) leads to a contradiction.

Suppose that $ES_n > 0$ (the case $ES_n < 0$ is similar). The Taylor expansion (55) for $v(u)$ and the hypothesis $EU = 0$ implies that

$$2Ev(U) = E\psi'(S_n + \tilde{v}(U))v(U)U. \tag{63}$$

The Edgeworth expansion (57) for $E\psi'(S_n + \tilde{v}(u))$ together with the independence of S_n and U and the inequalities (62), implies that for any $\epsilon > 0$, if n is sufficiently large then

$$\begin{aligned} &E\psi'(S_n + \tilde{v}(u)) \\ &\quad = \frac{1}{\sqrt{2\pi\text{var}(S_n)}} \int_{-\delta}^{\delta} \psi'(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2/2\text{var}(S_n)\}\,dx(1 \pm \epsilon). \end{aligned} \tag{64}$$

Now since ψ and ψ' have support $[-\delta, \delta]$, integration by parts yields

$$\begin{aligned} &\int_{-\delta}^{\delta} \psi'(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2/2\text{var}(S_n)\}\,dx \\ &\quad = \int_{-\delta}^{\delta} \psi(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2/2\text{var}(S_n)\} \frac{x + \tilde{v}(u) - ES_n}{\text{var}(S_n)}\,dx, \end{aligned} \tag{65}$$

and since $x + \tilde{v}(u)$ is of smaller order of magnitude than ES_n, it follows that for large n

$$E\psi'(S_n + \tilde{v}(u)) = -\frac{ES_n}{\text{var}(S_n)} E\psi(S_n)(1 \pm \epsilon). \tag{66}$$

But it now follows from the Taylor series for $2Ev(U_i)$ (by summing over i) that

$$2ES_n = -n\frac{ES_n}{\text{var}(S_n)} E\psi(S_n)Ev(U)U(1 \pm \epsilon), \tag{67}$$

which is a contradiction, because the right side is negative and the left side positive. This proves the assertion (61).

The proof of inequality (60) is similar. Suppose that for some $C > 0$ there were Nash equilibria along a sequence $n \to \infty$ for which $ES_n \geq C\sqrt{\text{var}(S_n)}$. In view of (61), this implies in particular that $ES_n \to \infty$, and also that $|y(x)| \geq C/2$ for all $x \in [-\delta, \delta]$. Thus, the Edgeworth approximation (64) remains valid, as does the integration by parts identity (65). Since $ES_n \to \infty$, the terms $x + \tilde{v}(u)$ are of smaller order of magnitude that ES_n, and so once again (66) and therefore (67) follow. This is, once again, a contradiction, because the right side of (67) is negative while the left side diverges to $+\infty$.

□

F Proof of Lemma 8

Lemma 8. *Fix $\delta > 0$. For any $\epsilon > 0$ and any $C < \infty$ there exists $\beta = \beta(\epsilon, C) > 0$ and $n' = n'(\epsilon, C) < \infty$ such that the following statement is true: if $n \geq n'$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables such that*

$$E|Y_1 - EY_1|^3 \leq C\text{var}(Y_1)^{3/2} \quad \textit{and} \quad \text{var}(Y_1) \geq \beta/n \tag{68}$$

then for every interval $J \subset \mathbb{R}$ of length δ or greater, the sum $S_n = \sum_{i=1}^n Y_i$ satisfies

$$P\{S_n \in J\} \leq \epsilon|J|/\delta. \tag{69}$$

Proof. It suffices to prove this for intervals of length δ, because any interval of length $n\delta$ can be partitioned into n pairwise disjoint intervals each of length δ. Without loss of generality, $EY_1 = 0$ and $\delta = 1$ (if not, translate and re-scale). Let g be a nonnegative, even, C^∞ function with $\|g\|_\infty = 1$ that takes the value 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and is identically zero outside $[-1, 1]$. It is enough to show that for any $x \in \mathbb{R}$,

$$Eg(S_n + x) \leq \epsilon.$$

Since g is C^∞ and has compact support, its Fourier transform is real-valued and integrable, so the Fourier inversion theorem implies that

$$Eg(S_n + x) = \frac{1}{2\pi} \int \hat{g}(\theta)\varphi(-\theta)^n e^{-i\theta x}\, d\theta,$$

where $\varphi(\theta) = Ee^{i\theta Y_1}$ is the characteristic function of Y_1. Because $EY_1 = 0$, the derivative of the characteristic function at $\theta = 0$ is 0, and hence φ has Taylor expansion

$$|1 - \varphi(\theta) - \frac{1}{2} E Y_1^2 \theta^2| \leq \frac{1}{6} E|Y_1|^3 |\theta|^3.$$

Consequently, if the hypotheses (20) hold then for any $\gamma > 0$, if n is sufficiently large,

$$|\varphi(\theta)^n| \leq e^{-\beta^2\theta^2/4}$$

for all $|\theta| \leq \gamma$. This implies (since $|\hat{g}| \leq 2$) that

$$Eg(S_n + x) \leq \frac{1}{\pi} \int_{|\theta|<\gamma} e^{-\beta^2\theta^2/4} \, d\theta + \frac{1}{2\pi} \int_{|\theta|\geq\gamma} |\hat{g}(\theta)| \, d\theta.$$

Since $\hat{g}$ is integrable, the constant γ can be chosen so that the second integral is less that $\epsilon/2$, and if β is sufficiently large then the first integral will be bounded by $\epsilon/2$. $\square$