Acquisition of/Stochastic Evidence[1]

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Abstract

We explore two highly interrelated models of "hard information." In the *evidence–acquisition model*, an agent with private information searches for evidence to show to the principal about her type. In the *signal–choice model*, a privately informed agent chooses an action which generates a random signal whose realization may be correlated with her type. The signal–choice model is a special case and, as we show, under certain conditions, a reduced form of the evidence–acquisition model. We develop tools for characterizing optimal mechanisms for these models by giving conditions under which some aspects of the principal's optimal choices can be identified only from the information structure, without regard to the utility functions or the principal's priors. We also give a novel result on conditions under which there is no value to commitment for the principal.

1 Introduction

The earliest work in economics on information transmission considered settings where any agent, regardless of her information, could send any "signal" or "message," though potentially at costs which depend on her private information. In Spence's (1973) classic signaling model, the cost of education depends on the agent's type, but she can get any amount of education she wants. In Crawford and Sobel's (1982) classic cheap–talk model, there are no costs, so nothing about the messages depend on the agent's type. In such models, effective communication relies on how the agent's incentives to take different actions or to induce different beliefs varies with her type.[1]

More recent work treats information as more strongly tied to the agent's type. For example, in the literature on hard evidence, the set of messages the agent has available can depend on her type, so that presentation of certain messages proves certain facts about her type. In the literature on career concerns, the agent's actions affect observable outcomes differently for different types. In the literature on Bayesian persuasion, the outcome of a given experiment carried out by the agent depends on her type.

We connect these and other approaches to information transmission through a model where the agent chooses among actions that generate random signals depending on her type. The agent can then choose which realizations to present to a principal who chooses an action affecting the utility of both the principal and the agent. We refer to this as the *evidence–acquisition model*. We also study a model that is a special case and, under some conditions, a reduced form of the evidence-acquisition model, called the *signal–choice model* in which the agent has no option about what evidence to present.

Most of the literature on evidence analyzes a principal–agent model where the agent is already endowed with evidence and the question is what evidence he will disclose. However, there are many situations of economic interest where an agent must take an action in order to generate evidence and in such a situation it is typically the case that the evidence that is generated is random. That is, when the agent takes an action that generates evidence there is uncertainty about what the outcome would be. For example, when one takes a test there is uncertainty about the outcome of the test.

We study two general issues that the literature on evidence has been concerned with. First, we identify conditions under which it is possible to restrict attention to a class of mechanisms that are relatively simple. Second, we identify conditions in which the outcome of the optimal mechanism can be obtained without commitment by the principal. The paper is organized as follows.

[1] This description omits the literature on cheap talk with type–independent preferences for the sender, largely initiated by Chakraborty and Harbaugh (2010). See Lipnowski and Ravid (2019) for a recent example.

In Section 2, we present the "technology" of the two models, relate them to the literature, and introduce a running example. In Section 3, we briefly discuss game–theoretic versions of the models and show that in a natural game, the signal–choice model is a reduced form of the evidence–acquisition model.

In Section 4, we turn to mechanism design. First, we provide an analog of the Revelation Principle for the evidence–acquisition model. The general class of mechanisms for these problems is quite complex, involving numerous steps of communication between the agent and the principal. We analyze conditions under which we can identify some of the principal's communication in the optimal mechanism, specifically, the principal's recommendation regarding what evidence he would like to see. When we can identify this recommendation, we can reduce the model to the signal–choice model. We also give conditions under which we can identify the principal's recommendation regarding what signal to choose, leading to a further simplication.

In Section 5, we show that under certain conditions, the optimal mechanism does not require commitment by the principal. That is, the best mechanism for the principal yields the same outcome as the best equilibrium of the game where the principal is not committed. This result can be thought of as a generalization of Ben-Porath, Dekel, and Lipman (2019) or earlier results such as Glazer and Rubinstein (2004, 2006), Sher (2011), or Hart, Kremer, and Perry (2017). The result here is more general in that it allows stochastic evidence. This extension requires stronger assumptions on the preferences of the principal than those in our previous work. However, our assumptions on the preferences are weaker than those assumed in Glazer and Rubinstein (2004, 2006), and not comparable to Sher (2011) or Hart, Kremer, and Perry (2017). Our result applies in a wide class of interesting problems. In particular, for binary decisions such as whether or not to hire (at a fixed wage), whether or not to adopt a project, whether or not to provide some benefit or resource, and so on, the result is completely general in the sense that it applies for any utility functions for the agent and the principal.

Proofs not contained in the text are in the Appendix.

2 Models

In this section, we discuss the "primitives" of the model, reserving discussion of the specifics of the game or mechanism for later sections.

Running Example, Part 1. Throughout the paper, we will use the following example to illustrate ideas and results. We have an employer, also referred to as *the principal*, and an employee, also called *the agent*. The agent's private information is her productivity for the principal. We consider two variations. First, we consider what we will call the *wage–setting version* of this problem. Here, as in Spence (1973), the principal sets a wage for the agent and his payoff is maximized by setting the wage equal to the agent's true productivity. By contrast, the agent's payoff is strictly increasing in the wage. Second, we consider the *hiring version* of the problem. Here there is a fixed wage, outside the control of the principal, and he can only decide whether or not to hire the agent. The principal prefers hiring to not hiring iff the agent's productivity is sufficiently high, while the agent strictly prefers being hired, regardless of her true type. Hence in both versions the agent wants the principal to think she has a high type and the principal wants to know the true type, but in the second, the decision is coarser. We consider various forms of evidence acquisition by the agent to try to persuade the principal she has high productivity. ▌

As in the running example, the players in the model are an agent and a principal. The agent has a finite set of types T where the realization $t \in T$ is the agent's private information. The principal's prior over T is denoted τ and is assumed to have full support. The principal has a finite set of actions X. An element of X specifies all aspects of the principal's action, including allocation of goods, monetary transfers, provision of public goods, or other activities. After information exchange between the agent and the principal, the principal chooses some $x \in X$. There is a set $\mathcal{L}$ of all possible evidence messages. Information exchange includes the transmission of an evidence message and may also include cheap talk between the principal and the agent.

We consider two ways of modeling information transmission, one of which is a special case and, under certain conditions, a reduced form of the other. The more general model is the *evidence–acquisition model*, a model where the agent searches to find evidence. The agent has a variety of options available to try to obtain evidence. This search process could be sequential or one–shot. Rather than model this process, we focus on outcomes of the search process by treating the agent as choosing a probability distribution over the evidence set she ultimately obtains. Formally, let A_t denote the set of evidence–gathering actions available to type t, with typical element $a \in A_t$, where we identify the action a with the probability distribution over evidence sets it generates. That is, $a \in \Delta(2^{\mathcal{L}} \setminus \{\emptyset\})$.[2] We denote a typical set of evidence as $M \subseteq \mathcal{L}$. Let $\mathcal{M}$ be the set of possible message sets M that can be produced. That is, $\mathcal{M}$ is the collection of M such that there exists t and $a \in A_t$ with $M \in \text{supp}(a)$. The assumption that $\emptyset \notin \mathcal{M}$ can be thought of as assuming the agent can always say *something*, even if it is not informative — e.g., "I have no evidence to present." If M is the realized set of messages, then the agent can present any one $m \in M$ to the principal.[3] We write the utility function of the

[2] For any set B, $\Delta(B)$ is the set of probability distributions over B.

[3] As in the usual deterministic evidence model, the assumption that the agent can present only one message is without loss of generality. For example, if the agent could present two messages, we would simply replace $\mathcal{L}$ with the set of pairs of messages.

agent and principal as $u : T \times A \times X \to \mathbf{R}$ and $v : T \times A \times X \to \mathbf{R}$ respectively, where $A = \cup_t A_t$.[4]

While we assume that the principal observes only the m sent by the agent and not the chosen evidence acquisition strategy a, the model (implicitly) allows observability of a as well. To see this, suppose that for every pair of distinct distributions, $a \neq a'$, if $m \in M \in \text{supp}(a)$, then there is no $M' \in \text{supp}(a')$ with $m \in M'$. Then upon observing the message m, the principal knows the evidence acquisition strategy. Similarly, we can assume that certain distribution choices are observable and others are not or that only some of the messages reveal a in this sense, so that whether the distribution is observed is itself random and/or in the control of the agent.

The model incorporates the important specific case where there is a set of tests, say Q, where each $q \in Q$ and $t \in T$ define a probability distribution over sets of evidence messages (test results). In some settings (e.g., college admissions tests), it is natural to assume that the principal observes the test q selected by the agent. Our model allows but does not require such observability.

Running Example, Part 2. For the purposes of the example, we assume a very stylized evidence–acquisition technology. To understand the idea, think of the agent of type t as able to choose a variety of ways to potentially demonstrate her ability. Each of these options gives a different probability distribution over an "outcome" she generates, where this outcome is, on average, equal to her true type. However, she can also withhold part of this "outcome" and show a lower realization than what she actually generates. More formally, $a \in A_t$ if and only if the following two statements are true. First, every $M \in \text{supp}(a)$ takes the form $[0, m]$ for some $m \in \mathbf{R}_+$. To state the second property, note that the first implies that any $a \in A_t$ corresponds to a probability distribution over $\mathbf{R}_+$ where if the realization of this random variable is m, this means the message set is $[0, m]$. The second property is that for any $a \in A_t$, the expectation of this associated random variable is t. That is, in the case where a has a finite support,

$$\sum_{[0,m] \in \text{supp}(a)} a([0, m])m = t.$$

In our example, the agent wants to persude the principal that her type is large, so it is natural to conjecture that the option of showing a lower outcome will never be used by the agent and hence is irrelevant. In fact, one of our results will be that only the upper bound of a given evidence set will be shown by the agent in an optimal mechanism. However, this result will be entirely independent of the preferences of the agent — the

[4]For some purposes, it is natural to also let the agent and/or principal's utility to depend on the realized evidence set. This allows the possibility that the agent's costs depend on the realized set and reflects the idea that the realization itself may be informative. We avoid adding this as it complicates the notation even further, but note that this would not affect Theorems 1 or 2.

same is true even in a problem where the agent wants to persuade the principal that her type is small. ▌

A special case of the evidence–acquisition model is where the agent has no choice of what message to send at the last step. More formally, this special case is when for every $t \in T$ and every $a \in A_t$, every $M \in \text{supp}(a)$ is a singleton. For convenience, we write this special case, the *signal–choice model*, differently. Instead of referring to agent's choices as evidence acquisition strategies, we refer to them as *signal distributions*. Specifically, the set of options available to type $t \in T$ is a nonempty set $S_t \subseteq \Delta(\mathcal{L})$. We refer to an $s \in \Delta(\mathcal{L})$ as a *signal distribution*. The interpretation is that if the agent chooses $s \in \Delta(\mathcal{L})$, then the principal sees message $m \in \mathcal{L}$ with probability $s(m)$. Equivalently, we can think of this as the singleton message in the realized evidence set. Letting $S = \cup_t S_t$, with some abuse of notation, we denote the utility functions of the agent and principal by $u : T \times S \times X \to \mathbf{R}$ and $v : T \times S \times X \to \mathbf{R}$ respectively. While we will vary the arguments of the utility functions across models, we will always denote the agent's utility function by u and the principal's by v.

Similarly to our comments above about the observability of a, the model allows the possibility that the realized m reveals the agent's choice of s always or reveals it with some probability or reveals it for some s choices but not others.

While we discuss the details of games or mechanisms below, the "technology" of evidence/signals imposes some timing structure. Specifically, in both models, we assume the agent knows her type at the outset. There may be cheap talk between the principal and the agent before the agent chooses a strategy of evidence gathering or a signal distribution. After this, the agent sees the realization of her strategy. In the evidence–acquisition case, this is a set of evidence messages and (perhaps after further cheap talk) she can then send one evidence message to the principal. In the signal–choice model, the principal also sees the realization. After this, the principal chooses $x \in X$.

Running Example, Part 3. For a signal–choice version of our running example, we can "convert" the same technology as in the evidence–acquisition model described in Part 2 into a signal–choice model. More precisely, note that the agent in the evidence–acquisition model can pick a distribution over evidence sets and decide what message she will use from each set. In other words, she can choose a particular distribution over sets of the form $[0, m]$ and decide for each upper bound m what message $m' \in [0, m]$ she will send to the principal. Recall that the agent of type t can only generate a distribution over sets of the form $[0, m]$ with the property that the expectation of the upper bound m is t. Given this, it is not hard to see that when we convert to signals, this generates the set of signal distributions with expected value less than or equal to t. In other words, for the signal–choice version of our running example, we assume that S_t, the set of signal distributions for type t, is the set of all probability distributions on $\mathbf{R}_+$ with expected value less than or equal to t. Thus signal distributions are either unbiased or biased "against" the agent. One can think of this as a stylized model where the agent can give the principal one name of a reference for the principal to contact. References cannot be systematically biased in the agent's favor, but the agent generally cannot predict exactly what a given reference will say. It is easy to see that this process generates a signal distribution, that is, a distribution on $\mathbf{R}_+$. ∎

Related literature: The usual model of evidence considers games or mechanism design problems where the agent's set of feasible messages depends on her type. Thus by presenting a message which is only feasible for a certain set of types, the agent proves that her type is in this set. For early contributions in game theory, see Grossman (1981), Milgrom (1981), and Dye (1985). For early contributions in mechanism design theory, see Green and Laffont (1986). For more recent examples of papers in game theory or mechanism design, see Shin (2003), Acharya, DeMarzo, and Kremer (2011), Ben-Porath and Lipman (2012), Kartik and Tercieux (2012), Guttman, Kremer, and Skrzypacz (2014), and Rappoport (2020). Finally, for closely related work related to both games and mechanisms, see Glazer and Rubinstein (2004, 2006), Sher (2011), Hart, Kremer, and Perry (2017), and Ben-Porath, Dekel, and Lipman (2019).

In these papers, the agent is endowed with evidence and only chooses which evidence to disclose. The evidence–acquisition and signal–choice models extend the usual model by considering decisions by the agent which generate evidence and where there is ex ante uncertainty regarding the evidence that will materialize. Both models are natural for applications. For an example of the evidence–acquisition model, consider a division within an organization which wants additional funding for a project it is developing, say, a new product. The division can develop and test a prototype or do other market research to obtain evidence regarding the profitability of the product. The research may be costly and the resulting evidence is random ex ante. The division may choose which parts of its results to share with the organization.

As an example of a signal–choice model in applications, consider a lawyer who have private information about the innocence or guilt of her client trying to persuade a judge. When the lawyer calls a witness to the stand, she may know more about what the witness will say than the judge does, but may not be able to perfectly predict the witness' testimony. In this sense, the witness is a random signal, the realization of which depends stochastically on the lawyer's private information. Similarly, as discussed above, when an agent gives the name of a recommender to the principal, she may not know exactly what the recommender will say.

Another interpretation of the signal–choice model is as a natural variation and generalization of the classic Holmstrom (1999) career–concerns model. In the usual career–concerns model, an agent of unknown ability chooses actions which have outcomes that depend on her ability. Both the agent and the principal (or market) learn about the agent's ability over time by observing these outcomes. Our model is different in that we assume the agent knows her type, while the usual model has symmetric uncertainty about the type.[5] Our model also allows partial observability of actions as explained above.

The signal–choice model can also be thought of as an "informed agent" version of the Bayesian persuasion model of Kamenica–Gentzkow (2011). As in the Bayesian persuasion model, the agent chooses an "experiment" which reveals information to the principal. Our model differs from Kamenica–Gentzkow in three ways. First, we do not necessarily allow for any possible signal to be created. Second, we assume the agent knows her type, even though she may not know the outcome of the experiment.[6] Third, we assume the principal does not observe the full experiment as in Kamenica and Gentzkow. Specifically, while we can allow the principal to observe the signal choice of the agent as discussed above, he cannot observe the signals that would have been chosen by other types.

This model also related to models of noisy signaling. For example, in a classic paper, Matthews and Mirman (1983) study a privately–informed firm that chooses an unobservable quantity of output. Because of stochastic demand, this leads to a stochastic price, the realization of which is observed by a potential entrant. Thus the firm effectively chooses a probability distribution over what the rival will see and the expected payoff of the firm depends on its type, its choice, and the rival's response. We can think of the choice of output as a choice of a signal distribution and the price as a realization. Our model assumes that realizations don't affect payoffs, but could be extended to allow this possibility.

Deb, Pai, and Said (2018) develop another model which can be thought of as a signal–choice model. A forecaster has private information about the quality of the signals she receives about some random variable. She sees a sequence of signals, announcing a prediction about the random variable after each such observation. After this, the realization of the random variable is observed. The principal updates his beliefs about the quality of her information. To embed this in a signal–choice model, the forecaster's "message" can be thought of as tuple giving the sequence of forecasts together with the realization of the random variable. A choice of a strategy by the forecaster giving her forecasts as a function of the signals she sees generates a probability distribution over such sequences and hence is a signal choice. Deb, Pai, and Said's result that the optimal mechanism in this setting does not require commitment by the principal is a special case of our results in Section 5. Their proof restricts attention to deterministic mechanisms; our results show that no such restriction is needed.

[5]See Chen (2015) or Halac and Kremer (2017) for career–concerns models where the agent has private information.

[6]For work on Bayesian persuasion with privately informed agents, see Perez–Richet (2014), Hedlund (2017), and Kosenko (2018).

Espinosa and Ray's (2020) model can also be thought of as a signal–choice model. They consider a principal and agent where the principal observes a noisy signal of the agent's type and the agent can partially control the variance of this signal. In their baseline model, the agent does not incur costs associated with this control. As with Deb, Pai, and Said, our results in Section 5 imply that the principal is not made better off in their model if he has commitment power in their baseline model.

Matthews and Postlewaite (1985), Che and Kartik (2009), DeMarzo, Kremer, and Skrzypacz (2019), Ball and Kattwinkel (2019), and Shishkin (2020) give models related to our evidence–acquisition model. In the first three of these papers, an uninformed agent chooses a test which may reveal information about her type. If the test does not produce a result, the agent's only option is to say there was no result. Otherwise, she can show the result or claim to have none. (In Matthews and Postlewaite, testing always yields a result, but the agent can claim not to have tested.) Thus the agent's choice of a test, like the choice of an action in our model, produces a probability distribution over a set of options for the agent to reveal. Shishkin takes an information–design approach to evidence acquisition. He assumes that an uninformed agent chooses an observable design of the evidence she will acquire about her type but where there is an exogenous probability that she receives no evidence. We discuss Ball and Kattwinkel's model and its relationship to ours at the end of Section 4.

3 Games

There are many timing assumptions one could consider in modeling the interaction between the agent and principal. We focus on the following sequential game.

First, the agent learns her type. In the evidence–acquisition model, she then chooses $a \in A_t$ and $M \subseteq \mathcal{L}$ is realized. She then chooses $m \in M$. If we consider the signal–choice model instead, the agent simply chooses $s \in S_t$ and the realization m is determined. Either way, the principal observes m but not the agent's type or other information. That is, in the evidence–acquisition model, the principal does not observe the agent's choice of a or the realization M. In the signal–choice model, the principal does not observe the agent's choice of s. The principal then chooses $x \in X$.

Given the way we have defined the game, it is straightforward to show that the signal–choice model is a reduced form of the evidence–acquistion model. In the evidence–acquisition model, we can think of the agent choosing a and simultaneously choosing her *messaging strategy* — that is, her strategy for which message m to send as a function of the realization of the message set M. As we vary the agent's choice of distribution and messaging strategy, we trace out a set of probability distributions over messages m that the principal will observe. Thus each distribution and messaging strategy is equivalent to a signal choice. This is exactly the conversion described in Part 3 of our running example. In light of this, we could analyze the game as an evidence–acquisition model or equivalently replace the set of actions and messaging strategies with the set of induced signal distributions and analyze the game as a signal–choice model.

Recall that the signal–choice model can also be thought of as an evidence–acquisition model where every set of evidence is a singleton. Thus in the context of the game considered here, these two models are equivalent — from any game in one class, we can construct an equivalent game in the other.

Running Example, Part 4. We illustrate the game with our running example. Given that the evidence–acquisition model reduces to the signal–choice model, we focus only on the latter. Assume the agent has two equally likely types, h and ℓ where $h > \ell > 0$. For the wage–setting version, we assume $X = \mathbf{R}_+$, $u(t, s, x) = x$, and $v(t, s, x) = -(t - x)^2$. That is, the principal chooses a wage, the agent's utility is equal to the wage, the principal wishes to set the wage equal to the agent's true productivity, and the signals are costless. For the hiring version, we assume $X = \{0, 1\}$, $u(t, s, x) = x$, and $v(t, s, x) = x(t - \bar{w})$ where $h > \bar{w} > \ell$. In other words, the agent wants to be hired ($x = 1$), while the principal wants to hire the high type but not the low type.

For either version of the model, the following strategies form a perfect Bayesian equilibrium. Type h chooses the signal distribution which puts probability 1 on h, while ℓ chooses a distribution with probability ℓ/h on h and $1 - (\ell/h)$ on 0. The principal's belief puts probability 1 on ℓ unless the signal he sees is h. If he sees signal h, his belief puts probability $h/(\ell + h)$ on type h, so the expected productivity is $(h^2 + \ell^2)/(h + \ell)$. Either way, he chooses his action accordingly. So in the wage–setting version, he chooses $x = \ell$ if he sees any message other than h and sets $x = (h^2 + \ell^2)/(h + \ell)$ otherwise. In the hiring version, he does not hire if he sees any $m \neq h$. If he sees $m = h$, then he hires if

$$\frac{h^2 + \ell^2}{h + \ell} > \bar{w},$$

doesn't hire if the reverse strict inequality holds, and can choose any probability of hiring otherwise. It is easy to see that, given the principal's strategy, both types want to maximize the probability on signal h and these signal choices do that. So these strategies form an equilibrium.

To introduce the next section on mechanism design, consider the case where the principal can commit to his reaction to the m he observes. In this case, he can achieve his best possible outcome in the wage–setting version. To be specific, suppose the principal commits to choosing $x = m$ if m is either h or ℓ and to choosing $x = 0$ otherwise. Given any s chosen by the agent, the agent's expected payoff is less than or equal to the expectation of m since the principal's choice of x is always weakly below m. Since every $s \in S_t$ has expectation weakly less than t, this implies that the agent's payoff must be weakly less than t. Since the agent can obtain a payoff of exactly t by choosing the degenerate s which produces $m = t$ with probability 1, we see that this is an optimal reply for the agent. Clearly, this enables the principal to set $x = t$ always, thus achieving his highest possible payoff. It is not hard to show that no equilibrium of the game yields the principal this payoff, so the ability to commit strictly improves the principal's payoff.

On the other hand, commitment does not help the principal in the hiring version. This is demonstrated in Section 4.3 and generalized in Section 5. ∎

4 Mechanism Design

While we can assume any sequencing of messages for the case of game theory, for mechanism design, it is more standard to identify the sequence of communication steps which allows the principal to obtain the highest possible payoff. Using standard Revelation Principle type arguments, one can show that we can restrict attention to a certain class of direct truth–telling mechanisms. However, these mechanisms are rather complex for the signal–choice model and quite involved for the evidence–acquisition model. Henceforth we use the term *protocol* to refer to the sequence of stages of communication in a mechanism.[7]

For the signal–choice model, we have, in effect, an adverse selection problem (the agent's private knowledge regarding her type), followed by moral hazard (the agent's unobserved choice of a signal distribution). Thus a variation on Myerson's Revelation and Obedience Principle identifies the appropriate protocol.[8] First, the agent reports a type. Then the principal recommends a signal distribution. Finally, the agent chooses some distribution, the principal observes m, and the principal chooses $x \in X$.

In the evidence–acquisition model, though, the problem is much more complex. In effect, we start with adverse selection (the agent's type), then have moral hazard (the agent's choice of a distribution over evidence sets), followed by more adverse selection (the realized set of evidence messages). Consequently, we start similarly to the signal choice case where the agent reports her type, the principal recommends an action, and the agent chooses an action. But after this, the agent makes a report of the realized evidence set,

[7]Gerardi and Myerson (2007) have shown that the Revelation Principle may not hold for sequential equilibrium in dynamic environments, raising questions about our multi–stage mechanisms. However, results in Sugaya and Wolitzky (2018)'s Section 4 show that such problems do not arise in our single–agent setting.

[8]For similar results in the evidence literature, see Bull and Watson (2007) and Deneckere and Severinov (2008).

the principal recommends a message choice from this set, and the agent sends a message. Only then does the principal choose $x \in X$. One can show by examples (omitted for brevity) that, in general, each of these steps may be necessary for the principal to obtain the highest possible payoff.

In this section, we give conditions under which we can identify the principal's recommendations in an optimal mechanism based only on the evidence/signal structure, using little to no information about preferences. Under these conditions, we can eliminate some of the above steps, greatly simplifying the class of mechanisms we need to consider and thus greatly simplifying the analysis.

We begin with the evidence–acquisition model. After stating the protocol formally, we give a condition under which we can identify for each feasible set of messages, a particular message that the principal can always request in an optimal mechanism. Under this condition, we do not need the agent to report the feasible set of messages since the principal's response to this report is known. Consequently, when this condition holds, we can reduce the evidence–acquisition model to a signal–choice model. In effect, this condition enables us to simplify the protocol to have the same features of the game we assumed in Section 3 which allowed the analogous reduction in that context. After showing this, we develop a condition under which we can identify the principal's recommended signal distribution for each type report, again largely independently of the preferences. Under this condition, we can then eliminate the principal's recommendation of a signal distribution, leaving us with a greatly simplified mechanism design problem.

We now define mechanisms for the evidence–acquisition model. We define only deterministic mechanisms, but the principal will typically choose a randomization over these mechanism, for reasons we explain below. We begin with a verbal description, collecting the relevant notation at the end.

The protocol for evidence–acquisition models has seven stages. We refer to this as the *full protocol for evidence–acquisition models*. Recall that $\mathcal{M}$ is the collection of M such that there exists t and $a \in A_t$ with $M \in \text{supp}(a)$.

Stage 1. The agent makes a report of a type $r \in T$.

Stage 2. Given the report, the principal requests a distribution a over evidence sets.

Stage 3. The agent chooses some feasible action a' and the evidence set M is realized.

Stage 4. The agent makes a report $\hat{M} \in \mathcal{M}$ of her realized message set.

Stage 5. The principal proposes a message $m \in \hat{M}$ for the agent to send.

Stage 6. The agent sends a message $\hat{m}$ from the set of messages she has available.

Stage 7. The principal chooses an action x as a function of the history he has observed.

The reader may prefer to skip the following notation on first reading. To state the mechanism protocol formally, we use b's to denote the agent's pure strategies at various stages and g's to denote the principal's pure strategies. The agent chooses three objects. For stage 1, the agent chooses a reporting strategy $b_T : T \to T$. For stage 3, the agent chooses an action strategy giving her action as a function of her true type, her report, and the principal's recommendation, so $b_A : T \times T \times A \to A$. For stage 5, the agent has a second reporting strategy, again a function of all she has seen and done, so $b_{\mathcal{M}} : T \times T \times A \times A \times \mathcal{M} \to \mathcal{M}$. Finally, for stage 6, the agent has an evidence presentation strategy, $b_{\mathcal{L}} : T \times T \times A \times A \times \mathcal{M} \times \mathcal{M} \times \mathcal{L} \to \mathcal{L}$. We require that $b_{\mathcal{L}}(t, r, a, a', M, \hat{M}, m) \in M$ — that is, if the agent's type is t, her report r, the recommended action a, her chosen action a', the realized message set M, the reported message set $\hat{M}$, and the requested message m, the evidence message the agent sends must be in M, the true message set. We let B_T, B_A, $B_{\mathcal{M}}$, and $B_{\mathcal{L}}$ denote the sets of these functions respectively.

Similarly, for stage 2, the principal chooses a recommendation strategy $g_A : T \to A$, giving his recommended action as a function of the reported type. For stage 5, he chooses a message request strategy $g_{\mathcal{L}} : T \times A \times \mathcal{M} \to \mathcal{L}$. We require that $g_{\mathcal{L}}(r, a, \hat{M}) \in \hat{M}$. That is, if the agent reported r, the principal requested action a, and the agent reported evidence set $\hat{M}$, the message the principal requests must be feasible for the agent given her reported evidence set. For stage 7, he chooses an action strategy $g_X : T \times A \times \mathcal{M} \times \mathcal{L} \times \mathcal{L} \to X$. Let G_A, $G_{\mathcal{L}}$, and G_X denote the sets of these functions.

Let the principal's set of pure mechanisms or pure strategies be denoted $G = G_A \times G_{\mathcal{L}} \times G_X$. Let $\Gamma = \Delta(G)$ with typical element γ. We let $(\gamma_A, \gamma_{\mathcal{L}}, \gamma_X)$ denote the equivalent behavior strategy to γ. Let $B = B_T \times B_A \times B_{\mathcal{M}} \times B_{\mathcal{L}}$ denote the agent's set of pure strategies. Let $\beta \in \Delta(B)$ denote a typical mixed strategy for the agent.

A version of the standard Revelation Principle for this class of models says that without loss of generality, we can restrict attention to mechanisms where it is optimal for the agent to report truthfully and to obey the principal's recommendations at every stage along the equilibrium path.

To define incentive compatibility more precisely, note that any (β, γ, t) induces a probability distribution over *complete outcomes* — that is, realized (a, x) pairs. We denote this distribution by $\mu(a, x \mid \beta, \gamma, t)$. Let $U(\beta, \gamma, t)$ denote the agent's expected utility in the mechanism γ given strategy β when her type is t or

$$U(\beta, \gamma, t) = \int_{(a,x) \in A \times X} u(t, a, x) \, d\mu(a, x \mid \beta, \gamma, t).$$

We say that a pure strategy $\hat{b} = (\hat{b}_T, \hat{b}_A, \hat{b}_{\mathcal{M}}, \hat{b}_{\mathcal{L}})$ is *truthful and obedient* if for all t, a, M, and m, we have $\hat{b}_T(t) = t$, $\hat{b}_A(t, t, a) = a$, $\hat{b}_{\mathcal{M}}(t, t, a, a, M) = M$, and $\hat{b}_{\mathcal{L}}(t, t, a, a, M, M, m) = m$. That is, the agent reports truthfully and obeys the principal at all stages. Throughout, we use $\hat{b}^*$ to denote any such honest and obedient strategy.[9]

We say that a mechanism γ for the evidence–acquisition model is *incentive compatible* if for every t,

$$U(\hat{b}^*, \gamma, t) \geq U(b, \gamma, t), \quad \forall b \in B$$

for any truthful and obedient strategy $\hat{b}^*$. (Clearly, this condition also implies that $\hat{b}^*$ is a better strategy for the agent than any mixed strategy $\beta \in \Delta(B)$.)

Given any incentive compatible γ, let $\mu^*(a, x \mid \gamma, t) = \mu(a, x \mid \hat{b}^*, \gamma, t)$. We refer to μ^* as the *mechanism outcome*.

4.1 Identifying the Recommended Message

Clearly, this is a complex protocol, giving us a complex set of mechanisms and incentive compatibility constraints. In the rest of this section, we introduce two ways to simplify the protocol and conditions under which these simplifications are without loss of generality.

In both cases, the idea is to identify some choices by the principal in a way which depends on the evidence structure but uses little or no information about the preferences of the principal or the agent. As we will see, the ability to identify such choices greatly reduces the complexity of the protocol and the mechanism design problem.

The idea behind the first simplification is to identify the principal's response at Stage 5. If for every possible $\hat{M}$, there is a specific $m \in \hat{M}$ that the principal will always ask for, regardless of the preferences or other details of the model, then we can delete Stage 5, taking as given that the principal will request this message. This enables us to eliminate Stage 4 since the agent's report of a message set is needed only to give the principal the opportunity to make such a recommendation. Hence we can combine Stages 3 and 6, skipping Stages 4 and 5.

One way to understand when we can identify the principal's response in this way is by comparison to the literature with exogenously given evidence. In such models, one may need the principal to randomize over which message to request in response to the agent's type report. The idea is to prevent the agent from knowing how the principal

[9] Note that there are many such strategies since we do not specify how the agent behaves on histories inconsistent with her strategy. Truth–telling and obedience are without loss of generality on path, but not necessarily off path.

will check various possible lies, thus deterring misreporting. See Glazer and Rubinstein (2004) for illustrative examples. As shown by Bull and Watson (2007), though, under a condition they call normality which Lipman and Seppi (1995) had previously called the full reports condition, this request by the principal is not needed. Normality or full reports says that the agent has available a message which reveals as much information as all the messages the agent has available. Thus asking for this message is the "best" way to deter lies.

We generalize this property to evidence–acquisition models as follows. We say that the evidence technology satisfies *normality* if for every $M \in \mathcal{M}$, there exists $m^*_M \in M$ such that for every $M' \in \mathcal{M}$, we have

$$m^*_M \in M' \iff M \subseteq M'.$$

We refer to the message m^*_M as the *maximal evidence* for M.

To understand this condition, note that $M \subseteq M'$ trivially implies $m^*_M \in M'$ since $m^*_M \in M$. However, we write the condition as an "if and only if," including this trivial direction, to emphasize the following idea. Intuitively, the only thing that presenting a particular message m proves to the principal is that the agent is able to present this message — that is, that the set of messages the agent has available includes m. In this sense, the presentation of m is evidence directly about M', the agent's set of evidence, not about t. It provides evidence only indirectly about t since types differ in terms of which evidence sets they are able or likely to obtain. The definition says that learning that m^*_M is feasible (i.e., that the true evidence set contains it) reveals exactly the same information about the agent's set of messages as learning that every message in M is feasible (i.e., is contained in the true evidence set). In this sense, showing m^*_M reveals exactly what showing every message in M would reveal.

To put it differently, note that if the true message set, say M, is contained in M', then nothing the agent could show would ever refute the possibility that the agent's message set is M'. However, if $M \not\subseteq M'$, then there is some message $m \in M \setminus M'$ which the agent could show and prove conclusively that M' is not the feasible set. Normality says that for every M, there is one message in M which could be used to simultaneously rule out *every* such M', proving to the principal that the true set of messages is either M or something which contains M.

Running Example, Part 5. In our example, $\mathcal{M}$ contains every interval of the form $[0, m]$ for $m \in \mathbf{R}_+$ since each such interval can be generated with positive probability by some (actually, by any) type. Hence it is easy to see that the most informative message, m^*_M, for the interval $[0, m]$ is the upper bound, m. That is, $m^*_{[0,m]} = m$ or, equivalently, $M = [0, m^*_M]$. This is true as for any $m' \in \mathbf{R}_+$, we have $m^*_M \in [0, m']$ if and only if $[0, m^*_M] \subseteq [0, m']$. Hence our running example satisfies normality. As Theorem 1 below will indicate, this means that only the upper bounds of the intervals will ever be used in an optimal mechanism, regardless of the preferences, as asserted earlier. ▌

To see more concretely that normality is about the information content of messages regarding the set of available messages, consider the following example.

Example 1. The agent has two types, t_1 and t_2. Each type has only one distribution over evidence sets. Type t_1 obtains evidence set $\{m_1\}$ with probability 1/3, $\{m_2\}$ with probability 1/3, and $\{m_1, m_2\}$ with probability 1/3. Type t_2 receives evidence set $\{m_2\}$ with probability 1. This evidence technology violates normality. First, note that any singleton evidence set trivially has a maximal evidence message since if $M = \{m\}$, then it is obviously true that for any M', $m \in M'$ iff $M \subseteq M'$. So if normality fails, it is because $\{m_1, m_2\}$ has no maximal evidence message. It is easy to see that this is the case. For either message $m' \in \{m_1, m_2\}$, the singleton $\{m'\}$ is also an element of $\mathcal{M}$. Clearly, then, m' cannot be maximal since $m' \in \{m'\}$ but $\{m_1, m_2\} \not\subseteq \{m'\}$. ▌

To see why this is surprising, note that if the agent presents m_1 to the principal, she proves that her type is t_1 as type t_2 never has this message available. Yet m_1 is not maximal evidence from $\{m_1, m_2\}$. Intuitively, presentation of m_1 proves the agent's type but presenting both m_1 and m_2 would prove more about the agent's available messages than m_1 proves.

One way to understand this is to observe that in standard deterministic evidence models, the agent's type identifies exactly her set of available messages. In a sense, the agent's *full* type is the pair (t, M) where M is the set of messages the agent has. So in this example, unlike in deterministic evidence models, proving that the "type" is t does not prove the agent's full type.[10]

The following theorem shows that normality will enable us to identify the principal's message recommendations, a result we can then use to simplify the protocol. Recall that a mechanism for the principal is a probability distribution γ over G with associated behavior strategy representation $(\gamma_A, \gamma_{\mathcal{L}}, \gamma_X)$.

Theorem 1. *In the evidence–acquisition model, fix any incentive compatible mechanism γ. If the evidence technology is normal, then there exists an incentive compatible mechanism $(\gamma^*_A, \gamma^*_{\mathcal{L}}, \gamma^*_X)$ with the following properties. First, $\gamma^*_{\mathcal{L}}(t, a, M)(m^*_M) = 1$. That is, the principal always recommends the maximal evidence message for any reported M. Second, for all t,*

$$\mu^*(\cdot \mid \gamma, t) = \mu^*(\cdot \mid \gamma^*, t),$$

so γ and γ^* have the same mechanism outcome for every $t \in T$.

[10] Another way to see this point is to redefine the type space to be the set of possible (t, M) and the set of feasible messages for "type" (t, M) to be M. Applying the standard definition of normality to this model yields our definition.

This simplification is, in general, not possible when the evidence technology is not normal. In Appendix B, we analyze the non–normal evidence technology in Example 1 above and show that we cannot identify the message the principal requests from $\{m_1, m_2\}$ independently of the preferences of the principal and agent. More specifically, we give an example of preferences for which it is better for the principal to request m_1 and an example where it is better for him to request m_2, *even though* m_1 *perfectly reveals the agent's type.*

Theorem 1 implies that we can simplify the protocol under normality. Since the principal can always recommend the maximal evidence message for any reported message set, we do not need to include the stage where he makes this recommendation. Similarly, this means we do not need the agent to report the message set since the mechanism does not depend on it.

Hence a corollary to Theorem 1 is that we can use a simpler protocol. We refer to the following as the *abbreviated protocol for evidence–acquisition models*:

Stage 1. The agent reports a $t \in T$.

Stage 2. Given the report, the principal recommends a distribution over evidence sets for the agent.

Stage 3. The agent chooses a distribution and the evidence set M is realized.

Stage 4. The agent sends a message m from the set of available messages M.

Stage 5. The principal chooses an action as a function of the history he has observed, namely the agent's report, the recommended distribution, and the message m.

We abuse notation by using the same notation to denote strategies for this protocol. Hence a pure strategy for the agent is now $b = (b_T, b_A, b_{\mathcal{L}})$ where $b_T : T \to T$ and $b_A : T \times T \times A \to A$ as before. Also, $b_{\mathcal{L}} : T \times T \times A \times A \times \mathcal{M} \to \mathcal{L}$ where $b_{\mathcal{L}}(t, r, a, a', M) \in M$ gives the message the agent sends as a function of her true type t, her reported type r, the principal's recommended distribution a, the distribution she actually chose a, and the realized set M. A pure strategy for the principal is $g = (g_A, g_X)$ where $g_A : T \to A$, with $g_A(t) \in A_t$ and $g_X : T \times A \times \mathcal{L} \to X$ gives the principal's choice of x as a function of the agent's report, the recommended distribution, and the observed message. Again, we denote the agent's pure strategies by $B = B_T \times B_A \times B_{\mathcal{L}}$ and the principal's pure strategies by $G = G_A \times G_X$.

The definition of incentive compatibility for this class of mechanisms is similar to the preceding. Briefly, incentive compatibility requires that an optimal strategy for the agent is to report t truthfully (so $b_T(t) = t$), to follow the principal's recommendation (so $b_A(t,t,a) = a$), and to use maximal evidence (so $b_{\mathcal{L}}(t,t,a,a,M) = m^*_M$).

We have the following corollary, proved in Appendix C:

Corollary 1. *Assume the evidence technology is normal. Then for any incentive compatible mechanism in the full protocol for evidence–acquisition models, there is an incentive compatible mechanism for the abbreviated protocol with the same mechanism outcome.*

4.2 Reduction to Signal Choice

The identification of the principal's recommended message under normality enables us to reduce the mechanism design problem for the evidence–acquisition model to the mechanism design problem for the signal–choice model. To show this, we first describe the latter. In this case, it is easy to see that we can assume the following *protocol for signal–choice*:

Stage 1. The agent reports a $t \in T$. Let a reporting strategy be denoted $b_T : T \to T$.

Stage 2. Given the report, the principal requests a signal distribution. A pure strategy is denoted $g_S : T \to S$.

Stage 3. The agent chooses a signal distribution s as a function of her type, her report, and the recommendation of the principal, with the resulting message seen by the principal. Let $b_S : T \times T \times S \to S$ with $b_S(t,r,s) \in S_t$ denote a typical pure strategy for the agent.

Stage 4. The principal chooses an outcome as a function of what has been said. Let $g_X : T \times S \times \mathcal{L} \to X$ denote a typical pure strategy for the principal at this stage.

Abusing notation, again let $B = B_T \times B_S$ denote the set of pure strategies for the agent and $G = G_S \times G_X$ the set of pure strategies for the principal in this protocol. By the Revelation Principle, we can focus on mechanisms $\gamma \in \Gamma$ with the property that any strategy $\hat{b}^* = (\hat{b}^*_T, \hat{b}^*_S)$ for the agent satisfying $\hat{b}^*_T(t) = t$ and $\hat{b}^*_S(t,t,s) = s$ is a best reply for the agent to γ. Again, we refer to any such $\hat{b}^*$ as truthful and obedient. Given an incentive compatible mechanism γ, we can define the mechanism outcome as the function mapping t to probability distributions over outcomes, here defined as (s,x) pairs. I.e., we can write $\mu^*(s,x \mid \gamma, t)$ as the probability distribution over (s,x) induced by the strategies $(\hat{b}^*, \gamma)$ given the agent's type is t.

Just as in our analysis of games in Section 3, we can think of the agent's strategy in the evidence–acquisition model as a choice of a distribution over evidence sets and a messaging strategy. Again, any given distribution and messaging strategy generates a probability distribution over the message the agent shows the principal. Thus we can replace the selection of a distribution and a messaging strategy with the selection of a signal distribution. In general, this change takes away some of the principal's ability to influence the agent's decisions and will lead to a less effective mechanism. However, under normality, the ability to reduce to the abbreviated protocol implies that this change does not harm the principal.

More formally, fix an evidence–acquisition model. We construct a signal–choice model from it as follows. For any $a \in A$ and any function $\sigma : \text{supp}(a) \to \mathcal{L}$ such that $\sigma(M) \in M$, we can define a signal $s \in \Delta(\mathcal{L})$ by

$$s(m) = a\left(\{M \mid \sigma(M) = m\}\right).$$

Let $\Sigma(a)$ denote the set of such σ functions given a and let $s_{(a,\sigma)}$ denote the distribution on $\mathcal{L}$ induced by (a, σ). Let

$$S_t = \{s_{(a,\sigma)} \mid a \in A_t,\ \sigma \in \Sigma(a)\}.$$

We can define utility functions for the signal–choice model by letting $u(t, s_{(a,\sigma)}, x) = u(t, a, x)$ and analogously for v. This is exactly the translation from evidence acquisition to signal choice discussed less formally in Section 3.

The following result explains the sense in which the signal–choice model so constructed is equivalent to the evidence–acquisition model under normality.

Theorem 2. *In the evidence–acquisition model, fix any incentive compatible mechanism γ. If the evidence technology is normal, then there exists an equivalent incentive compatible mechanism γ^* in the signal–choice model constructed from it in the following sense. For any truthful and obedient strategy $\hat{b}^*$ for the agent in the signal–choice model given γ^*, we have*

$$\mu^*(a, x \mid \gamma, t) = \mu(s_{(a,\sigma^*)}, x \mid \hat{b}^*, \gamma^*, t),$$

for all t where σ^* is the function $\sigma^*(M) = m^*_M$ for all $M \in \text{supp}(a)$. In this sense, γ and γ^* have equivalent mechanism outcomes for every $t \in T$.

In short, under the assumption of normality, any outcome that can be induced by a mechanism for the evidence–acquisition model can be induced by an incentive compatible mechanism in the protocol for the signal–choice model. This is analogous to our result on games in Section 3.

One can consider mechanisms with a variety of other timings. For example, perhaps the agent only comes to the principal *after* having generated evidence. Recognizing this, the optimal mechanism takes into account the way the rules of the mechanism affect these incentives. For example, this seems like a natural way to think about courts.

The lawyers know the rules of the court in advance and work to obtain evidence before bringing the case to court. It is easy to show the analog of Theorem 1, Corollary 1, and Theorem 2 for this model. More specifically, it is still true that under normality, one can restrict attention to mechanisms for which the principal always recommends the maximal evidence message for any evidence set, enabling us to use (an appropriately modified version of) the abbreviated protocol and reduce to a version of the signal–choice model.

4.3 Identifying the Recommended Signal

In this section, we focus on the signal–choice model, where this can be interpreted as a reduced form of the evidence–acquisition model under normality.

While normality greatly simplifies the mechanism design problem, the problem is still complex. We next turn to conditions under which we can identify the signal choice the principal requests as a function of the type.

For convenience, in this section, we assume $\mathcal{L}$ is finite and write a signal distribution $s \in S$ as a (row) vector in $\mathbf{R}_+^{\#\mathcal{L}}$. Fix t^* and $s^*, \hat{s}^* \in S_{t^*}$. We say that s^* is *more informative than* $\hat{s}^*$ if there exists an $\#\mathcal{L} \times \#\mathcal{L}$ Markov matrix Λ such that $s^*\Lambda = \hat{s}^*$ and for every t and every $s \in S_t$, $s\Lambda \in \text{conv}(S_t)$.[11]

In the case where each S_t is finite, we can give an equivalent statement which will aid in clarifying the intuition of this condition. Let $\mathcal{S}$ denote the matrix formed by "stacking" the signal distributions. In other words, this is a matrix with $\#\mathcal{L}$ columns and a number of rows equal to $\sum_t \#S_t$. The first $\#S_{t_1}$ rows are the signal distributions available to t_1, the next $\#S_{t_2}$ rows those available to t_2, etc. Note that if $s \in S_t \cap S_{t'}$ for $t \neq t'$, then s appears (at least) twice in the matrix. Then s^* is more informative than $\hat{s}^*$ if there exists a Markov matrix Λ such that $\mathcal{S}\Lambda = \hat{\mathcal{S}}$ where the matrix $\hat{\mathcal{S}}$ has $\hat{s}^*$ in the row corresponding to s^* in $\mathcal{S}$ and for any row s of $\hat{\mathcal{S}}$ corresponding to one of type t's signal distributions, we have $s \in \text{conv}(S_t)$.

To see the intuition, recall Blackwell–Girshick's (1954) (BG) comparison of experiments. In their model, there are n states of the world. An experiment gives a probability distribution over a finite set of observations as a function of the state of the world. If there are N possible observations, we can write this as an $n \times N$ matrix E where e_{ij} is the probability of observation j in state i. Suppose we have two experiments, E and F. BG say experiment E is more informative than experiment F if there exists a Markov matrix Λ such that $E\Lambda = F$. The matrix Λ defines a garbling of the results of experiment

[11] A matrix is Markov if all entries are non–negative and every row sum is 1.

E, so this says that F can be obtained from E by adding random noise.

Thus we can interpret our informativeness comparison as saying that the "experiment" $\mathcal{S}$ is more informative than "experiment" $\hat{\mathcal{S}}$ in the sense that we can obtain the latter by adding noise to the former. To understand the sense in which $\mathcal{S}$ and $\hat{\mathcal{S}}$ can be thought of as experiments, note that the rows in an experiment correspond to states of the world, while a row in $\mathcal{S}$ corresponds to a (type, signal distribution) pair. Intuitively, just as we can think of (t, M) as the (partly endogenous) "full type" in the evidence–acquisition model, it is natural to think of (t, s) as the "full type" in the signal–choice model.

To see the sense in which the existence of Λ implies s is more informative than s', suppose we have a mechanism in which the principal recommends s' if the agent reports that her type is t. Suppose the principal changes the mechanism to recommend s in this situation instead and changes no other recommendations. Suppose that the principal's response to messages he subsequently receives from the agent after this recommendation is to "garble" them according to the Markov matrix Λ and then to respond the way the original mechanism specified. If the agent uses signal s, then the resulting distribution over the garbled message will be $s\Lambda$. By hypothesis, this is s'. Thus the distribution over the principal's choice of x will be the same as in the original mechanism. Suppose that the agent's true type is $\hat{t}$ and that she uses some signal $\hat{s} \in S_{\hat{t}}$. Then the induced distribution over garbled messages will be $\hat{s}\Lambda$. By hypothesis, this is an element of $\text{conv}(S_{\hat{t}})$. In other words, in the original mechanism, type $\hat{t}$ could have generated this distribution over messages by a particular randomization over her available signals. Thus the expected outcome this type would generate is something she could have generated in the original mechanism. If the original mechanism was incentive compatible, then this deviation is not profitable. Thus the new mechanism is incentive compatible and generates the same outcome as the original one.

Of course, if the more informative signal is so costly that the agent can never be induced to choose it, then it is not useful. Hence to identify the signal the principal will ask the agent to send, we need a condition which identifies that signal as both informative and not excessively expensive. For simplicity, we eliminate the second issue by assuming signals are costless in the sense that neither u nor v depend on the agent's choice of s. To be precise, we say the model has *costless signals* if for every $t \in T$, $x \in X$, and $s, s' \in S_t$, we have $u(t, s, x) = u(t, s', x)$ and $v(t, s, x) = v(t, s', x)$.

Theorem 3. *In the signal–choice model with costless signals, fix any incentive compatible mechanism γ with marginal γ_S on G_S. If there exists t^* and $s^*, \hat{s}^* \in S_{t^*}$ such that s^* is more informative than $\hat{s}^*$, then there exists an incentive compatible mechanism (γ_S^*, γ_X^*)*

satisfying the following two properties. First,

$$\gamma_S^*(t)(s) = \begin{cases} \gamma_S(t)(s), & \text{if } t \neq t^* \text{ or } s \notin \{s^*, \hat{s}^*\}; \\ \gamma_S(t^*)(s^*) + \gamma_S(t^*)(\hat{s}^*), & \text{if } t = t^* \text{ and } s = s^*; \\ 0, & \text{if } t = t^* \text{ and } s = \hat{s}^*. \end{cases}$$

That is, γ^* moves any probability on recommending $\hat{s}^*$ for t^* to recommending s^* instead. Second, for all t,

$$\mu^*(\cdot \mid \gamma, t) = \mu^*(\cdot \mid \gamma^*, t),$$

so γ and γ^* have the same mechanism outcome for every $t \in T$.

Remark 1. Theorems 1 and 3 are connected in the following sense. Suppose we begin with an evidence–acquisition model satisfying normality. By Theorem 2, we can reduce this to a signal–choice model where each signal distribution corresponds to a particular choice of a distribution over evidence sets and a messaging strategy for which message to send as a function of the realized set. Fix a particular distribution over evidence sets and let s be a signal distribution generated from this choice and any messaging strategy which does *not* always select the maximal evidence message. Let s^* be the signal distribution generated from the same distribution over evidence sets and the message strategy which does always select the maximal evidence message. Then s^* is more informative than s in the sense defined above. (See Section F in the Appendix for proof.) Thus the result in Theorem 1 that we can restrict attention to mechanisms where the principal always induces use of maximal evidence can be thought of as an implication of the result in Theorem 3 that we can restrict to mechanisms where the principal always induces more informative signals. We present these results separately since the reduction of the evidence–acquisition model to the signal–choice model requires showing Theorem 1, so we cannot present only Theorem 3. ▌

Ball and Kattwinkel (2019) study a model where the agent reports her type and then the principal selects a probabilistic pass–fail test out of a given set of such tests. Ball and Kattwinkel's (2019) notion of more discerning tests is related to our notion of more informative signals but is not the same. In their model, a given test τ together with a type t and an effort choice by the agent determines a probability distribution over results where the set of results is $\{0, 1\}$. If the agent takes effort, the agent passes the test (gets an outcome of 1) with probability $\tau(t)(1)$ and fails with probability $\tau(t)(0)$. In what follows, we write $\tau(t) = (\tau(t)(0), \tau(t)(1))$ and write $\tau_0(t) = (1, 0)$. In other words, $\tau(t)$ is the distribution over results given that the agent puts in effort, while $\tau_0(t)$ is the distribution when she doesn't.

Ball and Kattwinkel say that a test $\hat{\tau}$ is more t–discerning than a test τ if there is a Markov matrix Λ such that $\hat{\tau}(t)\Lambda = \tau(t)$ and for all $t' \neq t$, $\hat{\tau}(t')\Lambda$ is dominated by $\tau(t')$ in the sense of first–order stochastic dominance. Equivalently, $\hat{\tau}(t)\Lambda = \tau(t)$ and for all

$t' \neq t$, $\hat{\tau}(t')\Lambda = p_{t'}\tau(t') + (1 - p_{t'})\tau_0(t')$ for some $p_{t'} \in [0, 1]$. That is, if we garble the result for t under test $\hat{\tau}$, we get the same result as for test τ given that the agent puts in effort, while for any other t', we get some result the agent could generate by a certain randomization over effort. This looks very similar to our notion, except that our notion applies the garbling Λ to all signal distributions. In our model, the agent could choose any distribution, while in Ball and Kattwinkel, the agent can only choose distributions that can be generated by a choice of an effort level through the test chosen by the principal. In effect, Ball and Kattwinkel assume the principal can "punish" the agent if she chooses a signal different from the one he specifies. Consequently, informativeness comparisons can ignore incentive constraints associated with signals that are not generated by the test specified by the principal. Since we don't give the principal this ability, our condition requires more. In particular, if for type t, a test τ given that the agent takes effort is more informative than any other test and effort selection in our sense, then it is more discerning in their sense.

Theorem 3 implies that in any model with costless signals, if type t has some signal distribution $s^* \in S_t$ which is more informative than any other $s \in S_t$, then the principal may as well always recommend s^* to t. If every t has such a most informative signal distribution, then Stage 2 of the mechanism protocol is not needed as we can restrict attention to mechanisms where every type of the agent is induced to choose her most informative signal distribution. In such a case, we can focus on the following *succinct protocol*:

Stage 1. The agent reports a $t \in T$ and chooses a signal distribution s. Let a reporting strategy be denoted $b_T : T \to T$ and a signal distribution strategy be $b_S : T \to S$ with $b(t) \in S_t$.

Stage 2. The principal observes the report, the realized m, and chooses an outcome. Let $g_X : T \times \mathcal{L} \to X$ denote a typical pure strategy for the principal.

Abusing notation yet again, let $B = B_T \times B_S$ denote the set of pure strategies for the agent and G the set of pure strategies for the principal in this protocol. When each type t has a most informative signal distribution s_t^*, we can focus on mechanisms $\gamma \in \Gamma$ with the property that the strategy $\hat{b}_T(t) = t$ and $\hat{b}_S(t) = s_t^*$ is a best reply for the agent to γ.

Running Example, Part 6. We showed in Part 5 of the example that the evidence–acquisition technology is normal. In particular, given any realized message set of the form $[0, m]$, the upper bound m is the most informative message for the set. Hence Theorem 2 implies that we can focus on the signal–choice model where for each t, S_t is the set of all distributions on $\mathbf{R}_+$ with expectation less than or equal to t. Since $\mathbf{R}_+$ is not finite, we need to adjust the example to apply our condition. So let $\mathcal{L}$ be any finite subset of $\mathbf{R}_+$ containing at least T, where we also generalize the example, now letting T be any finite subset of $\mathbf{R}_+$, not necessarily $\{\ell, h\}$. Assume S_t is the set of all probability distributions on $\mathcal{L}$ with expectation less than or equal to t.

We now show that the most informative signal distribution for type t is the degenerate distribution on t. Fix any type t^*. Let $s^* \in S_{t^*}$ denote the degenerate distribution putting probability 1 on $m = t^*$ and fix any other $s \in S_{t^*}$. Let the Λ matrix be an identity matrix but with the row corresponding to $m = t^*$ replaced by s. That is, we let

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ s(m_1) & s(m_2) & s(m_3) & \dots & s(m_{\#\mathcal{L}-1}) & s(m_{\#\mathcal{L}}) \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $s^*\Lambda = s$. Fix any other type t and any $\hat{s} \in S_t$. Let $\tilde{s} = \hat{s}\Lambda$. For $m \neq t^*$, we have $\tilde{s}(m) = \hat{s}(m) + \hat{s}(t^*)s(m)$. For $m = t^*$, we have $\tilde{s}(t^*) = \hat{s}(t^*)s(t^*)$. So

$$\begin{aligned}\sum_m \tilde{s}(m)m &= \sum_{m\neq t^*} [\hat{s}(m) + \hat{s}(t^*)s(m)]m + \hat{s}(t^*)s(t^*)t^* \\ &= \sum_{m\neq t^*} \hat{s}(m)m + \sum_{m\neq t^*} \hat{s}(t^*)s(m)m + \hat{s}(t^*)s(t^*)t^* \\ &= \sum_{m\neq t^*} \hat{s}(m)m + \hat{s}(t^*)\sum_m s(m)m \\ &\leq \sum_{m\neq t^*} \hat{s}(m)m + \hat{s}(t^*)t^* \\ &= \sum_m \hat{s}(m)m \leq t.\end{aligned}$$

The next–to–last line follows from $s \in S_{t^*}$ and therefore $\sum_m s(m)m \leq t^*$. The last inequality on the last line follows from $\hat{s} \in S_t$ and therefore $\sum_m \hat{s}(m)m \leq t$. Hence for every $\hat{s} \in S_t$, we see that $\hat{s}\Lambda$ is a probability distribution over $\mathcal{L}$ with expectation weakly less than t and hence is an element of S_t and therefore of $\text{conv}(S_t)$. Hence s^* is more informative than s.

Now that we have identified the signal choices for each type in the optimal mechanism, it is not difficult to compute the rest of the mechanism. We already showed that the principal can achieve his best possible outcome for each type when his utility function is $-(t-x)^2$, so consider the hiring version where the principal's choice is to hire the agent ($x = 1$) or not ($x = 0$) and his payoff is $x(t - \bar{w})$ where $\bar{w} \in (\ell, h)$. Recall that types are equally likely. The agent's payoff is x. Let $\gamma^*(t)$ denote the probability the principal chooses $x = 1$ when the agent reports type t and the realized message m also equals t.

Given that the mechanism will induce truthful reporting and will induce the agent to choose the degenerate distribution with $m = t$, the principal's expected payoff is

$$\frac{1}{2}\gamma^*(h)(h-w) + \frac{1}{2}\gamma^*(\ell)(\ell - w).$$

To make it easier to induce the agent to choose the appropriate degenerate distribution, the optimal mechanism has $x = 0$ if the message observed differs from the agent's type report. As we will see, type h never wishes to imitate ℓ, so we do not need to impose this incentive compatibility constraint. Hence the only incentive compatibility constraint we require is

$$\gamma^*(\ell) \geq \gamma^*(h)\frac{\ell}{h},$$

since the maximum probability ℓ can put on $m = h$ is when she chooses the distribution with probability ℓ/h on h and the remaining probability on 0. Since the principal's utility is decreasing in $\gamma^*(\ell)$ and increasing in $\gamma^*(h)$, the constraint is binding. Hence the principal chooses $\gamma^*(h)$ to maximize

$$\gamma^*(h)\left[\frac{1}{2}(h-\bar{w}) + \frac{1}{2}\frac{\ell}{h}(\ell - \bar{w})\right].$$

So if

$$\frac{h^2+\ell^2}{h+\ell} > \bar{w},$$

the optimal mechanism has $\gamma^*(h) = 1$. If we have the opposite strict inequality, it has $\gamma^*(h) = \gamma^*(\ell) = 0$. In both cases, type h has no incentive to imitate type ℓ, as asserted.

Also, in both cases, the outcome is the same as in the equilibrium we computed for this example in Section 3. In this sense, there is no value to the principal from commitment: he obtains the same outcome when he is able to commit to his responses to the agent and in a particular equilibrium of the game where he cannot commit. We present a generalization of the result of this example in the following section. ∎

5 Commitment

We just saw in Section 4 that there is no value to commitment in the hiring version of our running example. In this section, we generalize this result. More specifically, we introduce an assumption on the relationship between the preferences of the principal and of the agent and show that if evidence acquisition is costless, then this assumption implies there is no value to commitment. Thus, under our assumption on preferences, we generalize previous results for deterministic evidence models to a general structure of stochastic evidence.

More specifically, the result does not rely on assumptions regarding normality of the evidence technology or informativeness of signals. It also does not depend on the specific protocol used and so can allow for departures from the protocols discussed in the previous sections. For example, as discussed at the end of Section 4.2, it may be more natural in some settings to assume that the agent first makes a report to the mechanism *after* acquiring evidence. Our result allows this possibility and many others. The primary requirement for the protocol is that we compare "apples to apples" — that is, that we compare what the principal can obtain with commitment in a particular protocol to what he can obtain without commitment in the same protocol. Our primary substantive assumption on the protocol is finiteness, though, as we explain, this can be substantially relaxed. To tractably analyze sequential rationality, we impose further structure on the protocol, but this structure is for convenience, not because the result hinges on it.

In particular, our assumptions on preferences are weaker in some ways and stronger in other ways than the assumptions used in the previous work on value of commitment in games with evidence — see, in particular, Glazer and Rubinstein (2004, 2006), Sher (2011), Hart, Kremer, and Perry (2017), and Ben-Porath, Dekel, and Lipman (2019). With the exception of our previous paper, these models all assume require type–independent utility functions, which we do not require. After stating our result, we compare our assumptions to theirs in more detail.

We present our result in two parts. First, we show that under only a finiteness assumption on the protocol, there is a Nash equilibrium with the same outcome as the optimal mechanism whenever the preferences satisfy a certain condition. Second, we show that we can strengthen the conclusion to perfect Bayesian equilibrium under stronger conditions.

To state the first result, fix any protocol for the evidence–acquisition model (including the special case of the signal–choice model). Let B denote the set of pure strategies for the agent and G the set of pure strategies for the principal as before. As before, let $U(\beta, \gamma, t)$ denote the agent's expected payoff under the protocol given mixed strategy profile (β, γ) when her type is t. Let $U(\beta, \gamma) = \mathrm{E}_t\, U(\beta, \gamma, t)$. Let the principal's expected utility given that the agent is type t be denoted $V(\beta, \gamma, t)$ and let $V(\beta, \gamma) = \mathrm{E}_t V(\beta, \gamma, t)$. Given any $\gamma \in \Delta(G)$, let $BR(\gamma)$ denote the agent's set of best replies — i.e.,

$$BR(\gamma) = \{\beta \in \Delta(B) \mid U(\beta, \gamma) \geq U(\beta', \gamma),\ \forall \beta' \in \Delta(B)\}.$$

Let

$$V^* = \max_{\gamma \in \Gamma} \max_{\beta \in BR(\gamma)} V(\beta, \gamma).$$

In other words, V^* is the principal's maximal expected payoff when he can commit to any mixed strategy in the protocol *and* can choose the agent's best reply to his strategy.

If (β^*, γ^*) solves $V^* = V(\beta^*, \gamma^*)$ and $\beta^* \in BR(\gamma^*)$, we say (β^*, γ^*) *is optimal for the principal.*

For our Nash equilibrium result, the only assumption we make on the protocol and evidence structure is that B and G are finite. As the proof will make clear, it would not be difficult to weaken this finiteness assumption further. In particular, the protocol could have simultaneous moves by the principal and agent or any order of reports and actions.

We make two assumptions on preferences. First, we assume that evidence acquisition is costless. That is, we can write the utility function of the agent and principal as $u : T \times X \to \mathbf{R}$ and $v : T \times X \to \mathbf{R}$ respectively, dropping the a argument used in Section 2.

Our primary preference condition is that there is some function $\nu : T \to \mathbf{R}$ such that $v(t, x) = \nu(t)u(t, x)$ for all $(t, x) \in T \times X$. When this holds, we say the preferences are *semi–aligned.* In terms of the protocol, this holds iff $V(b, g, t) = \nu(t)U(b, g, t)$ for all $(b, g, t) \in B \times G \times T$. The critical implication of semi–aligned preferences — the only fact that will be used in the proof — is that they imply that if $U(b, g, t) = U(b', g', t)$ for all $t \in T$, then $V(b, g) = V(b', g')$. That is, if all types of the agent are indifferent between two outcomes, then the principal is as well.

While this assumption is nontrivial, it is without (further) loss of generality in settings where the principal has two actions available — i.e., where $\#X = 2$. Thus it holds in the hiring version of our running example. Other natural settings where the principal has two actions available are cases where the principal has to decide whether to fund or not fund a project the agent's project, lend or not lend funds, provide a public good or not, etc.

To see that preferences are semi–aligned when there are only two outcomes and evidence acquisition is costless, note that we can write the agent's utility function as $u(t, x)$. When there are only two x's, say, x_0 and x_1, we can always renormalize the agent's payoffs so that $u(t, x_0) = 0$ for all t, $u(t, x_1) = 1$ for types t who prefer x_1 to x_0, and $u(t, x) = -1$ for types who prefer x_0 to x_1.[12] We can also renormalize the principal's utility function so that $v(t, x_0) = 0$ for all t. Without loss of generality, assume $v(t, x_0) \neq v(t, x_1)$ for all t.[13] Hence, given these renormalizations, we can write $v(t, x) = \nu(t)u(t, x)$ by defining $\nu(t) = v(t, x_1)/u(t, x_1)$ for all t.

The following is our result on value of commitment relative to Nash equilibrium.

[12]Types who are indifferent between x_0 and x_1 do not affect the arguments, so we can assume without loss of generality that there are no such types.

[13]If this is violated for some t, then the principal's decisions are the same as those he would make if such t were impossible. Hence such types can be disregarded.

Theorem 4. *Fix any protocol for which B and G are finite. Assume evidence acquisition is costless and that preferences are semi–aligned. If (β^*, γ^*) is optimal for the principal, then there exists $\hat{\beta} \in \Delta(B)$ such that $(\hat{\beta}, \gamma^*)$ is a Nash equilibrium of the game induced by the protocol and $V(\hat{\beta}, \gamma^*) = V^*$.*

Proof. Consider the restricted game where the principal's set of pure strategies is G, but the agent's set of pure strategies is $B \cap BR(\gamma^*)$. By finiteness of B and G, the restricted game has a mixed equilibrium, say, $(\hat{\beta}, \hat{\gamma})$. We will show that $(\hat{\beta}, \gamma^*)$ is a Nash equilibrium of the unrestricted game with $V(\hat{\beta}, \gamma^*) = V^*$, establishing the claim.

By construction, $\hat{\beta}$ must put probability 1 on best replies to γ^* and so is itself a best reply to γ^*. This also implies that

$$U(\hat{\beta}, \gamma^*, t) = U(\beta^*, \gamma^*, t)$$

for all t. Since preferences are semi–aligned, this implies

$$\begin{aligned} V(\hat{\beta}, \gamma^*) &= \mathrm{E}_t[V(\hat{\beta}, \gamma^*, t)] \\ &= \mathrm{E}_t[\nu(t) U(\hat{\beta}, \gamma^*, t)] \\ &= \mathrm{E}_t[\nu(t) U(\beta^*, \gamma^*, t)] \\ &= V(\beta^*, \gamma^*) = V^*. \end{aligned}$$

We complete the proof by showing that γ^* is a best reply to $\hat{\beta}$.

Let

$$\bar{\gamma}_\varepsilon = \varepsilon \hat{\gamma} + (1 - \varepsilon) \gamma^*.$$

We claim that even in the unrestricted game where the agent can choose any $\beta \in \Delta(B)$, if ε is sufficiently small, then $\hat{\beta}$ is a best response to $\bar{\gamma}_\varepsilon$. To see this, for any $b \in B$, let $\hat{\Delta}_b = U(b, \hat{\gamma}) - U(\hat{\beta}, \hat{\gamma})$ and let $\Delta^*_b = U(\hat{\beta}, \gamma^*) - U(b, \gamma^*)$. Then $\hat{\beta} \in BR(\bar{\gamma}_\varepsilon)$ if for all $b \in B$,

$$\varepsilon U(\hat{\beta}, \hat{\gamma}) + (1 - \varepsilon) U(\hat{\beta}, \gamma^*) \geq \varepsilon U(b, \hat{\gamma}) + (1 - \varepsilon) U(b, \gamma^*)$$

or

$$(1 - \varepsilon) \Delta^*_b \geq \varepsilon \hat{\Delta}_b$$

or

$$\varepsilon(\hat{\Delta}_b + \Delta^*_b) \leq \Delta^*_b. \tag{1}$$

Recall that by the definition of the restricted game, $\hat{\beta}$ puts probability 1 on pure strategies that are best responses to γ^* and hence is itself a best response to γ^*. Hence $\Delta^*_b \geq 0$.

Suppose $\Delta^*_b = 0$. Then $b \in BR(\gamma^*)$. Recall that $\hat{\beta}$ is a best reply to $\hat{\gamma}$ within the restricted game. Since $b \in BR(\gamma^*)$, it is a feasible strategy in the restricted game, so $\hat{\beta}$

must be a better reply to $\hat{\gamma}$ than b. So $\hat{\Delta}_b \leq 0$. In this case, equation (1) holds for all $\varepsilon \in [0, 1]$.

So suppose $\Delta_b^* > 0$. If $\hat{\Delta}_b + \Delta_b^* \leq 0$, again, equation (1) holds for all $\varepsilon \in [0, 1]$.

Finally, suppose $\Delta_b^* > 0$ and $\hat{\Delta}_b + \Delta_b^* > 0$. Then equation (1) holds for all $\varepsilon \in [0, \varepsilon_b]$ where $\varepsilon_b \equiv \Delta_b^*/(\hat{\Delta}_b + \Delta_b^*) > 0$. Since B is finite, this implies that there exists $\bar{\varepsilon} > 0$ such that $\hat{\beta}$ is a best response to $\bar{\gamma}_\varepsilon$ for all $\varepsilon \in (0, \bar{\varepsilon})$. Fix any ε in this interval.

Since $\hat{\beta} \in BR(\bar{\gamma}_\varepsilon)$, we have

$$V^* = \max_{\gamma \in \Gamma} \max_{\beta \in BR(\gamma)} V(\beta, \gamma) \geq V(\hat{\beta}, \bar{\gamma}_\varepsilon) = \varepsilon V(\hat{\beta}, \hat{\gamma}) + (1 - \varepsilon) V(\hat{\beta}, \gamma^*). \quad (2)$$

As shown above, $V(\hat{\beta}, \gamma^*) = V^*$, so this implies that $V(\hat{\beta}, \gamma^*) \geq V(\hat{\beta}, \hat{\gamma})$.

Because $(\hat{\beta}, \hat{\gamma})$ is an equilibrium of the restricted game (and because this game does not restrict the principal's strategies), we have $V(\hat{\beta}, \hat{\gamma}) \geq V(\hat{\beta}, \gamma^*)$. Hence $V(\hat{\beta}, \hat{\gamma}) = V(\hat{\beta}, \gamma^*)$. Hence γ^* is a best reply to $\hat{\beta}$, completing the proof. ▌

Remark 2. The proof of this result does not use the assumption that evidence acquisition is costless, though the PBE result below does. On the other hand, the assumption that preferences are semi–aligned can be unnatural when evidence acquisition is costly. When preferences are semi–aligned and evidence acquisition is costly, the principal cares about the agent's costs. In some situations, this seems natural. For example, if the principal is a social welfare agency which requires potential aid recipients to provide documentation establishing their need, then the principal may well be concerned about not overburdening these recipients. On the other hand, if there are recipient types to whom the principal does not want to give aid, then we must model this by assuming $\nu(t) < 0$ for these types. In this case, our preference assumption would imply that the principal is made better off when such types bear high costs getting documents, an odd assumption.

To extend the result to perfect Bayesian equilibrium, we need to put more structure on the protocol. Otherwise, it is difficult to characterize what kind of choices the principal might have at certain information sets and therefore difficult to characterize sequential rationality at all information sets. We emphasize that the additional structure is to allow a relatively straightforward proof; we do not know of any counterexamples from protocols outside the class we consider. We see the particular protocol used here as a reasonably general but illustrative structure.

To avoid repetition, we state the definitions, result, and proof for the evidence–acquisition model, but it is not difficult to rewrite it for the signal–choice model instead. For simplicity, we assume the protocol is a multi–stage game with certain properties. To be specific, as in all our previous analysis, we assume that the agent learns her type first.

After this, we have some fixed finite number of stages. Each of these stages has one of two forms. The first possibility is that we have cheap talk messages, either one from the agent to the principal or one from the principal to the agent. The set of cheap–talk messages is fixed throughout, independently of the agent's type or any actions. At the end of such a stage, both players observe the message sent. The second possibility is that the agent chooses some unobserved action which may affect the set of evidence she'll end up with and she may privately observe some outcome of this action. At the end of such a stage, the principal does not observe either the agent's action or this outcome. For simplicity, we suppose that the order in which these various forms of stages occur is fixed exogenously, independently of the agent's type or actions.

After these stages, there's a last stage where the agent presents an evidence message to the principal and the principal responds by choosing an action from the set X. The set of evidence messages available to the agent depends stochastically on the agent's type and the sequence of actions and outcomes from the earlier stages.

We require, as above, that the principal and agent have finite sets of pure strategies. Hence we assume there is a finite K such that for all feasible histories of messages, no more than K stages are played. Similarly, we assume that the set of all possible cheap talk messages for either player is finite as is the set of actions and possible evidence messages for the agent.

We say that a protocol satisfying these properties is *allowable*.

Theorem 5. *Given any allowable protocol, under the assumptions of Theorem 4, there is a perfect Bayesian equilibrium* $(\hat{\beta}, \hat{\gamma})$ *with* $V(\hat{\beta}, \hat{\gamma}) = V^*$.

We conclude this section by discussing the strength and tightness of our assumptions. First, we compare them to those used in earlier results showing no value to commitment in mechanism design problems with evidence. As we noted at the outset, the previous literature all considered deterministic evidence, so the main way our result differs is in extending to a general model of stochastic evidence. The comparison of the assumptions on preferences is more involved.

The first results in the deterministic evidence literature were shown by Glazer and Rubinstein (2004, 2006). They considered the case where the principal chooses between two outcomes, called accept ($x = 1$) and reject ($x = 0$). The agent's utility is x. The principal's utility is x if the agent's type is in a certain set of types, $-x$ otherwise. They consider nonstochastic evidence — that is, each type has only a single degenerate distribution over evidence sets. They do not assume the deterministic evidence version of normality, just as we do not require normality, but, unlike us, they assume a particular protocol. As discussed above, when $\#X = 2$ and there are no costs to signals/distributions, our assumption of semi–aligned preferences is without loss of generality. Our preference structure is more general than that of Glazer–Rubinstein since we are not imposing their assumption that u is independent of t.

Sher (2011) generalizes Glazer–Rubinstein to allow a finite set of actions but where the principal's utility can be written as a concave function of the agent's utility. Hart, Kremer, and Perry (2017) generalize the preference conditions by assuming that the principal's utility can be written as a single–peaked function of the agent's utility given any belief over the agent's type. Both papers continue to assume that the agent's utility is independent of her type. Hart, Kremer, and Perry restrict the set of mechanisms to deterministic mechanisms. They also impose the version of normality for deterministic evidence models, unlike Sher and unlike us. Our assumptions on the protocol are weaker than theirs and our assumptions on preferences do not impose type–independence. On the other hand, if we specialize our assumptions to the type–independent case, then their assumptions on preferences are weaker than ours. Thus neither model is strictly more general than the other.

Our 2019 paper differs in part by considering the multi–agent case. Specializing to the single–agent case, we assumed that the principal's utility function could be written as $v(t,x) = \nu(t)u(t,x) + \psi(x)$, a slight generalization of our assumption here of semi–aligned preferences. With the more general evidence structure we use here, we cannot generalize to the preferences we used in our previous paper. The simplest way to see this is to note that if this were possible, then we could certainly extend to $v(t,x) = \nu(t)u(t,x) + \psi(x) + \varphi(t)$, since this just adds a term which does not affect any optimal choices. Put differently, we could renormalize the principal's payoff by subtracting the expectation of this function.

Part 4 of our running example shows that this is not possible. Recall that we considered a version where $u(t,x) = x$ and $v(t,x) = -(x-t)^2$ and showed that commitment enabled the principal to obtain a strictly higher payoff than in any equilibrium. Note that the principal's payoff function can be written as $2tx - t^2 - x^2 = 2tu(s,x,t) - t^2 - x^2$. So letting $\nu(t) = 2t$, $\varphi(t) = -t^2$, and $\psi(x) = -x^2$, we see that this shows the result does not extend.

Similarly, the result does not extend to multiple agents. For example, suppose we have two agents, $i = 1,2$. Suppose the principal's decision is which agent to give one unit of a good to. Let $X = \{0,1,2\}$ where $x = 0$ means the principal keeps the good and $x = i$ means the principal gives the good to agent i. Suppose agent i's utility function, $u_i(t_i,x)$ is 1 if i receives the good, 0 otherwise. Suppose the principal's payoff is $v_i(t_i)$ if he gives the good to agent i. Then we can write the principal's utility as $v(t,x) = \sum_i v_i(t_i)u_i(t_i,x)$, a natural generalization of our assumption of semi–aligned preferences for the multiple agent case. However, in Appendix H, we give an example showing that the no–value–to–commitment result does not hold for this model even though the principal has only two actions.[14] Again, this is in contrast to results in Ben-Porath, Dekel, and Lipman (2019) for deterministic evidence.

[14]One can show that there is still a value to commitment if we allow the principal to keep the good.

Appendix

A Proof of Theorem 1

Fix any incentive compatible mechanism $(\gamma_A, \gamma_{\mathcal{L}}, \gamma_X)$. We show how to construct an incentive compatible mechanism with the same mechanism outcome with the property that the principal always recommends m^*_M when the agent reports message set M.

Fix any profile $(\hat{t}, \hat{a}, \hat{M}, \hat{m})$ consisting of a type report $\hat{t} \in T$, a recommended distribution over evidence sets $\hat{a} \in \text{supp}(\gamma_A(\hat{t}))$, a reported message set $\hat{M} \in \mathcal{M}$, and a requested message $\hat{m} \in \text{supp}(\gamma_{\mathcal{L}}(\hat{t}, \hat{a}, \hat{M}))$ such that $\hat{m} \neq m^*_{\hat{M}}$. If there is no such tuple, then the principal always recommends maximal evidence, so there is nothing to prove. We construct an alternative mechanism which replaces the recommendation $\hat{m}$ with a recommendation of $m^*_{\hat{M}}$ in this situation and will show that this mechanism is incentive compatible and implements the same outcome as the original mechanism. For brevity, let $\hat{h} = (\hat{t}, \hat{a}, \hat{M})$, the history on which we are changing the recommendations. We use h to denote a typical element of $T \times A \times \mathcal{M}$.

Define the new mechanism, $(\gamma^*_A, \gamma^*_{\mathcal{L}}, \gamma^*_X)$, as follows. First, $\gamma^*_A = \gamma_A$. Let $\gamma^*_{\mathcal{L}}$ satisfy $\gamma^*_{\mathcal{L}}(h)(m) = \gamma_{\mathcal{L}}(h)(m)$ if $h \neq \hat{h}$. Similarly, let $\gamma^*_{\mathcal{L}}(\hat{h})(m) = \gamma_{\mathcal{L}}(\hat{h})(m)$ for $m \notin \{\hat{m}, m^*_{\hat{M}}\}$. Finally, let

$$\gamma^*_{\mathcal{L}}(\hat{h})(m) = \begin{cases} \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}}) + \gamma_{\mathcal{L}}(\hat{h})(\hat{m}), & \text{if } m = m^*_{\hat{M}}; \\ 0, & \text{if } m = \hat{m}. \end{cases}$$

In other words, the probability that was on recommendation $\hat{m}$ is moved to $m^*_{\hat{M}}$.

Let $\gamma^*_X(h, m, m')(x) = \gamma_X(h, m, m')(x)$ if $(h, m) \neq (\hat{h}, m^*_{\hat{M}})$. In other words, on histories other than $\hat{h}$ and on $\hat{h}$ if the principal did not request maximal evidence, we do not change the mechanism's outcome. Also, for all $m \in \mathcal{L} \setminus \{m^*_{\hat{M}}\}$, we set $\gamma^*_X(\hat{h}, m^*_{\hat{M}}, m)(x)$ equal to

$$\frac{\gamma_{\mathcal{L}}(\hat{h})(\hat{m})\gamma_X(\hat{h}, \hat{m}, m)(x) + \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})\gamma_X(\hat{h}, m^*_{\hat{M}}, m)(x)}{\gamma_{\mathcal{L}}(\hat{h})(\hat{m}) + \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})}.$$

Finally, we set $\gamma^*_X(\hat{h}, m^*_{\hat{M}}, m^*_{\hat{M}})(x)$ equal to

$$\frac{\gamma_{\mathcal{L}}(\hat{h})(\hat{m})\gamma_X(\hat{h}, \hat{m}, \hat{m})(x) + \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})\gamma_X(\hat{h}, m^*_{\hat{M}}, m^*_{\hat{M}})(x)}{\gamma_{\mathcal{L}}(\hat{h})(\hat{m}) + \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})}.$$

In other words, if $m^*_{\hat{M}}$ is requested and anything else is reported, then the response is the "average response" to this form of disobedience, averaging over the cases where $\hat{m}$

or $m^*_{\hat{M}}$ was requested in the original mechanism. On the other hand, if $m^*_{\hat{M}}$ is requested and reported, then the response is the average response to obedience in response to a request for either $\hat{m}$ or $m^*_{\hat{M}}$ in the original mechanism.

We first show that this change in the mechanism does not change the outcome if the agent is truthful and obedient. The only situation a truthful and obedient agent is affected by the change is when her type is $\hat{t}$, the principal recommends (and she chooses) action $\hat{a}$, and the resulting message set is $\hat{M}$. Conditional on history $\hat{h}$ and obeying the principal's recommendations, the probability of x in the new mechanism is

$$\begin{aligned}
&\sum_{m\in\mathcal{L}} \gamma^*_{\mathcal{L}}(\hat{h})(m)\gamma^*_X(\hat{h},m,m)(x) \\
&= \sum_{m\in\mathcal{L}\setminus\{\hat{m},m^*_{\hat{M}}\}} \gamma_{\mathcal{L}}(\hat{h})(m)\gamma_X(\hat{h},m,m)(x) \\
&\quad + 0 + \gamma^*_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})\gamma^*_X(\hat{h},m^*_{\hat{M}},m^*_{\hat{M}})(x) \\
&= \sum_{m\in\mathcal{L}\setminus\{\hat{m},m^*_{\hat{M}}\}} \gamma_{\mathcal{L}}(\hat{h})(m)\gamma_X(\hat{h},m,m)(x) \\
&\quad + [\gamma_{\mathcal{L}}(\hat{h})(\hat{m}) + \gamma_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})]\gamma^*_X(\hat{h},m^*_{\hat{M}},m^*_{\hat{M}})(x) \\
&= \sum_{m\in\mathcal{L}} \gamma_{\mathcal{L}}(\hat{h})(m)\gamma_X(\hat{h},m,m)(x).
\end{aligned}$$

Hence, as asserted, the outcome under truth–telling is the same in the new mechanism as in the original mechanism. Therefore, the agent's expected payoff from truth–telling and obedience is the same in the two mechanisms.

We now show that for any type t and any deviation feasible for t in the new mechanism, there is a deviation that is feasible for type t in the original mechanism which yields the same expected payoff. Since truth–telling is superior to any feasible deviation in the original mechanism, then, truth–telling is superior to any feasible deviation in the new mechanism.

To see this, fix any type t (which may equal $\hat{t}$) and consider any feasible deviation. Obviously, if the deviation involves reporting a type other than $\hat{t}$, this deviation is also available in the original mechanism and yields the same payoff in the new mechanism as in the original one since the way the mechanism responds to such a report has not changed. Hence we can restrict attention to deviations which involve reporting type $\hat{t}$. So fix any such deviation. Clearly, we may as well condition on the event that the principal requests the distribution $\hat{a}$, the agent chooses a (which may equal $\hat{a}$), the agent obtains message set M, and reports message set $\hat{M}$ (which may equal M). Let $z : \hat{M} \to M$ give the message the agent sends as a function of the message the principal requests from her.

Then the agent's expected payoff conditional on this event is

$$\sum_{(x,m)\in X\times\mathcal{L}} \gamma^*_{\mathcal{L}}(\hat{h})(m)\gamma^*_X(\hat{h},m,z(m))(x)u(t,a,x).$$

We can write this as

$$\sum_{(x,m)\in X\times(\mathcal{L}\setminus\{\hat{m},m^*_{\hat{M}}\})} \gamma_{\mathcal{L}}(\hat{h})(m)\gamma_X(\hat{h},m,z(m))(x)u(t,a,x)$$
$$+\gamma^*_{\mathcal{L}}(\hat{h})(m^*_{\hat{M}})\sum_{x\in X}\gamma^*_X(\hat{h},m^*_{\hat{M}},z(m^*_{\hat{M}}))(x)u(t,a,x).$$

We have two cases. First, suppose $z(m^*_{\hat{M}}) \neq m^*_{\hat{M}}$. In this case, the last term is equal to

$$\sum_{(x,m)\in X\times\{\hat{m},m^*_{\hat{M}}\}} \gamma_M(\hat{h})(m)\gamma_X(\hat{h},m,z(m^*_{\hat{M}}))(x)u(t,a,x).$$

Thus the conditional payoff to the deviation in the new mechanism is the same as the conditional payoff in the original mechanism where the agent responds to a request for *either* $\hat{m}$ or $m^*_{\hat{M}}$ by sending $z(m^*_{\hat{M}})$. So in this case, the payoff to the deviation in the new mechanism is the same as the payoff to a certain deviation which was also feasible in the original mechanism.

Second, suppose $z(m^*_{\hat{M}}) = m^*_{\hat{M}}$. In this case, the last term is equal to

$$\sum_{(x,m)\in X\times\{\hat{m},m^*_{\hat{M}}\}} \gamma_M(\hat{h})(m)\gamma_X(\hat{h},m,m)(x)u(t,a,x).$$

In other words, the payoff in the new mechanism is the same as the payoff in the old mechanism where the agent responds to a request for $\hat{m}$ with $\hat{m}$ and a request for $m^*_{\hat{M}}$ with $m^*_{\hat{M}}$. Note that we are assuming that the deviation in the new mechanism is feasible for the agent, so $m^*_{\hat{M}} \in M$. By the definition of normality, this implies $\hat{m} \in M$. Hence this deviation has the same payoff as a feasible deviation in the original mechanism.

In either case, then, the best deviation payoff in the new mechanism cannot exceed the best deviation payoff in the original mechanism, so the new mechanism is incentive compatible.

Clearly, we can repeat this argument as needed to obtain an incentive compatible mechanism which has the same mechanism outcome as γ and which has the property that $\gamma_{\mathcal{L}}(t,a,M)(m^*_M) = 1$ for all $(t,a,M) \in T \times A \times \mathcal{M}$.

B Details for Example 1

Recall that there are two types, each with prior probability 1/2, and each type has only one possible probability distribution over evidence sets. Type t_1 obtains evidence set $\{m_1\}$ with probability 1/3, $\{m_2\}$ with probability 1/3, and $\{m_1, m_2\}$ with probability 1/3. Type t_2 receives evidence set $\{m_2\}$ with probability 1.

We give two examples of preferences, one for which the principal is strictly better off requesting message m_1 when the agent reports evidence set $\{m_1, m_2\}$ and one where he is strictly better off requesting message m_2.

Before giving the examples, we note that we can simplify mechanisms for this setting in some innocuous ways. First, there is no need for the principal to recommend a choice of a distribution over evidence since each type has only one such distribution available. Similarly, if the agent reports a singleton evidence set, there is no need for the principal to recommend a particular message. Thus a mechanism starts with a type report by the agent. If the agent reports t_1, she then makes a report of the realized evidence set. If this set is $\{m_1, m_2\}$, the principal then recommends a message to send from this set. Finally, after the agent sends a message, there is an outcome as a function of all that the principal has seen. Incentive compatibility means that the agent reports truthfully and follows the principal's recommendation if she reports evidence set $\{m_1, m_2\}$.

First, we present preferences for which the principal is strictly better off requesting m_2 from $\{m_1, m_2\}$. For this example, let $X = \{x, y, z\}$. Because there is only one possible distribution over evidence sets for each type, we can write utility for the principal and the agent as a function of x and t only. Assume the agent's utility as a function of her type is given by

	t_1	t_2
x	1	0
y	0	1
z	2	1

while the principal's utility is given by

	t_1	t_2
x	0	M
y	1	0
z	-2	0

where $M > 0$ and large. We write a typical lottery over X in the form (q_x, q_y, q_z). We also write x interchangably with $(1, 0, 0)$ and similarly for other degenerate lotteries.

Consider the following mechanism. Regardless of the type/evidence set reports by the agent and the requested message (if any) by the principal, if the message sent by the agent is m_2, the outcome is x. Regardless of the reports and requested message, if the message sent by the agent is m_1, the outcome is y. The principal requests message m_2 when the agent reports type t_1 and message set $\{m_1, m_2\}$.

It is not hard to see that the agent may as well report truthfully and will obey the principal's recommendation. If the agent is type t_2, the only evidence message she can send is m_2, which necessarily generates outcome x. Hence she may as well report her type truthfully. If the agent is type t_1 and has evidence set $\{m_1\}$ or evidence set $\{m_2\}$, again her report of an evidence set does not affect the outcome, so she may as well report the evidence set truthfully. If her evidence set is $\{m_1, m_2\}$, she will want to send message m_2 so she will follow the principal's recommendation and has no incentive to misreport the set of evidence she has. Finally, t_1 has no incentive to misreport her type at the outset since the type report itself does not affect the outcome she will generate.

It is easy to see that this mechanism yields the principal an expected payoff of $(1/2)M + (1/6)$.

Now suppose the principal requests m_1 when the agent reports type t_1 and evidence set $\{m_1, m_2\}$. Without loss of generality, we can assume that if the principal observes a history inconsistent with obedience and truth–telling where the agent sends evidence message m_1, then he chooses y. This is because m_1 proves that the type is t_1 and y is the worst outcome for t_1. Hence this does the best possible job of preventing such deviations. We also may as well assume that there is some lottery δ which is the principal's response to any history inconsistent with obedience and truth–telling where the agent sends evidence message m_2. This is because all such deviations are feasible for any type and hence all are required to be worse for any type than truth–telling and obedience. Since the outcomes from these various situations have the same direct payoff consequences for the principal (none since they are off path) and are subject to the same constraints, there is no need to make them different.

Let α^i, $i = 1, 2$, denote the outcome when the agent reports type t_1, evidence set $\{m_i\}$, and sends evidence message m_i. Let α^3 denote the outcome when the agent reports t_1, evidence set $\{m_1, m_2\}$, and obeys the principal recommendation to send m_1. Finally, let β denote the outcome when the agent reports t_2 and sends evidence message m_2.

To write the incentive compatibility constraints succinctly, let $\succeq_i$ denote the preference relation for type t_i, $i = 1, 2$. We require that each type reports truthfully and obeys rather than doing some obvious deviation involving m_2 (e.g., reporting a type and/or message set that implies she must have a certain evidence message and then sending the other). Hence incentive compatibility requires $\alpha^2 \succeq_1 \delta$ (otherwise, when t_1 has $\{m_2\}$, she could claim evidence set $\{m_1\}$), $\alpha^3 \succeq_1 \delta$ (so when t_1 has evidence set $\{m_1, m_2\}$, she has no incentive to claim $\{m_1\}$ and the prove she lied), and $\beta \succeq_2 \delta$ (so t_2 does not lie and claim to be t_1 with evidence set $\{m_1\}$). Note that t_1 never has an incentive to lie in an obvious way and send m_1 since this leads to her least preferred outcome y.

There are only a few situations in which the agent can misreport without being automatically caught. First, if she is type t_2, she could claim to be t_1 and to have message set $\{m_2\}$. Hence we require $\beta \succeq_2 \alpha^2$. Second, if the agent is type t_1 and has evidence set $\{m_1, m_2\}$, she could claim either evidence set $\{m_1\}$ or $\{m_2\}$ and provide the evidence message consistent with this. Hence we require $\alpha^3 \succeq_1 \alpha^1, \alpha^2$. Third, because the agent knows she will be asked to report m_1 if she claims to have $\{m_1, m_2\}$, an agent of type t_1 with evidence set $\{m_1\}$ can claim to have evidence set $\{m_1, m_2\}$. Hence we require $\alpha^1 \succeq_1 \alpha^3$.

The only other constraint is that the agent reports her type honestly at the outset. For t_2, this is implied by $\beta \succeq_2 \delta$ and $\beta \succeq_2 \alpha^2$. For t_1, we require that $(1/3)(\alpha^1 + \alpha^2 + \alpha^3) \succeq_1 (1/3)y + (2/3)\beta$.

Summarizing, the constraints are

$$\alpha^1 \sim_1 \alpha^3 \succeq_1 \alpha^2 \succeq_1 \delta$$

$$\frac{1}{3}\alpha^1 + \frac{1}{3}\alpha^2 + \frac{1}{3}\alpha^3 \succeq_1 \frac{1}{3}\,y + \frac{2}{3}\,\beta$$

$$\beta \succeq_2 \alpha^2, \delta.$$

From here, we see that we may as well set $\delta = \alpha^2$ since this will satisfy the constraints and, since δ is off path, has no direct payoff consequences for the principal. Also, there is no need to distinguish between α^1 and α^3 since they enter entirely symmetrically. So we may as well take $\alpha^3 = \alpha^1$. This reduces the constraints to

$$\alpha^1 \succeq_1 \alpha^2$$

$$\frac{2}{3}\alpha^1 + \frac{1}{3}\alpha^2 \succeq_1 \frac{1}{3}\,y + \frac{2}{3}\,\beta$$

$$\beta \succeq_2 \alpha^2.$$

Rewriting in terms of expected utilities and doing some rearranging gives

$$\alpha_x^1 + 2\alpha_z^1 \geq \alpha_x^2 + 2\alpha_z^2 \tag{3}$$

$$\alpha_x^1 + 2\alpha_z^1 + \frac{1}{2}[\alpha_x^2 + 2\alpha_z^2] \geq \beta_x + 2\beta_z \tag{4}$$

$$\beta_x \leq \alpha_x^2. \tag{5}$$

It is not hard to see that we may as well set $\beta_z = 0$. If it is strictly positive, we can reduce it to zero, increasing β_y to compensate. Since the principal gets 0 utility from y

or z given type t_2, this has no direct payoff consequences but may relax the constraints. Hence we can rewrite equation (4) as

$$\alpha_x^1 + 2\alpha_z^1 + \frac{1}{2}[\alpha_x^2 + 2\alpha_z^2] \geq \beta_x.$$

But we have

$$\alpha_x^1 + 2\alpha_z^1 + \frac{1}{2}[\alpha_x^2 + 2\alpha_z^2] \geq \alpha_x^1 + 2\alpha_z^1 \geq \alpha_x^2 + 2\alpha_z^2 \geq \alpha_x^2 \geq \beta_x$$

where the first inequality is from non–negativity of the probabilites, the second from equation (3), the third from non–negativity, and the fourth from equation (5). Hence constraint (4) is redundant.

From the remaining constraints, then, we see that we must have $\beta_x = \alpha_x^2$. This is because the principal wants β_x as large as possible and it affects no constraints other than (5). Finally, then, we can write the principal's problem as maximizing

$$\frac{1}{2}\,\alpha_x^2 M + \frac{1}{3}[\alpha_y^1 - 2\alpha_z^1] + \frac{1}{6}[\alpha_y^2 - 2\alpha_z^2]$$

subject to equation (3). If we substitute for α_y^i using $\alpha_x^i + \alpha_y^i + \alpha_z^i = 1$ and rearrange, we can write the objective function as a constant plus

$$-\frac{1}{3}\alpha_x^1 - \alpha_z^1 - \frac{1}{2}\alpha_z^2 + \alpha_x^2\left(\frac{1}{2}\,M - \frac{1}{6}\right).$$

Clearly, the remaining constraint (3) must bind — otherwise, $\alpha_x^1 = \alpha_z^1 = \alpha_z^2 = 0$ and $\alpha_x^2 = 1$, contradicting the constraint. Hence

$$\alpha_x^1 = \alpha_x^2 + 2\alpha_z^2 - 2\alpha_z^1.$$

Substituting into the objective function gives

$$-\frac{1}{3}\,\alpha_z^1 - \frac{7}{6}\,\alpha_Z^2 + \alpha_x^2\left(\frac{1}{2}\,M - \frac{1}{2}\right).$$

Assuming $M > 1$, the solution is $\alpha_x^2 = 1$ and $\alpha_z^2 = \alpha_z^1 = 0$, implying $\alpha_x^1 = 1$ as well.

In short, the optimal mechanism which requests m_1 when the agent reports evidence set $\{m_1, m_2\}$ has outcome x regardless of the agent's type or evidence set. This gives the principal a payoff of $M/2$, strictly below what he obtained from the mechanism which requested m_2 from $\{m_1, m_2\}$.

The preferences for which the principal is strictly better off requesting m_1 are simpler. Here we let $X = \{x, y\}$, take the agent's utility function to be

	t_1	t_2
x	0	0
y	1	1

and assume the principal's utility is given by

	t_1	t_2
x	0	M
y	1	0

where $M > 0$ and large. First, consider the following mechanism where the principal induces the agent to send m_1 from evidence set $\{m_1, m_2\}$: the principal chooses y whenever the message sent is m_1 and x whenever it is m_2. Since both types of agents prefer y, the agent will send m_1 whenever possible. So the principal will get the outcome x when the type is t_2 and get x with probability 1/3, y otherwise, when the type is t_1. The principal's payoff is $(1/2)M + (1/3)$.

Next consider the best mechanism for the principal with the property that he induces the agent to send m_2 from $\{m_1, m_2\}$. As before, let α^i be the lottery chosen by the principal when the agent reports t_1, reports evidence set $\{m_i\}$, and sends message m_i. Let α^3 be the outcome when the agent reports t_1 and evidence set $\{m_1, m_2\}$ and then follows the principal's recommendation to set message m_2. Let β be the outcome when the type report is t_2, the evidence set report is $\{m_2\}$, and the message is m_2. Let δ^i denote the outcome when the history is clearly off path and the message sent is m_i. Then the incentive compatibility constraints are as follows. Type t_1 with evidence set $\{m_i\}$ must prefer α^i to the off path deviation she can generate of δ^i, $i = 1, 2$. Similarly, $\beta \succeq_2 \delta^2$. For undetectable deviations, we require $\alpha^2 \succeq_1 \alpha^3$ since t_1 with evidence set $\{m_2\}$ can pretend to have evidence set $\{m_1, m_2\}$. Also, we require $\alpha^3 \succeq_1 \alpha^1, \alpha^2$. Since t_2 could pretend to be t_1 with either $\{m_2\}$ or $\{m_1, m_2\}$, we require $\beta \succeq_2 \alpha^2, \alpha^3$. Finally, we require

$$\frac{1}{3}\alpha^1 + \frac{1}{3}\alpha^2 + \frac{1}{3}\alpha^3 \succeq_1 \frac{1}{3}\delta^1 + \frac{2}{3}\beta.$$

It is not hard to see that we may as well set $\alpha^2 = \delta^2$ since δ^2 has no direct payoff consequences and this will necessarily satisfy the constraints. Also, given the symmetry of the way they enter the problem, we may as well set $\alpha^2 = \alpha^3$. Finally, given that the constraints are weaker as δ^1 is made worse for t_1, we may as well set $\delta^1 = x$.

Using this to simplify, the constraints are

$$\alpha^2 \succeq_1 \alpha^1$$

$$\frac{1}{3}\alpha^1 + \frac{2}{3}\alpha^2 \succeq_1 \frac{1}{3}x + \frac{2}{3}\beta$$

$$\beta \succeq_2 \alpha^2.$$

Rewriting using the agent's utility function, we have

$$\alpha^2_y \geq \alpha^1_y$$

$$\frac{1}{3}\,\alpha_y^1 + \frac{2}{3}\,\alpha_y^2 \geq \frac{2}{3}\,\beta_y$$
$$\beta_y \geq \alpha_y^2.$$

Since the principal prefers x to y when the agent is type t_2, the last constraint must bind, so $\beta_y = \alpha_y^2$. This implies that the middle constraint does not bind. Hence we can reduce the problem to maximizing

$$\frac{1}{2}\,(1-\alpha_y^2)M + \frac{1}{2}\left[\frac{1}{3}\,\alpha_y^1 + \frac{2}{3}\,\alpha_y^2\right]$$

subject to $\alpha_y^2 \geq \alpha_y^1$. It is easy to see that if M is large enough, the solution is $\alpha_y^1 = \alpha_y^2 = 0$.

Hence for M large, the best mechanism for the principal which induces the agent to send m_2 from evidence set $\{m_1, m_2\}$ is the constant mechanism x, which gives the principal the payoff $M/2$. So the principal is strictly worse off with such a mechanism.

C Proof of Corollary 1

Fix an incentive compatible mechanism $\gamma = (\gamma_A, \gamma_{\mathcal{L}}, \gamma_X)$. By Theorem 1, we can assume without loss of generality that $\gamma_{\mathcal{L}}(t, a, M)(m_M^*) = 1$ for all $(t, a, M) \in T \times A \times \mathcal{M}$. We construct a mechanism (γ_A^*, γ_X^*) for the abbreviated protocol which is incentive compatible and has the same outcome as γ. To do so, first let $\gamma_A^* = \gamma_A$.

To construct γ_X^*, note that in the abbreviated protocol, $\gamma_X^* : T \times A \times \mathcal{L} \to \Delta(X)$, while in the full protocol, $\gamma_X : T \times A \times \mathcal{M} \times \mathcal{L} \times \mathcal{L} \to \Delta(X)$ since the choice of x can depend on the agent's report of an evidence set and the message the principal requests, in addition to the type report, distribution recommendation, and received message as in the abbreviated protocol.

Given any $m \in \mathcal{L}$, define $M^*(m)$ as follows. First, if there is any M such that $m = m_M^*$, then let $M^*(m)$ equal this message set M.[15] Otherwise, let $M^*(m)$ denote any $M \in \mathcal{M}$ such that $m \in M$. Given this, let

$$\gamma_X^*(t, a, m) = \gamma_X(t, a, M^*(m), m_{M^*(m)}^*, m).$$

In other words, if the agent reports t, the principal recommends a, and the agent shows message m, then the outcome is the same as in the original mechanism when the agent

[15]It is straightforward to show that if $m_M^* = m_{\hat{M}}^*$, then $M = \hat{M}$. That is, $M^*(m)$ is unambiguously defined in this case.

reports t, the principal recommends a, the agent reports evidence set $M^*(m)$, the principal requests the maximal evidence message for this set, and the agent provides message m.

If the agent truthfully reports her type, follows the principal's recommended distribution a, and provides the maximal evidence message from any evidence set she obtains, this construction implies that the resulting distribution over X in the new mechanism will be the same as in the original mechanism. Hence if this mechanism is incentive compatible, it yields the same outcome as the original mechanism.

So consider an agent of type t who reports type $\hat{t}$ (which may or may not equal t), has a recommended to her by the principal, chooses $\hat{a}$, obtains evidence set M, and sends message m from it. In this situation, the outcome under the new mechanism is $\gamma_X(\hat{t}, a, M^*(m), m^*_{M^*(m)}, m)$, exactly the same outcome the agent could have obtained by reporting $\hat{t}$, choosing $\hat{a}$, reporting $M^*(m)$ as her evidence set, and then sending m. That is, any outcome the agent can generate in the new mechanism using a strategy which deviates from truth–telling, obedience, and sending maximal evidence is an outcome she could have generated in the original mechanism using a certain strategy which deviated from truth–telling and obedience. Since the original mechanism was incentive compatible, truth–telling and obedience were superior to this deviation. Hence the agent prefers truth–telling, obedience, and maximal evidence in the new mechanism to any deviation, so the mechanism is incentive compatible.

D Proof of Theorem 2

Fix an incentive compatible mechanism for the evidence–acquisition model under normality. By Corollary 1, we can take this mechanism to be based on the abbreviated protocol. Hence it consists of a pair of functions $\gamma_A : T \to \Delta(A)$ and $\gamma_X : T \times A \times \mathcal{L} \to \Delta(X)$. For the signal choice model, a mechanism is a pair of functions $\gamma^*_S : T \to \Delta(S)$ and $\gamma^*_X : T \times S \times \mathcal{L} \to \Delta(X)$.

Given the incentive compatible mechanism for the abbreviated protocol, we construct an equivalent incentive compatible mechanism for the associated signal–choice model as follows. Let

$$\gamma^*_S(t)(s_{(a,\sigma^*)}) = \gamma_A(t)(a).$$

That is, given a report of t, the principal recommends the signal distribution generated by evidence distribution a followed by showing maximal evidence with the same probability he recommended a in the original mechanism. Let

$$\gamma^*_X(t, s_{(a,\sigma^*)}, m) = \gamma_X(t, a, m).$$

That is, if the agent report type t and the signal distribution the principal recommends is the one corresponding to a and maximal evidence, then the principal replies to message m in the new mechanism the same way he replied in the original mechanism given type report t and recommendation a.

It is easy to see that if the agent reports her type truthfully and follows the principal's recommended signal distribution, then the outcome is equivalent to that of the original mechanism as defined in the statement of the theorem. If the agent deviates, this corresponds directly to a particular deviation strategy in the original mechanism and hence cannot be profitable for her. In particular, if type t reports $\hat{t}$, receives the recommendation s_{a,σ^*}, and uses signal distribution $s_{(\hat{a},\hat{\sigma})}$ instead, she generates exactly the outcome she would have generated in the original mechanism if she reported $\hat{t}$, received the recommendation a, chose the distribution $\hat{a}$ instead, and selected a message to send using the function $\hat{\sigma}$. Hence the mechanism is incentive compatible.

E Proof of Theorem 3

Fix an incentive compatible mechanism (γ_S, γ_X) where $\gamma_S(t_1)(\hat{s}_1) = \hat{\alpha} > 0$. Let $\alpha = \gamma_S(t_1)(s_1)$ (where this can be 0). We construct an incentive compatible mechanism (γ_S^*, γ_X^*) with the same outcome where the principal recommends s_1 to t_1 with probability $\alpha + \hat{\alpha}$ and never recommends $\hat{s}_1$ to t_1.

For any $t \neq t_1$, $\gamma_S^*(t) = \gamma_S(t)$ and $\gamma_X^*(t, s, m) = \gamma_X(t, s, m)$ for all (s, m). For $s \neq s_1, \hat{s}_1$, we have $\gamma_S^*(t_1)(s) = \gamma_S(t_1)(s)$ and $\gamma_X^*(t_1, s, m) = \gamma_X(t_1, s, m)$. That is, if the agent reports a type other than t_1, the new mechanism is the same as the original one and if the agent reports t_1, the principal recommends signals other than s_1 or $\hat{s}_1$ with the same probability and treats them the same way as in the original mechanism.

Let $\gamma_S^*(t_1)(\hat{s}_1) = 0$ and $\hat{\gamma}_S^*(t_1)(s_1) = \alpha + \hat{\alpha}$. Since the principal never recommends $\hat{s}_1$ in response to a report of t_1 in this mechanism, we only need to specify $\gamma_X^*(t, s, m)$ for $(t, s) = (t_1, s_1)$. For notational convenience, we enumerate the messages as $\mathcal{L} = \{m_1, \ldots, m_L\}$ and for the Markov matrix Λ, we write the entry corresponding to (m_i, m_j) as λ_{ij} rather than λ_{m_i, m_j}.

Let

$$\gamma_X^*(t_1, s_1, m_i) = \frac{\alpha}{\alpha + \hat{\alpha}} \gamma_X(t_1, s_1, m_i) + \frac{\hat{\alpha}}{\alpha + \hat{\alpha}} \sum_j \lambda_{ij} \gamma_X(t_1, \hat{s}_1, m_j).$$

Because all the λ_{ij}'s are non–negative and because $\sum_j \lambda_{ij} = 1$ for every i, we see that $\gamma^*(t_1, s_1, m_i)$ is a convex combination of probability distributions over X and hence is a probability distribution over X.

Given this specification, suppose all types report honestly and obey the principal's recommendations. Obviously, if the true type $t \neq t_1$, we have the same outcome as before. So suppose $t = t_1$. Then the expected outcome is

$$(\alpha + \hat{\alpha}) \sum_i s_1(m_i)\gamma_X^*(t_1, s_1, m_i) + \sum_{s \in S_{t_1} \setminus \{s_1, \hat{s}_1\}} \gamma_S^*(t_1)(s) \sum_M s(m)\gamma_X^*(t_1, s, m). \qquad (6)$$

Substituting for γ_X^*, the first term in equation (6) is

$$\alpha \sum_i s_1(m_i)\gamma_X(t_1, s_1, m_i) + \hat{\alpha} \sum_i s_1(m_i) \sum_j \lambda_{ij}\gamma_X(t_1, \hat{s}_1, m_j)$$
$$= \alpha \sum_i s_1(i)\gamma_X(t_1, s_1, m_i) + \hat{\alpha} \sum_j \gamma_X(t_1, \hat{s}_1, m_j) \sum_i s_1(m_i)\lambda_{ij}.$$

But $s_1\Lambda = \hat{s}_1$, so that for every j, $\sum_i s_1(m_i)\lambda_{ij} = \hat{s}_1(m_j)$. Hence this is

$$= \alpha \sum_i s_1(m_i)\gamma_X(t_1, s_1, m_i) + \hat{\alpha} \sum_i \hat{s}_1(m_i)\gamma_X(t_1, \hat{s}_1, m_j).$$

Substituting this for the first term in equation (6) and substituting for γ_S^* and γ_X^* in the second term, we see that the expected outcome under truth–telling and obedience is the same as under the original mechanism.

To show that the new mechanism is incentive compatible, we show that any deviation from truth–telling and obedience by any type generates a distribution over outcomes that the same type could have generated in the original mechanism. Since the original mechanism was incentive compatible, this deviation is not profitable, so the new mechanism is incentive compatible.

To see that this holds, fix any type t and any signal $s' \in S_t$. If t makes any type report other than t_1, the mechanism has not changed, so the claim obviously holds. So suppose type t reports type t_1. If the mechanism makes any signal recommendation other than s_1, then, again, the mechanism is the same as before, so the claim holds. So suppose the mechanism recommends signal s_1 and the agent uses s'. The expected outcome times the probability of this event is

$$(\alpha+\hat{\alpha}) \sum_i s'(m_i)\gamma_X^*(t_1, s_1, m_i) = \alpha \sum_i s'(m_i)\gamma_X(t_1, s_1, m_i) + \hat{\alpha} \sum_i s'(m_i) \sum_j \lambda_{ij}\gamma_X(t_1, \hat{s}_1, m_j).$$

By assumption, $s'\Lambda \in \text{conv}(S_t)$. Hence we can write $s'\Lambda = \sum_k a_k s^k$ where $a_k \geq 0$ for all k, $\sum_k a_k = 1$, and $s^k \in S_t$ for all k. In particular, for every j,

$$\sum_i s'(m_i)\lambda_{ij} = \sum_k a_k s^k(m_j).$$

Hence we can rewrite the above as

$$\alpha \sum_i s'(i)\gamma_X(t_1, s_1, m_i) + \hat{\alpha} \sum_k a_k s^k(i)\gamma_X(t_1, \hat{s}_1, m_i).$$

This is exactly what t would generate in the original mechanism if she responded to a recommendation of s_1 with s' and a recommendation of $\hat{s}_1$ by randomizing with probability a_k on s^k. Thus, as asserted, any expected outcome t can generate in the new mechanism is identical to some outcome she could have generated in the original mechanism. Hence the new mechanism is incentive compatible.

F Proof of Remark 1

Let $s = s_{(a,\sigma)}$ and $s^* = s_{(a,\sigma^*)}$ where $\sigma^*(M) = m^*_M$ for all $M \in \text{supp}(a)$. Abusing notation, write $\sigma(M)$ not as the message s sends from M but as the probability distribution over M when M is realized. So write $\sigma(M)(m)$ as the probability that message m is sent from set M. Enumerate the messages as $m_1, \ldots, m_K$. If $m_i = m^*_M$, we write $M = M_i$. Since no message can be maximal evidence for more than one evidence set, we have $s^*(m_i) = a(M_i)$. Define a Markov matrix Λ as follows. If $s^*(m_i) = 0$, then $\lambda_{ii} = 1$ and $\lambda_{ij} = 0$ for $j \neq i$. If $s^*(m_i) > 0$, then $\lambda_{ij} = \sigma(M_i)(m_j)$. In other words, if s^* sends m_i with positive probability, then λ_{ij} is the probability that m_j is the message s sends given message set M_i.

Note that the jth element of $s^*\Lambda$ is

$$\sum_i s^*(m_i)\lambda_{ji} = \sum_{M \in \mathcal{M}} a(M)\sigma(M)(m_j) = s(m_j).$$

Hence $s^*\Lambda = s$, as required. For any other $\hat{s}$, the jth element of $\hat{s}\Lambda$ is

$$\sum_{i|s^*(m_i)>0} \hat{s}(m_i)\sigma(M)(m_j) + \sum_{i|s^*(m_i)=0} \hat{s}(m_i)\lambda_{ji}$$

or

$$\begin{cases} \sum_{i|s^*(m_i)>0} \hat{s}(m_i)\sigma(M)(m_j), & \text{if } s^*(m_j) > 0; \\ \sum_{i|s^*(m_i)>0} \hat{s}(m_i)\sigma(M)(m_j) + \hat{s}(m_j), & \text{otherwise.} \end{cases}$$

In other words, $\hat{s}\Lambda$ is constructed as follows. We choose a message, say m_i, according to distribution $\hat{s}$. If $s^*(m_i) = 0$, then this message is sent. If $s^*(m_i) > 0$, then instead we randomize the message to send according to the distribution $\sigma(M_i)$.

We now show that this must be feasible for any type for whom $\hat{s}$ is feasible. Clearly, if $\hat{s}$ generates a message m_i, it must be able to send that message. So we need to show that the randomization is feasible — that is, that whenever m_i could be sent, every message in M_i is also feasible. But this follows from the fact that $m_i = m^*_{M_i}$. By definition, this means that if the feasible set is M and $m_i \in M$, then $M_i \subseteq M$. So if $\hat{s} \in S_t$, then $\hat{s}\Lambda \in S_t$, completing the proof.

G Proof of Theorem 5

G.1 Lemma

The following result will be useful. Let W be a finite set of states of the world and A a finite set of actions. Let $u : A \times W \to \mathbf{R}$ be a utility function. Say that $\sigma \in \Delta(A)$ is a *best reply to* $p \in \Delta(W)$ if

$$\sum_w p(w) \sum_a \sigma(a) u(a,w) \geq \sum_w p(w) \sum_a \sigma'(a) u(a,w), \ \forall \sigma' \in \Delta(A).$$

Say that σ is a *best reply* if there exists $p \in \Delta(W)$ such that σ is a best reply to p. Say that σ is *strictly dominated* if there exists $\sigma' \in \Delta(A)$ such that

$$\sum_a \sigma'(a) u(a,w) > \sum_a \sigma(a) u(a,w), \quad \forall w \in W.$$

Standard results say that σ is a best reply if and only if it is not strictly dominated. (This is typically stated for pure strategies σ, but it applies to mixed as well.)

Lemma 1. *Suppose σ' is strictly dominated. Then there exists a mixed strategy $\hat{\sigma}$ which strictly dominates σ' and which is not itself strictly dominated.*

Proof. Suppose not. That is, suppose σ' is strictly dominated, but that there is no undominated strategy which strictly dominates it. Let Σ^* denote the set of undominated strategies in $\Delta(A)$. Equivalently, Σ^* is the set of all $\sigma \in \Delta(A)$ that are best replies. By finiteness of A, this set is nonempty.

Let

$$\mathcal{U} = \operatorname{conv}\left(\left\{u \in \mathbf{R}^W \mid \exists \sigma \in \Sigma^* \cup \{\sigma'\} \text{ with } u_w = \sum_a \sigma(a) u(a,w), \ \forall w\right\}\right).$$

$$\mathcal{U}^D = \left\{u \in \mathbf{R}^W \mid u_w \geq \sum_a \sigma'(a) u(a,w), \ \forall w\right\}.$$

By hypothesis, there is no mixed strategy in Σ^* which strictly dominates σ'. Hence $\mathcal{U} \cap \text{int}(\mathcal{U}^D) = \emptyset$, so the interiors of $\mathcal{U}$ and $\mathcal{U}^D$ are disjoint. Clearly, both sets are nonempty and convex. Hence there exists a separating hyperplane. That is, there is $p \in \mathbf{R}^W$ such that $p \neq 0$ and $p \cdot u \geq p \cdot \hat{u}$ for all $u \in \mathcal{U}^D$, $\hat{u} \in \mathcal{U}$.

Consider $\hat{u}$ defined by $\hat{u}_w = \sum_a \sigma'(a)u(a,w)$. Obviously, this is an element of $\mathcal{U}$. Consider u defined by $u_w = \hat{u}_w$ for $w \neq w'$ and $u_{w'} = \hat{u}_{w'} + \varepsilon$ for some $\varepsilon > 0$ and some w'. Clearly, this is an element of $\mathcal{U}^D$. Hence the separating hyperplane satisfies $p_{w'}\varepsilon \geq 0$. Since w' is arbitrary, $p_w \geq 0$ for all w. Since $p \neq 0$, we can renormalize by replacing p with $\hat{p}$ defined by $\hat{p}_w = p_w / \sum_{w'} p_{w'}$. Hence $\hat{p} \in \Delta(W)$.

Continuing with the same $\hat{u}$ as above, we see that we have $\hat{p} \in \Delta(W)$ such that

$$\sum_w \hat{p}(w) \sum_a \sigma'(a)u(a,w) \geq \sum_w \hat{p}(w) u_w \ \forall u \in \mathcal{U}.$$

In particular, for any $\sigma \in \Sigma^*$, we can let u be the vector defined by $u_w = \sum_a \sigma(a)u(a,w)$ to conclude that

$$\sum_w \hat{p}(w) \sum_a \sigma'(a)u(a,w) \geq \sum_w \hat{p}(w) \sum_a \sigma(a)u(a,w) \ \forall \sigma \in \Sigma^*.$$

By hypothesis, σ' is strictly dominated by some mixed strategy, say $\hat{\sigma} \notin \Sigma^*$. Hence

$$\sum_w \hat{p}(w) \sum_a \hat{\sigma}(a)u(a,w) > \sum_w \hat{p}(w) \sum_a \sigma'(a)u(a,w) \geq \sum_w \hat{p}(w) \sum_a \sigma(a)u(a,w) \ \forall \sigma \in \Sigma^*.$$

Hence no best reply to $\hat{p}$ is contained in Σ^*, a contradiction. ▌

G.2 Theorem 5

Let H_P denote the set of possible public histories — i.e., histories both the principal and the agent see. More specifically, H_P consists of the various possible sequences of cheap–talk messages as well as the possible complete public histories of all cheap–talk messages followed by the agent's evidence message. It will be convenient to write such a history in the form $h \cdot r \cdot h'$ where h is a history, r the next cheap talk message observed, and h' a continuation.

Let H_A denote the set of private histories for the agent and denote a typical element by h_A. Hence h_A lists what the agent observes that the principal does not — her type, her action choice at each evidence action stage, and the outcome of that action choice. The full history observed by the agent — the public plus the private — will be written as (h, h_A) and the set of these histories is denoted H_F.

As before, β is a behavior strategy for the agent and γ a behavior strategy for the principal. Thus β maps H_F to possible choices for the agent, while γ maps public histories H_P to actions for the principal. We let ρ denote a belief for the principal, where this is a function from H_P to beliefs over T.

Because the protocol is allowable, all information sets for the agent are singletons. Because there is no issue of beliefs for the agent, given any strategy γ for the principal, we can define the set of strategies for the agent which are sequentially rational best replies, denoted $BR^s(\gamma)$.

Fix (β^*, γ^*) with $V(\beta^*, \gamma^*) = V^*$ and $\beta^* \in BR(\gamma^*)$, so that (β^*, γ^*) is optimal for the principal. If we construct the restricted game used in the proof of Theorem 4 by restricting the agent to strategies in $BR^s(\gamma^*)$ instead of $BR(\gamma^*)$, nothing in the proof changes. So there exists $\hat{\beta} \in BR^s(\gamma^*)$ such that $(\hat{\beta}, \gamma^*)$ is a Nash equilibrium with $V(\hat{\beta}, \gamma^*) = V^*$. Since $\hat{\beta} \in BR^s(\gamma^*)$, the agent's strategy is sequentially rational at all information sets. Hence we can assume we have a Nash equilibrium (β^*, γ^*) satisfying $V(\beta^*, \gamma^*) = V^*$ such that the agent's strategy is sequentially rational at all information sets.

Without loss of generality, we can also assume that all possible cheap–talk messages have positive probability at every cheap–talk stage in equilibrium. In other words, we can assume without loss of generality that (β^*, γ^*) satisfy the following two properties. First, for every public history h leading to a stage where the principal sends cheap talk, for every feasible cheap talk message r at that stage, $\gamma^*(h)(r) > 0$. Second, for every public history h leading to a stage where the agent sends cheap talk, for every feasible cheap talk message r at that stage, there exists a private history for the agent h_A consistent with being at this stage[16] such that $\beta^*(h, h_A)(r) > 0$.

To show this, first consider the principal. Fix any stage where the principal chooses a cheap–talk message and a public history h leading up to this stage. Suppose cheap talk message $\bar{r}$ has zero probability — i.e., $\gamma^*(h)(\bar{r}) = 0$. Fix any $\hat{r}$ with $\gamma^*(h)(\hat{r}) > 0$. Then we change β^*, γ^*, and ρ^* to $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\rho}$ as follows. Let $\hat{\gamma}(h)(\bar{r}) = \hat{\gamma}(h)(\hat{r}) = \gamma^*(h)(\hat{r})/2$. For every other cheap talk message r that the principal could send at this stage, we let $\hat{\gamma}(h)(r) = \gamma^*(h)(r)$. In other words, we spread the probability the principal was putting on $\hat{r}$ across $\hat{r}$ and $\bar{r}$.

For any continuation public history h', let $\hat{\gamma}(h \cdot \bar{r} \cdot h') = \hat{\gamma}(h \cdot \hat{r} \cdot h') = \gamma^*(h \cdot \hat{r} \cdot h')$. Note that h' includes the "empty continuation." We set $\hat{\rho}(h \cdot \bar{r} \cdot h') = \hat{\rho}(h \cdot \hat{r} \cdot h') = \rho^*(h \cdot \hat{r} \cdot h')$. For any private history for the agent h_A such that full history (h, h_A) leads to this stage, we set $\hat{\beta}((h, h_A) \cdot \bar{r} \cdot (h', h'_A)) = \hat{\beta}((h, h_A) \cdot \hat{r} \cdot (h', h'_A)) = \beta^*((h, h_A) \cdot \hat{r} \cdot (h', h'_A))$ for all continuations (h', h'_A). For any history that does not start with the public history h, we

[16] To be clear, consistent simply means that the history is the right length.

make no changes.

This construction simply changes the "interpretation" of cheap talk. The "meaning" of $\bar{r}$ after public history h is not pinned down by equilibrium initially since it has zero probability, but the meaning of $\hat{r}$ is identified in terms of its effects on equilibrium beliefs and continuation strategies. Essentially, this change has both players interpret $\bar{r}$ after public history h the same way that they interpret $\hat{r}$ after public history h.

It is easy to see that these changes do not change the equilibrium outcome. If the principal was sequentially rational on history $h \cdot \hat{r} \cdot h'$, he still is and is also sequentially rational on history $h \cdot \bar{r} \cdot h'$. The agent was originally sequentially rational on all histories and still is. We can iterate this construction to handle every stage at which the principal sends cheap talk.

Turning to the agent, fix any stage where the agent sends a cheap–talk message. Fix any public history h up to this stage. Let $\hat{H}_A$ be the set of possible private histories of the agent up to this stage, that is, H_A minus histories that are the wrong length. Let $\bar{r}$ be a particular cheap–talk message available to the agent at this stage and suppose that $\beta(h, h_A)(\bar{r}) = 0$ for all $h_A \in \hat{H}_A$. Fix any $\hat{r}$ such that $\beta(h, h_A)(\hat{r}) > 0$ for some $h_A \in \hat{H}_A$. Change strategies as follows.

Let $\hat{\beta}(h, h_A)(\bar{r}) = \hat{\beta}(h, h_A)(\hat{r}) = \beta^*(h, h_A)(\hat{r})/2$ for all $h_A \in \hat{H}_A$. In other words, for private histories where the agent gives $\hat{r}$ zero probability under β^*, we make no change. For private histories where the agent gives $\hat{r}$ strictly positive probability, we divide this probability across $\bar{r}$ and $\hat{r}$. Since $\hat{r}$ has positive probability for some private history h_A, this ensures the desired property. For other cheap talk messages r, we have $\hat{\beta}(h, h_A)(r) = \beta^*(h, h_A)(r)$ for all consistent h_A.

As before, for any continuation public history h', let $\hat{\gamma}(h \cdot \bar{r} \cdot h') = \hat{\gamma}(h \cdot \hat{r} \cdot h') = \gamma^*(h \cdot \hat{r} \cdot h')$. Again, we set $\hat{\rho}(h \cdot \bar{r} \cdot h') = \hat{\rho}(h \cdot \hat{r} \cdot h') = \rho^*(h \cdot \hat{r} \cdot h')$. We do the same for the agent's strategy, setting $\hat{\beta}((h, h_A) \cdot \bar{r} \cdot (h', h'_A)) = \hat{\beta}((h, h_A) \cdot \hat{r} \cdot (h', h'_A)) = \beta^*((h, h_A) \cdot \hat{r} \cdot (h', h'_A))$. For any history that doesn't start with the public history h, we make no changes.

As before, this does not change the equilibrium outcome and it leaves the agent sequentially rational at all information sets. In addition, it makes the principal sequentially rational at weakly more information sets than before. With abuse of notation, continue to let (β^*, γ^*) denote the Nash equilibrium strategies and ρ^* the principal's beliefs.

Summarizing to this point, we know that β^* satisfies sequential rationality for the agent at all information sets. By Nash, the principal is sequentially rational at all positive probability information sets. By the construction above, we've ensured that this covers all information sets where the principal has only observed cheap talk. Hence if there is any information set where some player is not sequentially rational, it must be that the principal's strategy γ^* is not sequentially rational at an information set where he has observed an evidence message and has to choose x.

So fix any such public history h. Let T^* be the set of types for whom h is feasible (that is, can send the evidence message observed by the principal at h). Let $V(\beta^*, \gamma^* \mid t, h)$ denote the principal's expected utility at h when the strategies followed from h forward are (β^*, γ^*) and the agent's true type is t. (Note that t and h together determine the node of his information set that the principal is at.) Beliefs $\rho \in \Delta(T^*)$ make γ^* sequentially rational at this information set iff

$$\sum_{t \in T^*} \rho(t) \sum_{g \in G} \gamma^*(g) V(\beta^*, g \mid t, h) \geq \sum_{t \in T^*} \rho(t) \sum_{g \in G} \gamma(g) V(\beta^*, g \mid t, h)$$

for all $\gamma \in \Delta(G)$. If such a ρ exists, we can set the principal's beliefs at this information set to this ρ and we have sequential rationality at this information set.

So suppose no such ρ exists. By Lemma 1, γ^* is dominated with respect to T^* in the sense that there is some $\hat{\gamma} \in \Delta(G)$ such that

$$\sum_g \hat{\gamma}(g) V(\beta^*, g \mid t, h) > \sum_g \gamma^*(g) V(\beta^*, g \mid t, h), \ \forall t \in T^* \tag{7}$$

and such that $\hat{\gamma}$ is not itself dominated in this sense. Since $\hat{\gamma}$ is not dominated in this sense, there exists $\hat{\rho} \in \Delta(T^*)$ such that $\hat{\gamma}$ maximizes the principal's expected utility. Set the principal's belief at this information set to equal $\hat{\rho}$ and change his strategy at this information set to $\hat{\gamma}(h)$. Call $(\beta^*, \hat{\gamma}^*, \hat{\mu}^*)$ the resulting assessment. Because we have only changed the principal's strategy at a last information set, one with zero probability, we know that $\hat{\gamma}^*$ is sequentially rational at every information set where γ^* was sequentially rational as well as at the information set h.

We now show that for any full history (h', h_A) with positive probability under (β^*, γ^*) (or, equivalently, under $(\beta^*, \hat{\gamma}^*)$), the agent is sequentially rational under $(\beta^*, \hat{\gamma}^*, \hat{\rho}^*)$. To see this, suppose not.

Let $\hat{T}$ denote the (nonempty) set of t such that there is a full history of the form $(h', t \cdot h'_A)$ (i.e., the agent's type is t) such that β^* is not sequentially rational at $(h', t \cdot h'_A)$. It is easy to see that we must have $\hat{T} \subseteq T^*$ since no other type could play in such a way as to lead to information set h and hence no other type could be affected by the change in the principal's strategy. Also, for all $t \in \hat{T}$,

$$\sum_g \hat{\gamma}(g) U(\beta^*, g \mid t, h) > \sum_g \gamma^*(g) U(\beta^*, g \mid t, h),$$

where $U(\beta^*, g \mid t, h)$ is the agent's expected utility from strategies (β^*, g) conditional on the agent's type being t and the history h. Equivalently, this is conditional on the node identified by (t, h).

By equation (7) and the assumption that preferences are semi–aligned, we know that for all $t \in T^*$,

$$\sum_g \hat{\gamma}(g)\nu(t)U(\beta^*, g \mid t, h) > \sum_g \gamma^*(g)\nu(t)U(\beta^*, g \mid t, h),$$

so for any $t \in \hat{T}$, we must have $\nu(t) > 0$. Let $\hat{\beta}'$ denote the best reply of the agent which differs from $\hat{\beta}$ only in letting types $t \in \hat{T}$ deviate. By hypothesis, $\hat{T} \neq \emptyset$, so $\hat{\beta}' \neq \hat{\beta}$.

Note that

$$\begin{aligned}
V(\hat{\beta}', \hat{\gamma}) &= \mathrm{E}_t[\nu(t)U(\hat{\beta}', \hat{\gamma}, t)] \\
&= \Pr[\nu(t) < 0]\mathrm{E}_t[\nu(t)U(\hat{\beta}, \gamma^*, t) \mid \nu(t) < 0] + \Pr[\nu(t) > 0]\mathrm{E}_t[\nu(t)U(\hat{\beta}', \hat{\gamma}, t) \mid \nu(t) > 0] \\
&> \Pr[\nu(t) < 0]\mathrm{E}_t[\nu(t)U(\hat{\beta}, \gamma^*, t) \mid \nu(t) < 0] + \Pr[\nu(t) > 0]\mathrm{E}_t[\nu(t)U(\hat{\beta}, \gamma^*, t) \mid \nu(t) > 0] \\
&= V(\hat{\beta}, \gamma^*) = V^*.
\end{aligned}$$

The second equality uses the fact that only types in $\hat{T}$ deviate and these all have $\nu(t) > 0$. The strict inequality comes from the fact that the types who deviate in response to $\hat{\gamma}$ are made strictly better off than they were at $(\hat{\beta}, \gamma^*)$.

But $\hat{\beta}'$ is a best reply to $\hat{\gamma}$, so this is not possible, by definition of V^*.

Summarizing, $(\beta^*, \hat{\gamma}^*, \hat{\rho}^*)$ has the property that β^* is sequentially rational for the agent at every full history with positive probability given $(\beta^*, \hat{\gamma}^*)$. We now extend this to all full histories.

So suppose β^* is not sequentially rational at some full history (h', h_A) which has zero probability under $(\beta^*, \hat{\gamma}^*)$. By construction, the public history h' must have positive probability, so it must be that h' is inconsistent with h_A. That is, it must be that some of the cheap talk messages in h' are not supposed to be sent given the private history h_A. Since this node in the tree (recall that the agent always knows everything) has zero probability, *every* node which is a successor to this one has zero probability as well. With this in mind, change the agent's strategy at this history to anything which is sequentially rational and call $\hat{\beta}^*$ the resulting behavior strategy for the agent. Because we are changing the agent's strategy only at a history which her own strategy prevents her from reaching, this does not affect the sequential rationality of the principal's strategy or the consistency of his beliefs. Hence proceeding this way, we can change the agent's strategy at such full histories as needed to ensure sequential rationality for the agent at all full histories without affecting sequential rationality for the principal or changing the equilibrium outcome.

Summarizing, we have shown that if there is any public history h where γ^* is not sequentially rational, we can adjust the strategies at this history and possibly others to ensure sequential rationality at h, at all histories for the agent, and at all positive probability histories for the principal without changing the equilibrium outcome. Hence we can construct a perfect Bayesian equilibrium with the same outcome as the Nash equilibrium (β^*, γ^*).

H Multi–Agent Example

Assume $T_i = \{t_i^a, t_i^b\}$ where the types are equally likely. Assume $v_1(t_1^a) = 4$ and $v_1(t_1^b) = 1$, while $v_2(t_2^a) = 3$ and $v_2(t_2^b) = 0$. Assume each type of each agent has only one signal distribution available. Letting the unique signal choice available to t_i^k be denoted s_i^k, assume these distributions are

	s_1^a	s_1^b	s_2^a	s_2^b
m_h	1	3/4	1	0
m_ℓ	0	1/4	0	1

So 2's type *must* be revealed to the principal. If 1 is type t_1^b, then her type is revealed with probability 1/4.

It is not hard to show that the optimal mechanism for our signal–choice protocol is:

$v_2(t_2)$	1's Report	1's Signal	Prob(1)
0	$v_1(t_1) = 1$	any	1
0	$v_1(t_1) = 4$	m_h	1
3	$v_1(t_1) = 1$	any	0
3	$v_1(t_1) = 4$	m_h	1/3
any	$v_1(t_1) = 4$	m_ℓ	0

In other words, when agent 2's type is revealed to be such that the value of giving the good to her is 0, then for any report and any signal realization consistent with this report from agent 1, the principal gives the good to agent 1. If agent 2's type is revealed to be such that the $v_2 = 3$ and agent 1 reports that $v_1 = 1$, the principal gives the good to 2. If 2's type is revealed to be such that $v_2 = 3$, 1 claims that $v_1 = 4$, and the signal realization confirms this in the sense that the realization is m_h, the principal gives the good to 1 with probability 1/3. Finally, whenever 1 is revealed to have lied, 2 gets the good.

There is no equilibrium with this outcome. The reason is that type reports cannot convey information in equilibrium. More specifically, note that if the signal realizations reveal the types, then sequential rationality for the principal dictates his allocation decision. Also, if 2's type is revealed to give 0 value to the principal, then sequential rationality dictates that he give the good to 1, regardless of any type report or signal realization for 1.

The only information sets where the principal's allocation is not uniquely determined by sequential rationality is when the value of giving the good to 2 is revealed to be 3 and the realization of 1's signal does not reveal her type. If the principal is more likely to give the good to 1 in this situation for one of the two possible type reports than for the other, then 1 will always use this type report. Hence the type report cannot convey any useful information.

It is not hard to use this to show that the only equilibrium outcome is that the principal gives the good to agent 2 if $v_2 = 3$ and gives it to 1 otherwise.

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