Rank Efficiency: Investigating a Widespread Ordinal Welfare Criterion

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Abstract

Many institutions that allocate scarce goods based on rank-order preferences gauge the success of their assignments by the resulting rank distributions, that is, how many participants get their first choice, how many get their second choice, and so on. For example, San Francisco Unified School District, Teach for America, and Harvard Business School all evaluate assignments in this way. Preferences over rank distributions capture the practical (but non-Paretian) intuition that hurting one agent to help others might be desirable. Motivated by this, call an assignment rank efficient if its rank distribution cannot feasibly be stochastically dominated. Rank efficient mechanisms are simple linear programs that can either be solved all at once by a computer or through an intuitive sequential improvement process where at each step, the policy-maker executes a potentially non-Pareto-improving trade cycle. Both methods are used in the field. Rank efficiency also dovetails nicely with previous literature: it is a refinement of ordinal efficiency (and hence of ex post efficiency). Although rank efficiency is theoretically incompatible with strategy-proofness, rank efficient mechanisms can admit a truth-telling equilibrium in low information environments. Preference data from Featherstone and Roth (2011) show that if agents were to truthfully reveal their preferences, a rank efficient mechanism could significantly outperform commonly considered alternatives like random serial dictatorship and the probabilistic serial mechanism. Finally, a competitive equilibrium mechanism like that of Hylland and Zeckhauser (1979) generates a straightforward generalization of rank efficiency and sheds light on how rank efficiency interfaces with fairness considerations.

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1 Introduction

Mechanisms that map submitted ordinal preferences into an assignment of agents to objects are common, and institutions that use them often gauge success by looking at rank distributions, that is, at how many participants get their first choice, how many get their second choice, and so on. For example, when school districts announce the results of a choice-based student placement system, they report things like how many students were assigned to one of their top three choices and how many were unmatched (NYC Department of Education 2009, San Francisco Unified School District 2011).\footnote{SFUSD also made frequent use of rank distributions to help them understand how to compare different school choice mechanisms proposed by myself, Atila Abdulkadiro˘ glu, Muriel Niederle, Parag Pathak, and Al Roth over the course of the 2009-2010 school year in which we advised them.} Another example is the mechanism used by Teach for America to match teachers to regions, which explicitly selects an assignment based on rank distribution.\footnote{Teach for America is a nationwide non-profit that sends mostly new college graduates to teach in at-risk public schools. Al Roth and I are actively involved with helping the organization to streamline its assignment process. See Section 4.4 for more.} And finally, consider the strategy-proof random serial dictatorship used by Harvard Business School to match MBAs to the country in which they fulfill their global immersion requirement.\footnote{This paper discusses the HBS global immersion match in Section 7. Al Roth and I are actively involved with the design of the match. See Featherstone and Roth (2011).} In the first year the mechanism was run, administrators were so concerned about the number of students who, in spite of ranking it last, were assigned to an unpopular country, that they seriously considered exchanging those students with others who had only given the unpopular country a moderately low rank. After being reminded that such rearrangement would violate a promise of strategy-proofness, they reluctantly stayed with the original assignment. Indeed, a big part of strategy-proofness is making it safe for agents to reveal that they somewhat like objects that most other agents strongly dislike.

This last anecdote is not surprising in light of the fact that random serial dictatorship is merely Pareto efficient. It could easily yield an assignment where shifting one student from first to second choice could enable four other students to shift from second to first choice.\footnote{Consider the following five agent, five position example, where the circles and boxes represent two different assignments. A policy-maker looking at the boxed assignment might be tempted to implement a trade cycle to get the circled assignment instead.}

\[
\begin{align*}
1 & : a \succ c \\
2 & : a \succ b \\
3 & : b \succ r \\
4 & : c \succ d \\
5 & : d \succ e
\end{align*}
\]
feasible assignment. Rank efficiency formalizes the intuition that, if there is no reason to differentiate between one agent getting his $k^{th}$ choice or another, then it can be welfare improving to hurt one agent in order to help others.

Mechanisms that implement rank efficient assignments can also be characterized in terms of this intuition. Consider giving values to the different ranks, such that the value of a first choice is more than that of a second choice, and so on. To see if a potentially non-Pareto-improving trade cycle should be executed, simply calculate whether doing so will increase the overall “rank-value” of the market. For instance, when considering a two-person trade that moves one agent from his first to his third choice, and the other from his fourth to his second, the relevant change in rank-value would be $(v_3 - v_1) + (v_2 - v_4)$, which could be positive or negative, depending on what $v$ was chosen. I show that an assignment is rank efficient if and only if there is some valuation vector $v$ such that all possible trade cycles yield a non-positive value. In other words, starting with some assignment and sequentially improving it by executing trade cycles that are rank-value improving, but potentially not Pareto-improving, will eventually yield a rank efficient assignment. Teach for America used exactly this kind of process prior to Spring 2011. Such a process can also be easily automated with a simple linear program, and in fact, linear programming mechanisms, although not as prevalent as random serial dictatorships, can be found in the field: they are used to match house-officers\(^5\) to hospitals in Cambridge and London (Roth 1991), and since Spring 2011, by Teach for America as well.\(^6\)

Rank efficiency also dovetails nicely with two efficiency concepts that have dominated the theoretical literature on assignment markets. Ex post efficiency requires that there be no other deterministic assignment that improves all agents. Stepping back to lotteries over assignments (that is, to the interim stage), and thinking about agents as possessing bundles of object probability shares, ordinal efficiency (Bogomolnaia and Moulin 2001) requires that there be no other assignment of shares that stochastically improves all agents. Rank efficiency is a refinement of both concepts.\(^7\)

Looking to mechanisms, random serial dictatorship\(^8\) is ex post efficient, strategy-proof, and ubiquitous in the field, while the probabilistic serial mechanism\(^9\) is ordinally efficient, generically non-strategy-proof, and absent in the field. Given the theoretical appeal of ordinal efficiency, it was unclear why there are no ordinally efficient mechanisms in the field. Since rank efficient mechanisms are also ordinally efficient, however, their presence resolves this institutional puzzle.

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\(^5\)Roughly speaking, house-officers are a U.K. analog to medical residents in the American market (Roth 1984).

\(^6\)One might object that Al and I suggested this mechanism; however, in Section 4.4, I argue that Teach for America adopted a linear programming assignment method, not because it was something new, but because it duplicated the non-automated process of the past in much less time.

\(^7\)Given the limited information of an ordinal setting, one can interpret rank efficiency as ex ante efficiency if the $v$ vector from the previous paragraph is thought of as an assumption about the policy-maker’s beliefs.

\(^8\)That is, draw an ordering of the agents from some fixed distribution over orderings, and let them choose their objects in that order. See Section 3.

\(^9\)The probabilistic serial mechanism is the baseline ordinally efficient mechanism Bogomolnaia and Moulin (2001). See Online Appendix Section 3 for a formal definition.
In the field, the conflict between strategy-proofness and refined efficiency is framed as a decision between random serial dictatorship and linear programming. How does the market designer choose which mechanism is the right one? Do rank efficient mechanisms ever make sense, or are they “mistakes” made by uninformed policy-makers? Two exercises inform these concerns. First, in low information environments, like those of Roth and Rothblum (1999), rank efficient mechanisms admit a truth-telling equilibrium. This is how Teach for America justifies using a potentially manipulable linear programming mechanism: teachers only apply once, are geographically separated, and know little about what regions are popular and how the mechanism is run. The second exercise uses empirical data (Featherstone and Roth 2011) from Harvard Business School’s strategy-proof global immersion match. Under truth-telling, a rank efficient mechanism yields more than a 15% improvement in the number of MBAs who are assigned to a first or second choice country (out of 11) when compared to the assignment given by random serial dictatorship. Although these gains might be undermined by strategic preference manipulation, the exercise shows that in a naturally occurring environment, the costs of strategy-proofness can be quite large. Together, these exercises indicate that in certain environments, it would be a mistake to not at least consider using a rank efficient mechanism.

Finally, the paper considers a class of ordinal mechanisms based on Hylland and Zeckhauser (1979). These mechanisms assume cardinal preferences to rationalize the submitted ordinal preferences, give each agent a budget of artificial\(^{10}\) money, and then finds prices such that agents would buy optimal allocations of object probability shares that clear the market (in other words, calculates a competitive equilibrium assignment). Such mechanisms always yield ordinally efficient assignments, and in fact, they yield a refinement of ordinal efficiency that generalizes rank efficiency by allowing different agents to be weighted differently when the rank distribution is tabulated. Hence, there is a mapping between generalized rank efficient mechanisms\(^ {11}\) and competitive equilibrium mechanisms. Perhaps surprisingly, a competitive equilibrium where all agents have the same budget does not correspond to an assignment supported by the rank efficient mechanism that places equal weights on each agent. In this sense, procedural fairness means different things under the two types of mechanisms. Equal budgets in a competitive equilibrium mechanism correspond to justice based on envy-freeness (Varian 1974, Dworkin 1981), while equal agent weights in a rank efficient mechanism correspond to justice based on a utilitarian\(^ {12}\) version of the Rawlsian veil-of-ignorance (Harsanyi 1975).

The rest of the paper is organized into three parts. The first part (Sections 2-6) presents rank efficiency and rank efficient mechanisms. It then relates these mech-

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\(^{10}\)That is, the money only exists within the mechanism and is not intrinsically valued by the agents. Its only worth is to purchased probability shares within the mechanism.

\(^{11}\)As just discussed, the rank efficient mechanism gives an assignment \(v_k\) points for assigning any agent his \(k^{th}\) choice. The generalized rank efficient mechanism weights agents according to some vector \((\alpha)\): the value of assigning an agent \(a\) to his \(k^{th}\) choice is worth \(\alpha_a \cdot v_k\).

\(^{12}\)That is, maximizing expected utility from behind the veil. If one were to follow Rawls (1972), then the objective would be maximin, which would lead to a concept of justice that more closely resembles that of Dworkin.
anisms to how Teach for America assigns teachers and to the broader theory liter-

ature concerning efficiency in assignment markets. The second part (Sections 7-8) argues that rank efficient mechanisms could potentially yield large welfare gains by demonstrating two things. First, preference data from the strategy-proof HBS global immersion match demonstrates that if a rank efficient mechanism could access the true preferences of the agents, it could yield big efficiency gains. Second, although strategy-proofness and rank efficiency are theoretically incompatible, rank efficient mechanisms can admit a truth-telling equilibrium in low information environments. The third part (Sections 9-10) generalizes rank efficiency and shows that the generalization is closely related to the competitive equilibrium mechanisms of Hylland and Zeckhauser (1979). This result can be leveraged to interpret the theoretical incompatibility of rank efficiency and envy-freeness as a wedge between two important concepts of justice. The conclusion discusses how rank efficient mechanisms should enter into the discussion about the costs of strategy-proofness, and how they raise interesting questions about the costs of Pareto-agnosticism.

2 The model

Consider assigning each agent $a$ from set $A$ to exactly one object $o$ from set $O$. Further, let there be $q_o$ copies of each object. Sometimes the set of objects will include a special “null” object, $\emptyset$, that denotes an agent’s outside option; for that reason, it is modeled as a non-scarce good, that is, $q_\emptyset = |A|$. When agents have no outside option, the null object is omitted. Whether there is a null object or not, I will require that there are enough objects that every agent can be feasibly matched, that is $\sum_{o \in O} q_o \geq |A|$.

A **deterministic assignment** is a function that maps agents to objects feasibly, that is, each agent is only mapped to one object, and no more than $q_o$ agents are assigned to any given $o \in O$. Such an assignment can be represented as an $|A| \times |O|$ matrix $x$ where $x_{ao} \in \{0, 1\}$, $\sum_o x_{ao} = 1$, and $\sum_a x_{ao} \leq q_o$, for all $a \in A$ and $o \in O$. $x_{ao} = 1$ means that agent $a$ is assigned to object $o$; $x_{ao} = 0$ means that agent $a$ is not. A **random assignment** is a lottery over deterministic assignments, which can be represented as the corresponding convex combination over deterministic assignment matrices. As such, random assignment matrices will have a similar structure to deterministic assignment matrices, except $x_{ao} \in [0, 1]$. By the extension of the Birkhoff-von Neumann theorem (Birkhoff 1946) put forth in Budish et al. (2011), any such matrix represents some lottery over deterministic assignments. Call this lottery a **lottery representation** of the random assignment matrix, and the deterministic

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13 The obvious example is military postings. Perhaps a more subtle example is public school assignment. In San Francisco, students are not required to rank all schools, but if they cannot be assigned to a school they ranked, they are generally given an administrative assignment. In this sort of situation, failing to rank all schools is equivalent to the student saying to the school district, “beyond what I ranked, you can fill out the rest of my rank-order list for me.”

14 This lottery need not be unique. Generally, this is an unimportant detail, as all such lotteries over assignments induce the same lottery over objects for each agent, but sometimes the subtlety is important. See Remark 1.

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assignments the representation’s support. Most of the time, this paper will use the freedom afforded by the Budish et al. theorem to focus on matrix representations.

Moving to the individual agent, call the \( a \)’s allocation, \( x_a \). Each agent \( a \) is endowed with an ordinal preference \( \succsim_a \) over \( \mathcal{O} \); note that indifferences are allowed. It will often prove convenient to express agents’ preferences in terms of rank functions, \( r_a(\cdot) \), which are mappings from \( \mathcal{O} \) to \( \{1, \ldots, |\mathcal{O}| + 1\} \). When preferences are strict, there are several equivalent ways to define the standard rank function: \( r_a(o) = |\{o’ \in \mathcal{O} | o’ \succsim_a o\}| \), or \( r_a(o) = |\{o’ \in \mathcal{O} | o’ >_a o\}| + 1 \), or even \( r_a(o) = |\{o’ \in \mathcal{O} | o’ \succsim_a o\}| \).

With indifferences, however, these definitions are no longer equivalent. Consider \( \succsim_a = a \succ b \sim c \succ d \). The ranks of \((a, b, c, d)\) under these definitions are, respectively, \((1, 3, 3, 4)\), \((1, 2, 2, 4)\), and \((1, 2, 2, 3)\). Another ambiguity in how to think of the rank function concerns the null object, \( \emptyset \). One could of course treat \( \emptyset \) just like any other object; however, policy-makers often report the number of unassigned agents as a separate category that is, in some sense, worse than any other rank. In the context of the mechanism to be introduced in this paper, one can model this by setting \( r_a(\emptyset) = |\mathcal{O}| \) and for all \( o \prec_a \emptyset, r_a(o) = |\mathcal{O}| + 1 \). In Section 8, I show that the specifics of the mapping from \( \succsim_a \) to \( r_a(\cdot) \) (the ranking scheme) can affect incentives for truth-telling. For now, the theory can be built around any of these definitions so long as \( o’ \succsim_a o \iff r_a(o’) < r_a(o) \) and \( o’ \sim_a o \iff r_a(o’) = r_a(o) \).

Finaly, define an ordinal assignment mechanism to be a mapping from submitted preferences to random assignments.

3 Ex post efficiency

Before introducing rank efficiency, I introduce the baseline concept of efficiency for assignment markets, ex post efficiency. Informally, a “good” random assignment is a lottery such that, regardless of which deterministic assignment is realized, there won’t be a rearrangement of the objects that all agents weakly prefer (strictly for one).

**Definition 1.** A feasible deterministic assignment \( \bar{x} \) is said to ex post dominate another deterministic assignment \( x \) if \( \bar{x}_a \succsim_a x_a \) for all \( a \in A \), and there is some \( a \) where the preference is strict. A feasible deterministic assignment is ex post efficient if it is not ex post dominated. A random assignment \( \bar{x} \) is ex post efficient if it is a lottery over ex post efficient deterministic assignments.

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\(^{15}\)\(|\mathcal{O}| + 1\) is included for technical reasons that will become clear in the next paragraph.

\(^{16}\)\(|\mathcal{O}| \sim_a\) is the set of indifference classes of \( \mathcal{O} \) with respect to \( \sim_a \); hence, \(|\{o’ \in \mathcal{O} | \sim_a o’ \succsim_a o\}|\) is the number of indifference classes whose objects are weakly preferred to \( o \). When preferences are strict, the indifference classes are all singletons.

\(^{17}\)This can be clearly seen in the press releases from the school matches in San Francisco and New York City (NYC Department of Education 2009, San Francisco Unified School District 2011).

\(^{18}\)The rank-value mechanisms (see Section 4.2) introduced in this paper effectively ignore agents’ preferences below \( \emptyset \).

\(^{19}\)This requirement can be relaxed for objects ranked below \( \emptyset \) under mechanisms that do not consider the rankings of unacceptable objects.
Remark 1. Although non-intuitive, it is possible for the matrix representation of an ex post efficient random assignment to have a lottery representation whose support is not entirely ex post efficient (see Abdulkadiroğlu and Sönmez (2003a) for an example). This is one instance where the non-uniqueness of the Birkhoff-von Neumann decomposition requires careful handling. Matrix representations of rank efficient (and ordinal efficient) random assignments do not have this problem.

Remark 2. The definition of ex post efficiency for random assignments can be equivalently expressed in terms of a state-by-state domination concept. See the Section 1 in the Online Appendix for more.

To characterize the mechanisms that implement ex post efficiency, it is helpful to formally define serial dictatorships, which, informally, allow the agents to choose their objects in some predefined order.

**Definition 2. Serial dictatorship** maps the reported strict preferences of the agents, \((\succ_a)\), and a permutation \(\pi\) of \((1, \ldots, |A|)\) to a deterministic assignment according to the recursion relations \(O_0 = O\), \(x_{a_{\pi(k)}} = \max_{\succ_{\pi(k)}} O_{k-1}\), and \(O_k = O_{k-1} \setminus x_{a_{\pi(k)}}\). Denote the assignment that results from these recursions as \(SD[\succ; \pi]\).

Random serial dictatorships simply randomize the ordering used in the serial dictatorship.

**Definition 3. Random serial dictatorship** maps the reported strict preferences of the agents, \((\succ_a)\), and a distribution over permutations of \((1, \ldots, |A|)\), \(p(\pi)\), to the random assignment that results from a lottery that picks \(SD[\succ; \pi]\) with probability \(p(\pi)\). Denote the resulting random assignment as \(RSD[\succ; p]\).

Random serial dictatorships are fundamentally related to the class of ex post efficient assignments: they always generate such assignments, and any such assignment can be generated by some random serial dictatorship.

**Proposition 1.** An assignment \(x\) is ex post efficient relative to strict preferences \(\succ\) if and only if there exists a distribution over permutations \(p\) such that \(x = RSD[\succ; p]\).

*Proof.* Bogomolnaia and Moulin (2001) show that a deterministic assignment \(x\) is ex post efficient if and only if there exists an ordering \(\pi\) such that \(x = SD[\succ; \pi]\). The “if” part of the preceding Proposition follows because \(RSD[\succ; p]\) is a lottery over ex post efficient assignments, by construction. The “only if” part follows because the lottery representation of \(x\) whose support is ex post efficient gives us the distribution over orderings required. \(\square\)

**Remark 3.** Random serial dictatorships with indifferences have been discussed in the literature (Svensson 1994, 1999, Bogomolnaia, Deb and Ehlers 2005), albeit from a more theoretical perspective. For the sake of simplicity, I have relegated discussion of how indifferences can be practically incorporated into the framework of random serial dictatorship to Section 2 of the Online Appendix.

\(^{20}\)Abusing notation, for deterministic assignments, let \(x_a\) denote both the \(a^{th}\) row of \(x\) and the object that \(x_a\) represents.
Random serial dictatorship is also strategy-proof.

**Proposition 2** (Roth and Sotomayor 1992). *If the distribution over orderings does not depend on the submitted preferences of the agents, then random serial dictatorship is strategy-proof.*

Strategy-proofness, ex post efficiency, and the simplicity of the algorithm are strong reasons to expect the random serial dictatorships to be widely present in the field, which is indeed the case.

## 4 Rank efficiency and the rank-value mechanisms

Many policy-makers gauge their success by looking to the rank distribution of the assignment they choose, that is, at how many agents get their first choice, how many get their second choice, and so on. Such a heuristic is a natural way to deal with a difficulty of Pareto-agnosticism: it can overlook potentially welfare improving trade-offs in which one agent is hurt but others are helped. In this section, I formally introduce rank efficiency and show that it is equivalent to a specific method of determining whether a trade cycle is welfare improving. This method can be operationalized either by simple linear programs or by a sequential improvement process.

### 4.1 Rank efficiency

Consider the cumulative frequency distribution of ranks received by the agents in a market. Formally, define the **rank distribution of assignment** \( x \) to be

\[
N^x(k) = \sum_{a \in A} \sum_{o \in O} 1_{\{r_a(o) \leq k\}} \cdot x_{ao}
\]

\( N^x(k) \) is the expected number of agents who get their \( k^{th} \) choice or better under assignment \( x \).

**Definition 4.** A random assignment \( x \) is **rank-dominated** by a feasible assignment \( \tilde{x} \) if the rank distribution of \( \tilde{x} \) first-order stochastically dominates that of \( x \), that is, \( N^{\tilde{x}}(k) \geq N^x(k) \) for all \( k \) (strict for some \( k \)). A feasible random assignment is called **rank efficient** if it is not rank-dominated by any other feasible assignment.

In fact, rank efficiency is a refinement of ex post efficiency. Formally,

**Proposition 3.** *If \( x \) is rank efficient, then \( x \) is ex post efficient; however, the converse need not hold.*

**Proof.** The example from Footnote 4 shows that the converse need not hold, as the circles assignment rank dominates the boxes assignment, even though both are ex post efficient. For the forward implication, by way of contradiction, say that the support of \( x \) contains a deterministic assignment \( \xi \) that is ex post dominated by \( \tilde{\xi} \). It must be that \( N^{\tilde{\xi}}(k) \geq N^\xi(k), \forall k \) (strict for some \( k \)), since \( \tilde{\xi} \) weakly improves the
allocation of all agents (strict for one). Let $\tilde{x}$ be the random assignment in which $\xi$ is replaced with $\tilde{\xi}$ in the lottery representation of $x$. Now, by linearity, the rank distribution of a convex combination of assignments is just the corresponding convex combination of the rank distributions of those assignments. Hence, it must be that $N^{\tilde{x}}(k) \geq N^x(k), \forall k$ (strict for some $k$), a contradiction.

Note that it follows directly from the proof that, unlike with ex post efficient assignments, any lottery representation of a matrix representation of a rank efficient assignment must have a completely rank efficient support.

Claim 1. The support of any lottery representation of a random assignment matrix that represents a rank efficient assignment must consist entirely of rank efficient deterministic assignments.

The example in Figure 1a helps to make the concept of rank efficiency more concrete. Define $\underbrace{x}$ and $\underbrace{y}$ to be the assignments represented by the circles and boxes. Also consider the output of the uniform random serial dictatorship (all serial dictatorship orderings are equally likely), $x^{U-RSD}$. The rank distributions of the three assignments are listed in Figure 1c. $\underbrace{x}$ rank dominates the others since, column-by-column, it has a larger rank distribution. Hence, this example shows that random serial dictatorship can yield rank dominated assignments. Of course, this will not always be the case (rank efficiency only guarantees that no other assignment rank dominates); in this example, it is due to the following claim, whose proof illustrates how to show that an assignment is the unique rank efficient assignment.

In Section 3 of the Online Appendix, I will show that the assignment that results from the probabilistic serial mechanism (introduced in Section 6) is also rank-dominated by $\underbrace{x}$. 

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(a) Preferences

1: $a \succ c \succ d$
2: $a \succ b \succ d$
3: $b \succ c \succ d$

(b) Random assignments

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1/2</td>
<td>1/6</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>5/6</td>
<td>1/6</td>
<td>0</td>
</tr>
</tbody>
</table>

(c) Rank distributions

<table>
<thead>
<tr>
<th></th>
<th>$N^x(1)$</th>
<th>$N^x(2)$</th>
<th>$N^x(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\underbrace{x}$</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$\underbrace{y}$</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$x^{U-RSD}$</td>
<td>11/6</td>
<td>8/3</td>
<td>3</td>
</tr>
</tbody>
</table>

Figure 1: A concrete example
Claim 2. In the simplified example, \([\bar{x}]\) is the unique rank efficient assignment.\(^{22}\)

Proof. Claim 1 means that it is sufficient to show that \([\bar{x}]\) is the unique deterministic rank efficient assignment. The three agents’ first choices only cover two objects; hence, \(N^x(1) \leq 2\), and since there are only three agents, \(N^x(2) \leq 3\). \([\bar{x}]\) hits this upper bound. Any other assignment that attains the bound must give \(b\) to 3 and split \(a\) between 1 and 2. Since \(b\) was given to 3, if 2 gets any \(a\), \(N^x(2)\) will be strictly less than 3. Thus, \([\bar{x}]\) uniquely attains the bound, demonstrating that it is the unique rank efficient assignment.

A corollary of the previous claim is that any rank efficient mechanism will choose \([\bar{x}]\). The natural next step is to look for a method of calculating the rank efficient assignments for a more general setting.

### 4.2 The rank-value mechanisms

Begin by defining a **valuation** to be a sequence \((v_k)_{k=1}^{|\mathcal{O}|+1}\) of strictly positive real numbers such that \(v_k > v_{k+1}\) for all \(k \in \{1, \ldots, |\mathcal{O}|\}\). \(v_k\) will be, in some sense, the “value” that the rank-value mechanism places on \(k^{th}\) choice allocations, so there is no need for more dimensions than \(|\mathcal{O}|\).\(^{23}\) Also note that the strictness of the inequality is essential; \(k^{th}\) choice allocations must be strictly more valuable than \((k+1)^{th}\) choice allocations.

**Definition 5.** The rank-value mechanism with respect to valuation \(v\) (or the \(v\)-rank-value mechanism, for short) maps agent rank orderings to a maximizer of the following linear program.

\[
\begin{align*}
\max_x & \quad \sum_{a \in A} \sum_{o \in \mathcal{O}} v_{r_a(o)} \cdot x_{ao} \\
\text{s.t.} & \quad \sum_a x_{ao} \leq q_o, \forall o \in \mathcal{O} \\
& \quad \sum_o x_{ao} \leq 1, \forall a \in A \\
& \quad x_{ao} \geq 0, \forall o \in \mathcal{O}, \forall a \in A
\end{align*}
\]

Assignments in the arg max are said to be **supported by the \(v\)-rank-value mechanism.**

This mechanism gives a score to every possible agent-object pair and then finds a feasible assignment that maximizes the sum of those scores. Giving a \(k^{th}\) choice to

\(^{22}\)One might argue that this fact casts doubt on rank efficiency as a concept. What if I had good reason to care more about 1 than 2 and 3? Section 9.1 will define a generalized version of rank efficiency in which agents can be weighted differently when the rank distribution is tabulated. Everything shown thus far has an analog to a world in which the weighting over students is fixed, but not uniform.

\(^{23}\)See the end of Section 2 for a reminder about why one might want an \((|\mathcal{O}|+1)^{th}\) rank.
an agent is worth $v_k$, regardless of which agent gets it. The mechanism can also be interpreted as making an assumption about agent’s cardinal utilities and maximizing a welfare sum. See Section 5 for more on interpreting the rank-value mechanism. Finally, note that the linear program for the rank-value mechanism can be easily computed. Now, I turn to the properties of the mechanism.

**Claim 3.** For any assignment $x$ in the arg max, $\sum_o x_{ao} = 1$ for all $a \in A$.

**Proof.** All elements of the valuation vector are strictly positive by definition, so all agents will be assigned as many probability shares as possible. \hfill $\square$

Having the unit-demand constraint be an inequality instead of an equality allows the use of duality theorems later on, but the difference is not substantive. A rank-value mechanism will also never assign an agent to something he ranked lower than $\emptyset$, that is, it is **individually rational**.

**Claim 4.** The $v$-rank-value mechanism is **individually rational**.

**Proof.** Suppose there is an assignment in the arg max that assigns agent $a$ to something he likes less than $\emptyset$. Then, moving $a$ to $\emptyset$ increases the objective and is feasible, since $v_k > v_{k+1}$ and $q_\emptyset = |A|$. \hfill $\square$

Finally, the $v$-rank-value mechanisms won’t allow objects to be unassigned if they could be used to improve an agent’s welfare. Formally, an assignment is **non-wasteful** if $x_{ao} > 0 \Rightarrow [o' \succsim_a o ] \Rightarrow \sum_a x_{ao'} = q_{o'}$. A mechanism is non-wasteful if it always chooses a non-wasteful assignment.

**Claim 5.** The $v$-rank-value mechanism is non-wasteful.

**Proof.** By way of contradiction, assume otherwise. It is feasible for an agent to swap any shares he has with unclaimed shares, and doing so increases the objective, contradicting optimality. \hfill $\square$

Finally, although knowing which maximizer of the linear program is chosen is not always necessary for the theory, the selection is important for implementation. Consider the set of deterministic assignments. Since this set is finite and countable, its elements can be ranked. Let the **tiebreaker function**, $\tau(x)$, denote the rank assigned to the deterministic allocation $x$. Choose the deterministic assignment in the linear program’s arg max with the lowest $\tau$ value. The following claim shows that this is enough to yield a well-defined mechanism

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24 Many polynomial time algorithms for this program exist, such as the Lawler (1976) improvement of the Hungarian algorithm (Egerváry 1931, Kuhn 1955, Munkres 1957), which has computational complexity $O \left( (\sum_o q_o)^3 \right)$ (Burkard, Dell’Amico and Martello 2009). Also, the simplex algorithm can be used. Although it is worst-case exponential, smoothed analysis indicates that parameters that would force the simplex to the exponential worst case are, in some sense, of measure zero, that is, the simplex method has polynomial smoothed complexity (Spielman and Teng 2004). In the HBS and Teach for America applications mentioned in this paper, I used CPLEX’s implementation of the simplex algorithm; solve times were less than a minute.
Claim 6. The arg max of the $v$-rank-value mechanism is the convex hull of some set of deterministic assignments.

Proof. This is a linear optimization on a compact set, so the arg max is not empty. Consider a lottery representation of some assignment in the arg max. By the linearity of the problem and the convexity of the feasible set, it must be that the deterministic assignments in the support are also in the arg max; otherwise, the objective would increase if they were dropped. Hence, any assignment in the arg max is a convex combination of deterministic assignments that are also in the arg max. Also, a convex combination of any set of assignments in the arg max must also be in the arg max, since the convex combination will have the same objective value as its components. \hfill \square

So, the tiebreaker procedure will always choose some element of the arg max, and if the tie-breaker ordering is drawn from some distribution, the tie-breaker procedure is effectively implementing a lottery representation of some random assignment in the arg max. Of course how the tiebreaker is chosen can affect the incentives of the mechanism. In this paper, I will focus on tiebreaker functions that are drawn from a distribution that does not depend on the submitted preferences of the agents. The simplest example of this is to choose the tiebreaker uniformly at random from the set of all strict orderings over the deterministic assignments. This is the same as finding all deterministic assignments in the arg max and picking one uniformly at random, which in turn, is the same as implementing the random assignment that is the centroid of the arg max.

Regardless of the tiebreaker, however, the family of rank-value mechanisms and the set of rank efficient assignments are very closely related.

**Theorem 1.** $x$ is a rank efficient assignment if and only if there exists a valuation $v$ such that $x$ is supported by the $v$-rank-value mechanism.

The intuition of this result comes from thinking about assignments in rank distribution space, that is mapping an assignment $x$ to $(N^x(1), N^x(2), \ldots, N^x(|O|)) \in \mathbb{R}^{|O|}$. In rank distribution space, the proof of the theorem is similar to the proofs for the first and second welfare theorems. For instance, the “if” part comes from rewriting the objective of the $v$-rank-value mechanism as

$$\sum_{k=1}^{|O|-1} N^x(k) (v_k - v_{k+1}) + |A| \cdot v_0$$

By definition, if $x$ is rank-dominated by $\tilde{x}$, then $\tilde{x}$ will increase this objective, since $v_k > v_{k+1}$. The “only if” part comes from an argument that uses the non-standard polyhedral separating hyperplane theorem of McLennan (2002). The details of the proof, I relegate to the Proof Appendix.

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25Since the mechanism is individually rational, it will never assign any agents to their $(|O| + 1)^{th}$ choice, since $\emptyset$ is never ranked below $|O|^{|O|}$.

26$N^x(|O|) = |A|$ by definition, since all agents are assigned to some object (even if it is $\emptyset$).

27The non-standard theorem is necessary to ensure that $v_k$ is strictly larger than $v_{k+1}$ for all $k$. 

12
The rank-value mechanism is not strategy-proof; in fact, in Section 8 I show that no mechanism can be both rank efficient and strategy-proof. I will eventually address this shortcoming, but for now I merely state it as fact. Also note that linear programming mechanisms have been previously observed in the field (Roth 1991) and are presently used by Teach for America (see Section 4.4 for more).

4.3 Sequential improvements

As I have previously mentioned, ex post efficiency is a Paretian concept, that is, it only looks for improvements that make all agents weakly better off. Rank efficiency, on the other hand, is equipped with a method for looking at non-Pareto improvements and deciding whether the good outweighs the bad. In other words, rank efficiency can make “tough decisions” while ex post efficiency cannot. Thinking about rank efficiency in terms of trade cycles can help to make this intuition more precise. Formally,

Definition 6. A trade cycle on assignment $x$ is a sequence $\tau = ((a_1, o_1), \ldots, (a_m, o_m))$ such that $x_{a_k o_k} > 0$ for each $(a_k, o_k)$ in the sequence.

Definition 7. A claim-trade cycle on assignment $x$ is a special trade cycle where $a_1$ is a fictional agent who is indifferent between all objects, and $o_1$ is an object that has not been entirely claimed by the agents in $A$, that is, $q_{o_1} > \sum_a x_{ao_1}$.

Remark 4. The fictional agent can be interpreted as holding all object shares that are not claimed by any of the agents in $A$.

Definition 8. Implementing trade cycle $\tau$ on assignment $x$ yields a new assignment, $\tilde{x}$ such that all entries stay the same except for (interpreting $o_0 \equiv o_m$),

$\tilde{x}_{a_k o_{k-1}} = x_{a_k o_{k-1}} + \Xi$, for all $k \in \{1, \ldots, |\tau|\}$

$\tilde{x}_{a_k o_k} = x_{a_k o_k} - \Xi$, for all $k \in \{1, \ldots, |\tau|\}$

where $\Xi \equiv \min_{k \in \{1, \ldots, |\tau|\}} x_{a_k o_k}$.

Implementing a claim-trade cycle is done similarly, except only the allocations of agents $a_2, \ldots, a_{|\tau|}$ are changed, and $\Xi \equiv \min_{k \in \{2, \ldots, |\tau|\}} \{x_{a_k o_k} \cdot q_{o_1} - \sum_a x_{ao_1}\}$.

A trade cycle can be interpreted as a sequence of agents who hold probability shares of objects. Implementing a trade cycle means that $a_k$ gives some of his $o_k$ to $a_{k+1}$ and receives the same amount of $o_{k-1}$ from $a_{k-1}$, and so on. A claim-trade cycle is just a trade that involves unclaimed shares. In addition to the characterization of the previous section, rank efficiency can also be characterized in terms of trade and claim-trade cycles in which some agents are made worse off but others are made better off. A valuation vector determines whether the good outweighs the bad. Formally,

28The one-to-one nature of the trade is why the volume of trade through the cycle in the definition, $\Xi$, is limited by the smallest holding of an agent in the cycle.
Definition 9. A trade cycle or claim-trade cycle $\tau$ on assignment $x$ is a $v$-rank-improving cycle if $\sum_{k=1}^{\mid \tau \mid} [v_{ra_k}(o_{k-1}) - v_{ra_k}(o_k)] > 0$ (where $o_0 \equiv o_{\mid \tau \mid}$).

Remark 5. Note that the $k = 1$ term in the sum is zero for a claim-trade cycle, since the fictional agent is indifferent between all objects.

Theorem 2. An assignment $x$ is rank efficient if and only if there exists a valuation $v$ such that $x$ admits no $v$-rank-improving cycles.

The “only if” part is straightforward. Note that claim-trade cycles must be considered because $\mid A \mid$ could be strictly less than $\sum o q_o$. Trade cycles cannot remedy the bad situation in which a valuable object is unassigned. The “if” requires some subtlety. Intuitively, a change from assignment $\tilde{x}$ to assignment $\hat{x}$ can be decomposed into distinct trade cycles and acquisitions of unassigned objects. Now say that $\hat{x} \!$ rank dominates $\tilde{x}$. Then, for any $v$, the objective of the $v$-rank-value mechanism is bigger for $\hat{x}$ than for $\tilde{x}$, which by linearity, means that at least one of these cycles or acquisitions must improve the objective, which is exactly what the condition in the definition of a $v$-rank-improving cycle means.

Hence, the valuation vector $v$ provides the rule that the $v$-rank-value mechanism uses to decide if trade cycles that aren’t Pareto improving should be implemented. A corollary to this method of proof (and the finiteness of the market) is that the policymaker can find an element of the arg max of the $v$-rank-value mechanism through a sequential improvement process:

1. Start with a feasible assignment matrix.
2. If possible, find a $v$-rank improving cycle and implement it.
3. If there was no $v$-rank improving cycle, then terminate. Otherwise, go to Step 2 with the new assignment.

Proposition 4. If all elements of the initial assignment matrix are rational numbers, then the sequential improvement process will terminate in a finite number of steps with a rank efficient assignment that is in the arg max of the $v$-rank-value mechanism.

Remark 6. All entries of a deterministic assignment matrix are rational.

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29 Bogomolnaia and Moulin (2001) characterize ordinal efficiency (see Section 6) in terms of trade cycles that improve all agents. They are able to dispense with the fictional agent because their model assumes $\mid A \mid = \mid O \mid$. In Proposition 7, I extend their theorem to the setting of this paper.

30 At least theoretically. Although this algorithm will eventually terminate, it may take a long time: Teach for America would spend about two man-weeks to match around 2,000 teachers using a method like this (they did this 4 times a year). It is unclear if they stopped because they couldn’t find any more rank-improvement cycles, or if they were merely quitting due to time constraints. It is true, however, that each step of the process necessarily increases the value of the objective of the $v$-rank-value mechanism.
In other words, starting with a deterministic assignment, a process of sequential implementation of \( v \)-rank-improving cycles will converge to a rank efficient assignment.\(^{31}\) While many algorithms for solving the assignment problem are not intuitive enough for most policy-makers to come up with on their own, the sequential improvement process is something that many policy-makers consider naturally. This was exactly what HBS considered doing when it found that too many MBAs were assigned to a country that they ranked last (see the first paragraph of the Introduction), and it is exactly how Teach for America conducted their assignment process prior to 2011 (see the next subsection).

Finally, I will mention that these non-Paretian, “tough decisions” cycles are closely related to the concept of callousness brought up by Budish and Cantillon (forthcoming) and Budish (2009) in the context of multi-unit assignment. Consider the example of Figure 1a. Putting weight on the deterministic assignment \( x \) can be interpreted as the mechanism allowing 1 to callously take what he prefers without considering the greater good. Rank efficiency prevents such callousness, and in fact, implementing \( v \)-rank-improving cycles can be interpreted as correcting callousness ex post.

4.4 Teach for America: before and after

The story of Teach for America’s recent redesign of their assignment system sheds light on how the characterizations of rank efficiency fit together.\(^{32}\) Teach for America (TFA) is a nationwide non-profit that puts selected college graduates into at-risk schools to teach. In the 2011-2012 school year, TFA intends to assign around 8,000 teachers nationwide. The application process for potential TFA admits is arduous and extends over several rounds, including interviews and tests of teaching ability. Once the original pool of around 50,000 applicants has been reduced to about 20,000, the applicants submit a final round application, along with rankings over the regions to which TFA could potentially assign them.\(^{33}\) From these applications, the admissions team decides who should be made an offer without considering their regional preferences.\(^{34}\)

Before the 2010-2011 admissions cycle, TFA used an interesting system to assign the admits. First, a computer would choose an initial match. How exactly the computer did this is immaterial, but what is important is that the admissions team found the computer match to have an unacceptably bad rank distribution. To correct this,

\(^{31}\)This algorithm is a version of the Klein cycle-canceling algorithm, a primal method used more generally to solve minimum cost flow problems (Klein 1967). Although picking improvements randomly can have bad time-complexity properties, more sophisticated algorithms (Cunningham and Marsh 1978) based on this general idea can be \( O \left( \left( \sum o qo \right)^3 \right) \).

\(^{32}\)Al Roth and I have been helping Teach for America to redesign their assignment system since Spring of 2011.

\(^{33}\)There were 43 regions in the 2011-2012 assignment cycle.

\(^{34}\)When Al and I suggested that assignment and admissions might be integrated, the idea was quickly rejected. Teach for America separates admission from assignment for two reasons. The first is that they feel they should admit the best candidates, and that it would be unfair to an unpopular region to lower the bar in finding its teachers. The second is that Teach for America feels that it can successfully persuade many admits into accepting their assignment, even if it isn’t a top choice.
two staff members would spend about a week in a conference room looking for trading cycles that could improve the match. Almost all of these were cycles where some admits were made worse off, while the rest were made better off. The whole process closely resembles the sequential improvement characterization in Section A.2.\textsuperscript{35}

In the course of the redesign, we observed the old process and tried to recreate it with an linear program. This was quite successful, which is no surprise in the light of the previous characterizations. Of course, we worried about strategy-proofness and strongly cautioned TFA about the potential problems of manipulation, but they had a strong belief that TFA admits wouldn’t or couldn’t game the system. They are geographically separated, only rank the regions once, and know little about the relative popularities of the regions. I formalize a version of this line of reasoning in Section 8. Also, TFA staff found the rank distributions that resulted from random serial dictatorship simulations inferior. In Section 7, I show that rank efficient mechanisms can yield rank distributions that are markedly better than those yielded by random serial dictatorship.

Although the linear program was never intended to be what TFA should use, TFA pressed for it, not because it offered anything new, but because it accomplished in 30 seconds what had before taken two man-weeks.\textsuperscript{36} The perceived equivalence between the automated and non-automated processes used by TFA turns out to line up quite well with the theoretical results we have established. So, although TFA has only recently started using a linear program to run their match, the non-automated system they used in the past also seems to have been a rank efficient mechanism.

5 Choosing the valuation vector

For rank-value mechanisms to be used practically, it is important to know what the valuation vector represents. In this section, I will give several interpretations of the rank-value mechanism that shed some normative light on what the valuation vector should be.

5.1 Scoring interpretation

Rewriting the objective of the rank-value mechanisms as

$$\sum_k v_k \cdot \left[ \sum_a \sum_o 1_{\{r_a(o) = k\}} \cdot x_{ao} \right]$$

expected number of agents

who get their $k^{th}$ choice

\textsuperscript{35}TFA does not allow admits to rank $\emptyset$, and the number of admits was equal to the number of positions. In this environment, all feasible assignments are non-wasteful.

\textsuperscript{36}The old rank-improvement cycles method was mainly implemented by Teach for America’s Director of Assignment Strategy, Johann von Hoffman. The new linear program is known colloquially at TFA as the “auto-Johann”.
yields an easy interpretation. Score an assignment by giving it $v_1$ points for every agent (in expectation) who gets his first choice, $v_2$ points for every agent who gets his second choice, and so on. Then, look for the feasible assignment with the biggest score. This is an easy to understand explanation for non-economists, and it can have some justification depending on where $v$ comes from.

For instance, let the objects be job offers. If $v_k$ is the policy-maker’s belief about the probability that an agent will accept the offer to his $k$th choice job, then the rank-value mechanism is merely maximizing the expected number of agents who accept the offer. This is exactly the problem faced by Teach for America, where admits get notification of admission and assignment simultaneously (applicants submit preferences over the regions when they apply, before they are admitted) and are not allowed to rank regions as unacceptable. Currently, TFA is considering modeling acceptance probabilities for use in their mechanism, which indicates that they take this interpretation of the rank-value mechanism seriously.

Alternatively, a policy-maker might want to minimize the expected rank in the assignment market. If he sets $v_k = |O| - (k - 1)$, the rank-value mechanism will accomplish this, since such a valuation makes the objective equal to $|O| + 1 - \sum_k k \cdot \left[\sum_a \sum_{o} 1_{\{r_a(o)=k\}} \cdot x_{ao}\right]$, where the term in square brackets is the expected number of agents who get their $k$th choice under assignment $x$.

Although these interpretations might seem a bit ad hoc, they can be useful in appropriate situations. A more general interpretation of the valuation vector is discussed in the next subsection: $v$ can be thought of as an assumption about the cardinal utilities of the agents.

### 5.2 Ex ante welfare interpretation

When evaluating policy, it is often helpful to look at things from behind the veil-of-ignorance, that is, from the perspective of a fictional agent with no preferences of his own, who knows that he will randomly become one of the agents in the assignment market, inheriting her preferences and allocation (Rawls 1972). From this “original position”, Harsanyi (1975) proposes that a rational agent should act to maximize his expected utility. Assuming the von Neumann-Morgenstern axioms for the fictional agent, this gives an expected utility representation over agent-object pairs (Harsanyi 1955, 1986) which is functionally a weighted sum of the expected utilities of the individual agents, that is, a social welfare function, $W = \sum_a \alpha_a \sum_o u_a(o)$.

The $u_a$‘s encode how $a$ values gambles over objects; these are objective statements that can be falsified. The $\alpha$‘s, however, encode how the fictional agent will evaluate gambles concerning outcomes in which he gets the same allocation, but as different agents. Since the original position is merely a thought experiment about justice, the

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37 TFA does not allow this, as they feel they have some probability of convincing any agent to accept any assignment. In fact, after notification goes out, TFA regional directors call some admits directly to persuade them to accept.

38 For instance, one could ask agent $a$ the following question: “Option A is a lottery where $x\%$ of the time you get your 1st choice and $(100-x)\%$ of the time you get your 3rd. Option B is getting your 2nd choice with certainty. What is the lowest $x$ at which you prefer Option A?”
α’s encode ethical judgments that cannot be falsified. Generally, economists avoid statements about interpersonal utility comparison, not because they aren’t theoretically grounded, but because of an aversion to non-falsifiable ethical judgments. Market designers, however, often need to show policy-makers how to input their ethical judgements into a mechanism.\textsuperscript{39}

Considering the objective of the rank-value mechanism as a welfare function, the fact that all agents’ cardinal utility profiles are derived from the same valuation vector seems to assume that all agents have precisely the same cardinal utility functions (modulo differences in ordinal ranking). Fortunately, this is not the case. Let \( v_{ao} \) be agent \( a \)'s von Neumann-Morgenstern utility value for object \( o \), and let \( F \) be a distribution over these values for all agents and objects. \( F \) encodes the planner’s beliefs about cardinal utilities and interpersonal utility comparison, once he has seen the submitted ordinal preferences. Given \( F \), to maximize welfare from behind the veil-of-ignorance is to maximize

\[
\int \sum_a \sum_o \sum_k v_{ak} \cdot 1_{\{r_a(o)=k\}} \cdot x_{ao} \cdot dF(v)
\]

Note that it is without loss of generality to index \( v \) by agent, \( a \), and rank, \( k \). The following proposition clarifies what assumptions justify running a rank-value mechanism.

**Proposition 5.** Assume the planner wants to maximize welfare subject to beliefs encoded by the distribution \( F \). If, relative to \( F \), the unconditional expectation of \( v_{ak} \) is independent of \( a \), then the planner can do so by running the \( \tilde{v} \)-rank-value mechanism, where \( \tilde{v}_k = \int v_{ak} \cdot dF(v) \) (for any agent \( a \)).\textsuperscript{40}

**Proof.** Changing the order of summation (and integration) in the welfare, we find

\[
\sum_a \sum_o \sum_k \left[ \int v_{ak} \cdot dF(v) \right] \cdot 1_{\{r_a(o)=k\}} \cdot x_{ao} = \sum_a \sum_o \sum_k \tilde{v}_k \cdot 1_{\{r_a(o)=k\}} \cdot x_{ao} = \sum_a \sum_o \tilde{v}_{r_a(k)} \cdot x_{ao}
\]

Note that the condition of the Proposition can be interpreted either as a direct assumption, or as an informational limitation, \( i.e. \) the planner would have different beliefs for the agents if he knew more about them, but he doesn’t. Also, note that the rank-value mechanism weights all agents the same, that is, it places the same social value on any agent getting a \( k^{th} \) choice. This ethical judgment, while reasonable, is replaceable (see Footnote 39).

\textsuperscript{39}A more general version of the rank-value mechanism, which would allow for a policy-maker to make whatever assumption about interpersonal utility comparison he likes, is discussed in Section 9.1.

\textsuperscript{40}See the Proof Appendix for a generalization of this proposition to different assumptions about how agents’ utilities compare.
A1: $a \succ b \succ c$

A2: $a \succ b \succ c$

B1: $b \succ a \succ c$

B2: $b \succ a \succ c$

(a) Preferences

A1: $5/12$ $1/12$ $1/2$

A2: $5/12$ $1/12$ $1/2$

B1: $1/12$ $5/12$ $1/2$

B2: $1/12$ $5/12$ $1/2$

(b) Uniform RSD assignment

(c) Improvement

Figure 2: Example from Che and Kojima (2010)

Of course, different assumptions could also have been made concerning the underlying cardinal utilities. If the policy-maker could observe each agent’s “type”, then he might want to assume different valuation vectors for each type. Maximizing ex ante welfare would still be solved by a linear program under such assumptions, however, the resulting assignment would not necessarily be rank efficient. In fact, this deviation from rank efficiency could be desirable under some informational assumptions. For instance, rank efficiency requires implementation of a bilateral trade that takes $a_1$ from 1st choice to 2nd and $a_2$ from 3rd to 1st. But if it were known that $a_1$ values his 1st choice much more than his 2nd and that $a_2$ is close to indifferent between his top three choices, this trade would be bad for welfare. In this sense, rank efficiency makes more intuitive sense under symmetric informational assumptions.

6 Relation to ordinal efficiency

Thus far, I have considered rank efficiency as a refinement of ex post efficiency. In this section, I will introduce another refinement commonly discussed in the literature, ordinal efficiency, and show that rank efficiency is a refinement of it as well. I will then present two propositions that highlight how ordinal efficiency differs from rank efficiency.

Consider the example in Figure 2a, adapted from Che and Kojima (2010). There is one copy each of objects $a$ and $b$, while there are two copies of object $c$. Random serial dictatorship relative to the uniform distribution over all orderings yields the random assignment in Figure 2b which, by construction, is ex post efficient.42 If each row of the assignment matrix represents a bundle of probability shares, however, there is a mutually beneficial trade: $A_1$ and $A_2$ can trade their shares in $b$ for the shares of $a$ held by $B_1$ and $B_2$, yielding the assignment in Figure 2c.43 Bogomolnaia

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41For instance, see McLennan (2002), Abdulkadiroğlu and Sönmez (2003a), Manea (2008), Manea (2009), Che and Kojima (2010), and Kojima and Manea (2010).

42To derive this random assignment matrix, note that only the first two agents in the ordering get $a$ or $b$; hence, every agent gets $c$ half the time. For an agent $a$ not to get his preferred object, he must be second in the ordering ($1/4$ of the time) and, conditional on that, the first agent in the ordering must take $a$’s first choice ($1/3$ of the time). $(1/4) \times (1/3) = 1/12$.

43The Budish et al. (2011) extension of the Birkhoff-von Neumann theorem guarantees that this new assignment can also be resolved into a lottery over deterministic assignments. In fact, so long as the trades are one-for-one, this is true; the one-for-one requirement ensures that no agent ends
and Moulin (2001) first noticed this problem and suggested a refinement of ex post efficiency which remedies it. Define the **personal rank distribution of agent** \( a \) by

\[
N^{x,a}(k) \equiv \sum_{o \in O} 1_{\{r_a(o) \leq k\}} \cdot x_{ao}.
\]

**Definition 10.** A feasible random assignment \( \tilde{x} \) **ordinally dominates** another assignment \( x \) if for all agents \( a \), \( \tilde{x}_a \) weakly stochastically dominates \( x_a \) with respect to \( \succsim_a \), that is, \( N^{\tilde{x},a}(k) \geq N^{x,a}(k) \) for all \( a \in A \) and \( k \) (strict for some \( a \) and \( k \)). An assignment is called **ordinally efficient** if there is no other feasible assignment that ordinally dominates it.

Bogomolnaia and Moulin (2001) showed that ordinal efficiency is a natural interim\(^{44}\) refinement of ex post efficiency, just as I earlier showed that rank efficiency can be interpreted as an ex ante refinement of ex post efficiency. It is also true that rank efficiency is a refinement of ordinal efficiency.

**Proposition 6.** If \( x \) is rank efficient, then \( x \) is ordinally efficient; however, the converse need not hold.

**Proof.** The example from Figure 1a shows that the converse need not hold. For the forward implication, note that weakly first-order stochastically improving all agents (strict for one) will necessarily lead to a first-order stochastic improvement in the rank distribution, since the global rank distribution is a sum over the agents’ individual rank distributions.

Just as the random serial dictatorships always produce ex post efficient assignments, the **simultaneous eating mechanisms** always yield ordinally efficient assignments (and can generate any ordinally efficient assignment). The formal description of this class of mechanisms is in Section 3 of the Online Appendix, but, essentially, agents simultaneously claim object shares (“eat”) at an agent-specific, predefined rate from their favorite (remaining) object, changing objects only when their current favorite is exhausted.\(^{45}\) The **probabilistic serial mechanism** (PS) is the member of this class where all agents “eat” at a uniform rate; it is the anonymous simultaneous eating mechanisms, just as uniform random serial dictatorship is the anonymous random serial dictatorship. It is important to note that simultaneous eating mechanisms are generically non-strategy-proof and are absent in the field. Since ordinal efficiency is such an attractive theoretical concept, it was unclear why there were no ordinally efficient mechanisms in the field.\(^{46}\) The previous proposition, coupled with the fact

\(^{44}\)Interim denotes the information set from which agent types are known, but the realized deterministic assignment is not. In the context of assignment markets, this is the perspective where agents’ ordinal preferences are known and the lottery over deterministic assignments has yet to be resolved.

\(^{45}\)In Section 3 of the Online Appendix, the assignment resulting from running PS on the example from Figure 1a is derived as an example. It is also rank-dominated by \( \tilde{x} \).

\(^{46}\)Serial dictatorships (one fixed dictatorship ordering) are ordinally efficient, but vacuously so, as all ex post deterministic assignments are also ordinally efficient (see Section A.2). More precisely then, the puzzle is why there are no ordinally efficient mechanisms that satisfy equal treatment of equals, that is, that give the same allocation to all agents who submit the same preference.
that rank efficient mechanisms do exist in the field resolves the puzzle. Instead of ordinal efficiency most broadly, institutional evolution seems to have chosen the rank efficiency refinement. Table 1 summarizes the relationships between ex post efficiency, ordinal efficiency, and rank efficiency.

Although ordinal and rank efficiency are closely related, two propositions highlight the differences. First, much like rank efficient assignments can be characterized in terms of rank-improving cycles, ordinally efficient assignments can be characterized in terms of improvement cycles, that is, trade cycles in which $o_{k-1} \succeq_a o_k$ for all $k$ (strict for some $k$).

**Proposition 7** (Bogomolnaia and Moulin 2001). An assignment $x$ is ordinally efficient if and only if $x$ is non-wasteful and admits no improvement cycles.

The proof is a straightforward extension of Bogomolnaia and Moulin (2001) and is left to the Online Appendix. Comparing this characterization to that of Section 4.3 highlights the fact that rank efficiency is a non-Paretian concept.

Second, conditional on an assignment being deterministic, ordinal efficiency is equivalent to ex post efficiency, while rank efficiency remains a refinement. Formally,

**Proposition 8.** Let $x$ be a deterministic assignment. Then,

(i) $x$ is ordinally efficient if and only if it is ex post efficient.

(ii) $x$ is rank efficient implies it is ex post efficient; however the converse need not hold.

Motivating rank efficiency as ex ante efficiency, modulo a few assumptions, is theoretically pleasing, but policy-makers tend to think in terms of deterministic assignments. If this is true, then the preceding distinction might shed some light on why simultaneous eating mechanisms aren’t seen in the field. Rank efficiency is a strict refinement even when only deterministic assignments are being compared.

### 7 The potential gains of the rank-value mechanism

As mentioned previously, both the ordinally efficient and rank efficient mechanisms are generically non-strategy-proof. This could be a major problem, as agent manipulation of reported preferences could yield assignments that look good relative to

<table>
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<th>Ex post</th>
<th>Interim</th>
<th>Ex ante</th>
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<tbody>
<tr>
<td>Random serial dictatorships</td>
<td>Ex post efficiency $\iff$ Ordinal efficiency $\not\iff$ Rank efficiency</td>
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<td>Simultaneous eating mechanisms /probabilistic serial</td>
<td>Simultaneous eating mechanisms $\iff$ Rank-value mechanisms</td>
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Table 1: Relationships between efficiency concepts
the submitted preferences, but are quite bad relative to the true preferences. In this section, I consider a simple empirical exercise that sheds light on this risk. Running uniform random serial dictatorship (the baseline strategy-proof mechanism) gives an idea of how efficient a match can be without sacrificing strategy-proofness. The preferences submitted to a strategy-proof mechanisms should be truthful, so to get an idea of the potential for gains, I can take those preferences and use them to run a non-strategy-proof alternative, such as the probabilistic serial mechanism or a rank-value mechanism. The efficiency of these counterfactuals is a sort of best-case analysis: if there is not significant improvement even without manipulation, then sacrificing strategy-proofness is likely not worth considering.

7.1 Assigning students to overseas programs at Harvard Business School

At Harvard Business School (HBS), first year MBAs must participate in a global immersion program. They are assigned to a foreign company and remotely work on a project with that company during the first semester. The program culminates in a two-week trip over the winter break in which the MBA will present her work in person and be given the opportunity to make foreign business contacts.

In 2011, at the beginning of their first semester at HBS, 900 MBAs were asked to rank the 11 different countries to which they could be assigned. The mechanism we used to match the students was strategy-proof, so the preferences submitted to it can be taken as truthful. Note that we allowed the MBAs to express indifferences; the strategy-proof adaptation of random serial dictatorship that we used is briefly described in the Online Appendix. In the interests of simplicity, however, I randomly broke student indifferences and considered only strict preference assignment mechanisms. All results are qualitatively the same regardless of the method used to break student indifferences, and regardless of whether indifferences are broken. Results from these alternate specifications are included in the Online Appendix.

7.2 The data

The data used for the analysis in this section is borrowed from Featherstone and Roth (2011). Figure 3 shows the rank distributions yielded by uniform random serial dictatorship, probabilistic serial (note that these first two overlap on the graph), and the \( v \)-rank-value mechanism, where the valuation vector is

\[
v = (100, 80, 50, 35, 15, 10, 5, 3, 2, 1, 0.5)
\]

---

47 At HBS, it is known as the FIELD 2 program.
48 Students only rank countries; once they are assigned to a country, the company they get is administratively assigned without their input.
49 I designed the HBS match jointly with Al Roth.
50 Other valuation vectors yield similar results; the main purpose I document the specific valuation used is to assure the reader that the analysis in this section doesn’t depend on a particularly extreme choice of \( v \).
The first thing to notice is that the rank distribution from probabilistic serial is essentially identical to that of uniform random serial dictatorship. The underlying assignments are also very similar, which means that, in the HBS context, there is little reason to favor the more complicated probabilistic serial mechanism over the simpler uniform random serial dictatorship. Che and Kojima (2010) show that, theoretically, the random assignment generated by the probabilistic serial mechanism asymptotically converges to the random assignment generated by uniform random serial dictatorship in the large market limit. Often with asymptotics, however, it is hard to know how large is large enough. In the case of HBS, 900 students and 11 countries seems sufficient.

Perhaps more striking is how well the $v$-rank-value mechanism performs. The number of students who get their first or second choice is increased by more than 15% when we move from the uniform random serial dictatorship to the rank-value mechanism. This corresponds to about 120 students. So, the gains from moving to a rank-value mechanism might indeed be large enough to justify backing away from strategy-proofness. However, these figures come from a best-case analysis. It could be that manipulation completely undermines the gains, or even makes the rank-value mechanism perform worse than random serial dictatorship.\textsuperscript{51} The analysis of this section simply makes the case that backing away from strategy-proofness for the gains given by probabilistic serial does not make sense for the HBS match, but that doing so for the gains given by the rank-value mechanism might well be worthwhile.

### 7.3 Strategy-proofness versus improved efficiency

The previous exercise illustrates that the concept of rank efficiency and its associated rank-value mechanisms fit well with previous literature that has considered the costs of strategy-proofness in stable matching (Erdil and Ergin 2008, Abdulkadiroğlu, Pathak and Roth 2009, Azevedo and Leshno 2010). Papers on the costs of strategy-proofness in non-stable assignment have mostly focused on how the Boston mechanism (Abdulkadiroğlu and Sönmez 2003b) can outperform random serial dictatorship. Featherstone and Niederle (2011) show this in the context of a truth-telling equilibrium, while Abdulkadiroğlu, Che and Yasuda (2011) focus on a non-truth-telling equilibrium. Both papers rely on stylized assumptions about how preferences are distributed. The present paper differs in two ways. First, it offers a new approach to get at cardinal utility in the context of an ordinal mechanism: make reasonable assumptions and maximize welfare accordingly. Second, in past literature, the leading contender with random serial dictatorship has been the class of Boston mechanisms. This paper offers a new mechanism that is based on the well-established empirical fact that policy-makers care about rank distributions.\textsuperscript{52,53} As I have begun to show in

\textsuperscript{51}Note that the work we do in Section 8 indicates that in at least some environments, we expect truth-telling.

\textsuperscript{52}Note that the Boston mechanism is not entirely ad hoc: it has been axiomatized by Kojima and Unver (2011).

\textsuperscript{53}Also, notice that the Boston mechanism is not a $v$-rank-value mechanism for any $v$. To see this, consider the example of Figure 1a. The Boston mechanism would give object $a$ to $1$ half the time, and
Figure 3: Rank distributions for our mechanisms
(U-RSD is Uniform Random Serial Dictatorship, PS is Probabilistic Serial, and RVM is Rank-Value Mechanism. Note that U-RSD and PS overlap.)
this section, the rank-value mechanisms might be an important part of the discussion about the costs of strategy-proofness in ordinal assignment markets.

8 Incentives for truth-telling

I begin this section by showing that efficiency and strategy-proofness are theoretically incompatible. There are two ways that rank-value mechanisms could still work in light of this. First, there could be environments in which rank-value mechanisms admit manipulating equilibria in which the efficiency gains don’t disappear, much like in Abdulkadiroğlu, Che and Yasuda (2011). Second, there could be environments in which truth-telling can be supported in equilibrium. Although, the first possibility is interesting, in this paper, I will focus on the second.

8.1 An impossibility result

A mechanism is called strategy-proof if the allocation it gives to any agent when she truthfully reveals her ordinal preference stochastically dominates the allocation it would give her if she revealed anything else. A mechanism is called weakly strategy-proof if the allocation it gives to an agent when she deviates from truth-telling never strictly stochastically dominates what it gives her when she truthfully reveals. Another way to phrase the difference between these two concepts is in terms of the cardinal preferences that rationalize the true ordinal preferences. Strategy-proof means that truth-telling is a dominant strategy regardless of the rationalizing cardinal utilities. Weakly strategy-proof means that for any beliefs and any potential lie, there exist rationalizing cardinal utilities that make truth-telling a better-response.

Theorem 3. No rank efficient mechanism is strategy-proof. In fact, no rank efficient mechanism is even weakly strategy-proof.

Proof. Consider a four agent, four object example.

\begin{align*}
1 & : b \succ e \\
2 & : b \succ c \succ e \\
3 & : c \succ d \succ e \\
4 & : d \succ b
\end{align*}

The unique\textsuperscript{54} rank efficient allocation is \{(1, e), (2, b), (3, c), (4, d)\}. So under any rank efficient mechanism, 1 must be assigned to e. Now, consider what happens if 1 deviates from truth-telling and submits \(b \succ c \succ d \succ e\) instead. Now, the unique rank efficient allocation is \{(1, b), (2, e), (3, c), (4, d)\}. Hence, under any rank efficient to 2 the other half of the time. But, Claim 2 shows that this is not a rank efficient assignment, and Theorem 1 shows that rank-value mechanisms must yield rank efficient assignments. The Boston mechanism is not forward-looking (i.e. it is a greedy algorithm), which is a requirement for any algorithmic solution to the defining linear program of a rank-value mechanism.

\textsuperscript{54}For an example of how to show that an assignment is the unique rank efficient assignment, see Claim 2 above.
mechanism, 1 gets $b$ with the deviation, which first-order stochastically dominates the
certain $e$ he would get with the truth.

Intuitively, someone has to get stuck with $e$. Under the true preferences, the
mechanism pays the smallest price for assigning $e$ to Agent 1. When he submits
$b \succ c \succ d \succ e$ instead, he is exaggerating about how bad $e$ is for him, making it
better for the mechanism to give it to Agent 2 instead. Still, this example required
quite a bit of specific knowledge. Next, I will show that truth-telling is a natural
response in some environments where agents have much less information.

8.2 A possibility result

Following Roth and Rothblum (1999), for some preference profile $\succsim$, define $\succsim_{o \leftrightarrow o'}$ to be the same preference profile, except all agents have switched the objects $o$ and $o'$ in their ordinal rankings. Define $q^{o \leftrightarrow o'}$ to switch the capacities of $o$ and $o'$, and let $x^{o \leftrightarrow o'}$ denote the (possibly infeasible) assignment where everyone assigned to $o$ in $x$ is reassigned to $o'$ and vice-versa. Finally, define the action of the $o \leftrightarrow o'$ operator on the tiebreaker by $\tau(x)^{o \leftrightarrow o'} \equiv \tau(x^{o \leftrightarrow o'})$. Note that $x^{o \leftrightarrow o'}$ will be feasible if the capacities are switched to $q^{o \leftrightarrow o'}$. An agent $a$’s beliefs are then summarized by a vector of random variables that range over potential ordinal preferences of the other agents ($\succsim_{-a}$, $A$, and $O$), capacities of the objects ($q$), and tie-breaker numberings, ($\tau$). We treat $\succsim_{a}$, $A$, and $O$ as known.

Definition 11. An agent’s beliefs are $\{o,o'\}$-symmetric if the distributions of $(\succsim_{-a},q,\tau)$ and $(\succsim_{-a},q,\tau)^{o \leftrightarrow o'}$ coincide. If the beliefs are $\{o,o'\}$-symmetric for all $o,o' \in O \setminus \{\emptyset\}$, then we simply call the beliefs symmetric.

One interpretation of this condition is that it represents a very symmetric envi-
ronment. The interpretation I favor, however, is that it represents an environment in
which agents have very little specific information, that is, an environment in which,
to the agents, nothing really distinguishes one object from the other. Even when this
is not globally true, it is often the case that preferences are tiered and that, within a
tier, beliefs are close to symmetric. Note that this condition is well defined even for
preferences that have indifferences. Under this informational assumption, switching
the order of two objects in the submitted preference is not profitable. Formally,

Proposition 9. Under a rank-value mechanism, if agent $a$’s beliefs are $\{o,o'\}$-symmetric,
and $o' \succ_a o$, then for $a$, the allocation resulting from any submitted preference that
declares $o \succ_a o'$ is weakly stochastically dominated by the allocation resulting from a
submitted preference that does not. If the beliefs are symmetric, then this holds for
all $o,o' \in O \setminus \{\emptyset\}$.

The proof for the theorem leverages the symmetry assumption to show that for
every state of the world in which the lie is profitable, there is at least one equally
likely state of the world in which it is not. This result is stated in a way that allows
for indifference, but from this point on in the main text, for expositional ease, I will only consider assignment markets where all preferences are strict. For the interested reader, the more general results for indifferences are in the Online Appendix. An immediate corollary of Proposition 9 is that if an agent has strict preferences and must rank all objects, then he does best to truth-tell if his beliefs are symmetric. Formally,

**Theorem 4.** Under a rank-value mechanism, if an agent’s beliefs are symmetric, his preferences are strict, and he is required to rank all objects, then the allocation he receives under truth-telling weakly stochastically dominates the allocation he receives under any other strategy.

**Corollary to Theorem 4.** Under the conditions of Theorem 4, all agents truth-telling is an equilibrium.

Although the requirement to rank all objects may seem unrealistic, it is actually quite prevalent in the field. Consider public school assignment. Students are not required to rank all schools, but if they cannot be assigned to a school they ranked, they are generally given an administrative assignment. In this sort of situation, failing to rank all schools is equivalent to the student saying to the school district, “beyond what I ranked, you can fill out the rest of my rank-order list for me.” As an aside, Teach for America also does not allow agents to rank the null object: any admit will get some assignment with their offer of admission.

### 8.3 Characterizing exhaustive manipulation classes

Nevertheless, one might feel that forcing agents to rank all objects is too onerous an assumption. If I relax this requirement, I can still tightly characterize classes of manipulations with which an agent can always best respond. The surprise of this exercise is that it hinges on an otherwise unremarkable detail of the design. Implementing mechanisms in the field often focuses attention on things that might otherwise be assumed away, and the results in this section are a result of carefully considering what a rank-value mechanism should look like in the context of the HBS global immersion match discussed in Section 7. How the null object is treated in the rank scheme turns out to be of great import.

To understand manipulation when agents are not required to rank all objects, I must first resolve the ambiguity concerning rank functions mentioned near the end of Section 2. First, define a rank scheme to be upward-looking if \( r_a(o) \) is independent of the preferences among objects to which \( o \) is strictly preferred. Now, consider how to deal with the outcome of being unmatched in the case of an agent whose true preferences are \( a \succ b \succ \emptyset \). One natural alternative is to treat \( \emptyset \) like any other object,

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55I know that this is the policy in the San Francisco Unified School District because I and co-authors (Atila Abdulkadiroğlu, Muriel Niederle, Parag Pathak, and Al Roth) spent a year redesigning the assignment system; however, it is hard to imagine a school district that doesn’t work in this way, as all students legally must be able to claim some seat in a public school, even after the match has run.
that is \( r(a) = 1, r(b) = 2, \) and \( r(\emptyset) = 3. \) Often, however, policy-makers think of being unmatched as a distinct worst element in the overall rank distribution,\(^{56}\) that is, \( r(\emptyset) = |\mathcal{O}|, \) regardless of how many objects the agent declared acceptable. To help in thinking about this, consider inserting a new indifference class, \( I, \) which contains only \( o, \) below all acceptable objects in \( \succeq_a \) (but ahead of \( \emptyset \)). Call this new preference \( \succeq_a^{+I}. \) If \( r_{\succeq_a^{+I}}(o) = r_{\succeq_a}(\emptyset), \) then call the ranking scheme unmatched-neutral, as \( \emptyset \) is treated just as any other object. If \( r_{\succeq_a^{+I}}(\emptyset) = r_{\succeq_a}(\emptyset), \) then call the ranking scheme unmatched-distinct. In this language, the example where \( \emptyset \) is treated just like any other object is unmatched-neutral, and the example where \( \emptyset \) is always given the lowest rank possible is unmatched-distinct.

Now, I define two important classes of manipulations. Call a submitted preference \( \succeq_a^{+I} \) a truncation of the true preference \( \succeq_a \) if there is some object \( o \succ_a \emptyset \) such that \( \succeq_a^{o \succ \emptyset} \) is equivalent\(^{57}\) to \( \succeq_a^{+I} \) above \( \emptyset. \) Similarly, call a submitted preference \( \succeq_a^{o \succ \emptyset} \) an extension of the true preference \( \succeq_a \) if there is some object \( \emptyset \succ_a o \) such that \( \succeq_a^{o \succ \emptyset} \) is equivalent to \( \succeq_a^{o \succ \emptyset}. \) Under these definitions, truth-telling is both a truncation and an extension (when \( o = \emptyset \)).

**Theorem 5.** If an agent’s beliefs are symmetric, his preferences are strict, and the rank scheme is upward-looking and...

- ...unmatched-distinct, then his allocation from any other strategy is weakly stochastically dominated by his allocation from playing some truncation.

- ...unmatched-neutral, then his allocation from any other strategy is weakly stochastically dominated by his allocation from playing some extension.

Note that I can’t leverage both parts of the theorem simultaneously to find a condition for truth-telling.

**Claim 7.** A ranking scheme cannot be both unmatched-neutral and unmatched-distinct.

**Proof.** If it is unmatched-neutral, then \( r_{\succeq_a^{+I}}(o) = r_{\succeq_a}(\emptyset), \) which means that \( r_{\succeq_a^{+I}}(\emptyset) > r_{\succeq_a}(\emptyset), \) contradicting the assumption that the ranking scheme was unmatched-distinct.

Truncations are a familiar class of strategies,\(^{58}\) and the intuition for why they can be profitable is simple. When the ranking scheme is unmatched-distinct, a truncation can be interpreted as an agent threatening the linear program: “give me something I like, or pay the price of giving me \( \emptyset \)”. When the ranking scheme is unmatched-neutral, this threat no longer works, as truncating actually relaxes the optimization. So why extensions? Intuitively, consider an agent whose preference is \( a \succ b \succ \emptyset \) in a

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\(^{56}\)See Footnote 17.

\(^{57}\)In the rank-value mechanism, how things are listed below \( \emptyset \) does not matter because \( \emptyset \) is never scarce.

\(^{58}\)The paper in which our informational assumption was first used, Roth and Rothblum (1999), found that, in symmetric environments, agents on the proposed-to side of a deferred acceptance mechanism can always do just as well with a truncation.
market with 100 other agents whose preferences are \( a \succ b \succ c \succ d \succ e \succ f \succ \emptyset \). If the agent doesn’t extend, the linear program will assign him to \( \emptyset \), since doing so is worth \( v_3 - v_7 \) more than the alternative of putting some other agent in \( \emptyset \).

So even in low information environments, agents can gain, but it is notable that they gain by using a class of strategies very similar to those that can theoretically be used to game two-sided deferred acceptance in a low information environment. In the lab, Featherstone and Mayefsky (2011) show that subjects can fail to truncate under a deferred acceptance mechanism, leaving a significant amount of money on the table by doing so. It would be interesting to see if a similar result held for rank-value mechanisms, and in fact, the experimental and computational work of Ünver (2001) and Ünver (2005) indicate that this is a real possibility.

9 Connection to competitive equilibrium mechanisms

In order to talk about the fairness properties of rank-value mechanisms, I must first generalize rank efficiency by allowing agents to be weighted differently in the rank distribution. In doing so, I will show that some ordinally efficient assignments are not rank efficient relative to any weights. This generalization is also of independent interest because it strongly relates to an ordinal adaptation of the much-discussed pseudomarket mechanism of Hylland and Zeckhauser (1979), in which agents are given a budget of artificial money and a competitive equilibrium is computed.

9.1 Generalizing rank efficiency

Until now, all agents have been equally weighted in the rank distribution, which is equivalent to saying that the utility an agent gets from a \( k^{th} \) choice is directly comparable to the utility any other agent gets from a \( k^{th} \) choice. I will weaken this assumption by allowing for a general weighting scheme in the rank distribution. Define the rank distribution of assignment \( x \) with respect to weights \( \alpha \) (or the \( \alpha \)-rank distribution for short) to be

\[
N_\alpha^x(k) \equiv \sum_{a \in A} \sum_{o \in O} \alpha_a \cdot 1_{\{r_a(o) \leq k\}} \cdot x_{ao}
\]

\( N_\alpha^x(k) \) is the weighted expected number of agents who get their \( k^{th} \) choice or better under assignment \( x \).

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59 The theorems just proved can help in the design of experiments that deal with the large strategy space of ordinal mechanisms, by allowing the experimenter to focus only on truncations or extensions. For a taste of how this might work, see Featherstone and Mayefsky (2011), which uses the “truncations are exhaustive” result of Roth and Rothblum (1999) to yield a more analytically tractable design.

60 The interpersonal utility comparison is actually more subtle than this; see Section 5.2 for a discussion.
Definition 12. A feasible random assignment \( x \) is \( \alpha \)-rank dominated by another feasible assignment \( \tilde{x} \) if the \( \alpha \)-rank distribution of \( \tilde{x} \) first-order stochastically dominates that of \( x \), that is, \( N^\tilde{x}_\alpha(k) \geq N^x_\alpha(k) \) for all \( k \) (strict for some \( k \)). A random assignment is called \( \alpha \)-rank efficient if it is not \( \alpha \)-rank-dominated by any other feasible assignment.

The mechanisms that correspond to \( \alpha \)-rank efficiency are a similarly modified version of the \( v \)-rank-value mechanisms.

Definition 13. The rank-value mechanism with respect to weights \( \alpha \) and valuation \( v \) (or the \((\alpha, v)\)-rank-value mechanism for short) maps agent rank orderings to a maximizer of the following linear program.

\[
\max_x \sum_{a \in A} \sum_{o \in O} \alpha_a \cdot v_{r_a(o)} \cdot x_{ao}
\]

s.t. \[
\sum_a x_{ao} \leq q_o, \forall o \in O
\]
\[
\sum_o x_{ao} \leq 1, \forall a \in A
\]
\[
x_{ao} \geq 0, \forall o \in O, \forall a \in A
\]

We say that the assignments in the arg max are supported by the \((\alpha, v)\)-rank-value mechanism.

Everything proven thus far can be easily adapted to the generalization; proofs in the Proof Appendix are done with respect to a non-specific \( \alpha \).

9.2 Supportable rank efficiency

Of course, one might not want to make any assumption about how agents’ utilities compare. The following concept makes this idea precise.

Definition 14. An assignment \( x \) is called supportably rank efficient if there exists an \( \alpha \) such that \( x \) is \( \alpha \)-rank efficient.

Supportable rank efficiency implies ordinal efficiency;\(^61\) however, it is perhaps surprising that ordinally efficient assignments can fail to be supportably rank efficient.

Theorem 6. Supportable rank efficiency implies ordinal efficiency. The converse need not hold.

The intuition behind the negative statement is rooted in appropriate generalization of the “tough decisions” trade cycles of Section A.2. For an assignment to be supportably

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\(^61\)Supportable rank efficiency implies \( \alpha \)-rank efficiency (for some \( \alpha \)) which, by the easy generalization of Proposition 6, implies ordinal efficiency.
rank efficient, there must be one \((\alpha, v)\) pair such that all trade cycles fail to be \((\alpha, v)\)-rank improving.\(^{62}\) For every trade cycle then, there is a (possibly empty) region of \((\alpha, v)\) space such that it is not \((\alpha, v)\)-rank improving. This region can be characterized by a simple inequality. The proof of the theorem starts with an ordinally efficient assignment and then defines trade cycles until the corresponding set of inequalities has no solution. There is nothing special about the counterexample; as an assignment grows large, it has many trade cycles, so unless special provision is made to keep them all from being \((\alpha, v)\)-rank improving relative to some common \((\alpha, v)\), it is no surprise that the system of inequalities cannot be satisfied. The proof suggests a natural conjecture (which I do not prove in this paper): as the market grows, the size of the set of supportably rank efficient assignments becomes small when compared to the size of the set of ordinally efficient assignments. Also, note that there is no reason that probabilistic serial assignments are special; in fact, the proof to Theorem 6 uses a probabilistic serial assignment to get the ordinally efficient assignment that forms the counterexample.

**Corollary to the proof.** Probabilistic serial does not always yield supportably rank efficient assignments.

This result might seem especially surprising in the light of McLennan (2002) and Manea (2008), who both find (using non-constructive and constructive methods, respectively) that any ordinally efficient assignment is ex ante efficient relative to some set of cardinal preferences that rationalize the ordinal preferences. The difference here is that, as shown in Section 5.2, the concept of rank-efficiency imposes the added constraint that all agents must share the same cardinal utility (in expectation) across ranks.

### 9.3 Competitive equilibrium mechanisms

An assignment is supportably rank efficient if and only if it can be supported by the \((\alpha, v)\)-rank-value mechanism for some \((\alpha, v)\). This is a straightforward corollary of the Generalization of Theorem 1 proved in Section A.1 of the Proof Appendix. It is also true, however, that an assignment is supportably rank efficient if and only if it can be generated by a certain adaptation of the pseudomarket mechanisms of Hylland and Zeckhauser (1979). The idea of these mechanisms is to 1) use a valuation vector to convert agents’ ordinal preferences to cardinal preferences, 2) give each agent a budget of artificial money, and 3) calculate a competitive equilibrium in the market where goods are probability shares in the objects. Formally,

**Definition 15.** The prices and random assignment \((\tilde{x}, \tilde{p})\), form a budget equilibrium with respect to valuation \(v\) and budgets \(B\) if, for all \(a\),

\[
\sum_k \left[ v_{rak}(o_{k-1}) - v_{rak}(o_k) \right] > 0 \text{ with } \sum_k \alpha_{ak} \cdot \left[ v_{rak}(o_{k-1}) - v_{rak}(o_k) \right] > 0.
\]

\(^{62}\)In Definition 9, replace \(\sum_k \left[ v_{rak}(o_{k-1}) - v_{rak}(o_k) \right] > 0\) with \(\sum_k \alpha_{ak} \cdot \left[ v_{rak}(o_{k-1}) - v_{rak}(o_k) \right] > 0\).
\[ \tilde{x}_a \in \arg\max_{x_a \geq 0} \sum_o v_{r(a)} \cdot x_{ao} \]
\[
\text{s.t.} \quad \sum_o \hat{p}_o \cdot x_{ao} \leq B_a \]
\[
\quad \sum_o x_{ao} \leq 1
\]

and
\[
\tilde{x}_{ao} \in \arg\max_{x_{ao} \geq 0} \sum_a \sum_o \hat{p}_o \cdot x_{ao} \\
\text{s.t.} \quad \sum_a x_{ao} \leq q_o \quad \forall o
\]

A \((B, v)\)-budget mechanism maps ordinal preferences to an assignment that is part of a \((B, v)\)-budget equilibrium, and the assignment is said to be supported by the \((B, v)\)-budget mechanism.

The first optimization can be thought of as the agent’s problem, and the second as the producer’s problem. Note that calculating a budget equilibrium is more difficult than solving the rank-value linear program.\(^\text{63}\)

**Remark 7.** The producer’s problem is equivalent to \(\sum_a x_{ao} \leq q_o\) and \(\hat{p}_o > 0 \Rightarrow \sum_a x_{ao} = q_o\), for all \(o \in O\).

The resemblance\(^\text{64}\) of budget equilibria to the more familiar competitive equilibrium suggests that there should also be results resembling the first and second welfare theorems. Supportable rank efficiency is the efficiency concept needed to complete the analogy. Formally,

**Theorem 7.** \(x\) is supportably rank efficient if and only if there exist budgets and a valuation, \((B, v)\), such that \(x\) is supported by the \((B, v)\)-budget mechanism.

The intuition for this proof comes from considering the linear programming duals of the optimizations involved in the \((\alpha, v)\)-rank-value mechanism and the \((B, v)\)-budget mechanism. If \(x\) is a supportably rank efficient assignment, then it must be in the \(\arg\max\) of the \((\alpha, v)\)-rank-value mechanism for some \((\alpha, v)\). Keeping the same valuation \(v\) for the budget equilibrium, the shadow prices associated with the object quota constraints will decentralize \(x\), so long as each agent \(a\) can afford his allocation, \(x_a\), which can be guaranteed by giving him a budget \(B_a = \sum_o \hat{p}_o \cdot x_{ao}\). For the reverse implication, start with an assignment \(x\) that is a \((B, v)\)-budget equilibrium, keep the

\(^\text{63}\)Budget equilibria can be approximately solved using a method laid out in the appendix of Hylland and Zeckhauser (1979), which in turn grapples with the known-to-be-difficult problem of approximating Kakutani fixed points (Scarf 1967, Scarf and Hansen 1973, Chen and Deng 2008). Polynomial time methods are known for markets in which agents are not limited by the \(\sum_o x_{ao} \leq 1\) constraint (Nisan 2007), but it is unclear whether such results can be generalized to budget equilibria.

\(^\text{64}\)Note that the \((\alpha, v)\)-rank-value mechanism can be decentralized to look like a competitive equilibrium as well. This decentralization, which I call an \((\alpha, v)\)-discount equilibrium is defined and characterized in Section D of the Proof Appendix.
same valuation, and set $\alpha_a$ equal to the inverse of the budget shadow price in agent $a$’s optimization.\footnote{Extra care must be taken when the shadow price is zero. This case is worked out in more detail in the Proof Appendix.}

An important corollary to the proof of this proposition is that, for a given valuation $v$ and set of preferences $\succeq$, there is a natural mapping between weights $\alpha$ and budgets $B$. Although not immediately obvious, the proof also shows that assignments in the arg max of the procedurally fair (all agents are weighted the same) $v$-rank-value mechanism are supported by a budget equilibrium that is not procedurally fair (budgets are not guaranteed to be equal). Similarly, budget equilibrium from equal budgets gives assignments that are supported by an $(\alpha, v)$-rank-value mechanism in which the weights are not guaranteed to be equal. In this sense, procedural fairness is fundamentally different between rank-value mechanisms and budget mechanisms. In the next section, I will show how this tension sheds light on the theoretical incompatibility of rank efficiency and envy-freeness.

10 Justice and rank-efficiency

Looking at the example from Figure 1a, one might object that Agent 1 has been unfairly singled out. Envy-freeness is the idea that makes this intuition precise. Let an assignment be strongly envy-free if $x_a$ weakly first-order stochastically dominates $x_{a'}$ relative to $\succeq_a$, for all $a, a' \in A$. Let an assignment be weakly envy-free if there is no $a' \in A$ such that $x_{a'}$ strongly first-order stochastically dominates $x_a$ relative to $\succeq_a$, for all $a \in A$. Finally, let an assignment be envy-free relative to some cardinal preferences if each agent weakly prefers his own allocation to all other agents’ allocations.

**Theorem 8.** No mechanism is rank efficient and even weakly envy-free

**Proof.** Consider the example from Figure 1a. $\blacklozenge$ is the unique rank efficient assignment, so any rank efficient mechanism must choose it. But 1 would prefer 2’s allocation, regardless of the cardinal preferences that rationalize his ordinal preference. \hfill $\square$

This theorem implicitly deals with two philosophical concepts of justice. Dworkin argues for the dual requirements of ex post efficiency and envy-freeness, which he calls “equality of resources” (Dworkin 1981).\footnote{In fact, Dworkin (1981) literally suggests a cardinal version of this paper’s budget equilibrium from equal budgets as an embodiment of the idea.} Harsanyi, in contrast, argues for policy to be chosen as if by an expected utility maximizer in the Rawlsian original position\footnote{Although Harsanyi’s conception of justice is based on Rawls’ veil-of-ignorance, the Difference Principle (i.e. insistence on a maximin objective from the original position) places Rawls’ conception more in line with Dworkin’s (Rawls 1972).}, a concept which I will call “Harsanyian justice” (Harsanyi 1975). In terms
of these concepts, Theorem 8 says that in assignment markets, *equality of resources* and *Harsanyian justice* cannot be jointly guaranteed. This is no surprise in view of the proof of Theorem 7. An assignment implemented by a budget equilibrium from equal budgets is, in some sense, envy-free, since all agents have the same opportunity set. But this same assignment might also be implementable by a rank-value mechanism with unequal weights, contradicting *Harsanyian justice*. To move forward from Theorem 8 then, one must make a choice. As discussed in Section 5.2, the rank-value mechanism with equal weights guarantees Harsanyian justice (modulo a few assumptions). But how does one guarantee *equality of resources*?

**Proposition 10.** If there exist prices \( \hat{p} \), a valuation \( v \), and budgets \( B \), such that \( B_a = B_{a'} \) for all \( a, a' \in A \) and \( (x, \hat{p}) \) is a \((B, v)\)-budget equilibrium, then \( x \) is weakly envy-free and supportably rank efficient. Further, \( x \) is envy-free relative to the cardinal preferences encoded by \( v \).

**Proof.** Theorem 7 established that budget equilibria are supportably rank efficient. Now, assume that \( x_{a'} \) strongly first-order stochastically dominates \( x_a \) relative to \( \succsim_a \). Then, for any \( v \), agent \( a \)'s objective in the definition of the \((B, v)\)-budget equilibrium must be higher at \( x_{a'} \) than at \( x_a \). But \( a \) could have afforded \( x_{a'} \) by the equal budgets assumption. This contradicts optimality. Envy-freeness relative to \( v \) follows by similar logic.

So, budget equilibrium from equal budgets yields weak envy-freeness and supportable rank efficiency (which implies ex post efficiency). Although this provides a partial workaround to the impasse of Theorem 8, weak envy-freeness is not a strong condition: it only guarantees that there exist rationalizing cardinal utilities that make the assignment envy-free. Still, the exercise is very similar to how the \( v \)-rank-value mechanism can be justified as maximizing social welfare: in the absence of cardinal information, make assumptions and push forward. Under such assumptions, budget equilibrium from equal budgets yields an envy-free assignment. Finally, note that the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) is strongly envy free, but as shown in Section 9.2, it may not be ex ante efficient in a world where beliefs about agents’ cardinal preferences meet the conditions of Proposition 5.\(^{69}\)

### 11 Conclusion

Rank efficiency is a natural concept that is used in the field, and under truth-telling, rank efficient mechanisms can yield significant welfare gains. Unfortunately, rank efficiency is also theoretically incompatible with strategy-proofness. This does not mean, however, that the market designer should freely dismiss it. In the words of Day and Milgrom (2008),

\[^{69}\text{Note that the wedge between fairness and efficiency exists in the cardinal setting as well. Competitive equilibrium from equal incomes (Varian 1974, Hylland and Zeckhauser 1979) gives Pareto efficiency, but Negishi’s theorem (Negishi 1960) does not guarantee that the corresponding weights in the planner’s problem will be equal.}\]
“...it is customary in mechanism design theory to impose incentive constraints first, investigating other aspects of performance only later. It is, of course, equally valid to begin with other constraints and such an approach can be useful. To the extent that optimization is only an approximation to the correct behavioral theory for [agents], it is interesting to investigate how closely incentive constraints can be approximated when other constraints are imposed first.”

The situation in this paper is not unique: there is a growing body of literature that points to the fact that the costs of strategy-proofness can be quite high. So, when then should the market designer pay this cost? In this paper, I show that truth-telling is an equilibrium of rank-value mechanism in a stylized, low-information environment. Such a theorem highlights the important intuition that rank-value mechanisms are more likely to work when agents know little about the popularity of the objects. Even so, it seems likely that there exist environments in which the performance of rank efficient mechanisms will be unacceptably poor. Unfortunately, any such characterization is again likely to be limited to a stylized model, and characterizations that truly generalize across environments seem intractable. In addition, it is certainly plausible that in more complicated situations, equilibrium predictions will be not be born out in the field (see Featherstone and Niederle (2011) and Featherstone and Mayefsky (2011)). Thus, to understand when the costs of strategy-proofness are too dear, theoretical results will need to be complemented with more empirical approaches, such as experiments and learning models.

Regardless of such evidence however, it may be that in the long run, in any environment in which a mechanism fails to admit a truth-telling equilibrium, agents will eventually converge to a manipulating equilibrium. Even if this is true, it might still be worthwhile to consider how much the short run can be extended. Consider the HBS match discussed in Section 7. In the first year of the match, had we run a rank efficient mechanism, it is quite plausible that agents would have truthfully revealed their preferences. We didn’t do this, however, because we were worried about the MBAs learning to manipulate over time. Taking extra efficiency in the first year, at the expense of subsequent years, seemed like too much of a “smash-and-grab” strategy. But what if the learning process were slower, such that the efficiency gains could be reaped for the first five years? In such a setting, the “smash-and-grab” strategy starts to sound like a good idea; in fact, one could even imagine running a rank efficient mechanism for four years and then switching to a strategy-proof mechanism as learning began to cause problems. Thinking about ways to slow the speed with which agents learn to manipulate a non-strategy-proof mechanism seems like an interesting, and thus far, unpursued line of research. When the efficiency cost of strategy-proofness is high, as I have shown can be the case, this line of inquiry could prove quite fruitful.

While there are many papers on the costs of strategy-proofness in matching mechanisms, the costs of Pareto-agnosticism have not been as carefully considered. In this paper, I show that if the policy-maker truly considers all k-th choice assignments as interchangeable, then by allowing him to express this, the rank-value mechanism can
yield big welfare gains. For more complicated welfare considerations, the \((\alpha, v)\)-rank efficient mechanisms or the generalizations of rank-value mechanisms mentioned at the end of Section 5.2 could help. The lessons from the mechanisms in this paper are then twofold. First, although economists are not in a position to make ethical judgements about interpersonal utility comparison, policy-makers are. It makes sense then for market designers to use mechanisms that have a place for policy-maker’s judgements to be inputted, like the \(\alpha\) in the \((\alpha, v)\)-rank-value mechanisms. Second, armed with the power to make such judgements, market designers can significantly improve welfare.

The fact that there are rank efficient mechanisms in the field could mean that they are successful in some situations. Alternatively though, it might be that well-intentioned policy-makers, ignorant of incentive concerns, are implementing bad policies. Market designers need to know whether they should be correcting the impulse to depart from strategy-proofness or sometimes suggesting that it is a small price to pay for potentially big welfare gains. Which advice is good very likely depends on the environment in which the mechanism is meant to serve and on the dynamics of the social learning process. Understanding when (if ever) market designers can comfortably suggest a rank efficient mechanism seems like an important open question.
References


Proof Appendix

A  Rank efficiency

A.1  Characterization of rank-value mechanisms

Here, I generalize Theorem 1 to the $\alpha$-rank efficiency concept defined in Section 9.

**Generalization of Theorem 1.** $x$ is an $\alpha$-rank efficient assignment if and only if there exists a valuation $v$ such that $x$ is supported by the $(\alpha, v)$-rank-value mechanism.

**Proof.** Start by proving the “if” part. The objective of the defining linear program can be rewritten as $\sum_{k=1}^{\lvert \mathcal{O} \rvert - 1} N_{\alpha}^x(k) (v_k - v_{k+1}) + |\mathcal{A}| \cdot v_{\lvert \mathcal{O} \rvert}$. By way of contradiction, let $\tilde{x}$ be a random assignment that $\alpha$-rank-dominates the allocation yielded by the $(\alpha, v)$-rank-value mechanism, $x^*$. Then, the following is true:

$$\sum_{k=1}^{\lvert \mathcal{O} \rvert - 1} N_{\alpha}^x(k) \cdot (v_k - v_{k+1}) + |\mathcal{A}| \cdot v_{\lvert \mathcal{O} \rvert} > \sum_{k=1}^{\lvert \mathcal{O} \rvert - 1} N_{\alpha}^{x^*}(k) \cdot (v_k - v_{k+1}) + |\mathcal{A}| \cdot v_{\lvert \mathcal{O} \rvert}$$

because the $\alpha$-rank dominance of $\tilde{x}$ over $x^*$ means that the left-hand side is weakly greater than the right-hand side, term by term, and strictly so for at least one term. But this is a contradiction of the optimality of the $(\alpha, v)$-rank-value mechanism’s defining linear program.

Now, I prove the “only if” part. Let $A_{\alpha} \subset \mathbb{R}^{\lvert \mathcal{O} \rvert - 1}$ denote the set of $\alpha$-rank distributions that can be generated some random assignment, where the first index of the vector corresponds to $N_{\alpha}^x(1)$, the second to $N_{\alpha}^x(2)$, and so on. By the definition of an $\alpha$-rank distribution, $A_{\alpha}$ is convex. Now, let $b \in A_{\alpha}$ be the $\alpha$-rank distribution generated by $x$, the $\alpha$-rank efficient assignment from the hypothesis of the theorem. The set of $\alpha$-rank distributions that would $\alpha$-rank dominate $b$ if they were generated by feasible random allocations is $U(b) \equiv \{a \mid a \in \mathbb{R}^{\lvert \mathcal{O} \rvert - 1}; a_i \geq b_i, \forall i; \exists i', a_i > b_{i'}\}$. $U(b)$ is also convex. Any member of both $A_{\alpha}$ and $U(b)$ would $\alpha$-rank dominate $x$, so it must be that $A_{\alpha} \cap U(b) = \emptyset$. The separating hyperplane theorem then tells us that there is some $p \neq 0$ and some $c$ such that $p \cdot u \geq c, \forall u \in U(b)$ and $p \cdot a \leq c, \forall a \in A_{\alpha}$. By construction, $p \cdot b = c$. Without loss of generality, assume $c \geq 0$. Now, let $e_i$ denote the unit vector that points along the $i^{th}$ axis of $\mathbb{R}^{\lvert \mathcal{O} \rvert - 1}$. For any $\delta > 0$, it must be that $b + \delta e_i \in U(b)$. This means that $p \cdot (b + \delta e_i) = c + \delta p_i \geq c$, which means that $p_i \geq 0$. Consider the valuation $v$ where $v_i = 1 + \sum_{j=i}^{\lvert \mathcal{O} \rvert - 1} p_j, \forall i \in \{1, \ldots, \lvert \mathcal{O} \rvert - 1\}$, and $v_{\lvert \mathcal{O} \rvert} = 1$. By construction, the $(\alpha, v)$-rank-value mechanism supports $x$. But because I have yet to prove that $p_i > 0, \forall i$, I am not sure that $v$ is a valid valuation, that is $v_k > v_{k+1}, \forall k$.

To show this will require the stronger Polyhedral Separating Hyperplane Theorem of McLennan (2002). Adopting the notation of that paper’s Theorem 2, let $P = A_{\alpha} - \mathbb{R}^{\lvert \mathcal{O} \rvert - 1}$ (which is a polyhedron by Theorem 1.2 of Ziegler (1995)) and $b = p \in P$. 

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The $\alpha$-rank efficiency of $x$ tells us that $x$ is not in the relative interior\textsuperscript{70} of $P$, so the Theorem lets us conclude that there is a hyperplane $H$ such that $P$ is contained in one of its half-spaces, $P \subset H^{-}$, and $F \equiv P \cap H$ is the smallest face of $P$ that contains $b$. Pick $p$ and $c$ such that $H = \{u \in \mathbb{R}^{m-1} | p \cdot u = c\}$. First, note that for $\delta > 0$, $b - \delta e_i \in P$, by definition. Now, say that $p_i = 0$. Then, $b - \delta e_i \in H$ as well, which means that $b - \delta e_i \in F$. Lemma 2 of McLennan (2002) with $S = \{b\}$ tells us that the relative interior of $S$, that is $b$,\textsuperscript{71} is contained in the relative interior of the smallest face that contains $S$, that is $F$. Hence, for some $\delta > 0$ small enough, $b + \delta e_i \in F$. But since $F \subset P$, this contradicts our original assumption of the $\alpha$-rank-efficiency of $x$. Hence, $p_i > 0, \forall i$.

\[\Box\]

### A.2 Rank-improving cycles and improvement cycles

Without changing the number of copies of the objects (including $\emptyset$), consider adding $\sum_o q_o - |A|$ dummy agents to the market that are indifferent between all the objects. Call this the \textit{augmented market}. Note that the dummy agents are different from the fictional agent introduced in the definition of claim-trade cycles.\textsuperscript{72} The \textit{extension} of $x$ to the augmented market is an assignment $\tilde{x}$ in which all real agents’ allocations stay the same and the remaining objects are feasibly assigned to the dummy agents.

**Lemma 1.** Let $\tilde{x}$ be the extension of $x$ to the augmented market. Then, $x$ is non-wasteful if and only if there does not exist an improvement cycle in $\tilde{x}$ that includes a dummy agent.

**Proof.** ("only if") If there is an improvement cycle that includes a dummy agent, then there is also a two agent improvement cycle where one of the agents is the dummy. Since the dummy is indifferent between all the objects and is holding something that was not assigned in $x$, this cycle is equivalent to the real agent claiming something that was not assigned, contradicting the assumption that $x$ was non-wasteful. ("if") If $x$ is wasteful, then $\tilde{x}$ must have a two person improvement cycle between the real agent who wants to claim an object unassigned under $x$ and the dummy agent who was assigned that object in $\tilde{x}$.

\[\Box\]

**Proposition 7** (Bogomolnaia and Moulin 2001). An assignment $x$ is ordinally efficient if and only if $x$ is non-wasteful and admits no improvement cycles.

**Proof.** This is proven with $|A| = \sum_o q_o$ for preferences with indifferences in Katta and Sethuraman (2006), and without indifferences by Bogomolnaia and Moulin (2001). Ordinal efficiency in a market with $|A| > \sum_o q_o$ is equivalent to ordinal efficiency in the augmented market, so ordinal efficiency is equivalent to no cycles in the augmented market. A cycle in the augmented market either contains a dummy or doesn’t, so

\textsuperscript{70}The relative interior of a set $S$ is its interior in the relative topology of its affine hull. The affine hull of $S$ is the set of all affine combinations of elements of $S$. An affine combination of a set of elements $\{s_1, \ldots, s_k\}$ of $S$ is a sum $\sum_i \alpha_i \cdot s_i$, where $\alpha_i \in \mathbb{R}, \forall i$ and $\sum_{i=1}^{k} \alpha_i = 1$.

\textsuperscript{71}The affine hull of $\{b\}$ is $\{b\}$.

\textsuperscript{72}The fictional agent is, in some sense, a “super” dummy that owns the shares held by all of the smaller dummies.
by Lemma 1, a cycle in the augmented market is equivalent to either a cycle or wastefulness in the regular market.

Before addressing \( \alpha \)-rank efficiency, I re-introduce the generalization of a \( v \)-rank-improving cycle that was relegated to Footnote 62 in the main text.

**Definition 16.** A trade cycle \( \tau \) on assignment \( x \) is an \((\alpha, v)\)-rank-improving cycle  
\[
|\tau| \sum_{k=1}^{|\tau|} \alpha_{a_k} \cdot [v_{r_{a_k}(o_k)} - v_{r_{a_k}(o_{k-1})}] > 0 \text{ (with } o_0 \equiv o_{|\tau|}).
\]

**Lemma 2.** Let \( \tilde{x} \) be the extension of \( x \) to the augmented market. Then, \( x \) admits no \((\alpha, v)\)-rank improving cycles if and only if \( \tilde{x} \) admits no \((\alpha, v)\)-rank improving cycles.

**Remark 8.** Note that while there can claim-trade cycles in \( x \), there can’t in \( \tilde{x} \), since by construction, all object shares are claimed.

**Proof.** (“only if”) By way of contradiction, assume that \( \tau \) is an \((\alpha, v)\)-rank improving cycle in \( \tilde{x} \). If \( \tau \) has exactly one dummy, then it is already an \((\alpha, v)\)-rank improving claim-trade cycle in \( x \), a contradiction. If \( \tau \) has multiple dummies, then it can be partitioned into several claim-trade cycles in \( x \) by taking the parts between any dummy and the next dummy in \( \tau \). Because \( \tau \) is overall \((\alpha, v)\)-rank improving, at least one of the elements of the partition must be an \((\alpha, v)\)-rank improving claim-trade cycle in \( x \), a contradiction. Obviously, if \( \tau \) has no dummies, then it is also an \((\alpha, v)\)-rank improving trade cycle in \( x \), a contradiction. (“if”) By way of contradiction, assume that \( \tau \) is an \((\alpha, v)\)-rank improving claim-trade cycle in \( x \). Then an \((\alpha, v)\)-rank improving trade cycle can be constructed in \( \tilde{x} \) by letting \( a_1 \) be a dummy that is holding \( o_1 \), a contradiction. Now, assume that \( \tau \) is an \((\alpha, v)\)-rank improving trade cycle in \( x \). It must also be an \((\alpha, v)\)-rank improving trade cycle in \( \tilde{x} \), a contradiction. \( \square \)

**Generalization of Theorem 2.** \( x \) is \( \alpha \)-rank efficient if and only if there exists a valuation \( v \) such that \( x \) admits no \((\alpha, v)\)-rank-improving cycles.

**Proof.** (“only if”) By way of contradiction, assume that for any valuation \( v \), there exists an \((\alpha, v)\)-rank improving cycle. This would yield an improvement in the objective of the \((\alpha, v)\)-rank-value mechanism, which means that \( x \) cannot be generated by any \((\alpha, v)\)-rank-value mechanism, which, by the Generalization of Theorem 1 in Section A.1 of the Proof Appendix, means that \( x \) is not \( \alpha \)-rank efficient.

(“if”) Consider the extension of \( x \) to the augmented market, \( \tilde{x} \). Now, by way of contradiction, let \( \hat{x} \) \( \alpha \)-rank dominate \( \tilde{x} \) in that market. Let \( \xi \equiv \hat{x} - \tilde{x} \). Necessarily, \( \sum_a \xi_{ao} = 0 \) and \( \sum_o \xi_{ao} = 0 \). Construct a cycle by the following procedure. First, find some \((a_1, o_1)\) such that \( \xi_{a_1o_1} < 0 \). Then, find some \( a_2 \) such that \( \xi_{a_1o_1} > 0 \). Next, find some \( o_2 \) such that \( \xi_{a_1o_1} < 0 \). In words, this process starts by picking a negative element in \( \xi \), then goes down the column to a positive element, then across the row to a negative element, and so on. Continue building a sequence of agents and objects in this way, until an agent is visited twice, i.e., a cycle is found. This process is well-defined because any row or column with a non-zero element must have another non-zero element of the opposite sign, since all rows and columns sum to zero. The
process will eventually terminate because the market is finite. By construction, the
process finds sequence of agents and objects that is a trade cycle. Denote it by \( \tau \).
Now, let \( \Xi = \min_{(a,o) \in \tau} |\xi_{ao}| \) and define:

\[
\xi_{ao}^1 = \begin{cases} 
\Xi & \exists k \text{ s.t. } (a, o) = (a_k, o_k) \\
-\Xi & \exists k \text{ s.t. } (a, o) = (a_{k+1}, o_k) \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \xi - \xi^1 \) has at least one fewer non-zero entries than \( \xi \). Hence, this process can be
repeated to decompose \( \xi \) into a sum of trade cycles, that is \( \xi = \sum_i \xi^i \) where each \( \xi^i \)
represents a trade cycle in the augmented market. Then, \( \hat{x} = \tilde{x} + \sum_i \xi^i \). Now, since
\( \hat{x} \) \( \alpha \)-rank dominates, the objective of the \( (\alpha, v) \)-rank-value mechanism is bigger at \( \hat{x} \),
regardless of the valuation. By the linearity of the objective of the \( (\alpha, v) \)-rank-value
mechanism, it cannot be that the value of the objective evaluated at \( \xi^i \) is weakly
negative for all \( i \). In other words, regardless of the valuation, at least one of the \( \xi^i \)'s
must be \( (\alpha, v) \)-rank improving, which by Lemma 2 means that there is an \( (\alpha, v) \)-rank
improving cycle in \( \tilde{x} \), which by Lemma 2 is equivalent to an \( (\alpha, v) \)-rank improving
cycle in \( x \), a contradiction.

**Proposition 4.** If all elements of the initial assignment matrix are rational numbers,
then the sequential improvement process will terminate in a finite number of steps with
a rank efficient assignment that is in the arg max of the \( v \)-rank-value mechanism.

**Proof.** Implementing an \( (\alpha, v) \)-rank-improving cycle transforms deterministic assign-
ments into deterministic assignments. It also necessarily increases the objective of
the \( (\alpha, v) \)-rank-value mechanism, so the algorithm cannot cycle. Since the set of de-
terministic assignments is finite, the algorithm must terminate. Note that the same
logic works if all entries in the initial matrix are rational: simply multiply the starting
matrix by the least common denominator.

**Lemma 3.** Let \( x \) be a deterministic assignment. Then \( x \) is ex post efficient if and
only if it is non-wasteful and admits no improvement cycles.

**Proof.** The “only if” follows from the fact that implementing any improvement cycle
in the augmented market immediately yields an ex post dominator. For the “if” part,
consider the extension of \( x \) to the augmented market, \( \tilde{x} \) and assume it has an ex post
dominator, \( \hat{x} \). Using the decomposition strategy of the proof to the Generalization of
Theorem 2, one can move from \( \hat{x} \) to \( \hat{x} \) through a series of trade cycles. But, because
both \( \tilde{x} \) and \( \hat{x} \) are deterministic assignments, for all the trade cycles, \( \Xi \) will equal 1.
Hence, each agent can only be involved in, at most, one of the trade cycles, and hence,
all of the trade cycles must be improvement cycles. By Lemma 1, these cycles map
back to either an improvement cycle, or wastefulness in \( x \), a contradiction.

**Proposition 8.** Let \( x \) be a deterministic assignment. Then,

(i) \( x \) is ordinally efficient if and only if it is ex post efficient.
(ii) $x$ is rank efficient implies it is ex post efficient; however the converse need not hold.

**Proof.** Part (i) follows directly from Proposition 7 and Lemma 3. Now, I prove (ii). Proposition 3 establishes that rank efficiency implies ex post efficiency. To see that the converse need not hold, consider $x$ and $y$ from Figure 1a. Both are ex post efficient deterministic assignments, but only one is rank efficient. \qed

### A.3 Ex ante efficiency interpretation

The proof in the main text only addresses uniform weights. To justify an $(\alpha, v)$-rank-value mechanism, I need

$$\alpha \cdot \int v_{a}k \cdot F(v) = \alpha' \cdot \int v_{ak} \cdot F(v).$$

In other words, the unconditional expectations of $v_{a}$ for all agents should be proportional, and the constants of proportionality should be the appropriate weights from the $\alpha$ vector.

### B Truth-telling in low information environments

I allow for indifferences and a general weighting of the agents in all theorems I will be deriving, so when I generalize relative to a statement in the main text, I will note that I have done so. The general scheme of the proofs in this section borrow heavily from Roth and Rothblum (1999), but the proofs themselves are new, as they must be adapted to the rank-value mechanism, which is quite different from the more algorithmically defined deferred acceptance algorithm. Let $RV_M^{\alpha,v}(\succsim_{a}: q; \tau)$ denote $a$’s allocation under the $(\alpha, v)$-rank-value mechanism when the submitted preferences are $\succsim_{a}$, the capacities are $q$, and the tie-breaker ordering is $\tau$. Let the objective function of the $(\alpha, v)$-rank-value mechanism evaluated at assignment $x$ and submitted preferences $\succsim$ be denoted by

$$V^{\alpha,v}(\succsim;x) = \sum_{a''} \sum_{o''} \alpha_{a''} \cdot v_{r_{a''}(o'')} \cdot x_{a''o''}$$

**Lemma 4.** Let $o' \succ_{a} o$. Then,

$$[RV_{M_{a}}^{\alpha,v}(\succsim_{a}: q; \tau) = o] \Rightarrow [RV_{M_{a}}^{\alpha,v}(\succsim_{a}^{{o'\rightarrow o}}, \succsim_{-a}^{-}: q; \tau) = o]$$

**Proof.** Swapping $o$ and $o'$ changes the objective in the following way:

$$V^{\alpha,v}(\succsim_{a}^{{o'\rightarrow o}}, \succsim_{-a}^{-}; x) = V^{\alpha,v}(\succsim_{a}; x) + \alpha_{a} \cdot (v_{r_{a}(o')} - v_{r_{a}(o)}) \cdot (x_{ao} - x_{ao'})$$

(1)

Now, let $y = RV_{M_{a}}^{\alpha,v}(\succsim_{a}: q; \tau)$, which tells us that $y_{ao} = 1$ and $y_{ao'} = 0$, which means

$$V^{\alpha,v}(\succsim_{a}; y) - V^{\alpha,v}(\succsim_{a}^{{o'\rightarrow o}}, \succsim_{-a}^{-}; y) = -\alpha_{a} \cdot (v_{r_{a}(o')} - v_{r_{a}(o)}) < 0$$

Now consider another assignment $z \neq y$ that is a member of the arg max of the linear program when $o$ and $o'$ are switched, that is

$$V^{\alpha,v}(\succsim_{a}^{{o'\rightarrow o}}, \succsim_{-a}^{-}; z) \geq V^{\alpha,v}(\succsim_{a}^{{o'\rightarrow o}}, \succsim_{-a}^{-}; y)$$

(2)
First, I show that $z_{ao} = 1$. Assume otherwise. Then, by Equation 1, I derive

$$V^{\alpha,v}(z; z) - V^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; z) \geq 0 > V^{\alpha,v}(z; y) - V^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; y)$$

If I add this inequality to Inequality 2, I find that $y$ was not an optimum to start with, a contradiction. Hence, $a$ must receive $o$ when he switches. Now, if $z_{ao} = 1$, then following the same logic, I derive

$$V^{\alpha,v}(z; z) - V^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; z) = V^{\alpha,v}(z; y) - V^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; y)$$

and combine it with Inequality 2 to find that either $y$ was not an optimum to start with (a contradiction), or that $z$ is a member of the arg max under both normal and switched preferences. But then, the only way that $z$ could be chosen under the switch is if $\tau(z) < \tau(y)$, which would contradict $y$ being chosen under the truth. □

**Corollary** (to Lemma 4). Let $o' \succ_a o$. Then,

$$\left[ RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; \tau) = o^\prime \right] \Rightarrow \left[ RV M_{a}^{\alpha,v}(z; \tau) = o^\prime \right]$$

**Proposition 9.** Under a rank-value mechanism, if agent $a$’s beliefs are $\{o, o^\prime\}$-symmetric, and $o' \succ_a o$, then for $a$, the allocation resulting from any submitted preference that declares $o \succ_a o^\prime$ is weakly stochastically dominated by the allocation resulting from a submitted preference that does not. If the beliefs are symmetric, then this holds for all $o, o' \in \mathcal{O} \setminus \{\emptyset\}$.

**Proof.** First, note that since the linear program does not depend on labels,

$$[RV M_{a}^{\alpha,v}(z; q; \tau) = v] \Leftrightarrow [RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; q^{o \preceq o'}; \pi^{o \preceq o'}) = v]$$

$$[RV M_{a}^{\alpha,v}(z; q; \tau) = o] \Leftrightarrow [RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; q^{o \preceq o'}; \pi^{o \preceq o'}) = o']$$

where $o, o' \in \mathcal{O}$ and $v \in \mathcal{O} \setminus \{o, o'\}$. Then, it must also be that

$$[RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; q; \tau) = v] \Leftrightarrow [RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; q^{o \preceq o'}; \pi^{o \preceq o'}) = v]$$

$$[RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; q; \tau) = o] \Leftrightarrow [RV M_{a}^{\alpha,v}(\pi_a^{o \preceq o'}; \pi_{-a}; q^{o \preceq o'}; \pi^{o \preceq o'}) = o']$$

Now, using Lemma 4 and its corollary, one can think about the gains and losses an agent $a$ who prefers $o'$ to $o$ gets from switching those two objects. The possible cases are shown in Table 2. Using these cases, one can then think about what our agent gets when everyone else’s preferences switch $o$ and $o'$, which is equally likely due to the assumption of $\{o, o'\}$-symmetric beliefs. This analysis is shown in Table 3. In each of the cases, considering that everyone else switching $o$ and $o'$ is equally likely, truth-telling yields an allocation for $a$ that weakly stochastically dominates what she would have gotten by switching $o$ and $o'$. □
Lie: $RVM_a^{α, v} (\succsim_a^{o+o'}; \succsim_{-a}; q; τ) = u ∉ \{o, o'\} = o' = o$

Truth: $RVM_a (\succsim_a; \succsim_{-a}; q; τ) = o'$ Case A Impossible Case B

$= o$ Case C Case D Case E

Impossible Impossible

Table 2: Cases

<table>
<thead>
<tr>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
<th>Case D</th>
<th>Case E</th>
<th>Case F</th>
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</table>

Table 3: Assignments under the different cases

The condition in Proposition 9 is important enough to warrant a name.

**Definition 17.** A reported preference $\succsim_a'$ is called **weakly** $\{o, o'\}$-order-preserving with respect to $\succsim_a$ if $[o' \succsim_a o] \Rightarrow [o' \succsim_a' o]$. If this is true for all $o, o' \in O \setminus \emptyset$, then just call the reported preference **weakly order-preserving with respect to** $\succsim_a$.

Notice that if I am dealing with a world where only strict preferences are allowed, then this property basically says that objects besides $\emptyset$ are listed in their true order, which is where I get Theorem 4. If I allow for indifferences, then I am not insisting on truthful ordering, but instead are insisting that a true strict ordering of two objects is never reported reversed. Before moving on, I review a notation that was only briefly introduced in the main text. Let the true preference be $\succsim_a$ and the preference that adds an indifference class $I$ of objects right above $\emptyset$ be $\succsim_a^{+I}$. For some object $o$, let $\succsim_a^{+o}$ denote $\succsim_a^{+\{o\}}$.

**Lemma 5.** If the ranking scheme is upward-looking and unmatched-distinct then, submitting a preference where the bottom declared-acceptable indifference class consists entirely of truly unacceptable objects is weakly stochastically dominated by the submitting the same preference with the bottom declared-acceptable indifference class dropped.

**Proof.** By way of contradiction, assume that $I$ is a set of unacceptable objects, and that submitting $\succsim_a^{+I}$ makes $a$ better off, that is $RVM_a (\succsim_a^{+I}; \succsim_{-a}; q; τ) \equiv z$, \[ \]
Define Proof. is a truly acceptable object that is declared unacceptable in Lemma 6. If the ranking scheme is upward-looking and unmatched-neutral, then if $\emptyset$ lowest indifference class of $\succsim^+_a$, but it could either be in a higher indifference class, or it could be $\emptyset$.

**Case 1.** ($y_a$ is ranked above $\succsim^+_a$) $z_a$ must also be ranked above the bottom acceptable indifference class of $\succsim^+_a$, so (since I have assumed that the rankings are upward-looking) $V(\succsim^+_a); z) = V(\succsim^+_a); y)$ and $V(\succsim^+_a); z) = V(\succsim^+_a); y)$. Combined with the optimality inequalities, these equalities imply that $y$ and $z$ are both in the arg max under both preferences. This cannot be, since the tiebreaker $\tau$ would choose one or the other.

**Case 2.** ($y_a = \emptyset$) $z_a$ is necessarily not in the lowest acceptable indifference class of $\succsim^+_a$, so $V(\succsim^+_a); z) = V(\succsim^+_a); y)$. Now since the ranking scheme is unmatched-distinct, $V(\succsim^+_a); y) = V(\succsim^+_a); y)$. Stringing inequalities together, I find that $y$ and $z$ are both in the arg max under both preferences. This cannot be, since the tiebreaker $\tau$ would choose one or the other.

**Lemma 6.** If the ranking scheme is upward-looking and unmatched-neutral, then if $o$ is a truly acceptable object that is declared unacceptable in $\succsim_a$, then

$$RV M_a^{\alpha,v}(\succsim, q; \tau) \succsim_a RV M_a^{\alpha,v}(\succsim^+_a; \succsim^+_a; q; \tau)$$

**Proof.** Define $y \equiv RV M_a^{\alpha,v}(\succsim, q; \tau)$ and $z \equiv RV M_a^{\alpha,v}(\succsim^+_a; \succsim^+_a; q; \tau)$. Since $y$ is an arg max under $\succsim$, I have $V^{\alpha,v}(\succsim, y) \geq V^{\alpha,v}(\succsim, z)$, and since $z$ is an arg max under $(\succsim^+_a; \succsim^+_a)$, I have $V^{\alpha,v}(\succsim^+_a, \succsim^+_a; y) \geq V^{\alpha,v}(\succsim^+_a, \succsim^+_a; y)$.

Now, by way of contradiction, assume that $y_a \succsim a z_a$. There are three cases.

**Case 1.** ($z_a = \emptyset$) The ranking scheme is unmatched-neutral, so $V^{\alpha,v}(\succsim, z) > V^{\alpha,v}(\succsim^+_a, \succsim^+_a; z)$. Since $y_a$ is above the part of the rank-order that has changed, I know $V^{\alpha,v}(\succsim^+_a, \succsim^+_a; y) = V^{\alpha,v}(\succsim, y)$ by the upward-looking assumption. Combined with the optimality inequalities, I quickly arrive at a contradiction.

**Case 2.** ($z_a = o$) Denote by $z|^{(a, b)}$ the assignment I get by starting with $z$ and only moving $a$ from $o$ to $\emptyset$. This is feasible, so since $y$ is an arg max under $\succsim$, I know that $V^{\alpha,v}(\succsim; y) \geq V^{\alpha,v}(\succsim; z|^{(a, b)})$ By unmatched-neutrality, $V^{\alpha,v}(\succsim; z|^{(a, b)}) = V^{\alpha,v}(\succsim^+_a; \succsim^+_a; z)$ Again, since $y_a$ is above any changes to the rank orders, $V^{\alpha,v}(\succsim; y) = V^{\alpha,v}(\succsim^+_a, \succsim^+_a; y)$. Combined with the fact that $z$ is an arg max under $(\succsim^+_a, \succsim^+_a)$, I can deduce that all of these objective values must be equal. But if this were true, there would no way that the same tiebreaker $\tau$ could have chosen two different assignments.

**Case 3.** ($z_a \succsim a o$) Both $z_a$ and $y_a$ are above any changes in the rank orders. Hence, by the upward-looking assumption, $z$ and $y$ must be in the arg max for both submitted preferences, and since I break ties by choosing the lowest
\[ \therefore \text{it is a contradiction that I choose } y \text{ under the true preferences and } z \text{ under the truncations.} \]

**Proposition 11.** Assume that agent \( a \)'s beliefs are symmetric. If the ranking scheme is upward-looking and unmatched-neutral, then truly acceptable objects must be declared acceptable. If the ranking scheme is upward-looking and unmatched-distinct, then any unacceptable objects that \( a \) declares acceptable must be in the bottom indifference class and declared indifferent to some acceptable objects.

**Proof.** The previous lemma establishes the first part of the theorem. For the second part, note that symmetry requires that the objects are submitted in a weakly order preserving way. Hence, there can exist only one indifference class that contains both unacceptable and acceptable objects. Below that indifference class, all classes must be filled with unacceptable objects. By Lemma 5, I can drop them.

This combined with Proposition 9 gives us the generalization of Theorem 5 to preferences with indifferences.

**Generalization of Theorem 5.** If an agent’s beliefs are symmetric and the ranking scheme is upward-looking and...

- \( \ldots \) unmatched-distinct, then his expected allocation from any other strategy is weakly stochastically dominated by his expected allocation from submitting some weakly order preserving preference in which any truly unacceptable objects that are declared acceptable are declared indifferent to some truly acceptable object and are declared in the lowest acceptable indifference class.

- \( \ldots \) unmatched-neutral, then his expected allocation from any other strategy is weakly stochastically dominated by his expected allocation from submitting some weakly order preserving preference in which all truly acceptable objects are declared acceptable.

**C Supportable rank efficiency**

**Theorem 6.** Supportable rank efficiency implies ordinal efficiency. The converse need not hold.

**Proof.** To show the positive statement, note that supportable rank efficiency implies, for some \( \alpha \), \( \alpha \)-rank efficiency, which in turn, for any \( \alpha \), implies ordinal efficiency. To see that the converse need not hold, start by defining \( \Delta^{(\alpha,v)}_{\tau} = \sum_{m=1}^{|\tau|} \alpha_{a_m} \cdot (v_{ram(o_{m-1})} - v_{ram(o_m)}) \) for a trade cycle \( \tau \) (where \( o_0 \equiv o_m \)). By the Generalization of Theorem 2 in Section A.2, an assignment is only supportably rank efficient there exists some \( (\alpha, v) \) such that \( \Delta^{(\alpha,v)}_{\tau} V \leq 0 \) for all trade cycles \( \tau \). Now, consider a 6
an agent example, where there is one seat at a through d and 2 seats at e.

\[
\begin{align*}
1 : & \quad a \succ d \succ e \\
2 : & \quad a \succ c \succ e \\
3 : & \quad a \succ b \succ e \\
4 : & \quad b \succ c \succ e \\
5 : & \quad c \succ d \succ e \\
6 : & \quad b \succ e
\end{align*}
\]

Probabilistic serial yields the random allocation

\[
\begin{array}{cccccc}
 & a & b & c & d & e \\
1 : & 1/3 & 0 & 0 & 17/27 & 1/27 \\
2 : & 1/3 & 0 & 7/27 & 0 & 11/27 \\
3 : & 1/3 & 1/9 & 0 & 0 & 15/27 \\
4 : & 0 & 4/9 & 4/27 & 0 & 11/27 \\
5 : & 0 & 0 & 16/27 & 10/27 & 1/27 \\
6 : & 0 & 4/9 & 0 & 0 & 15/27 \\
\end{array}
\]

which I know to be ordinally efficient. Relative to the trade cycles

\[
\begin{align*}
\tau_1 & = ((1, a), (2, e)) \\
\tau_2 & = ((1, e), (2, a)) \\
\tau_3 & = ((1, a), (3, b), (4, c), (5, d)) \\
\tau_4 & = ((1, a), (2, e), (3, b), (4, c), (5, d))
\end{align*}
\]

I can calculate how \(\Delta_{\tau}^{(\alpha, v)} V\) will change:

\[
\begin{align*}
\Delta_{\tau_1}^{(\alpha, v)} V & = (\alpha_2 - \alpha_1) \cdot (v_1 - v_3) \\
\Delta_{\tau_2}^{(\alpha, v)} V & = (\alpha_1 - \alpha_2) \cdot (v_1 - v_3) \\
\Delta_{\tau_3}^{(\alpha, v)} V & = (\alpha_3 + \alpha_4 + \alpha_5 - \alpha_1) \cdot (v_1 - v_2) \\
\Delta_{\tau_4}^{(\alpha, v)} V & = -\alpha_1 \cdot (v_1 - v_2) + \alpha_2 \cdot (v_1 - v_3) - \alpha_3 \cdot (v_2 - v_3) + (\alpha_4 + \alpha_5) \cdot (v_1 - v_2)
\end{align*}
\]

Now, I show how there is no \((\alpha, v)\) such that one of these trade cycles isn’t \((\alpha, v)\)-rank improving, that is \(\Delta_{\tau}^{(\alpha, v)} V \leq 0\). Since \(v_1 > v_3\), I need \(\alpha_1 = \alpha_2\) to prevent either \(\tau_1\) and \(\tau_2\) from being \((\alpha, v)\)-rank-improving. To prevent \(\tau_3\) from being \(\alpha\)-rank-improving, I need \(\alpha_1 \geq \alpha_3 + \alpha_4 + \alpha_5\). Finally, since \(\alpha_1 = \alpha_2\), I can simplify \(\Delta_{\tau_4}^{(\alpha, v)} V\) to \((\alpha_1 - \alpha_3) \cdot (v_2 - v_3) + (\alpha_4 + \alpha_5) \cdot (v_1 - v_2)\). The second term is definitely positive, so it must be that \(\alpha_1 < \alpha_3\). But combining the last two inequalities gives us \(\alpha_3 > \alpha_3 + \alpha_4 + \alpha_5\), which cannot be. Hence, there is no \((\alpha, v)\) such that \(\Delta_{\tau}^{(\alpha, v)} V \leq 0\) for all four of the trade cycles I have listed. Thus the ordinally efficient assignment I started with is not supportably rank efficient.  
\(\Box\)
D Competitive equilibrium mechanisms

To establish Theorem 7, I will be using the concept of a discount equilibrium.

Definition 18. Prices and an assignment, \((x^*, \hat{p})\), form a discount equilibrium with respect to weights \(\alpha\) and valuation \(v\) (or just \((\alpha, v)\)-discount equilibrium) if, for all \(a\),

\[
x_a^* \in \arg \max_{x_a \geq 0} \sum_o \left( v_{ra(o)} - \frac{\hat{p}_o}{\alpha_a} \right) \cdot x_{ao} \\
\text{s.t.} \quad \sum_o x_{ao} \leq 1
\]

and

\[
x_{ao}^* \in \arg \max_{x_{ao} \geq 0} \sum_a \sum_o \hat{p}_o \cdot x_{ao} \\
\text{s.t.} \quad \sum_a x_{ao} \leq q_o, \forall o
\]

An \((\alpha, v)\)-discount equilibrium decentralizes the linear program that defined the \((\alpha, v)\)-rank-value mechanism. Formally,

Proposition 12. \(x\) is \(\alpha\)-rank efficient if and only if there exist prices \(\hat{p}\), and a valuation \(v\), such that \((x^*, \hat{p})\) form an \((\alpha, v)\)-discount equilibrium

Proof. First, I prove the “only if” part. Since \(x\) is \(\alpha\)-rank efficient, there must be a valuation such that it is in the arg max of the \((\alpha, v)\)-rank-value mechanism, by the Generalization of Theorem 1 in Section A.1.

The LP dual of the \((\alpha, v)\)-rank efficient mechanism is

\[
\min_{u_a \geq 0, p_o \geq 0} \left\{ \sum_{o \in O} q_o \cdot p_o + \sum_{a \in A} \alpha_a \cdot u_a \right\} \\
\text{s.t.} \quad u_a \geq v_{ra(o)} - \frac{p_o}{\alpha_a} \quad \forall o \in O, a \in A
\]

I will support the discount equilibrium with the prices from the dual of the rank efficient mechanism. Now, the LP dual of the agent \(a\)'s problem in the \((\alpha, v)\)-discount equilibrium with these prices is

\[
\min_{u_a} u_a \\
\text{s.t.} \quad u_a \geq v_{ra(o)} - \frac{p_o}{\alpha_a} \quad \forall o \in O
\]

By nested optimization, I see that the \(u_a\) from the rank efficient mechanism’s dual also solves the discount equilibrium duals. Now, let \(x_{ao}\) be the assignment I are considering from the arg max of the rank efficient mechanism. The LP duality theorem tells us that

\[
\sum_{o \in O} q_o \cdot p_o + \sum_{a \in A} \alpha_a \cdot u_a = \sum_{a \in A} \sum_{o \in O} \alpha_a \cdot v_{ra(o)} \cdot x_{ao}
\]
From the constraint in the dual of the rank efficient mechanism, I can then derive that, for any \( \xi \) such that \( \sum_o \xi_{ao} \leq 1 \), \( \forall a \in \mathcal{A} \), I have

\[
\sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o \in \mathcal{O}} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot \xi_{ao} \leq \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{O}} \alpha_a \cdot v_{ra(o)} \cdot x_{ao} \quad (3)
\]

If I plug in \( x \) for \( \xi \), then this reduces to \( \sum_{o \in \mathcal{O}} p_o \cdot \left( q_o - \sum_{a \in \mathcal{A}} x_{ao} \right) \leq 0 \). Since all elements of that sum are weakly positive, I conclude that \( p_o \cdot \left( q_o - \sum_{a \in \mathcal{A}} x_{ao} \right) = 0, \forall o \in \mathcal{O} \). This, coupled with a constraint from the rank efficient mechanism, is the second condition of the discount equilibrium. Note that I have also shown that

\[
\sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o \in \mathcal{O}} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot x_{ao} = \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{O}} \alpha_a \cdot v_{ra(o)} \cdot x_{ao}
\]

Finally, I show that the \( x_{ao} \) from the arg max of the rank efficient mechanism must also be part of the arg max for the agents’ problems in the discount equilibrium. By way of contradiction, say that they aren’t, and let \( \xi_a \) be the bundle chosen by agent \( a \) under prices \( p \). Then, for all agents, \( \sum_o \xi_{ao} \leq 1 \) and \( \sum_o \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot \xi_{ao} \geq \sum_o \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot x_{ao} \), with the second inequality strict for some agent \( a' \). Thus,

\[
\sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot \xi_{ao} > \sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot x_{ao}
\]

But I have shown

\[
\sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot x_{ao} = \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{O}} \alpha_a \cdot v_{ra(o)} \cdot x_{ao},
\]

which means that I have derived

\[
\sum_{o \in \mathcal{O}} q_o \cdot p_o + \sum_{a \in \mathcal{A}} \alpha_a \cdot \sum_{o} \left( v_{ra(o)} - \frac{p_o}{\alpha_a} \right) \cdot \xi_{ao} > \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{O}} \alpha_a \cdot v_{ra(o)} \cdot x_{ao}
\]

which stands in direct contradiction to something I have already proven, Inequality 3. Therefore, the assignment from the rank efficient mechanism solves the agents’ problems in the discount equilibrium. Thus I have shown that any member of the arg max of the rank efficient equilibrium can be supported as a discount equilibrium.

Now, I prove the “if” part. Consider the optimization

\[
\max_x \left\{ \sum_o \sum_a \alpha_a \cdot v_{ra(o)} \cdot x_{ao} + \sum_o p_o \cdot \left( q_o - \sum_a x_{ao} \right) \right\}
\]

s.t.

\[
\sum_o x_{ao} \leq 1
\]

\[
\sum_a x_{ao} \leq q_o
\]
By the positivity of the prices and the object quota constraints, the value of this optimization is greater than or equal to the value of
\[
\max_x \sum_o \sum_a \alpha_a \cdot v_{ra(o)} \cdot x_{ao} \\
\text{s.t.} \quad \sum_o x_{ao} \leq 1 \\
\quad \sum_a x_{ao} \leq q_o
\]
Now, if I start with the assignment \(x\) from some discount equilibrium, it will also solve the first optimization. To see this, relax the optimization by removing the object capacity constraints. The relaxation can then be separated into an optimization for each agent that is equivalent to the agent’s problem from the discount equilibrium, so \(x\) solves the relaxation. But, since \(x\) is part of a discount equilibrium, it also meets the object capacity constraints, and is thus an optimizer of the full problem as well.

By the seller’s problem in the discount equilibrium (see Remark 7), the value of this assignment in the first optimization is attained in the second as well. Hence, \(x\) must solve the second optimization, since it attains the upper bound at \(x\). Hence, \(x\) is in the arg max of the \((\alpha,v)\)-rank-value mechanism’s defining program, which means that \(x\) is \(\alpha\)-rank efficient.

This is a strange decentralization, as it distorts prices in a way that one might expect it to not yield any sort of efficiency. At the end of this section, I will discuss the intuition for why the discount equilibrium decentralization remains efficient.

Proposition 13. There exist weights \(\alpha\) such that \((x,p)\) is an \((\alpha,v)\)-discount equilibrium if and only if there exist budgets \(B\) such that \((x,p)\) is a \((B,v)\)-budget equilibrium.

Proof. I start by showing that if I start with an \((\alpha,v)\)-discount equilibrium \((x^*,\hat{p})\), then it is also a \((B,v)\)-budget equilibrium for \(B_a = \sum_o \hat{p}_o \cdot x_{ao}^*\). By way of contradiction, assume that \((x^*,\hat{p})\) is not a \((B,v)\)-budget equilibrium assignment for \(B\). By construction, it is feasible, so it must be that there is some other feasible \(\tilde{x}\) such that \(\sum_o v_{ra(o)} \cdot \tilde{x}_{ao} > \sum_o v_{ra(o)} \cdot x_{ao}^*\). Since it is feasible, I also know that \(\sum_o \hat{p}_o \cdot \tilde{x}_{ao} \leq \sum_o \hat{p}_o \cdot x_{ao}^*\). Combining the last two inequalities (multiplying the second by \(1/\alpha_a\)), I find \(\sum_o (v_{ra(o)} - \frac{\hat{p}_o}{\alpha_a}) \cdot \tilde{x}_{ao} > \sum_o (v_{ra(o)} - \frac{\hat{p}_o}{\alpha_a}) \cdot x_{ao}^*\), a contradiction of the original assumption that \((x^*,\hat{p})\) was an \((\alpha,v)\)-discount equilibrium.

Now, I show that if I start with a \((B,v)\)-budget equilibrium, \((\tilde{x},\tilde{p})\), then it is also an \((\alpha,v)\)-discount equilibrium for some choice of \(\alpha\). Before I define our weights, however, I first introduce a few LP duals. The dual of the agent optimization problem in the budget equilibrium is
\[
\left(\lambda_a, \tilde{u}_a\right) \in \arg \min_{\lambda_a,u_a \geq 0} u_a + \lambda_a \cdot B_a \\
\text{s.t.} \quad u_a \geq v_{ra(o)} - \hat{p}_o \cdot \lambda_a, \quad \forall a
\]
The LP dual of the agent optimization problem in an \((\alpha, v)\)-discount equilibrium is

\[
\begin{align*}
    u^*_a & \in \arg\min_{u_a \geq 0} \quad u_a \\
    \text{s.t.} & \quad u_a \geq v_{r_a(o)} - \frac{\hat{p}_o}{\alpha} \quad \forall a
\end{align*}
\]

I now show that each agent \(a\) can be made to optimize by choosing \(\tilde{x}_a\) under the discount equilibrium for some choice of \(\alpha\). Tildes indicate optimal choices from the budget equilibrium that I start with. From here I continue by cases.

Case 1. \((\tilde{\lambda}_a = 0):\) By inspection of the dual, I can see that \(\tilde{u}_a = v_1\). By the LP duality theorem, I then know that \(\sum_o v_{r_a(o)} \cdot \tilde{x}_{ao} = \tilde{u}_a = v_1\). The only way this can happen is if \(\tilde{x}_{ao} = 1\{r_a(o) = 1\}\), that is, if agent \(a\) is able to buy a full share of his most preferred object, \(r_a^{-1}(1)\). Without exhausting his budget. Clearly, if \(\alpha_a \to \infty\), then \(a\) will also take a full share of his most preferred object in the discount equilibrium, as the influence of prices vanishes in the limit. This can be done with a finite \(\alpha_a\) as well: simply set \(\alpha_a\) high enough that \(v_k - \frac{p_{\hat{r}_a^{-1}(k)}}{\alpha_a}\) is a maximum at \(k = 1\), that is set \(\alpha_a = \max_k \left\{ \frac{p_{\hat{r}_a^{-1}(1)} - p_{\hat{r}_a^{-1}(k)}}{v_1 - v_k} \right\}\). I know this is finite because \(v_1 > v_k, \forall k\) and prices must be finite in a budget equilibrium.\(^{73}\)

Case 2. \((\tilde{\lambda}_a > 0):\) Set \(\alpha_a = \frac{1}{\tilde{\lambda}_a}\) and let stars indicate optimal choices from agent \(a\)’s problem in the corresponding discount equilibrium. Our choice of \(\alpha_a\) nests our two LP duals, so that \(u^*_a = \tilde{u}_a\). The LP duality theorem then gives us that \(\sum_o \left( v_{r_a(o)} - \frac{\hat{p}_o}{\alpha_a} \right) \cdot x^*_{ao} = u^*_a = \tilde{u}_a = \sum_o v_{r_a(o)} \cdot \tilde{x}_{ao} = B_a\). Feasibility of our starting budget equilibrium requires that \(\sum_o v_{r_a(o)} \cdot \tilde{x}_{ao} \leq B_a\), which, when combined with our duality equality, tells us that \(\sum_o \left( v_{r_a(o)} - \frac{\hat{p}_o}{\alpha_a} \right) \cdot x^*_{ao} \leq \sum_o v_{r_a(o)} \cdot \tilde{x}_{ao}\). A strict inequality here contradicts \(x^*_a\) being agent \(a\)'s optimum under the discount equilibrium. Hence, \(\tilde{x}_a\) is an optimum choice for \(a\) under the discount equilibrium with the \(\alpha_a\) I have chosen.

Hence I have shown that agents optimizing according to the problem in the discount equilibrium can choose \(\tilde{x}\), given an appropriately chosen \(\alpha\) vector. \(\square\)

**Theorem 7.** \(x\) is supportably rank efficient if and only if there exist budgets and a valuation, \((B, v)\), such that \(x\) is supported by the \((B, v)\)-budget mechanism.

**Proof.** The previous two proofs establish the result. \(\square\)

The efficiency of the budget equilibria lines up well with a textbook understanding of general equilibrium and the welfare theorems, but it is less clear why the discount equilibrium should yield an efficient outcome, as distorting prices with the \(\alpha\)-discounts seems like it should cause trouble in the same way that distortionary taxation does.

\(^{73}\)If there were an infinite price, no agent could purchase the object, since budgets are finite, this would necessarily violate the \(p_o > 0 \Rightarrow \sum_o \tilde{x}_{ao} = q_o\) requirement.
So what is going on? The agent’s problem in the discount equilibrium can be thought of as the problem of an agent with enough money to buy whatever he wants, but who values money quasi-linearly. In the artificial economy where money factors into the agent’s utility, the discount equilibrium will not be efficient. The efficiency concepts in this paper, however, ignore the artificial money and rely only on the distribution of objects among the agents. In short, in the discount equilibrium, distortions in the relative cost of keeping money serve a similar role to the budgets in the budget equilibrium, but they do so in a way that allows more precise control over what Pareto weights will support the chosen optimal assignment (at the cost of sacrificing control over agents’ opportunity sets).