Ambiguity under Growing Awareness*

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Abstract

In this paper, we study choice under growing awareness in the wake of new discoveries. The decision maker's behavior is described by two preference relations, one before and one after new discoveries are made. The original preference admits a *subjective expected utility* representation. As awareness grows, the original decision problem expands and so does the state space. Therefore, the decision maker's original preference has to be extended to a larger domain, and consequently the new preference might exhibit ambiguity aversion. We propose two consistency notions that connect the initial and new preferences. *Unambiguity Consistency* requires that the original states remain unambiguous while new states might be ambiguous. This provides a novel interpretation of ambiguity aversion as a systematic preference to bet on old states than on newly discovered states. *Likelihood Consistency* requires that the relative likelihoods of the original states are preserved. Our main results axiomatically characterize a *maxmin expected utility* representation of the new preference that satisfies the two consistency notions. Moreover, we introduce a comparative notion of ambiguity aversion under growing awareness and characterize a parametric version of our model.

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1 Introduction

When modeling choice behavior under uncertainty, economists take for granted that the description of the underlying decision problem – including the states of nature, actions and consequences – is fixed. However, in many real life situations, a decision maker (henceforth, DM) makes new discoveries that might change the decision problem. New scientific insights, novel technologies, new medical treatments, new financial instruments, or new goods/services emerge on a almost daily basis. Such discoveries might reveal contingencies of which the DM was unaware. As awareness grows, the DM's universe (i.e., state space) expands and this might affect her preferences. In this paper, we explore how the DM's beliefs and tastes evolve after discovering new acts or new consequences.

In particular, we provide a theory of choice under growing awareness in which a subjective expected utility (SEU) preference extends to a maxmin expected utility (MEU) preference. While the extended preference captures ambiguity aversion, it inherits some properties of the initial SEU preference. Our theory provides a novel interpretation of ambiguity aversion. In particular, ambiguity arises because the DM treats new and old states differently. There is no exogenous information about states. In contrast, in the Ellberg experiments exogenous information about states is provided and ambiguity arises since the DM treats states with known and unknown probabilities differently.

To illustrate changes in beliefs and tastes due to growing awareness, consider a patient who suffers from a disease and needs to choose an appropriate treatment. There are two standard treatments, A and B. Each treatment leads to one of two possible outcomes: a success or a failure. The patient knows that there are two health factors that determine the outcome of each treatment, factor x and factor y, each being either good or bad. Treatment A is successful only when factor x is good, while treatment B works out only when her factor y is good. The patient believes that both factors are good with probability 0.5.

Suppose now that the patient discovers that there is a new treatment C. While consulting her doctor, she becomes aware of a third factor z that matters: treatment C is successful when factor z is good. Since treatment C and factor z are new to the patient, she cannot come up with a unique probability that factor z is good and becomes pessimistic about the novel treatment. Moreover, the discovery of the new factor z causes the patient to reevaluate the standard treatments A and B and her original beliefs regarding factors x and y.

Given the discovery of treatment C, the patient's preferences might change fundamentally. To discipline the effect of new discoveries, we consider two consistency notions between the patient's behavior before and after the discovery.

Behaviorally, our consistency notions can be described by the way the old treatments are evaluated after awareness has changed. To illustrate the main idea of the first consistency notion, suppose that the new treatment C can be seen as a combination of the standard treatment A and another new treatment D. Then it requires that the evaluation of A is independent of D. Since the patient evaluated A previously and is familiar with it, the new component D should not affect how A is evaluated. However, this consistency notion allows the values of A and B to change as the patient's beliefs change after the discovery.

The second consistency notion requires that the ranking over the old treatments A and B does not change as awareness grows. The two consistency notions are independent in general, but the second notion implies the first notion when the discovery is a new consequence. Our goal is to axiomatically characterize each consistency notion by connecting the patient's initial and new preferences.

Recently, Karni and Vierø (2013) have introduced an elegant theory of choice under growing awareness called *reverse Bayesianism*. They focus on SEU preferences and characterize the evolution of subjective beliefs in the decision theoretic framework of Anscombe and Aumann (1963). However, in reverse Bayesianism growing awareness does not affect the SEU form of preferences and thus it precludes ambiguity in an expanded state space.

In contrast, we allow the DM's behavior to change fundamentally as awareness grows.[1] In our theory, while being originally a SEU maximizer, the DM might become ambiguity averse in an expanded universe. More specifically, the DM's behavior is described by two preference relations, one before and one after a discovery is made. The initial preference takes the SEU form. One can think of this assumption as follows. The DM is relatively familiar with the original decision problem and therefore she came up with a (unique) probability measure over the states. As awareness grows, the new extended preference admits a MEU representation of Gilboa and Schmeidler (1989) since the DM faces a new decision problem.

Our main results behaviorally characterize the evolution of an original SEU preference to a new, extended MEU preference under two consistency notions. The first consistency notion, called *Unambiguity Consistency*, requires that growing awareness does not interfere with unambiguity of the original preference. More precisely, the new events that correspond to the old states are unambiguous; only the new states may be ambiguous.[2]

[1] There is empirical evidence suggesting that individuals' awareness and ambiguity are related. For instance, Giustinelli and Pavoni (2017) ask Italian middle schoolers about their likelihoods to successfully graduate from different high school tracks. The authors find that students who were initially unaware of some school tracks (but learn their existence during the survey) perceive significant ambiguity about the success of alternative curricula that have these tracks. Related to our patient story, there is a growing body of evidence reporting people's ambiguity averse attitude when they face new medical tests and treatments (e.g., see Han et al. (2009) and Taber et al. (2015)).

[2] Unambiguity Consistency builds on the notion of unambiguous events in the sense of Nehring (1999) and Ghirardato et al. (2004). An event is unambiguous if all probabilities assign the same value to the event.

An extended MEU preference that satisfies Unambiguity Consistency is characterized by a novel axiom called *Negative Unambiguity Independence* (NUI) in Theorem 1. NUI states that the DM can hedge against the ambiguity of the new acts (e.g., the new treatment C). However, she cannot use the old acts (e.g., the standard treatments A, B) to hedge since such acts are always evaluated independently of new components (e.g., D).[3]

Under Unambiguity Consistency, new discoveries might affect the DM's ranking over the old acts since her beliefs (as well as her risk preferences) might change. For instance, after the discovery of the new health factor z, the patient might believe that factor x is more likely to be good than factor y, leading her to strictly prefer the standard treatment A over B.

Our second consistency notion, called *Likelihood Consistency*, requires that the new extended preference maintains the relative likelihoods of the original states. Preserving the original likelihoods might be a reasonable property in choice situations where the original preferences are supported by hard facts or objective information. Likelihood Consistency is characterized by *Binary Awareness Consistency* (BAC), which requires that the DM's ranking of old acts (standard treatments A and B) is not affected by growing awareness. Theorem 2 shows that under BAC, the relative likelihoods of old states remain preserved by the extended MEU preference. Moreover, the DM's risk preferences (i.e., her ranking of constant acts) do not change as awareness grows.

The theory of extended MEU preferences satisfying Unambiguity Consistency and Likelihood Consistency is called *generalized reverse Bayesianism*. In the special case in which all the newly discovered contingencies are unambiguous, the extended preference is SEU and our theory coincides with the theory of reverse Bayesianism of Karni and Vierø (2013).[4]

Interestingly, the theory of generalized reverse Bayesianism can be behaviorally disentangled between choice situations in which awareness grows due to discoveries of new acts versus discoveries of new consequences. When the new treatment C is discovered, each original state is extended by indicating whether factor z is good or bad (equivalently, whether treatment C leads to a success or failure). In this context, our theory implies that the old treatments A and B are unambiguous acts while the new treatment C is ambiguous. Consequently, an ambiguity averse DM tends to prefer the old and unambiguous acts to the newly discovered ones.

However, when a new consequence is discovered, ambiguity aversion will be exhibited differently. Suppose that the patient discovers that the standard treatments A and B might

[3]The spirit of our axiom is reminiscent of the Negative Certainty Independence axiom introduced by Dillenberger (2010) and used by Cerreia-Vioglio et al. (2015) to derive the Cautious Expected Utility model in the context of choice under risk. However, in our framework, NUI has different behavioral implications since we allow for ambiguity.

[4]Notice that Unambiguity Consistency is trivially satisfied in the setup of Karni and Vierø (2013).

cause a health complication. In this case, the original state space expands differently. The original state space is extended by new states indicating whether treatments A and B lead to the health complication or not. If the patient perceives ambiguity about the new states, then the standard treatments become ambiguous acts. Therefore, the patient prefers combinations of the old treatments A and B as a hedge against the ambiguity of A and B.

At a fundamental level, our theory provides a novel interpretation of the widely-studied ambiguity phenomenon. Typically, as in the classical Ellsberg experiments, ambiguity is exogenously created. That is, subjects are informed about exogenous probabilities for some events in a given state space; for other states such information is missing. The task is to elicit subjects' attitudes towards ambiguity. A preference for betting on known probability events rather than betting on unknown probability events is understood as ambiguity aversion.

In our theory, the DM perceives ambiguity about states of which she was originally unaware. In other words, an expanding universe can be seen as a "source" of ambiguity. As awareness grows, the DM might be unable to extend her subjective beliefs so that her new beliefs cannot be represented by a unique probability measure. Therefore, in our theory ambiguity aversion is displayed differently as a preference for betting on old, familiar states rather than betting on the newly discovered states.[5]

The rest of the paper is organized as follows. Section 2 presents the basic setup and illustrates how new discoveries expand the original state space. In Section 3, we discuss the SEU and MEU representations of the original and new preferences, and provide our definitions of consistent evolution of beliefs. In Section 4, we provide representation theorems that characterize our two consistency notions. In Section 5, we develop a comparative notion of ambiguity under growing awareness in the spirit of Ghirardato and Marinacci (2002). In Section 6, we also derive a parametric version of our MEU representation. A brief overview about the literature on choice under (un)awareness is provided in Section 7. All proofs are collected in Appendix A. The online appendix discusses evolution of beliefs with a fixed level of awareness.

2 State Space Construction

To explore how growing awareness affects preferences of a decision maker (henceforth, DM), we adopt the formal setup developed in Karni and Vierø (2013).

[5]Indeed, Daniel Ellsberg describes ambiguity as a much broader phenomenon than a comparison between known and unknown probabilities. In his words (Ellsberg (1961, p. 657)), ambiguity refers to "*a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's degree of 'confidence' in an estimate of relative likelihoods.*" In this paper, we introduce a specific scenario where ambiguity might arise even if there are no objective probabilities.

There are a nonempty, finite set F of *feasible acts* and a nonempty, finite set C of *feasible consequences*. These two sets together determine a *conceivable state space*, $S \equiv C^F$. The elements of S represent all the possible resolutions of uncertainty. That is, a state s specifies a unique consequence associated with each feasible act, thereby resolving all uncertainty.[6]

Once the set of conceivable states is fixed, the set of acts expands to include a set of *conceivable acts*. That is, it is assumed that the DM can imagine acts whose outcomes are lotteries with consequences in C as prizes, the so-called Anscombe-Aumann acts. Denote by $\Delta(C)$ the set of lotteries on C, i.e., functions $p : C \to [0, 1]$ such that $\sum_{c \in C} p(c) = 1$. The set of conceivable acts $\hat{F}$ contains all functions that map states to lotteries:

$$\hat{F} \equiv \{f : S \to \Delta(C)\}. \tag{1}$$

A collection $\mathcal{D} \equiv \{C, F, S, \hat{F}\}$ describes a choice problem under uncertainty, called the *original decision problem*. The DM's behavior is modeled via a preference relation on $\hat{F}$, denoted by $\succcurlyeq_{\hat{F}}$. As usual, $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ refers to the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, respectively.

Our primary goal is to examine how the DM's behavior might change in response to growing awareness in the wake of new discoveries. In other words, we investigate how discoveries of new acts or new consequences affect the DM's original preference relation $\succcurlyeq_{\hat{F}}$. In order to do that, the original decision problem needs to be reformulated to incorporate new discoveries into a new choice problem.

We begin our analysis by describing how the original decision problem and the set of conceivable states expand when a new act or a new consequence is discovered. For simplicity, we focus on a single discovery throughout the paper.

The example below illustrates how the construction of the conceivable state space works.

Original model. Let us consider the choice problem discussed in the introduction. Let $C = \{c_1, c_2\}$ be the set of consequences and $F = \{f_1, f_2\}$ be the set of feasible acts. The corresponding conceivable state space S consists of four conceivable states.

	$x^g\,y^g$	$x^g y^b$	$x^b\,y^g$	$x^b y^b$
$F \setminus S$	s_1	s_2	s_3	s_4
f_1	c_1	c_1	c_2	c_2
f_2	c_1	c_2	c_1	c_2

Table 1: Original state space

[6]This way to construct a state space was suggested by Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).

In the language of our example, consequences c_1 and c_2 correspond to a success and a failure, respectively. Acts f_1 and f_2 correspond to treatment A and treatment B, respectively. Recall that f_1 (treatment A) is successful only when the patient's health factor x is good and f_2 (treatment B) is successful only when her health factor y is good.

In Table 1, x^g and x_b (likewise, y^g and y_b) represent the cases in which factor x (resp., y) is good and bad, respectively. State s_1 corresponds to the case where x is good and y is good, state s_2 corresponds to the case where x is good and y is bad, and so on. Hence, both treatments lead to a success in state s_1 and to a failure in state s_4. In state s_3, f_1 (treatment A) will be a failure while f_2 (treatment B) will be successful.

Discoveries of new acts. When the DM discovers a new act $\bar{f}$, the set of acts expands to $F_{\bar{f}} \equiv F \cup \{\bar{f}\}$ and so does the original state space.

Coming back to our example, suppose that the patient discovers $\bar{f}$, the new treatment C. The original set of acts F expands to $F_{\bar{f}} \equiv \{f_1, f_2, \bar{f}\}$. The expanded state space, $S_{\bar{f}} \equiv C^{F_{\bar{f}}}$, consists of eight (i.e., 2^3) conceivable states:

	$x^g y^g z^g$	$x^g y^b z^g$	$x^b y^g z^g$	$x^b y^b z^g$	$x^g y^g z^b$	$x^g y^b z^b$	$x^b y^g z^b$	$x^b y^b z^b$
$F_{\bar{f}} \setminus S_{\bar{f}}$	s_1^1	s_2^1	s_3^1	s_4^1	s_5^1	s_6^1	s_7^1	s_8^1
f_1	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_2	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
$\bar{f}$	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2

Table 2: Extended state space: new act $\bar{f}$

Since the outcome of the new treatment C depends on the health factor z, each state will now determine whether each of the three relevant factors, x, y and z is good or bad. For instance, state s_1^1 corresponds to the case that all factors are good and state s_5^1 corresponds to the case that health factors x, y are good, but factor z is bad.

It should be noted that the discovery of a new feasible act changes the state space in a specific way. The expanded state space $S_{\bar{f}}$ represents a *uniform refinement (filtration) of the original state space* S (see Karni (2015, p. 8)). To put it differently, each original state s in S corresponds to an event E_s in $S_{\bar{f}}$ that refines s; i.e., $E_s = \{s^1 \in S_{\bar{f}} : \exists c \in C \text{ s.t. } s^1 = (s, c)\}$. For example, consider state $s_1 = (c_1, c_1)$. This state corresponds to the new event $E_{s_1} = \{s_1^1, s_5^1\}$ that refines s_1 by incorporating consequences c_1 and c_2 associated with the new act $\bar{f}$, i.e., $s_1^1 = (c_1, c_1, c_1)$ and $s_5^1 = (c_1, c_1, c_2)$. In other words, state s_1, originally indicating that both factors x, y are good, expands now to two states, s_1^1 and s_5^1. In state s_1^1 all factors x, y and z are good while in state s_5^1 only x, y are good and factor z is bad. Likewise, the original state $s_2 = (c_1, c_2)$ corresponds to the new event $E_{s_2} = \{s_2^1, s_6^1\}$, and so on. The collection of events $\{E_s\}_{s\in S}$ forms a partition of the expanded state space $S_{\bar{f}}$. Therefore, $S_{\bar{f}} = E(S) \equiv \cup_{s\in S} E_s$ when a new act is discovered.

Given the set of newly discovered contingencies, the set of conceivable acts is defined by

$$\hat{F}_{\bar{f}} \equiv \{f : S_{\bar{f}} \to \Delta(C)\}. \tag{2}$$

As awareness grows, the patient's original preference $\succcurlyeq_{\hat{F}}$ has to be extended to a new preference relation $\succcurlyeq_{\hat{F}_{\bar{f}}}$ defined over the new set of conceivable acts $\hat{F}_{\bar{f}}$.

Discoveries of new consequences. When the DM discovers a new consequence $\bar{c}$, the set of consequences expands to $C_{\bar{c}} \equiv C \cup \{\bar{c}\}$. The new consequence needs to be incorporated into the range of the feasible acts in F, thus the original set of feasible acts changes. Denote by $F_{\bar{c}}$ the set of feasible acts with extended range due to the discovery of $\bar{c}$.

Consider the original decision problem with the state space S depicted in Table 1. Suppose that the patient discovers that a health complication $\bar{c}$ is possible. Since $C_{\bar{c}} \equiv \{c_1, c_2, \bar{c}\}$, the new state space $S_{\bar{c}} \equiv C_{\bar{c}}^{F_{\bar{c}}}$ consists of nine (i.e., 3^2) states:

$F_{\bar{c}} \setminus S_{\bar{c}}$	s_1^1	s_2^1	s_3^1	s_4^1	s_5^1	s_6^1	s_7^1	s_8^1	s_9^1
f_1	c_1	c_1	c_2	c_2	c_1	$\bar{c}$	c_2	$\bar{c}$	$\bar{c}$
f_2	c_1	c_2	c_1	c_2	$\bar{c}$	c_1	$\bar{c}$	c_2	$\bar{c}$

Table 3: Extended state space: new consequence $\bar{c}$

After discovering the new consequence $\bar{c}$, the patient becomes aware of new states. In states s_6^1, s_8^1 and s_9^1 , f_1 (treatment A) leads to the health complication. In states s_5^1, s_7^1 and s_9^1, f_2 (treatment B) leads to the health complication.

When the new consequence $\bar{c}$ is discovered, the original state space $S = C^F$ genuinely expands. While states in $C^{F_{\bar{c}}} \equiv \{s_1^1, s_2^1, s_3^1, s_4^1\}$ correspond to the original states, the states in $S_{\bar{c}} \setminus C^{F_{\bar{c}}}$ are new. In other words, an event E_s in $S_{\bar{c}}$ that corresponds to an original state $s \in S$ is the state s itself; i.e., $E_s = \{s\} = \{s^1\}$ for some $s^1 \in S_1$. For instance, $E_{s_1} = \{s_1\} = \{s_1^1\} = \{(c_1, c_1)\}$. The new event $S_{\bar{c}} \setminus C^{F_{\bar{c}}}$ is the set of conceivable states in which the health complication can happen. Therefore, $S_{\bar{c}} \supset E(S) \equiv \cup_{s\in S} E_s$ when a new consequence is discovered.

After discovering $\bar{c}$, the patient's original preference $\succcurlyeq_{\hat{F}}$ has to be extended to a new preference relation $\succcurlyeq_{\hat{F}_{\bar{c}}}$ defined over the set of conceivable acts with an extended range, i.e.,

$$\hat{F}_{\bar{c}} \equiv \big\{f : S_{\bar{c}} \to \Delta(C_{\bar{c}})\big\}. \tag{3}$$

The primary goal of this paper is to formally link preferences $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_{\bar{f}}}$ (resp., $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_{\bar{c}}}$) in the wake of the DM's growing awareness.

For any acts $f, g \in \hat{F}$ and any event $E \subseteq S$, denote by $f_{-E}\, g$ the act in $\hat{F}$ that returns $g(s)$ in state $s \in S$ and $f(s')$ in state $s' \in S \setminus E$. A state $s \in S$ is said to be *null* if $f_{-s}\, p \sim_{\hat{F}} f_{-s}\, q$ for all $p, q \in \Delta(C)$, otherwise s is *nonnull.* For notational simplicity, we will assume that all states in S, $S_{\bar{f}}$, and $S_{\bar{c}}$ are nonnull throughout the paper.[7]

3 Preferences and Consistency Notions

In this section, we illustrate our main idea of how new discoveries affect the DM's preferences.

Although the state space expansion depends on whether the discovery is an act or consequence, we provide unified notations, definitions, and characterization theorems that work for both cases. We denote by $\mathcal{F}$ a family of sets of conceivable acts corresponding to increasing levels of awareness. As reference to the original decision problem $\mathcal{D}$, we fix $\hat{F} \in \mathcal{F}$ and call $\hat{F}$ the *initial set of conceivable acts* (i.e., the set of acts before a new discovery is made). The DM's *initial preference relation* on $\hat{F}$ is denoted by $\succsim_{\hat{F}}$. When a new act or a new consequence is discovered, the original decision problem expands and the preference relation $\succsim_{\hat{F}}$ has to be extended to a larger domain. We denote by $\succsim_{\hat{F}_1}$ the *extended preference relation* on a set of acts $\hat{F}_1$ in an extended decision problem $\mathcal{D}_1 \equiv \{C_1, F_1, S_1, \hat{F}_1\}$. When a new act $\bar{f}$ is discovered, $C_1 \equiv C$, $F_1 \equiv F_{\bar{f}} = F \cup \{\bar{f}\}$, $S_1 \equiv S_{\bar{f}} = C^{F_{\bar{f}}}$, and the extended preference relation is defined on the expanded set of conceivable acts, i.e., $\succsim_{\hat{F}_1} \equiv \succsim_{\hat{F}_{\bar{f}}}$. Likewise, when a new consequence $\bar{c}$ is discovered, $C_1 \equiv C_{\bar{c}} = C \cup \{\bar{c}\}$, $F_1 \equiv F_{\bar{c}}$, $S_1 \equiv S_{\bar{c}} = (C_{\bar{c}})^{F_{\bar{c}}}$, and the extended preference relation is defined on the set of conceivable acts with an extended range, i.e., $\succsim_{\hat{F}_1} \equiv \succsim_{\hat{F}_{\bar{c}}}$.

As awareness grows, the representation of the DM's preferences might change fundamentally. To capture the idea of changing preferences formally, it will be taken for granted that the initial preference is of the subjective expected utility form.

Definition 1 (Initial Preference). The initial preference relation $\succsim_{\hat{F}}$ on $\hat{F}$ is said to admit a subjective expected utility (SEU) representation if there exist a probability measure

[7] Allowing for null states does not change our results. In fact, in the online appendix we consider a case where a DM discovers that some links between acts and consequences become feasible whereas other links become infeasible. Therefore, in this case, we explicitly study how the DM's beliefs change when a nonnull state becomes a null state and vice versa. Moreover, we discuss how a MEU preference consistently evolves to another MEU preference after a discovery of new links. Therefore, the online appendix also illustrates that our analysis can be extended to three periods in which the MEU preference in the third period inherits properties of the MEU preference in the second period.

$\mu \in \Delta(S)$ and an expected utility functional $U : \Delta(C) \to \mathbb{R}$ such that for any $f \in \hat{F}$,

$$(4) \qquad V^{SEU}(f) = \sum_{s \in S} U(f(s))\mu(s).$$

However, a DM who behaves as a SEU maximizer might become ambiguity averse after new discoveries. To accommodate ambiguity under growing awareness, the extended preference relation takes the familiar maxmin expected utility (MEU) form of Gilboa and Schmeidler (1989).

Definition 2 (Extended Preference). The extended preference $\succsim_{\hat{F}_1}$ on $\hat{F}_1$ is said to admit a maxmin expected utility (MEU) representation if there exist a nonempty, convex, and compact set of probability measures $\Pi_1 \subseteq \Delta(S)$ and an expected utility functional $U_1 : \Delta(C_1) \to \mathbb{R}$ such that for any $f \in \hat{F}_1$,

$$(5) \qquad V^{MEU}(f) = \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1(f(s^1))\pi(s^1).$$

Instead of a unique probability measure, the DM's beliefs over the expanded state space are represented by a set of priors. A DM whose preferences are governed by the MEU functional in Equation (5) is said to be *ambiguity averse.* In contrast, when the min-operator is replaced by a max-operator, a DM is said to be *ambiguity loving.*[8]

Our goal is twofold. First, we want to behaviorally underpin the representations (4) and (5). Second, we will connect the initial preference $\succsim_{\hat{F}}$ and the extended preference $\succsim_{\hat{F}_1}$ via axioms characterizing how the DM's beliefs and tastes evolve as awareness grows.

Notice that both preference relations $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ are fully characterized by tuples (μ, U) and (Π_1, U_1) from the respective representations (4) and (5). Therefore, to link the initial and extended preference, we will relate (μ, U) and (Π_1, U_1). In order to make sharp conclusions about how the initial preferences evolve in response to growing awareness, we will impose two consistency conditions between μ and Π_1.

Our first consistency notion requires that the extended MEU preference inherits unambiguity property of the initial SEU preference across the original states. We denote by $\{E_s\}_{s \in S}$ the family of events in the extended state space S_1, each E_s corresponding to an original state s in S. The first consistency, called *Unambiguity Consistency*, requires that each event E_s is revealed to be unambiguous by the extended preference relation $\succsim_{\hat{F}_1}$. Following Nehring (1999) and Ghirardato et al. (2004), an event E_s is *unambiguous* if each probability measure in Π_1 assigns the same value to E_s; i.e., $\pi(E_s) = \pi'(E_s)$ for all $\pi, \pi' \in \Pi_1$.

[8] The results of this paper also hold when the DM is ambiguity loving.

Our first consistency notion is formalized as follows.

Definition 3 (Unambiguity Consistency). Let $\succsim_{\hat{F}}$ be a SEU preference relation on $\hat{F}$ with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference relation on $\hat{F}_1$ with (Π_1, U_1). Then, $\succsim_{\hat{F}_1}$ is said to be an *unambiguity consistent extension* of $\succsim_{\hat{F}}$ to $\hat{F}_1$, if each event E_s that corresponds to the original states is unambiguous according to $\succsim_{\hat{F}_1}$, i.e., for all $\pi, \pi' \in \Pi_1$ and $s \in S$,

$$\pi(E_s) = \pi'(E_s). \tag{6}$$

Unambiguity Consistency provides a novel view on ambiguity. In particular, ambiguity appears since the DM treats new and old states differently.

Unambiguity Consistency implies that old acts are unambiguous. In the context of our example, Unambiguity Consistency implies that the evaluation of the old treatment A is independent of another new treatment D when evaluating treatment C, which is a combination of A and D. However, under Unambiguity Consistency, the DM's old belief μ on S and her new belief Π_1 on $\{E_s\}_{s\in S}$ to be unrelated. Likewise, the DM's risk preferences, U and U_1, may change as awareness grows.

Our second consistency notion, called *Likelihood Consistency*, directly connects μ and Π_1 by requiring that the DM's new beliefs Π_1 preserve the relative likelihoods of the old belief μ. This consistency notion is formalized below.

Definition 4 (Likelihood Consistency). Let $\succsim_{\hat{F}}$ be a SEU preference relation on $\hat{F}$ with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference relation on $\hat{F}_1$ with (Π_1, U_1). Then, $\succsim_{\hat{F}_1}$ is said to be a *likelihood consistent extension* of $\succsim_{\hat{F}}$ to $\hat{F}_1$, if the new beliefs in Π_1 preserve the relative likelihoods of μ on S; i.e., for all $s, s' \in S$ and $\pi \in \Pi_1$:

$$\frac{\mu(s)}{\mu(s')} = \frac{\pi\big(E_s\big)}{\pi\big(E_{s'}\big)}. \tag{7}$$

Likelihood Consistency implies that the rankings over old acts are preserved. In the context of discovering new acts, Likelihood Consistency implies Unambiguity Consistency because of $\sum_{s\in S}\pi\big(E_s\big) = 1$. However, in the context of discovering new consequences, the state space genuinely expands and $\sum_{s\in S}\pi\big(E_s\big) < 1$. Therefore, the two consistency notions are independent in this context.

3.1 Illustrations and Behavioral Implications

To illustrate our consistency notions, consider again the patient example. When confronted with the original model (see Table 1), suppose she believes that each state is equally likely.

That is, her belief μ on S is given by

$$\mu(s_1) = \mu(s_2) = \mu(s_3) = \mu(s_4) = \frac{1}{4}. \tag{8}$$

Thus the patient with a SEU preference $\succsim_{\hat{F}}$ is indifferent between the two treatments (f_1 and f_2) and any mixture thereof (i.e., $\alpha f_1 + (1-\alpha) f_2 \sim_{\hat{F}} f_1 \sim_{\hat{F}} f_2$).

Consider now the patient's extended preference $\succsim_{\hat{F}_1}$ after the new treatment $\bar{f}$ is discovered. Recall, in this case, each original state $s \in S$ admits a finer description depending on whether her factor z is good or bad. That is, $S_1 \equiv S_{\bar{f}}$ (see Table 2).

The patient might not be able to "split" her initial belief $\mu(s_i)$ across the new states s_i^1 and s_{i+4}^1 with $i = 1, \ldots, 4$. For example, she might consider the following set of priors:

$$\Pi_1 = \left\{ \pi \in \Delta(S_1) : \begin{array}{ll} \pi(s_i^1) + \pi(s_{i+4}^1) = \pi(E_{s_i}) = \frac{\beta}{2} & \text{for } i = 1, 2; \\ \pi(s_i^1) + \pi(s_{i+4}^1) = \pi(E_{s_i}) = (1-\beta)\frac{1}{2} & \text{for } \ i = 3, 4, \end{array} \right. \tag{9}$$

where $\beta \in [0, \frac{1}{2}]$. Notice that each event $E_{s_i} = \{s_i^1, s_{i+4}^1\}$ in S_1 that corresponds to the original state $s_i \in S$ is unambiguous while each newly discovered state $s_i^1 \in S^1$ is ambiguous. Thus, for any $\beta \in [0, \frac{1}{2}]$, the patient's extended preference $\succsim_{\hat{F}_1}$ preserves unambiguity of the initial preference $\succsim_{\hat{F}}$.

When $\beta = \frac{1}{2}$, the extended MEU preference is likelihood consistent since the set of priors maintains the relative likelihoods of μ as $\mu(s_i)/\mu(s_j) = \frac{1}{4}/\frac{1}{4} = \pi(E_{s_i})/\pi(E_{s_j}) = \frac{1}{4}/\frac{1}{4}$. Moreover, the ambiguity averse patient is still indifferent between f_1, f_2, and any of their mixtures. However, she strictly prefers either of the standard treatments to the new treatment f_3 (i.e., $\alpha f_1 + (1-\alpha) f_2 \sim_{\hat{F}_1} f_1 \sim_{\hat{F}_1} f_2 \succ_{\hat{F}_1} f_3$). As remarked before, Likelihood Consistency implies Unambiguity Consistency.

Consider now the case in which the new consequence $\bar{c}$ is discovered. See Table 3. Since the states s_5^1 through s_9^1 are newly discovered, the patient might not be able to form a unique prior over S_1. Instead, she might consider a set of priors Π_1. For instance, consider the following set.

$$\Pi_1 = \left\{ \pi \in \Delta(S_1) : \pi(s_i^1) = \frac{\gamma}{16} \text{ and } \gamma \in [1, \overline{\gamma}] \ \text{ for all } i = 1, \ldots, 4 \right\}. \tag{10}$$

When new consequences are discovered, Likelihood Consistency and Unambiguity Consistency are independent. For example, when $\overline{\gamma} = 1$, the extended preference with respect to Π_1 reveals that the original states s_1^1 through s_4^1 are unambiguous while the newly discovered states $s_5^1, s_6^1, s_7^1, s_8^1$ and s_9^1 are ambiguous. However, when $\overline{\gamma} = 2$, the original states s_1^1, s_2^1, s_3^1, and s_4^1 are ambiguous; i.e., Unambiguity Consistency is violated. Nevertheless, Π_1 still preserves the relative likelihoods of μ since $\mu(s_i)/\mu(s_j) = \frac{1}{4}/\frac{1}{4} = \pi(E_{s_i})/\pi(E_{s_j}) = \frac{\gamma}{16}/\frac{\gamma}{16}$.[9] Moreover, the patient's preference has different behavioral implications compared to the case where a new act is discovered. Specifically, the patient, who is still indifferent between old treatment f_1 and f_2, strictly prefers any mixture of them over each of f_1 and f_2 alone. In other words, the patient reveals ambiguity aversion in the standard sense (i.e., $\alpha f_1 + (1-\alpha) f_2 \succ_{\hat{F}_1} f_1 \sim_{\hat{F}_1} f_2$).

To sum up, ambiguity arises differently depending on whether a new act or consequence is discovered. When new acts are discovered, new acts are ambiguous, while when new consequences are discovered, old acts are ambiguous. However, regardless of what is discovered, (i) Unambiguity Consistency implies that new states are ambiguous while old states are unambiguous, and (ii) Likelihood Consistency implies that the rankings over old acts are preserved, $f_1 \sim_{\hat{F}} f_2$ and $f_1 \sim_{\hat{F}_1} f_2$.

4 Behavioral Foundations

In this section, we axiomatically characterize Unambiguity and Likelihood Consistency. Although the discovery of new acts expands the original state space differently than the discovery of new consequences, our results are unified in a way that characterizing axioms are the same in the both contexts.

4.1 Basic Preference Structure

We have the initial preference $\succcurlyeq_{\hat{F}}$ from the initial decision problem $\mathcal{D} = (C, F, S, \hat{F})$ and the extended preference $\succcurlyeq_{\hat{F}_1}$ from the expanded decision problem $\mathcal{D}_1 = (C_1, F_1, S_1, \hat{F}_1)$. We first introduce the basic axioms on $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ to obtain the representations (4) and (5).

For all $f, g \in \hat{F}$, and $\alpha \in [0,1]$, $\alpha f + (1-\alpha) g \in \hat{F}$ is the act $h \in \hat{F}$ defined by $h(s) = \alpha f(s) + (1-\alpha) g(s)$ for any $s \in S$. Then, $\hat{F}$ is a convex subset in a linear space.

First, we assume that both $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ satisfy the following basic axioms:

(A.1) (**Weak order**) For all $\hat{F} \in \mathcal{F}$, the preference relation $\succcurlyeq_{\hat{F}}$ is transitive and complete.

(A.2) (**Archimedean**) For all $\hat{F} \in \mathcal{F}$ and $f, g, h \in \hat{F}$, if $f \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} h$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha) h \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} \beta f + (1-\beta) h$.

(A.3) (**Monotonicity**) For all $\hat{F} \in \mathcal{F}$ and $f, g \in \hat{F}$, if $f(s) \succcurlyeq_{\hat{F}} g(s)$ for all $s \in C^F$, then $f \succcurlyeq_{\hat{F}} g$.

[9] Similar to (9), it is not difficult to construct an example in which s_1^1, s_2^1, s_3^1, and s_4^1 are unambiguous, but Π_1 does not preserve the relative likelihoods of μ.

(A.4) (**Nondegeneracy**) For all $\hat{F} \in \mathcal{F}$, there are $f, g \in \hat{F}$ such that $f \succ_{\hat{F}} g$.

To capture our idea that the DM's behavior might change fundamentally as awareness grows, we allow for $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ to belong to different families of preferences. In particular, we assume that the initial preference relation $\succsim_{\hat{F}}$ satisfies the Independence Axiom:

(A.5) (**Independence**) For all $f, g, h \in \hat{F}$, and $\alpha \in (0, 1]$, $f \succsim_{\hat{F}} g$ if and only if $\alpha f + (1 - \alpha)h \succsim_{\hat{F}} \alpha g + (1 - \alpha)h$.

That is, the initial preference relation $\succsim_{\hat{F}}$ is assumed to admit the SEU representation (4) with respect to a unique probability distribution μ on S and an expected utility functional $U : \Delta(C) \to \mathbb{R}$ (e.g., see Anscombe and Aumann, 1963).

So far, we only request that $\succsim_{\hat{F}_1}$ satisfies axioms (A.1) through (A.4). Therefore, the extended preference $\succsim_{\hat{F}_1}$ might violate the Independence Axiom allowing for ambiguity.

4.2 MEU and Unambiguity Consistency

In this subsection, we obtain the MEU representation of the extended preference $\succsim_{\hat{F}_1}$ that is an unambiguity consistent extension of the initial SEU preference $\succsim_{\hat{F}}$.

We introduce an axiom, called *Negative Unambiguity Independence* (henceforth, NUI). Roughly speaking, NUI specifies how the new and old acts are evaluated by the extended preference $\succsim_{\hat{F}_1}$. The axiom has two parts. The first part states that if a new act f is weakly preferred to a lottery q, then mixing the act with another act g is at least as good as mixing the lottery with the new act g. The second part directly connects the new acts with the old, binary acts (called bets). Specifically, it requires that bets on the events that correspond to original states cannot be used to hedge against ambiguity of the new acts.[10]

Recall that, for each initial state $s \in S$, E_s denotes the event in S_1 which corresponds to $s \in S$.[11] A bet on E_s is an act $p_{E_s}r$ that yields a lottery p when E_s obtains and q otherwise.

We can now state NUI formally.

(A.6) (**Negative Unambiguity Independence (NUI)**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0, 1]$,

$$\text{if} \quad f \succsim_{\hat{F}_1} q, \quad \text{then} \quad \alpha f + (1 - \alpha)g \succsim_{\hat{F}_1} \alpha q + (1 - \alpha)g,$$

[10] In the context of the patient example, the second part of our axiom can be stated in the following way: the evaluation of the old treatment A is independent of D when evaluating the new treatment C, which is a combination of A and another new treatment D.

[11] When a new act is discovered, E_s is the set of all new states that are obtained from s by extending it by a consequence c in C. Formally, $E_s = \{s^1 \in S_1 : \exists c \in C \text{ s.t. } s^1 = (s, c)\}$. When a new consequence is discovered, E_s is the original state itself, i.e., $E_s = \{s\} = \{s^1\}$ for some $s^1 \in S_1$.

and when $g = p_{E_s} r$ for some $s \in S$ and $p, r \in \Delta(C_1)$,

$$\text{if } \; f \sim_{\hat{F}_1} q, \;\; \text{then } \; \alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g.$$

The spirit of our axiom is reminiscent of Negative Certainty Independence axiom introduced by Dillenberger (2010) and used by Cerreia-Vioglio et al. (2015) to characterize the Cautious Expected Utility theory in the context of choice under risk.[12] However, in our setup, NUI has different behavioral implications since we allow for ambiguity.

In the context of growing awareness, NUI has two behavioral consequences: First, the axiom guarantees that the extended preference relation admits a MEU representation. Second, NUI implies that all the events that correspond to the original states are unambiguous.

We can now formalize our main representation theorem.

Theorem 1. *Let $\succcurlyeq_{\hat{F}}$ by an initial preference from $\mathcal{D} = (C, F, S, \hat{F})$ and $\succcurlyeq_{\hat{F}_1}$ be an extended preference from $\mathcal{D}_1 = (C_1, F_1, S_1, \hat{F}_1)$. Then, the following two statements are equivalent:*

(i) $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and NUI.

(ii) There exist a non-constant and affine function $U : \Delta(C) \to \mathbb{R}$, and a probability measure $\mu \in \Delta(S)$, such that for all $f, g \in \hat{F}$:

$$f \succcurlyeq_{\hat{F}} g \;\; \textit{if an only if} \;\; \sum_{s \in S} U\big(f(s)\big)\mu(s) \geqslant \sum_{s \in S} U\big(g(s)\big)\mu(s);$$

and there exist a non-constant and affine function $U_1 : \Delta(C_1) \to \mathbb{R}$, and a nonempty, convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$, such that for all $f, g, \in \hat{F}_1$:

$$f \succcurlyeq_{\hat{F}_1} g \;\; \textit{if an only if} \;\; \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) \geqslant \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(g(s^1)\big)\pi(s^1).$$

Moreover, U and U_1 are unique up to a positive linear transformation, μ and Π_1 are unique, and for every $s \in S$, the corresponding event $E_s \subset S_1$ is unambiguous, i.e.,

$$\pi\big(E_s\big) = \tilde{\pi}\big(E_s\big) \;\; \textit{for all} \;\; \pi, \tilde{\pi} \in \Pi_1. \tag{11}$$

[12] Riella (2015) also uses the Negative Certainty Independence axiom in the context of choice under uncertainty. In particular, he extends the main result of Cerreia-Vioglio et al. (2015) and obtains a single-prior expected multiple-utility representation for incomplete preferences. The idea behind NUI is also similar to the Caution axiom of Gilboa et al. (2010). In their setup, the Caution axiom connects two preferences, called objective and subjective rationality relations. The former relation admits a MEU representation while the latter relation admits a representation à la Bewley (2002).

Theorem 1 characterizes the SEU and MEU representations of $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ and simultaneously establishes that the extended preference $\succsim_{\hat{F}_1}$ preserves unambiguity of $\succsim_{\hat{F}}$ as awareness grows.

Remark 1. Unambiguity Consistency has a stronger behavioral implication in the context of discovering new consequences. In particular, when a new consequence $\bar{c}$ is discovered, the event $E(S) = \cup_{s\in S} E_s$ in S_1 corresponds to the original state space S whereas $S_1 \setminus E(S)$ is the set of newly discovered states. In this context, the extended MEU preference admits an additive decomposition across the unambiguous event $E(S)$ and its complement $S_1 \setminus E(S)$. This observation is formally stated in the following corollary.

Corollary 1. Suppose that $C_1 = C \cup \{\bar{c}\}$ and $F_1 = F_{\bar{c}}$. Let $\succsim_{\hat{F}}$ be a SEU preference and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U_1) as in Theorem 1. Then, there are $\delta \in (0,1)$, $\tilde{\mu} \in \Delta(S)$, and a nonempty, convex and compact set $\tilde{\Pi} \subset \Delta(S_1 \setminus E(S))$ such that for any $f \in \hat{F}_1$,

$$\min_{\pi\in\Pi_1} \sum_{s^1\in S_1} U_1\big(f(s^1)\big)\pi(s^1) = \delta\Big(\sum_{s\in S}\tilde{\mu}(s)U_1\big(f(E_s)\big)\Big) + (1-\delta)\Big(\min_{\tilde{\pi}\in\tilde{\Pi}} \sum_{s^1\in S_1\setminus E(S)} U_1\big(f(s^1)\big)\tilde{\pi}(s^1)\Big).$$

Under Unambiguity Consistency, the DM who discovers a new consequence shifts $(1-\delta)$ of the original probability mass to the set of newly discovered states, $S_1 \setminus E(S)$. In other words, $(1-\delta)$ is the subjective probability that one of the newly discovered states will occur. However, the DM might not know how to "split" the probability mass $(1-\delta)$ across the newly discovered states in $S_1 \setminus E(S)$ and thus she may perceive the new states as ambiguous.

However, when new acts are discovered, Unambiguity Consistency does not imply that $\succsim_{\hat{F}_1}$ is additively separable across the unambiguous events $\{E_s\}_{s\in S}$.[13]

Notice that under Unambiguity Consistency, the DM solely inherits unambiguity property of her original beliefs in response to growing awareness. The old and new beliefs might be unrelated. To link μ and Π_1 or U and U_1, additional axioms are required.

To ensure that risk attitudes are not affected by awareness (i.e., $U = U_1$ on $\Delta(C)$), we impose an axiom called *Invariant Risk Preferences*.[14] It requires that the DM's rankings of lotteries remain the same at any awareness level. Formally,

(A.7) (**Invariant Risk Preferences**) For all $p, q \in \Delta(C)$, $p \succsim_{\hat{F}} q$ if and only if $p \succsim_{\hat{F}_1} q$.

By requesting that $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ jointly satisfy (A.7), we get the following lemma.

[13] When new acts are discovered, the additive decomposition of the extended MEU preference is obtained in Theorem 4 under a stronger version of NUI.

[14] The axiom was introduced by Karni and Vierø (2013) in their axiomatization of reverse Bayesianism.

Lemma 1. *Let $\succsim_{\hat{F}}$ be a SEU preference with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U_1). If $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ jointly satisfy Invariant Risk Preferences, then there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $U(p) = \alpha\, U_1(p) + \beta$ for any $p \in \Delta(C)$.*

As a consequence, when $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ jointly satisfy Invariant Risk Preferences, in addition to the axioms of Theorem 1, then the extended MEU preference $\succsim_{\hat{F}_1}$ preserves both unambiguity and the risk attitude of the initial SEU preference $\succsim_{\hat{F}}$.[15]

4.3 MEU and Likelihood Consistency

In this subsection, we behaviorally characterize Likelihood Consistency. In order to do that, we impose another axiom called *Binary Awareness Consistency* (BAC) in addition to a weak version of NUI. Recall that, for any $f, g \in \hat{F}$ and $E \subseteq S$, $f_{-E}\, g$ is the act in $\hat{F}$ that returns $g(s)$ in state $s \in S$ and $f(s')$ in state $s' \in S \setminus E$. BAC directly connects the initial preference $\succsim_{\hat{F}}$ and the extended preference $\succsim_{\hat{F}_1}$ in the following way.

(A.8) (**Binary Awareness Consistency (BAC)**) For all $p, p', q, q', r \in \Delta(C)$ and all $s \in S$,

$$(p_{-s}\, q) \succsim_{\hat{F}} (p'_{-s}\, q') \;\text{ if and only if }\; \big(r_{-E(S)}(p_{-E_s}\, q)\big) \succsim_{\hat{F}_1} \big(r_{-E(S)}(p'_{-E_s}\, q')\big).$$

Roughly speaking, BAC requires that rankings over the old binary acts $(p_{-s}\, q)$ and $(p'_{-s}\, q')$ are not affected by growing awareness. Since $E(S) = S_1$ in the context of discovering new acts, $\big(r_{-E(S)}(p_{-E_s}\, q)\big)$ is a "projection" of the old act $(p_{-s}\, q)$ on $\hat{F}_1$. When the new discovery is a consequence, BAC is also reminiscent of the Sure-Thing Principle constrained to binary acts. We also weaken NUI in the following way.

(A.9) (**Weak Negative Unambiguity Independence (WNUI)**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0,1]$,

$$\text{if }\; f \succsim_{\hat{F}_1} q, \;\text{ then }\; \alpha f + (1-\alpha) g \succsim_{\hat{F}_1} \alpha q + (1-\alpha) g,$$

and when $g \in \Delta(C_1)$,

$$\text{if }\; f \sim_{\hat{F}_1} q, \;\text{ then }\; \alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g.$$

Our second representation result is stated below.

[15] Recently, Ma and Schipper (2017) run an experiment in which subjects make risky choices under different levels of awareness. They found no evidence for varying risk attitudes across different levels of awareness.

Theorem 2. *Let $\succcurlyeq_{\hat{F}}$ be an initial preference from $\mathcal{D} = (C, F, S, \hat{F})$ and $\succcurlyeq_{\hat{F}_1}$ be an extended preference from $\mathcal{D}_1 = (C_1, F_1, S_1, \hat{F}_1)$. Then, the following two statements are equivalent:*

(i) *$\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and WNUI, and $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy BAC.*

(ii) *There exist a non-constant and affine function $U : \Delta(C_1) \to \mathbb{R}$, and a probability measure $\mu \in \Delta(S)$, such that for all $f, g \in \hat{F}$:*

$$f \succcurlyeq_{\hat{F}} g \quad \text{if and only if} \quad \sum_{s \in S} U\big(f(s)\big)\mu(s) \geqslant \sum_{s \in S} U\big(g(s)\big)\mu(s);$$

and there exists a nonempty, convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$, such that for all $f, g, \in \hat{F}_1$:

$$f \succcurlyeq_{\hat{F}_1} g \quad \text{if and only if} \quad \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U\big(f(s^1)\big)\pi(s^1) \geqslant \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U\big(g(s^1)\big)\pi(s^1).$$

Moreover, U is unique up to a positive linear transformation, μ and Π_1 are unique, and $\succcurlyeq_{\hat{F}_1}$ is a likelihood consistent extension of $\succcurlyeq_{\hat{F}}$, i.e., for all $s, s' \in S$ and all $\pi \in \Pi_1$,

$$\frac{\mu(s)}{\mu(s')} = \frac{\pi\big(E_s\big)}{\pi\big(E_{s'}\big)}. \tag{12}$$

Theorem 2 characterizes the representations (4) and (5) with $U = U_1$ on $\Delta(C)$, and simultaneously establishes that $\succcurlyeq_{\hat{F}_1}$ is a likelihood consistent extension of $\succcurlyeq_{\hat{F}}$.

Remark 2. Theorem 2 shows that BAC implies Invariant Risk Preferences. This is because BAC requires that the rankings over old acts including constant acts are preserved. Moreover, if all the newly discovered states are unambiguous, Theorem 2 provides an alternative foundation of reverse Bayesianism of Karni and Vierø (2013, Theorems 1-2).[16]

Interestingly, depending on whether the discovery is an act or consequence, Theorem 2 might have different implications about Unambiguity Consistency. In the context of discovering new acts, Theorem 2 also characterizes Unambiguity Consistency. Formally,

[16] Karni and Vierø (2013) require two axioms in addition to SEU axioms; Invariant Risk Preferences and Projection Consistency in the context of discovering acts and Awareness Consistency in the context of discovering consequences. Our BAC is weaker than both Awareness Consistency and Projection Consistency. In the online appendix, we discuss implications of Projection Consistency and Awareness Consistency under MEU axioms.

Corollary 2. Suppose that $C_1 = C$ and $F_1 = F \cup \{\bar{f}\}$. Let $\succsim_{\hat{F}}$ be a SEU preference with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U) as in Theorem 2. Then $\succsim_{\hat{F}_1}$ is an unambiguity and likelihood consistent extension of $\succsim_{\hat{F}}$, i.e., for all $s \in S$ and all $\pi, \tilde{\pi} \in \Pi_1$,

$$\mu(s) = \pi\big(E_s\big) = \tilde{\pi}\big(E_s\big). \tag{13}$$

Therefore, in the context of discovering new acts, the extended MEU preference inherits *all* the properties of the initial SEU preferences, including the DM's old beliefs as well as her old risk attitude. For this reason, Theorem 2 establishes a behavioral foundation of the theory of *generalized reverse Bayesianism* in the family of MEU preferences.

However, when the discovery is a consequence, the extended preference $\succsim_{\hat{F}_1}$ in Theorem 2 is not necessarily unambiguity consistent. The following corollary of Theorem 2 shows that events in $\{E_s\}_{s \in S}$ (even $E(S) = \cup_{s \in S}$) are possibly ambiguous.

Corollary 3. Suppose that $C_1 = C \cup \{\bar{c}\}$ and $F_1 = F_{\bar{c}}$. Let $\succsim_{\hat{F}}$ be a SEU preference with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U) as in Theorem 2. Then there is a set $[\underline{\delta}, \overline{\delta}] \times \tilde{\Pi} \subset [0,1] \times \Delta(S_1 \setminus S)$ such that for any $f \in \hat{F}_1$,

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U\big(f(s^1)\big)\pi(s^1) = \min_{(\delta, \tilde{\pi}) \in [\underline{\delta}, \overline{\delta}] \times \tilde{\Pi}} \left\{ \delta \sum_{s \in S} \mu(s) U\big(f(s)\big) + (1-\delta) \sum_{s^1 \in S_1 \setminus S} U\big(f(s^1)\big)\tilde{\pi}(s^1) \right\}.$$

We conclude this section by summarizing our characterization results in Table 4. Unambiguity Consistency is characterized in Theorem 1 by NUI. Likelihood Consistency is characterized in Theorem 2 by WNUI and BAC. Since Likelihood Consistency implies Unambiguity Consistency when a new act is discovered, Theorem 2 also characterizes both consistency notions. Finally, Theorems 1 and 2 characterize Unambiguity and Likelihood Consistency by NUI and BAC in the context of discovering new consequences.

	new act	new consequence
Unambiguity Consistency	Theorem 1 (NUI)	Theorem 1 (NUI)
Likelihood Consistency	Theorem 2 (WNUI+BAC)	Theorem 2 (WNUI+BAC)
Unambiguity and Likelihood	Theorem 2 (WNUI+BAC)	Theorems 1, 2 (NUI+BAC)

Table 4: Summary of Characterization Results

5 Comparative Ambiguity under Growing Awareness

In this section, we explore how two distinct discoveries might affect ambiguity. In other words, we establish a comparative notion of ambiguity allowing an outside observer (a researcher or a social planner) to elicit which discovery induces "more" ambiguity aversion. For example, think of a firm that plans to introduce several new products to the market. Since new discoveries may lead to ambiguity aversion among potential customers, and this may lower their demand, the firm wants to know which product causes less ambiguity.

Our goal is to extend the comparative notion of ambiguity of Ghirardato and Marinacci (2002) to decision problems under growing awareness. The idea is to compare two preference relations in the following sense: "*If a DM prefers a constant act to an ambiguous one, a more ambiguity averse one will do the same.*" However, in Ghirardato and Marinacci (2002), each preference is associated with the same decision problem (i.e., the same set of acts). In our setup, we compare two extended preferences, each corresponding to a new, distinct discovery.

The main challenge is that two different discoveries generate distinct expansions of the original decision problem. To make the comparative notion applicable to our setup, few modifications are necessary. Discoveries of distinct acts or consequences require two separate analyses. Since the two analyses are similar, we focus here on discoveries of two distinct acts.

Let us consider two expanded decision problems $\mathcal{D}_1$ and $\mathcal{D}_2$ be generated by discoveries of two different acts, i.e., $F_1 = F \cup \{\bar{f}\}$ and $F_2 = F \cup \{f^*\}$ such that $\bar{f} \neq f^*$. Suppose that a researcher wants to know which discovery induces more ambiguity aversion. Below, we first illustrate that the standard method of ranking sets of priors with respect to set inclusion is still intuitive in the context of growing awareness. Then we define a bijection that formally connects the new state spaces S_1 and S_2 and the new sets of conceivable acts $\hat{F}_1$ and $\hat{F}_2$.

Set Inclusion: To illustrate the main idea, consider a DM facing the decision problem $\mathcal{D}_1$ with $F_1 = \{f_1, f_2, \bar{f}\}$ and S_1 depicted in Table 1. Let $\succsim_{\hat{F}_1}$ be her extended MEU preference with (Π_1, U) such that

$$\Pi_1 = \Big\{\pi \in \Delta(S_1) : \pi(s^1_i, s^1_{i+4}) = \frac{1}{4} \text{ for all } i = 1, \dots, 4\Big\}. \tag{14}$$

Suppose that there is another DM facing the decision problem $\mathcal{D}_2$ where $F_2 \equiv \{f_1, f_2, f^*\}$ and f^* is a newly discovered act such that $\bar{f} \neq f^*$. The discovery of f^* induces the new state space $S_2 \equiv C^{F_2}$ as presented in Table 5. Let $\succsim_{\hat{F}_2}$ be his extended MEU preference with (Π_2, U) such that

$$\Pi_2 = \Big\{\pi \in \Delta(S_2) : \pi(s^2_1) = \pi(s^2_5) = \frac{1}{8} \text{ and } \pi(s^2_i, s^2_{i+4}) = \frac{1}{4} \text{ for all } i = 2, 3, 4\Big\}. \tag{15}$$

$F_2 \setminus S_2$	s^2_1	s^2_2	s^2_3	s^2_4	s^2_5	s^2_6	s^2_7	s^2_8
f_1	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_2	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
f^*	c_2	c_2	c_2	c_2	c_1	c_1	c_1	c_1

Table 5: Extended state space: new act f^*

Discoveries of $\bar{f}$ and f^* induce two extended preferences, $\succcurlyeq_{\hat{F}_1}$ with (Π_1, U) and $\succcurlyeq_{\hat{F}_2}$ with (Π_2, U). Our goal is to show that a comparison of the sets of priors Π_1 and Π_2 with respect to their sets inclusion determines whether $\succcurlyeq_{\hat{F}_1}$ is more (or less) ambiguity averse than $\succcurlyeq_{\hat{F}_2}$.

From Equation (15), we can conclude that each state in $E_{s_1} = \{s^2_1, s^2_5\}$ is unambiguous with probability $\frac{1}{8}$ for the DM with $\succcurlyeq_{\hat{F}_2}$. However, by Equation (14), for the DM with $\succcurlyeq_{\hat{F}_1}$, both states in $E_{s_1} = \{s^1_1, s^1_5\}$ are ambiguous with probabilities ranging from 0 to $\frac{1}{4}$. Moreover, both DMs have the same beliefs on all states in events E_{s_2} through E_{s_4}. Intuitively, the researcher can infer that $\succcurlyeq_{\hat{F}_1}$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$. In fact, although Π_1 and Π_2 belong to different state spaces, the researcher can compare Π_1 and Π_2 with respect to set inclusion. More formally, in order to compare Π_1 and Π_2 by set inclusion, we need to show that S_1 and S_2 are "semantically" equivalent.

Semantic Equivalence: Notice that discoveries of $\bar{f}$ and f^* expand the original state space S differently. However, since $C_1 = C_2 = \{c_1, c_2\}$, the expanded state spaces $S_1 \equiv C^{F_1}$ and $S_2 \equiv C^{F_2}$ are "semantically" equivalent. Consider events $\{s^1_1, s^1_5\}$ in S_1 and $\{s^2_1, s^2_5\}$ in S_2. Both events correspond to the original state $s_1 \in S$. Moreover, they contain exactly the same states since $s^1_1 = (c_1, c_1, c_1) = s^2_5$ and $s^1_5 = (c_1, c_1, c_2) = s^2_1$. In other words, state s^1_1 is "semantically" equivalent to state s^2_5. Likewise, s^1_5 is "semantically" equivalent to s^2_1.

Recall, when a new act is discovered, each original state s is extended by adding a consequence $c \in C$ (i.e., (s, c)). One can thus identify each state s^1 in S_1 with a state s^2 in S_2 such that both states s^1 and s^2 are generated by extending some original state s in S by adding the same consequence c in C (i.e., $s^1 = s^2 = (s, c)$).

Formally, define a bijection $\varphi : S_1 \to S_2$ such that $\varphi(s^1) = s^2 \in S_2$ if and only if $s^1 = s^2 = (s, c)$ for some $s \in S$ and $c \in C$. States s^1 and $\varphi(s^1) = s^2$ are called *semantically equivalent*. Notice that due to the construction of the extended state spaces S_1 and S_2, the bijection φ is well-defined and unique.

Given the bijection φ, each act f in $\hat{F}_1$ can be associated with an act f_φ in $\hat{F}_2$ such that f_φ ascribes the lotteries associated with f to the semantically equivalent states. More specifically, if q is a lottery that act f returns in state $s^1 \in S_1$ (i.e., $f(s^1) = q$), then also act f_φ returns the lottery q in the semantically equivalent state $\varphi(s^1)$ (i.e., $f(\varphi(s^1)) = q$). Thus, given the bijection φ, for each act $f \in \hat{F}_1$ there is a *semantically equivalent act* f_φ in $\hat{F}_2$.

Finally, we can define our comparative notion of ambiguity aversion under growing awareness after discovering new but distinct acts.

Definition 5 (Comparative Ambiguity Aversion). Let $\succsim_{\hat{F}_1}$ be a preference relation on $\hat{F}_1$ and $\succsim_{\hat{F}_2}$ be a preference relation on $\hat{F}_2$ where $F_1 = F \cup \{\bar{f}\}$ and $F_2 = F \cup \{f^*\}$. Given a bijection $\varphi : S_1 \to S_2$, $\succsim_{\hat{F}_1}$ is said to be *more ambiguity averse than* $\succsim_{\hat{F}_2}$ if, for all lotteries $p, q \in \Delta(C)$ and an act $f \in \hat{F}_1$, it is true that

$$p \succsim_{\hat{F}_2} q \text{ if and only if } p \succsim_{\hat{F}_1} q, \text{ and} \tag{16}$$

$$p \succsim_{\hat{F}_2} f \text{ implies } p \succsim_{\hat{F}_1} f_\varphi \text{ and } p \succ_{\hat{F}_2} f \text{ implies } p \succ_{\hat{F}_1} f_\varphi. \tag{17}$$

Let Π_1^φ be the set of priors in S_2 that is semantically equivalent to Π_1. That is, $\Pi_1^\varphi \equiv \{\tilde{\pi} \in \Delta(S_2) : \exists \pi \in \Pi_1 \text{ s.t. } \pi(s^1) = \tilde{\pi}(\varphi(s^1))\ \forall s^1 \in S_1\}$. Now, our next result can be stated.

Theorem 3. *Suppose that* $\succsim_{\hat{F}_1}$ *is a MEU extension of* $\succsim_{\hat{F}}$ *to* $\hat{F}_1$ *with* (Π_1, U)*, and that* $\succsim_{\hat{F}_2}$ *is a MEU extension of* $\succsim_{\hat{F}}$ *to* $\hat{F}_2$ *with* (Π_2, U) *where* $F_1 = F \cup \{\bar{f}\}$ *and* $F_2 = F \cup \{f^*\}$*. Then,* $\succsim_{\hat{F}_1}$ *is more ambiguity averse than* $\succsim_{\hat{F}_2}$ *if and only if* $\Pi_2 \subseteq \Pi_1^\varphi$.

Theorem 3 shows that the comparative notion of ambiguity can naturally be extended choice problems with growing awareness. Since two different decision problems $\mathcal{D}_1$ and $\mathcal{D}_2$ – generated by distinct discoveries – can be connected via semantically equivalent states and acts, the comparative notion of ambiguity under growing awareness can be characterized in a standard way by comparing the sets of priors Π_1^φ and Π_2 with respect to set inclusion.

6 Parametric Approach

In this section, we characterize a parametric version of our MEU representation that satisfies Unambiguity Consistency and Likelihood Consistency. The suggested parametric MEU representation makes our theory tractable for broad economic applications and empirical studies. We also show that our parametric MEU model is particularly convenient for comparative statics analysis.

6.1 Parametric MEU for Discoveries of New Acts

Consider a choice situation in which the DM discovers a new act $\bar{f}$. Under Unambiguity Consistency and Likelihood Consistency, the extended MEU preference $\succsim_{\hat{F}_1}$ with respect to a set of priors Π_1 inherits the old beliefs, i.e., $\mu(s) = \pi(E_s)$ for each $s \in S$ and $\pi \in \Pi_1$.[17] In the previous sections, we have argued that the DM perceives ambiguity because she does not know how to "split" her old belief $\mu(s)$ across the newly discovered states of the event E_s. In this section, we suggest the following procedural way to "split" $\mu(s)$.

Suppose that the DM comes up with a probability measure $\eta_s \in \Delta(E_s)$ on the new states in E_s. However, the DM might not be confident that η_s truthfully describes the likelihoods of the newly discovered states. Therefore, the DM might "distort" η_s by a parameter $\alpha_s \in [0, 1]$. For a given $\alpha_s \in [0, 1]$ and $\eta_s \in \Delta(E_s)$, the DM forms her beliefs over E_s defined as a convex mixture between η_s and the set of all possible probability measures $\Delta(E_s)$. That is,

$$\Pi_{E_s}^{(\eta_s,\alpha_s)} = \alpha_s\{\eta_s\} + (1-\alpha_s)\Delta(E_s). \tag{18}$$

The parameter α_s might be interpreted as the DM's degree of confidence in η_s. When $\alpha_s = 1$, the DM is confident that η_s accurately represents the likelihoods of the new states. When $\alpha_s = 0$, she is not confident at all and her beliefs are represented by the set of all priors $\Delta(E_s)$. In this case, the DM is said be *completely ignorant.*

For example, in our patient example, when the patient is told by her doctor that the probability that factor z is good is 0.7; i.e., $\eta_{s_1} = (\eta(s_1^1), \eta(s_5^1)) = (0.7, 0.3)$ is a probability measure over $E_{s_1} = \{s_1^1, s_5^1\}$. When her degree of confidence is α_1 in E_{s_1},

$$\Pi_{E_{s_1}}^{(\eta_1,\alpha_1)} = \alpha_1\,\eta_1 + (1-\alpha_1)\,\Delta(E_{s_1}) = \Big\{\big(\pi(s_1^1), \pi(s_5^1)\big) : \pi(s_1^1) \in [\frac{\alpha_1}{2}, 1 - \frac{\alpha_1}{2}]\Big\}. \tag{19}$$

We allow the DM's belief η_s on E_s as well as her degree of confidence α_s in η_s to vary across the events in $\{E_s\}_{s\in S}$. In other words, different degrees of confidence α_s might reflect her perception that each original state $s \in S$ is affected differently by the discovery of act $\bar{f}$. For example, suppose that the patient also considers $\eta_4 = (0.7, 0.3)$ on E_{s_4}. However, the patient might be more cautious about factor z when she considers event E_{s_4}, in which both factors x, y are bad, as compared to event E_{s_1} in which both factors x and y are good. As a result, her degree of confidence α_1 might be larger than α_4.

To characterize our parametric MEU model, we need to strengthen NUI as follows.

(A.10) (**Extreme Negative Unambiguity Independence (Extreme NUI)**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0, 1]$,

$$\text{if} \;\; f \succcurlyeq_{\hat{F}_1} q, \;\; \text{then} \;\; \alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)g,$$

and if for each $s \in S$, there is $s^1 \in E_s$ such that $f(\tilde{s}^1) \succcurlyeq_{\hat{F}_1} f(s^1)$ and $g(\tilde{s}^1) \succcurlyeq_{\hat{F}_1} g(s^1)$

[17] Recall, E_s is the event corresponding to the old state $s \in S$ (i.e., $E_s = \{s^1 \in S_1 : \exists c \in C \text{ s.t. } s^1 = (s, c)\}$).

for all $\tilde{s}^1 \in E_s$, then

$$f \sim_{\hat{F}_1} q \text{ implies } \alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g.$$

Recall that NUI requires that bets $g = p_{E_s} r$ cannot be used to hedge against ambiguity. Notice that for any act f and a bet $g = p_{E_s} r$, f and g satisfy the following property in Extreme NUI: for any $s \in S$, there exists $s^1 \in E_s$ with $f(\tilde{s}^1) \succcurlyeq f(s^1)$, $g(\tilde{s}^1) \succcurlyeq g(s^1)$ for all $\tilde{s}^1 \in E_s$. Therefore, Extreme NUI is stronger than NUI.

Under Extreme NUI, the extended MEU preference satisfies Unambiguity Consistency and Likelihood Consistency. Moreover, the extended MEU representation admits an additive decomposition across the events $\{E_s\}_{s \in S}$.

Theorem 4. *Let $\succcurlyeq_{\hat{F}}$ by an initial preference from $\mathcal{D} = (C, F, S, \hat{F})$ and $\succcurlyeq_{\hat{F}_1}$ be an extended preference from $\mathcal{D}_1 = (C_1, F_1, S_1, \hat{F}_1)$. Suppose that $F_1 = F \cup \{\bar{f}\}$ and $C_1 = C$. Then, the following two statements are equivalent:*

(i) $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and Extreme NUI, and $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy BAC.

(ii) There exist a non-constant and affine function $U : \Delta(C) \to \mathbb{R}$, and a probability measure $\mu \in \Delta(S)$, such that for all $f, g \in \hat{F}$:

$$f \succcurlyeq_{\hat{F}} g \iff \sum_{s \in S} U\big(f(s)\big)\mu(s) \geqslant \sum_{s \in S} U\big(g(s)\big)\mu(s);$$

and there exists $\{(\eta_s, \alpha_s)\}_{s \in S} \in \prod_{s \in S} (\Delta(E_s) \times [0,1])$ such that for all $f, g, \in \hat{F}_1$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if

$$(20) \quad \sum_{s \in S} \mu(s) \Big(\min_{\pi \in \Pi_{E_s}^{(\eta_s, \alpha_s)}} \sum_{s^1 \in E_s} U\big(f(s^1)\big)\pi(s^1)\Big) \geqslant \sum_{s \in S} \mu(s) \Big(\min_{\pi \in \Pi_{E_s}^{(\eta_s, \alpha_s)}} \sum_{s^1 \in E_s} U\big(g(s^1)\big)\pi(s^1)\Big),$$

where

$$\Pi_{E_s}^{(\eta_s, \alpha_s)} = \alpha_s \{\eta_s\} + (1-\alpha_s)\Delta(E_s).$$

Consistent with Corollary 2, Theorem 4 (specially, Equation (20)) implies that $\succcurlyeq_{\hat{F}_1}$ is an unambiguity and likelihood consistent extension of the original preference $\succcurlyeq_{\hat{F}}$. Moreover, the DM's new beliefs Π_1 on the expanded state space S_1 take the following form:

$$(21) \qquad \Pi_1 = \times_{s \in S}\, \mu(s)\, \Pi_{E_s}^{(\eta_s, \alpha_s)} = \times_{s \in S} \Big\{ \mu(s)\, \alpha_s \{\eta_s\} + \mu(s)(1-\alpha_s)\Delta(E_s) \Big\}.$$

Remark 3. The idea behind the second part of Extreme NUI is adopted from Eichberger and Kelsey (1999). The authors characterize the concept of *E-capacities*. Since E-capacities are convex, Choquet expected utility preferences with respect to E-capacities constitute a special case of MEU preferences (see Schmeidler, 1989). Our parametric MEU model is more general and derived by imposing different axioms in a different choice theoretic context.

Remark 4. The parametric model in Theorem 4 has an equivalent representation. It can be shown that there exist a probability measure μ on $\{E_s\}_{s\in S}$, a probability measure $\eta \in \Delta(S_1)$ that agrees with μ, a collection of real numbers $\{\alpha_s \in [0,1]\}_{s\in S}$, and an expected utility functional $U : \Delta(C) \to \mathbb{R}$ such that every act $f \in \hat{F}_1$ is evaluated via the functional:

$$V(f) = \sum_{s\in S}\left[\sum_{s^1\in E_s} \alpha_s\, U\big(f(s^1)\big)\,\eta(s^1) + (1-\alpha_s)\,\mu(E_s)\,m_s(f)\right], \tag{22}$$

where $m_s(f) := \min\big\{U(f(s^1)) : s^1 \in E_s\big\}$, the smallest expected utility of f on E_s.

6.2 Comparative Statics

In this subsection, we apply our parametric MEU representation to derive a parametric version of comparative ambiguity aversion.

In light of Theorem 4, the extended MEU preference $\succcurlyeq_{\hat{F}_1}$ is entirely characterized by $\big(U, \{(\eta_s, \alpha_s)\}_{s\in S}\big)$. Now suppose that there are two distinct discoveries, i.e., $F_1 = F \cup \{\bar{f}\}$ and $F_2 = F \cup \{f^*\}$. Then, the comparative statics across different sources of growing awareness reduces to comparing the parameters $\{(\eta_s^1, \alpha_s^1)\}_{s\in S}$ and $\{(\eta_s^2, \alpha_s^2)\}_{s\in S}$. Our next result shows that comparing the respective degrees of confidence $(\alpha_s^1)_{s\in S}$ and $(\alpha_s^2)_{s\in S}$ is essentially sufficient to establish which discovery, $\bar{f}$ or f^*, induces more ambiguity aversion.

Theorem 5. *Suppose $F_1 = F \cup \{\bar{f}\}$ and $F_2 = F \cup \{f^*\}$. Let $\succcurlyeq_{\hat{F}_1}$ be a (parametric) MEU preference with $(U, \{(\eta_s^1, \alpha_s^1)\}_{s\in S})$ and $\succcurlyeq_{\hat{F}_2}$ be a (parametric) MEU preference with $(U, \{(\eta_s^1, \alpha_s^1)\}_{s\in S})$. Then, $\succcurlyeq_{\hat{F}_1}$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$ if and only if*

$$\frac{\alpha_s^2}{\alpha_s^1} \geq \max_{s^1\in E_s}\left\{\frac{\eta_s^1(s^1)}{\eta_s^2(\varphi(s^1))}; \frac{1-\eta_s^1(s^1)}{1-\eta_s^2(\varphi(s^1))}\right\} \text{ for all } s \in S.$$

Theorem 5 has two immediate implications. First, if $\succcurlyeq_{\hat{F}_1}$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$, then $\alpha_s^2 \geq \alpha_s^1$ for all $s \in S$. Second, if $\eta_s^1(s^1) = \eta_s^2(\varphi(s^1))$ for any $s \in S$ and $s^1 \in E_s$, then $\succcurlyeq_{\hat{F}_1}$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$ if and only if $\alpha_s^2 \geq \alpha_s^1$ for all $s \in S$.

6.3 Parametric MEU for Discoveries of New Consequences

Let us conclude this section by briefly discussing a parametric MEU representation in the context of discovering new consequences. Similar to Theorem 4, we can obtain the following parametric representation by slightly modifying Extreme NUI: there exist $\delta, \alpha \in [0,1]$ and $\eta \in \Delta\big(S_1 \setminus E(S)\big)$ such that every act $f \in \hat{F}_1$ is evaluated via the functional:

$$V(f) = \delta\Big(\sum_{s \in S} \mu(s) U(f(E_s))\Big) + (1-\delta)\Big(\min_{\pi \in \Pi^{(\eta,\alpha)}} \sum_{s^1 \in S_1 \setminus E(S)} U\big(f(s^1)\big)\pi(s^1)\Big),$$

where

$$\Pi^{(\eta,\alpha)} = \alpha\,\{\eta\} + (1-\alpha)\,\Delta\big(S_1 \setminus E(S)\big) \text{ and } E(S) = \cup_{s \in S} E_s.$$

Note that $\succcurlyeq_{\hat{F}_1}$ is characterized by a tuple (δ, α, η). With this representation, we can derive a parametric version of comparative ambiguity aversion. Similar to Theorem 5, we can show that $\succcurlyeq_{\hat{F}_1}$ with $(\delta_1, \alpha_1, \eta_1)$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$ with $(\delta_2, \alpha_2, \eta_2)$ if and only if

$$\delta_1 = \delta_2 \text{ and } \frac{\alpha^2}{\alpha^1} \geq \max_{s^1 \in S_1 \setminus E(S)} \left\{ \frac{\eta^1(s^1)}{\eta^2(\varphi(s^1))}; \frac{1-\eta^1(s^1)}{1-\eta^2(\varphi(s^1))} \right\}.$$

7 Related Literature

In this section, we overview the related literature on choice under unawareness. Following Schipper (2014a,b,c), "*unawareness refers to lack of conception rather than lack of information.*" Under lack of information, the DM does not know which conceivable states occur. However, under lack of conception, she cannot even conceive that there are other states.

There are two main approaches on modeling unawareness and growing awareness: the preference-based approach and the epistemic approach. The goal of the preference-based approach is to investigate behavioral implications of unawareness and growing awareness. This approach has been taken by Karni and Vierø (2013, 2015, 2017), Hayashi (2012), and Schipper (2013, 2014b). In this paper, we also follow the preference-based approach.[18]

Since Dekel et al. (1998), economists acknowledge that the standard state space approach used for modeling private information – via a given state space with a partition – cannot capture unawareness. To model unawareness, the state space approach has to be augmented

[18] The epistemic approach develops formal models of unawareness and studies epistemic properties of unawareness. This approach has been taken by Fagin and Halpern (1988), Dekel et al. (1998), Modica and Rustichini (1994, 1999), Modica and Rustichini (1994, 1999), Halpern (2001), Heifetz et al. (2008), Halpern and Rêgo (2009, 2013), Heifetz et al. (2006, 2008), Li (2009), Galanis (2011), Heinsalu (2012), and Piermont (2017) among others. For comprehensive surveys of the epistemic literature and applications of unawareness models to economic theory and game theory, see Schipper (2014a,c).

with a structure that accommodates an expansion of the original state space, referring to growing awareness (e.g., in our setup, S expands to S_1).

Heifetz et al. (2006, 2008) derive an elegant model of unawareness. Instead of one state space, unawareness is accommodated via a lattice of disjoint state spaces. Each space corresponds to one level of awareness. As awareness grows, the DM discovers a new state space with a higher level of awareness. Our model adopts the state space expansion procedure developed by Karni and Vierø (2013) which admits the lattice structure of unawareness. In our setup, discoveries induce a lattice of state spaces ordered by increasing levels of awareness.

The preference-based approach explores how unawareness and growing awareness affect choice under uncertainty. The first rigorous study in this context is Karni and Vierø (2013). As mentioned in previous chapters, the authors develop the theory of reverse Bayesianism that characterizes a belief-consistent evolution of SEU preferences under growing awareness due to discoveries of new acts, new consequences, or links between them.

Karni and Vierø (2015) extend the theory of reverse Bayesianism to a family of probabilistically sophisticated preferences in the sense of Machina and Schmeidler (1992, 1995). Beliefs are still represented by a unique probability measure as in the SEU theory. However, preferences do not need to be linear in probabilities allowing for violations of SEU in the spirit of the Allais paradox.[19]

Hayashi (2012) studies the evolution of subjective probabilities from the point of view of dynamic behavior. In his setup, the state space expansion follows a product structure. By imposing a form of dynamic consistency between choices made before and after a state space expansion, he characterizes a consistent evolution of beliefs in the sense that the marginal distribution of the new belief induced over the old state space coincides with the old belief.

Schipper (2013) characterizes awareness-dependent SEU preferences. He shows that unawareness has a different behavioral meaning than the notion of null events. In particular, a DM is unaware of an event if and only if the event and its complement are null events. However, Schipper's (2013) theory is silent about how beliefs across different levels of awareness may be related to each other.

Karni and Vierø (2017) provide another interesting extension of reverse Bayesianism to choice situations in which a DM is aware of her unawareness. Roughly speaking, the DM anticipates discovery of unknown consequences. She assigns utilities to these unspecified consequences even if these consequences may not even exist. Under the assumption that preferences take the SEU form, they characterize the principle of reverse Bayesianism in

[19] Notice that our approach goes beyond this paradigm. As awareness grows, the DM's preferences take the form of MEU preferences allowing for ambiguity-sensitive behavior. Ambiguity averse behavior is inconsistent with probabilistically sophisticated preferences.

this context. Vierø (2017) generalizes Karni and Vierø (2017) to more than two periods. In particular, she axiomatically characterizes a recursive dynamic model of growing awareness that nests discounted expected utility model and the model of Karni and Vierø (2017).

Grant et al. (2017) provide a model of learning under unawareness in which there is incomplete information about the structure of the state space. A DM learns about the unknown states through sequential experimentation. At the initial stage, the DM is completely ignorant and her beliefs are represented by the set of all priors. The DM's beliefs are successively updated while discovering new acts and new consequences. Grant et al. (2017) and our paper complement each other. We axiomatically characterize two consistency notions for belief updating under growing awareness and ambiguity while Grant et al. (2017) study the underlying stochastic process of learning and belief updating using the imprecise Dirichlet process.

Other related studies focus on situations in which a DM has a limited understanding of the conceivable states or feasible acts rather than being unaware of them. For example, Ahn and Ergin (2010) propose a model of choice under uncertainty in which the DM's beliefs depend on descriptions of relevant contingencies. Descriptions are represented by partitions of a fixed state space. In this context, they derive a partition-dependent expected utility representation. Although their model accommodates situations in which the DM may receive better descriptions through refining the original partition, there are two main differences. First, in their model the refinement process is tacit while in our setup either the original states are refined due to discoveries of new acts or the state space genuinely expands due to discoveries of new consequences. Second, in their model the representation of preferences do not change as the DM's understanding improves. However, in our model, growing awareness may lead the DM to change his behavior and to become ambiguity averse.

Lehrer and Teper (2014) study rules that extend restricted complete SEU preferences (defined over a restricted set of acts) to unrestricted but incomplete preferences (defined over the entire domain of acts). Under the so-called prudent rule, the extended preferences are incomplete à la Bewley (2002).[20] They also discuss how to complete the Bewley representation and, under a modified prudent rule, the completion takes a restricted MEU form.

Similar to us, Lehrer and Teper allow preferences to change as the set of acts expands. Besides that, there are several substantial differences from our approach. First, in their setup the evolution of beliefs is not adressed while our theory characterizes consistent evolution of beliefs and preferences. Second, in their setup the state space is intact. That is, although the set of acts expands, it does not affect the description of the original states. Finally, the

[20]In the model of Bewley, an act f is preferred to another act g if and only if the expected utility of f is greater than the expected utility of g under any probability distribution in the DM's set of priors.

set of priors takes a particular form (the form of complete ignorance) and the existence of such set is not triggered by new discoveries per se. In particular, the set of priors is induced by the initial preference relation; it is the set of probability measures that rationalize the DM's (original) preference over the restricted set of acts.

Alon (2015) derives a choice model in which the DM is aware of her unawareness, which is represented by an imaginary, "unforeseen event," extending the exogenous state space. While evaluating an act, the imaginary event is associated with the worst consequence leading to the worst-case expected utility representation, as a special case of the neo-additive capacity model of Chateauneuf et al. (2007).

8 Conclusion

In this paper, we study how new discoveries cause fundamental changes in people's behavior. In particular, we study how SEU preferences evolve to MEU preferences due to growing awareness. To discipline the effect of growing awareness, we introduce two consistency notions that connect SEU and MEU beliefs and preferences, and we axiomatically characterize them. Moreover, our framework provides a novel interpretation of ambiguity aversion, where ambiguity arises because the DM treats old and new states differently.

We focus on the case where the DM with SEU preferences becomes ambiguity averse. One might be interested in how MEU preferences evolve to different MEU preferences or MEU preferences evolve to SEU preferences due to growing awareness. The former case (i.e., MEU to MEU) is studied in the online appendix in the context of discovering links between acts and consequences. We leave the latter case for future research.

A Appendix: Proofs

A.1 A Useful Lemma for Theorems 1-2

In Theorems 1-2, we need to show that $\succcurlyeq_{\hat{F}_1}$ admits a MEU representation. Since $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4), it is suffices to show that $\succcurlyeq_{\hat{F}_1}$ satisfies the following two key axioms of Gilboa and Schmeidler (1989):

(A.11) (**Certainty Independence**) For all $f, g \in \hat{F}_1$, $c \in C_1$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if $\alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha g + (1-\alpha)c$ for all $\alpha \in (0, 1]$.

(A.12) (**Ambiguity Aversion**) For all $f, g \in \hat{F}_1$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if $\alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} g$ for all $\alpha \in (0, 1]$.

Lemma 2 shows that WNUI implies Certainty Independence and Ambiguity Aversion.

Lemma 2. *If $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI, then it satisfies (A.11)-(A.12).*

Proof of Lemma 2. Suppose $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI. First, we prove that Certainty Independence is satisfied. Take any $f, g \in \hat{F}_1$, $c \in C_1$, and $\alpha \in (0, 1]$ with $f \succcurlyeq_{\hat{F}_1} g$. Moreover, take a lottery $q \in \Delta(C_1)$ such that $g \sim_{\hat{F}_1} q$. Weak NUI implies that for any $\alpha \in [0, 1]$,

$$\text{if } \ f \succcurlyeq_{\hat{F}_1} q, \ \text{ then } \ \alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)c$$

and

$$\text{if } \ g \sim_{\hat{F}_1} q, \ \text{ then } \ \alpha g + (1-\alpha)c \sim_{\hat{F}_1} \alpha q + (1-\alpha)c.$$

Therefore, by Transitivity, $f \succcurlyeq_{\hat{F}_1} g$ implies $\alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha g + (1-\alpha)c$. The opposite direction of Certainty Independence is obvious.

Second, we will prove that Ambiguity Aversion is satisfied. Take any $f, g \in \hat{F}_1$ and $\alpha \in (0, 1]$ with $f \succcurlyeq_{\hat{F}_1} g$. Moreover, take a lottery $q \in \Delta(C_1)$ such that $g \sim_{\hat{F}_1} q$. Weak NUI implies that for any $\alpha \in [0, 1]$,

$$\text{if } \ f \succcurlyeq_{\hat{F}_1} q, \ \text{ then } \ \alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)g.$$

By Certainty Independence, $\alpha g + (1-\alpha)q \sim_{\hat{F}_1} \alpha q + (1-\alpha)q = q$. Therefore, by Transitivity, if $f \succcurlyeq_{\hat{F}_1} q$, then $\alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} g$. The opposite direction of Ambiguity Aversion is also immediate. This completes the proof.

Since $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5) and $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI in Theorems 1-2, from now we assume that $\succcurlyeq_{\hat{F}}$ has a SEU representation and $\succcurlyeq_{\hat{F}_1}$ has a MEU representation. That is, there exist a non-constant and affine function $U : \Delta(C) \to \mathbb{R}$, and a probability measure μ on S, such that for all $f, g \in \hat{F}$:

$$f \succcurlyeq_{\hat{F}} g \iff \sum_{s \in S} U\big(f(s)\big)\mu(s) \geqslant \sum_{s \in S} U\big(g(s)\big)\mu(s).$$

and there exist a non-constant and affine function $U_1 : \Delta(C_1) \to \mathbb{R}$, a convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$, such that for all $f, g, \in \hat{F}_1$:

$$f \succcurlyeq_{\hat{F}_1} g \iff \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) \geqslant \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(g(s^1)\big)\pi(s^1).$$

The uniqueness of U, U_1, μ, and Π_1 are straightforward. Since the necessity parts of Theorems 1-2 are straightforward, we only prove their sufficiency parts.

A.2 Proof of Theorem 1

Suppose $\succsim_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succsim_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4) and NUI. By the discussion in Section A.1, suppose $\succsim_{\hat{F}}$ has a SEU representation with (μ, U) and $\succsim_{\hat{F}_1}$ has a SEU representation with (Π_1, U_1). Without loss of generality, let $U_1(b) = 1$ and $U_1(w) = 0$ where b and w are the best and worst consequences in C_1, respectively.

Finally, we shall prove that $\succsim_{\hat{F}_1}$ is an unambiguity consistent extension of $\succsim_{\hat{F}}$ to $\hat{F}_1$. Let us fix $s \in S$. NUI implies that for all $f, g \in \hat{F}_1$ with $g = p_{E_s} w$ for some $p \in \Delta(C_1)$ and for all $q \in \Delta(C_1)$ and $\alpha \in [0,1]$,

$$\text{if} \;\; f \sim_{\hat{F}_1} q, \;\; \text{then} \;\; \alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g.$$

In terms of the MEU representation (5) for $\succsim_{\hat{F}_1}$, if

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) = U_1(q), \;\text{then} \tag{23}$$

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} \Big(\alpha\, U_1\big(f(s^1)\big) + (1-\alpha) U_1\big(g(s^1)\big)\Big)\, \pi(s^1) = \alpha\, U_1(q) + (1-\alpha) \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(g(s^1)\big)\pi(s^1). \tag{24}$$

Since $g = p_{E_s} w$, Equation (24) is equivalent to

$$\min_{\pi \in \Pi_1} \Bigg\{ \alpha \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) + (1-\alpha)\, \pi(E_s)\, U_1(p) \Bigg\} = \alpha\, U_1(q) + (1-\alpha) \min_{\pi \in \Pi_1} \Big\{ \pi(E_s) U_1(p) \Big\}.$$

Therefore, by combining (23) and (24), we have that for any $f \in \hat{F}_1$, $p, r \in \Delta(C)$, and $\alpha \in [0,1]$,

$$\min_{\pi \in \Pi_1} \Bigg\{ \alpha \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) + (1-\alpha)\pi(E_s)\, U_1(p) \Bigg\} \tag{25}$$

$$= \alpha \min_{\pi \in \Pi_1} \Bigg\{ \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) \Bigg\} + (1-\alpha) \min_{\pi \in \Pi_1} \Big\{ \pi(E_s) \Big\} U_1(p).$$

Let us now assume that $\alpha = \frac{1}{2}$ and $f = q_1 E_s p$ for some $q_1 \in \Delta(C_1)$. Then (25) is equivalent to

$$\min_{\pi \in \Pi_1} \{\pi(E_s)\} U_1(q_1) + U_1(p) = \min_{\pi \in \Pi_1} \{\pi(E_s) U_1(q_1) + (1 - \pi(E_s) U_1(p))\} + \min_{\pi \in \Pi_1} \{\pi(E_s)\} U_1(p).$$

Suppose that $U_1(p) > U_1(q_1)$. Then the above equality is equivalent to

$$\left(\max_{\pi\in\Pi_1}\left\{\pi(E_s)\right\} - \min_{\pi\in\Pi_1}\left\{\pi(E_s)\right\}\right)U_1(q_1) = \left(\max_{\pi\in\Pi_1}\left\{\pi(E_s)\right\} - \min_{\pi\in\Pi_1}\left\{\pi(E_s)\right\}\right)U_1(p). \tag{26}$$

Since (26) is satisfied for any $q_1, p \in \Delta(C)$ with $U_1(p) > U_1(q_1)$, we have $\min_{\pi\in\Pi_1}\left\{\pi(E_s)\right\} = \max_{\pi\in\Pi_1}\left\{\pi(E_s)\right\}$, i.e., E_s is an unambiguous event.

A.3 Proof of Theorem 2

Suppose $\succsim_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succsim_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and WNUI, and they jointly satisfy BAC. By the discussion in Section A.1, suppose $\succsim_{\hat{F}}$ has a SEU representation with (μ, U) and $\succsim_{\hat{F}_1}$ has a SEU representation with (Π_1, U_1). Without loss of generality, let $U(b) = U_1(b) = 1$ and $U(w) = U_1(w) = 0$ where b and w are the best and worst consequences in C, respectively. We prove Theorem 2 in the following two steps.

Step 1: $U = U_1$ on $\Delta(C)$.

For any $p, p' \in \Delta(C)$, by BAC, we have

$$(p_{-s}\, p) \succsim_{\hat{F}} (p'_{-s}\, p') \text{ if and only if } \big(w_{-E(S)}(p_{-E_s}\, p)\big) \succsim_{\hat{F}_1} \big(w_{-E(S)}(p'_{-E_s}\, p')\big);$$

equivalently,

$$U\big(p\big) \geq U\big(p'\big) \text{ iff } \min_{\pi\in\Pi_1}\left\{\pi(E(S))\right\}U_1(p) \geq \min_{\pi\in\Pi_1}\left\{\pi(E(S))\right\}U_1(p') \text{ iff } U_1\big(p\big) \geq U_1\big(q\big).$$

Therefore, $U = U_1$ on $\Delta(C)$, i.e, risk attitudes of $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ are the same.

Step 2: $\succsim_{\hat{F}_1}$ is a likelihood consistent extension of $\succsim_{\hat{F}}$.

For any $s \in S$, $p, q, p' \in \Delta(C)$, by BAC, we have

$$(p_{-s}\, q) \succsim_{\hat{F}} (p'_{-s}\, p') = p' \text{ if and only if } \big(p'_{-E(S)}(p_{-E_s}\, q)\big) \succsim_{\hat{F}_1} \big(p'_{-E(S)}(p'_{-E_s}\, p')\big) = p';$$

equivalently,

$$(1-\mu(s))U(p) + \mu(s)U(q) = U\big(p'\big) \text{ iff}$$

$$\min_{\pi\in\Pi_1}\left\{\pi(E(S)\backslash E_s)U(p) + \pi\big(E_s\big)U(q) + (1-E(S))U\big(p'\big)\right\} = U(p')$$

Therefore, we have for any $s \in S$ and $p, q \in \Delta(C)$, $(1-\mu(s))U(p) + \mu(s)U(q)$ is equal to

$$\min_{\pi\in\Pi_1}\left\{\pi(E(S)\backslash E_s)U(p) + \pi(E_s)U(q) + \big(1-\pi(E(S))\big)\big((1-\mu(s))U(p) + \mu(s)U(q)\big)\right\}.$$

Therefore, for any $s \in S$ and $p, q \in \Delta(C)$,

$$\min_{\pi \in \Pi_1} \left\{ \big(\pi(E_s) - \pi(E(S))\mu(s)\big)\big(U(q) - U(p)\big) \right\} = 0.$$

The above equation implies $\min_{\pi \in \Pi_1} \{\pi(E_s) - \pi(E(S))\mu(s)\} = 0$ when $U(p) < U(q)$ and $\max_{\pi \in \Pi_1} \{\pi(E_s) - \pi(E(S))\mu(s)\} = 0$ when $U(p) > U(q)$. Therefore, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$.

A.4 Proofs of Corollaries 2-3

Proof of Corollary 2. As shown in Theorem 2, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Since $\pi(E(S)) = \pi(S_1) = 1$ in the context of discovering acts, $\pi(E_s) = \mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Therefore, Unambiguity Consistency is satisfied.

Proof of Corollary 3. As shown in Theorem 2, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Let $\bar{\delta} = \max_{\pi \in \Pi_1}\{\pi(E(S))\}$ and $\underline{\delta} = \min_{\pi \in \Pi_1}\{\pi(E(S))\}$. Since $E_s = s^1$ for some $s^1 \in S_1$, we obtain $\pi(s^1) = \delta\mu(s)$ where $\delta \in [\underline{\delta}, \bar{\delta}]$.

A.5 Proof of Theorem 3

Suppose that $\hat{F}_1 = F \cup \{\bar{f}\}$ and $\hat{F}_2 = F \cup \{f^*\}$. Let $\succcurlyeq_{\hat{F}_1}$ is an umambiguous and likelihood consistent extension of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$ with (Π_1, U), and that $\succcurlyeq_{\hat{F}_2}$ be an umambiguous and likelihood consistent extension of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_2$ with (Π_2, U). Given $S_1 = C^{F_1}$ and $S_2 = C^{F_2}$, let $\varphi : S_1 \to S_2$ be a bijection such that $\varphi(s^1) = s^2 \in S_2$ if and only if there are $s \in S$ and $c \in C$ such that $s^1 = s^2 = (s, c)$. Given φ, for each act $f \in \hat{F}_1$, $f_\varphi \in \hat{F}_2$ is an act that returns the lottery $f(\varphi(s^1)) = q$ in state $\varphi(s^1) \in S_2$ when f returns q in $s^1 \in S_1$ (i.e., $f(s^1) = q$). Then, for each lottery $p \in \Delta(C)$ and act $f \in \hat{F}_1$, we have

$$p \sim_{\hat{F}_2} f \;\text{ implies }\; p \succcurlyeq_{\hat{F}_2} f_\varphi, \tag{27}$$

which is equivalent to

$$U(p) = \min_{\pi \in \Pi_2} \sum_{s^1 \in S_1} U\big(f(\varphi(s^1))\big)\pi(\varphi(s^1)) \geq \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U\big(f(s^1)\big)\pi(s^1) = \min_{\pi \in \Pi_1^\varphi} \sum_{s^1 \in S_1} U\big(f(\varphi(s^1))\big)\pi(\varphi(s^1)).$$

Therefore, $\succcurlyeq_{\hat{F}_1}$ is more ambiguity averse than $\succcurlyeq_{\hat{F}_2}$ if and only if $\Pi_2 \subseteq \Pi_1^\varphi$.

A.6 Proof of Theorem 4

We only prove the sufficiency part. Since Extreme NUI is stronger than NUI, by Theorem 2, there are (μ, U) and (Π_1, U) such that $\succcurlyeq_{\hat{F}}$ admits a SEU representation with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ admits a MEU representation with (Π_1, U). Moreover, $\succcurlyeq_{\hat{F}_1}$ is an unambiguity consistent and likelihood consistent extension of $\succcurlyeq_{\hat{F}}$. That is, $\pi(E_s) = \mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Without loss of generality, let $U(w) = 0$ where w is the worst consequence in C. We prove the theorem in three steps.

Step 1: For any $f, h \in \hat{F}_1$ and $s \in S$,

$$\min_{\pi\in\Pi_1}\Big\{\sum_{s\in E_s}U(f(s)\pi(s)+\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} = \min_{\pi\in\Pi_1}\sum_{s\in E_s}U(f(s)\pi(s)+\min_{\pi\in\Pi_1}\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s).$$

Let us fix $f, h \in \hat{F}_1$ and $s \in S$. Take any $p \in \Delta(C)$ such that $fE_s w \sim_{\hat{F}_1} pE_s w$. Notice that $fE_s w$ and $pE_s h$ as well as $pE_s w$ and $pE_s h$ agree on the worst state of each event in $\{E_{s'}\}_{s'\in S}$. Therefore, by Extreme NUI, for any $\alpha \in [0, 1]$,

$$(\alpha f + (1-\alpha)p)E_s(\alpha w + (1-\alpha)h) \sim_{\hat{F}_1} pE_s(\alpha w + (1-\alpha)h); \tag{28}$$

equivalently

$$V^{MEU}\big(\underbrace{\alpha f + (1-\alpha)p)E_s(\alpha w + (1-\alpha)h}_{g}\big) = V^{MEU}\big(\underbrace{pE_s(\alpha w + (1-\alpha)h)}_{\tilde{g}}\big). \tag{29}$$

Since E_s is unambiguous and $U(w) = 0$,

$$\begin{aligned} V^{MEU}(g) &= \min_{\pi\in\Pi_1}\Big\{\alpha\sum_{s\in E_s}U(f(s))\pi(s) + (1-\alpha)\pi(E_s)U(p) + (1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} \\ &= (1-\alpha)U(p)\pi(E_s) + \min_{\pi\in\Pi_1}\Big\{\alpha\sum_{s\in E_s}U(f(s))\pi(s) + (1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} \end{aligned}$$

and

$$V^{MEU}(\tilde{g}) = \min_{\pi\in\Pi_1}\Big\{U(p)\pi(E_s) + (1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} = U(p)\pi(E_s) + (1-\alpha)\min_{\pi\in\Pi_1}\Big\{\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\}$$

Thus, when $\alpha = \frac{1}{2}$, we get $V(g) = V(\tilde{g})$ is and only if

$$\min_{\pi\in\Pi_1}\Big\{\sum_{s\in E_s}U(f(s)\pi(s) + \sum_{s\in E_s}U(h(s))\pi(s)\Big\} = U(p)\pi(E_s) + \min_{\pi\in\Pi_1}\sum_{s\in E_s}U(h(s))\pi(s).$$

Moreover, since $fE_sw \sim_{\hat{F}_1} pE_sw$ and $U(w) = 0$, we have

$$\min_{\pi\in\Pi_1} \sum_{s\in E_s} U(f(s)\pi(s) = U(p)\pi(E_s).$$

Combining the last equalities we obtain

$$\min_{\pi\in\Pi_1} \Big\{ \sum_{s\in E_s} U(f(s)\pi(s) + \sum_{s\in S_1\setminus E_s} U(h(s))\pi(s) \Big\} = \min_{\pi\in\Pi_1} \sum_{s\in E_s} U(f(s)\pi(s) + \min_{\pi\in\Pi_1} \sum_{s\in S_1\setminus E_s} U(h(s))\pi(s).$$

Step 1 essentially proves the extended MEU preference $\succsim_{\hat{F}_1}$ is additively separable across the events $\{E_s\}_{s\in S}$. That is,

$$\min_{\pi\in\Pi_1} \left\{ \sum_{s\in S} \mu(s) \sum_{s_l^1\in E_s} U(h(s_l^1)) \frac{\pi(s_l^1)}{\pi(E_s)} \right\}$$

is minimized at each event E_s, separately. Therefore, we obtain the following representation:

$$V^{MEU}(f) = \sum_{s\in S} \mu(s) \min_{\pi_s\in\Pi_s^{F_1}} \left\{ \sum_{s_k^1\in E_s} \pi_s(s_k^1) U\big(f(s_k^1)\big) \right\}, \tag{30}$$

where $\Pi_s^{F_1} \subseteq \Delta(E_s)$.

In next steps, we prove that beliefs on E_s take the form $\Pi_s^{F_1} = \beta_s \{\eta_s\} + (1-\beta_s)\,\Delta(E_s)$ for some $\beta_s \in [0,1]$ and $\eta_s \in \Delta(E_s)$.

Fix an event E_s. Take any $s_t^1 \in E_s$. Let us take acts $f, g, h \in \hat{F}_1$ such that for any $s_j^1 \in E_s$, $f(s_j^1) \succsim_{\hat{F}_1} f(s_t^1)$, $g(s_j^1) \succsim_{\hat{F}_1} g(s_t^1)$, and $h(s_j^1) \succsim_{\hat{F}_1} h(s_t^1)$, and $f(E_{-s}) = g(E_{-s}) = h(E_{-s}) = w$. Then, by Extreme NUI, for any $\alpha \in (0,1]$,

$$f \succsim_{\hat{F}_1} g \text{ iff } \alpha f + (1-\alpha)h \succsim_{\hat{F}_1} \alpha g + (1-\alpha)h;$$

equivalently,

$$\min_{\pi_s\in\Pi_s^{F_1}} \left\{ \sum_{s_k^1\in E_s} \pi_s(s_k^1) U\big(f(s_k^1)\big) \right\} = \min_{\pi_s\in\Pi_s^{F_1}} \left\{ \sum_{s_k^1\in E_s} \pi_s(s_k^1) U\big(g(s_k^1)\big) \right\} \text{ iff} \tag{31}$$

$$\min_{\pi_s\in\Pi_s^{F_1}} \left\{ \sum_{s_k^1\in E_s} \pi_s(s_k) U\big(\alpha f(s_k^1) + (1-\alpha)h(s_k^1)\big) \right\} = \min_{\pi_s\in\Pi_s^{F_1}} \left\{ \sum_{s_k^1\in E_s} \pi_s(s_k^1) U\big(\alpha g(s_k^1) + (1-\alpha)h(s_k^1)\big) \right\}.$$

Step 2: Suppose $s_j^1 \in E_s \setminus s_t^1$, $f(s_j^1) \succcurlyeq_{\hat{F}_1} f(s_t^1)$, $g(s_j^1) \succcurlyeq_{\hat{F}_1} g(s_t^1)$, and $h(s_j^1) \succcurlyeq_{\hat{F}_1} h(s_t^1)$ with $f(s_t^1) = g(s_t^1) = h(s_t^1) = w$.

Then (31) is equivalent to

$$\min_{\pi_s \in \Pi_s^{F_1}} \left\{ \sum_{s_k^1 \in E_s \setminus s_t^1} \pi_s(s_k^1) U\big(f(s_k^1)\big) \right\} = \min_{\pi_s \in \Pi_s^{F_1}} \left\{ \sum_{s_k^1 \in E_s \setminus s_t^1} \pi_s(s_k^1) U\big(g(s_k^1)\big) \right\} \text{ iff}$$

$$\min_{\pi_s \in \Pi_s^{F_1}} \left\{ \sum_{s_k^1 \in E_s \setminus s_t^1} \pi_s(s_k) U\big(\alpha f(s_k^1) + (1-\alpha) h(s_k^1)\big) \right\} = \min_{\pi_s \in \Pi_s^{F_1}} \left\{ \sum_{s_k^1 \in E_s \setminus s_t^1} \pi_s(s_k^1) U\big(\alpha g(s_k^1) + (1-\alpha) h(s_k^1)\big) \right\}.$$

The above equivalence is in fact the Independence Axioms on $E_s \setminus s_t^1$. Therefore, there is $\overline{\pi}_s^t \in \Delta(E_s \setminus s_t^1)$ such that

$$\min_{\pi_s \in \Pi_s^{F_1}} \left\{ \sum_{s_k^1 \in E_s \setminus s_t^1} \pi_s(s_k^1) U\big(f(s_k^1)\big) \right\} = \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s^t(s_k^1) U\big(f(s_k^1)\big).$$

Let $f(s_k^1) = w$ for any $s_k^1 \neq s_j^1$. The above equation implies that $\min_{\pi_s \in \Pi_s^{F_1}} \big\{\pi_s(s_j^1)\big\} = \overline{\pi}_s^t(s_j^1)$. Since $\min_{\pi_s \in \Pi_s^{F_1}} \big\{\pi_s(s_j^1)\big\}$ is independent of s_t, we shall write $\overline{\pi}_s(s_j^1)$ instead of $\overline{\pi}_s^t(s_j^1)$.

Let $\pi_s^*(s_t^1) \equiv \max_{\pi_s \in \Pi_s^{F_1}} \{\pi_s(s_t^1)\}$.

Step 3: Suppose $s_j^1 \in E_s \setminus s_t^1$, $f(s_j^1) \succ_{\hat{F}_1} f(s_t^1)$, $g(s_j^1) \succ_{\hat{F}_1} g(s_t^1)$, and $h(s_j^1) \succ_{\hat{F}_1} h(s_t^1)$, and $f(s_t^1) = g(s_t^1) = h(s_t^1)$. Moreover, suppose $f(s_j^1) = f(s_k^1)$ for any $s_j^1, s_k^1 \neq s_t^1$.

Suppose $U(f(s_t^1)) = U(g(s_t^1)) = U(h(s_t^1))$ is small enough relative to $U(f(s_k^1))$, $U(g(s_k^1))$, and $U(g(s_k^1))$. Then (31) is equivalent to

$$\pi_s^*(s_t^1) U(f(s_t^1)) + (1 - \pi_s^*(s_t^1)) U(f(s_k^1)) = \big(1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)\big) U(g(s_t^1)) + \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1) U\big(g(s_k^1)\big)$$

if and only if

$$(1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)) U\big(\alpha f(s_t^1) + (1-\alpha) h(s_t^1)\big) + \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1) U\big(\alpha f(s_k^1) + (1-\alpha) h(s_k^1)\big) =$$

$$(1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)) U\big(\alpha g(s_t^1) + (1-\alpha) h(s_t^1)\big) + \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1) U\big(\alpha g(s_k^1) + (1-\alpha) h(s_k^1)\big)$$

The above equivalence implies $\pi_s^*(s_t^1) = 1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)$. Similarly, we have $\pi_s^*(s_{t'}^1) = 1 - \sum_{s_k^1 \in E_s \setminus s_{t'}^1} \overline{\pi}_s(s_k^1)$. These two equalities imply that $\pi_s^*(s_t^1) - \overline{\pi}_s(s_t^1) = \pi_s^*(s_{t'}^1) - \overline{\pi}_s(s_{t'}^1)$.

Since $\pi_s^*(s_t^1) \geq \overline{\pi}_s(s_t^1)$, let $1 - \alpha_s \equiv \pi_s^*(s_t^1) - \overline{\pi}_s(s_t^1)$. Finally, since $\pi_s^*(s_t^1) + \sum_{s_k^1 \neq s_t^1} \overline{\pi}_s(s_k^1) = 1$, we have $\sum_{s_k^1 \in E_s} \overline{\pi}_s(s_k^1) = \alpha_s$. Let

$$\eta_s(s_t^1) \equiv \frac{\overline{\pi}_s(s_t^1)}{\sum_{s_k^1 \in E_s} \overline{\pi}_s(s_k^1)} = \frac{\overline{\pi}_s(s_t^1)}{\alpha_s}.$$

Then we have $\Pi_s^{F_1} = \{\overline{\pi}_s\} + (1 - \alpha_s)\,\Delta(E_s) = \alpha_s\,\{\eta_s\} + (1 - \alpha_s)\,\Delta(E_s)$ where $\eta_s \in \Delta(E_s)$.

A.7 Proof of Theorem 5

Let $\succsim_{\hat{F}}$ be a SEU preference relation with (μ, U). Suppose that $\succsim_{\hat{F}_1}$ admits the parametric MEU representation (20) with $(\{(\eta_s^1, \alpha_s^1)\}_{s \in S}, U)$ and $\succsim_{\hat{F}_2}$ also admits the parametric MEU representation (20) with $(\{(\eta_s^2, \alpha_s^2)\}_{s \in S}, U)$. In other words, $\succsim_{\hat{F}_1}$ be a MEU extension on $\hat{F}_1$ with (Π_1, U) where

$$\Pi_1 = \times_{s \in S} \Pi_{E_s}^{(\eta_s^1, \alpha_s^1)} = \times_{s \in S} \{\alpha_s^1 \{\eta_s^1\} + (1 - \alpha_s^1)\Delta(E_s)\},$$

and $\succsim_{\hat{F}_2}$ be a MEU extension on $\hat{F}_2$ with (Π_2, U) where

$$\Pi_2 = \times_{s \in S} \Pi_{E_s}^{(\eta_s^2, \alpha_s^2)} = \times_{s \in S} \{\alpha_s^2 \{\eta_s^2\} + (1 - \alpha_s^2)\Delta(E_s)\}.$$

By Theorem 3, $\succsim_{\hat{F}_1}$ is more ambiguity averse than $\succsim_{\hat{F}_2}$ if and only if

$$\Pi_2 = \times_{s \in S} \Pi_{E_s}^{(\eta_s^2, \alpha_s^2)} \subseteq \Pi_1^{\varphi} = \left(\times_{s \in S} \Pi_{E_s}^{(\eta_s^1, \alpha_s^1)}\right)^{\varphi}.$$

Equivalently, $\succsim_{\hat{F}_1}$ is more ambiguity averse than $\succsim_{\hat{F}_2}$ if and only if

$$\Pi_{E_s}^{(\eta_s^2, \alpha_s^2)} = \alpha_s^2 \{\eta_s^2\} + (1 - \alpha_s^2)\Delta(E_s) \subseteq (\Pi_{E_s}^{(\eta_s^1, \alpha_s^1)})^{\varphi} = (\alpha_s^1 \{\eta_s^1\} + (1 - \alpha_s^1)\Delta(E_s))^{\varphi} \text{ for all } s \in S;$$

equivalently,

$$\left[\alpha_s^2\, \eta_s^2(\varphi(s^1)), \alpha_s^2\, \eta_s^2(\varphi(s^1)) + (1 - \alpha_s^2)\right] \subseteq \left[\alpha_s^1\, \eta_s^1(s^1), \alpha_s^1\, \eta_s^1(s^1) + (1 - \alpha_s^1)\right] \text{ for all } s \in S \text{ and } s^1 \in E_s;$$

Therefore, by a direct calculation, we obtain

$$\frac{\alpha_s^2}{\alpha_s^1} \geq \max\left\{ \frac{\eta_s^1(s^1)}{\eta_s^2(\varphi(s^1))}; \frac{1 - \eta_s^1(s^1)}{1 - \eta_s^2(\varphi(s^1))} \right\} \text{ for all } s \in S \text{ and } s^1 \in E_s.$$

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