Interpreted and Generated Signals

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Abstract

Incomplete information models in economics typically model private information as signals. The relationships between individuals’ signals and between the signals and the relevant variables are captured by a joint probability distribution function. In this paper, we demonstrate that standard assumptions about the joint probability distribution constrain in a substantive way the set of environments that could produce these signals. In particular, we define and contrast two types of signals: generated signals and interpreted signals. Generated signals are distortions of the true values. Interpreted signals are predictions based on attributes of a state. We show that even though conditional independence is a reasonable benchmark assumption for generated signals, it imposes a specific, and unlikely structure on interpreted signals. We also show that independent interpreted signals are negatively correlated in their correctness, be it conditional or unconditional and that the amount of negative correlation is uniquely determined in important cases. In contrast, generated signals can be independent conditional on the true value. Thus, our findings may limit the contexts in which many well known models of information aggregation and strategic choices in auctions, markets, and voting apply.

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1 Introduction

Economic agents often make decisions based on incomplete information. Social scientists model that incomplete information in the form of signals. Typically, a joint probability distribution characterizes the statistical relationships between these signals and the variable of interest: the better the information, the tighter the link. This abstraction, though mathematically convenient, avoids details of the variable of interest in relation to the state of the world and particulars of how signals are produced. In this paper, we distinguish between two models of signal formation, what we call generated and interpreted signals. We show that common assumptions about joint probability distributions hold widely for generated signals but do not hold for interpreted signals, with the exception of a single case.

In nearly all applications of incomplete information models, the private information of agents has the following components: a commonly known prior probability distribution, a commonly known signal generating process, and a privately observed signal generated independently conditional on the variable of interest for each agent. For example, suppose the private information concerns that status of a firm which can be classified as either "good" ($G$) or "bad" ($B$). Agents do not know the true status, but they have a common prior, say, $P(G) = P(B) = \frac{1}{2}$, each agent observes a binary signal, whose value is either $g$ or $b$. Most often, these signals would be assumed to be generated independently, i.e. their values would be independent conditional on the true status of the firm. This statistical property follows from the assumptions made about the signals. In this paper, we refer to signals produced by draws from assumed distributions as generated signals.

We contrast the statistical properties of generated signals with a second class of signals that we call interpreted signals. Unlike generated signals which are passively received by the agents, interpreted signals result from active cognitive effort on the part of agents. To create an interpretive signal, an agent filters reality into a set of categories. In some cases, these categories may rely on subsets of the full set of attributes, such as when an investor evaluates a company based on its price earning ratio. The notion that people use categories when interpreting the world is well established in psychology (see Fryer and Jackson 2008, Nisbett 2003, Page 2007 for summaries)¹ Using these categories, agents then make predictions about the value of the variable of interest. These predictions can be thought of as interpreted signals.²

For generated signals, the assumption that draws occur independently produces

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¹Organizations also differentially filter reality with categories. See for example, Stinchcombe 1990.

²Similarities exist between our interpreted signal framework and models of causal inference (Pearl 2000), fact free learning (Aragonès, Gilboa, Postlewaite, and Schmeidler 2005), and complexity (Al-Najjar, Casadesus-Masanell and Ozdenoren 2003). Interpreted signals resemble predictions from PAC learning (Valiant 1984) and classification theory (Barwise and Seligman 1997). The acronym PAC refers to Probably Approximately Correct. We differ from PAC learning and classification models in that we consider multiple agents each with his/her own classification rule.
signal variation. For interpreted signals, two agents' signals differ only if the agents rely on different predictive models. This can only occur if agents differ in how they categorize or classify objects, events, or data, if agents possess different data, or if agents make incorrect inferences. This last possibility we rule out in this first paper.\(^3\) This distinction is crucial. In generated signals, signal heterogeneity stems from randomness or disturbances. In interpreted signals, signal heterogeneity stems from cognitive diversity. Whereas increased randomness would generally thought of as harmful, increased cognitive diversity is generally seen as good. We return to this point in the conclusion.

Our focus in this paper is to analyze the statistical similarities and differences in the two types of signals. Among our results, we show that the ubiquitous assumption of conditional independence holds naturally for generated signals but does not hold for interpreted signals except in a single, unrealistic case. This result is troubling given that most economic situations are better captured with interpreted signals than with generated signals. To see why, consider the case of real estate valuation. One widely used model considers the price of housing relative to the rental cost. Another common model considers home affordability. These two models rely on different variables; therefore, they categorize the possible states of the world differently and make heterogeneous predictions. Those predictions are interpreted signals: agents acquire information through learning; their signals result from predictive models.

The interpreted signal framework that we present here models the relation between the variable of interest, what we call the \textit{outcome}, and the state of the world by a function, which we call the \textit{outcome function}. Agents make predictions about the outcome based on their categorization, which we call an \textit{interpretation}, of the state. This predictive model, acting as a private signal, captures the result of his learning about the outcome function. Therefore, the statistical properties of the reduced form signals from this interpreted framework is a result of the relationship between agents’ different interpretations/predictions and their relation to the outcome function.

In this paper, we consider the special case where the outcome of interest takes on binary values. The model generalizes, but the binary case provides the cleanest entry into the issues of concern. We establish two main classes of results regarding interpreted signals. First, we investigate whether interpreted signals can satisfy the standard assumption of independence conditional on the outcome. We show that interpreted signals satisfy the conditional independence assumption if and only if the outcome function and the associated interpreted signals are isomorphic to a structure that requires overlapping interpretations in a particular way: each agent must ignore a different piece of information. Intuitively, this runs counter to incentive effects, since by knowing that piece of information, the agent would have complete information. This result provides a powerful argument that even though conditional independence

\(^3\)Al-Najjar (2007) has an intriguing steady-state model with the implication that rational agents confronting an environment when learning is hard can interpret their environment differently even though they know each other’s model and base their inferences on identical data.
is a reasonable benchmark assumption for generated signals, it imposes a specific, and unlikely structure on interpreted signals making it an improper benchmark assumption for interpreted signals.

Second, we derive the core statistical properties of interpreted signals given independent interpretations—interpretations that consider distinct attributes of a common representation, or perspective (Hong and Page 2001, Page 2007). Independent interpretations apply to situations in which agents look at distinct but relevant pieces of information when making predictions. Later, we outline a logic for why, if people were to make strategic choices of interpretations, we might expect those interpretations to be independent. Among the implications of our findings are that several convenient distributional assumptions are inconsistent with independent interpreted signals. For example, many models of information aggregation in common value auctions, voting, and in information cascades assume the signaling structure mentioned above that involves two outcomes \( G \) (good) and \( B \) (bad) and two signal values \( g \) and \( b \): The two outcomes are equally likely, each signal value occurs with equal probability, the signals are independently correct (i.e., knowing that one signal is correct tells nothing about the probability that the other is correct) and the signals are informative, i.e., they are correct more than half of the time. Signals of this structure can not be independent interpreted signals. Instead, independent interpreted signals, must be negatively correlated in their correctness both unconditionally and conditionally on at least one outcome. Moreover, we show a related result that independent interpreted signals must be negatively correlated conditional on at least one outcome. Therefore, the standard assumptions are not consistent with a world in which these signals are, for example, predictions of the values of firms based on independent interpretations.

The negative correlation result can be made more precise, if as often assumed, good and bad predictions are equally likely. In that case, the probability of a correct interpreted signal uniquely determines the extent of this negative correlation. Thus, we have a benchmark correlation assumption for interpreted signals that differs from zero. Note that the uniqueness of this correlation has an unexpected corollary, namely that the complexity of the outcome function (loosely defined here as the extent of nonlinearity and interaction terms) has no effect on pairwise correlation beyond that captured by signal accuracy.

Our results call into question the generality of existing results that rely on an abstract model of signals—the auction, voting, or signaling literatures are replete with models that include assumptions based on tractability and not on descriptive realism. In those models, the statistical properties of signals influence strategic choices and thus outcomes. For example, in that part of the auction literature that relates to the derivation of optimal bidding strategies, the assumptions align with our interpreted signal framework (Klemperer 2004). However, in that part concerned with information aggregation in common value auctions, particularly those papers with large numbers of bidders, the assumptions are inconsistent with interpreted signals. Similarly, the bulk of the political science literature, including almost all jury models and election models (Ladha 1992, Feddersen and Pesendorfer 1997), makes assump-
tions consistent with generated signals, not interpreted signals, even though jurors by law must make predictions based on models using information presented at trial, i.e. they must use interpreted signals.

The remainder of the paper is organized as follows. In Section 2, we provide examples of generated and interpreted signals and compare their properties. In Section 3, we develop the interpreted signal framework by defining the concepts of interpretation and prediction. In Section 4, we prove our first main result that the environment leading to conditionally independent interpreted signals is unique (up to isomorphism) and that the unique specific structure is unrealistic. In Section 5, we establish the negative correlation result for independent interpreted signals. In Section 6, we present an example in the context of voting to illustrate how our interpreted signal framework can be used to investigate information aggregation properties of the majority rule voting mechanism. We demonstrate why using the interpreted signal framework requires a different technique and produces results distinct from standard information aggregation results. We conclude with a discussion of the implications of our results and future research questions.

2 Generated and Interpreted Signals: Examples

We begin with an example that highlights the statistical differences between generated and interpreted signals. Consider two venture capitalists who receive signals about the quality of an investment, say a restaurant. This restaurant can either be a good investment ($V = G$) or a bad one ($V = B$). Each outcome occurs with equal probability. We first describe the standard generated signal model.

2.1 Generated Signals

Generated signals can be thought of as noisy glimpses or distortions of an outcome value $V$. Imagine, for example, that two potential investors eat meals prepared at the restaurant. We can think of those meals as generated signals denoted by $s_1$ and $s_2$ that either take value $g$ (for good) or $b$ (for bad). Conditional on the restaurant being a good investment, i.e. conditional on outcome $G$, we assume that the probability of getting a good meal, i.e. of receiving $g$, equals $2/3$. Similarly, conditional on $B$, the probability of $b$ equals $2/3$. Each meal is an independent draw from this distribution. Thus, we can write the joint probability distributions of signals conditional on the restaurant’s value as follows:

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<th>Generated Signals Conditional on $G$</th>
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<tr>
<td>$s_1/s_2$</td>
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<td>$b$</td>
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<td>$g$</td>
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Generated Signals Conditional on $B$

$$
\begin{array}{c|cc}
  s_1/s_2 & b & g \\ 
  \hline 
  b & 4/9 & 2/9 \\ 
  g & 2/9 & 1/9 \\
\end{array}
$$

Using this information, we can compute the probability that the restaurant is a good investment conditional on the signals of the two investors.

Probability of a Good Investment Conditional on Generated Signals

$$
\begin{array}{c|cc}
  s_1/s_2 & b & g \\ 
  \hline 
  b & 1/5 & 1/2 \\ 
  g & 1/2 & 4/5 \\
\end{array}
$$

The above table can be read as follows. If both investors get the signal $b$ then the probability that the restaurant is a good investment equals $1/5$.

### 2.2 Interpreted Signals

We now turn to interpreted signals. Interpreted signals are predictions based on interpretations. Interpretations create partitions (or categorizations) of the set of possible restaurants. Interpretations partition the set of attributes that define a restaurant. In this example, we consider those attributes to be the restaurant’s location and its prices. We assume that each investor sees only one of these attributes and bases her prediction on that attribute’s value.

Interpreted signals require an outcome function, $V$, that maps the restaurant’s attributes, into a probability that the restaurant is a good investment. Here, we assume that the location $\ell$ and the prices $\$ can be either good 1 or bad 0, and that each combination of attribute pairs is equally likely. We assume the following functional form for the outcome function.$^4$

$$V(\ell, \$) = \begin{cases} 1/3 & \text{if } (\ell + \$) \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

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$^4$ In the paper, we consider deterministic outcome functions. Any probabilistic outcome function can be transformed into a deterministic outcome function by adding attributes. In this example, we need only add a third independent attribute $h$ that takes three values, say 0, $\frac{1}{2}$, and 1, with equal probability. The outcome function can then be written as a deterministic function as follows:

$$V(\ell, \$, h) = \begin{cases} 1 & \text{if } h = 1 \text{ or } (\ell + \$) > 1 \\ 0 & \text{otherwise} \end{cases}$$
In this example, we assume that the first investor looks only at the location and the second looks only at prices. The investors then construct predictive models based on their attributes’ values. In brief, if the attribute’s value equals 1 (resp. 0), the investor predicts the restaurant will be good (bad). An investor’s interpreted signal is the investor’s attribute. In the general framework, predictions can be based on multiple attributes and therefore are not identical to the attributes themselves. Given these interpreted signals, we can now write a joint probability distribution for the signals and the value of the outcome function.

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We can then calculate the joint probability distribution of the interpreted signals conditional on the restaurant’s quality. As with the generated signals, the probability of an agent getting the good (bad) signal conditional on the restaurant being of good (bad) quality equals $\frac{2}{3}$.

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Comparing the generated and interpreted signals reveals several differences. First, the interpreted signals are not conditionally independent. Conditional on the restaurant being a bad investment, having a good location reduces the likelihood that the prices are also good. In fact, the probability of having good prices equals zero. Second, the generated signals are independently correct, but the interpreted signals are not. Again, conditional on the restaurant being a bad investment, if one interpreted signal is incorrect, the other must be correct. Third, with interpreted signals, the correlation between the signals and values depends on the outcome function. With
generated signals, whatever correlation exists is just assumed. Finally, in the case of interpreted signals, we are limited to two attributes: location and prices. The only constraint on the number of generated signals is the chef’s time.\footnote{This example creates a clean distinction between generated and interpreted signals. In other examples, the differences can be subtle. Consider a university whose quality \( q \) in \( \{0, 1\} \) depends on its faculty’s abilities: each faculty member \( i \) has an ability \( a_i \) in the set \( \{0, 1\} \) drawn from some distribution \( F \). Assume that this university produces students at regular intervals and that each graduate \( j \) receives an added value \( x_j \) in \( [0, 1] \). The graduating students’ added values would be generated signals and the faculty’s abilities would be interpreted signals. Thus, their statistical properties would differ in the ways that we describe.}

3 The Interpreted Signal Framework

We now present the interpreted signal framework. This framework relies on a set of states of the world and an outcome function that assigns to each state an outcome. Let \( \Omega \) be a finite set of states with a cardinality \( N \). We denote the set of possible outcomes by \( S \) and the deterministic mapping from states to outcomes with the outcome function \( F : \Omega \to S \).\footnote{As previously mentioned, the framework extends to include probabilistic mappings.}

The environments that we consider are equivalent to classification problems in which an agent has to place the \( N \) objects into \( |S| \) bins representing possible outcome values and are related to the problem of selecting regressors (Aragonès, et al 2005). In this paper, we restrict attention to cases in which the cardinality of \( S \) equals two, \( S = \{G, B\} \). To make a prediction, an agent first partitions the state space into non-overlapping sets. We denote the partition of agent \( i \), \( \Pi^i \), to be the sets \( \{\pi^i_1, \pi^i_2, \ldots, \pi^i_{n_i}\} \), where \( n_i \) is the number of sets in agent \( i \)’s partition. \( \Pi^i \) is agent \( i \)’s understanding of reality and it is incomplete as long as not all sets in the partition are singletons. Given a state, the agent can only associate it with the set in her partition that contains it. Following Page (2007) we call these partitions \emph{interpretations}.

Let \( P \) denote the prior probability distribution over \( \Omega \). Given a prior distribution and an interpretation, an agent makes predictions about the outcome. For example, an agent might use a Bayesian approach to making these predictions and assume that the most probable outcome arises conditional on the set in her interpretation. We refer to these as \emph{experience generated predictions}. At this point, we do not specify how predictions are generated. Agent \( i \)’s \emph{prediction}, \( \phi_i \), is defined as a function from \( \Omega \) to \( S \) with the restriction that \( \phi_i \) is measurable with respect to agent \( i \)’s interpretation \( \Pi^i \).\footnote{Here, we do not allow predictions to be probability distributions over outcomes in order to be consistent with the generated signal model.} We call \( \phi_i \) agent \( i \)’s \emph{interpreted signal}.\footnote{In this paper, we will use the two terms, interpreted signals and predictions, interchangeably depending on our need.}

Following \( \Pi^i \) we call these partitions \emph{interpretations}.
Example 1  \( \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\} \). All states are equally likely, \( P(\omega_i) = \frac{1}{6} \). The set of outcomes, \( S = \{G, B\} \). The outcome function maps the first three states to \( G \) and the rest to \( B \). Let \( \Pi = \{\pi_1, \pi_2\} \) be an agent’s interpretation where \( \pi_1 = \{\omega_1, \omega_2, \omega_4\} \) and \( \pi_2 = \{\omega_3, \omega_5, \omega_6\} \). If this agent makes experience generated predictions, then her interpreted signal can be described by the following function

\[
\phi(\omega_i) = \begin{cases} 
G & \text{for } \omega_i \in \pi_1 \\
B & \text{for } \omega_i \in \pi_2 
\end{cases}
\]

In this example and generally, an outcome function \( F : \Omega \to S \) together with an interpreted signal \( \phi : \Omega \to S \) induces a function, \( \delta \), called the correctness of an interpreted signal, defined as follows: \( \delta : \Omega \to \{c, i\} \) such that

\[
\delta(\omega) = \begin{cases} 
c & \text{for } \omega \in \{\omega' \in \Omega : \phi(\omega') = F(\omega')\} 
\ i & \text{for } \omega \in \{\omega' \in \Omega : \phi(\omega') \neq F(\omega')\} 
\end{cases}
\]

where \( c \) means "correct" and \( i \) means "incorrect". The accuracy of an interpreted signal \( \phi \) is defined as the probability of its associated \( \delta \) taking value \( c \). In the above example, \( \delta(\omega) = c \) for \( \omega \in \{\omega_1, \omega_2, \omega_5, \omega_6\} \) and \( \delta(\omega) = i \) for \( \omega \in \{\omega_3, \omega_4\} \). Thus, with probability \( \frac{2}{3} \), the interpreted signal \( \phi \) is correct.

In the next two sections, we examine whether the standard statistical assumptions about signals are appropriate in the context of interpreted signals. We first establish a result that argues the inappropriateness of the conditional independence property as a benchmark assumption in this context. We then take interpreted signals based on independent interpretations as a benchmark model of the interpreted signal framework, study its properties, and compare them to the often assumed properties of generated signals.

4 Conditionally Independent Interpreted Signals

The most common assumption on signals in incomplete information models in the economics and political science literature is independence conditional on outcomes. Here we show that this assumption applied to interpreted signals imposes a specific structure on the outcome function and its associated interpreted signals. Unfortunately, that structure for reasons that will be clear is also unrealistic in cases that involve strategic choice of interpretations.

We first construct a particular outcome function and a collection of interpreted signals that are independent conditional on outcomes. We then prove that any outcome function and its associated collection of conditionally independent interpreted signals can be mapped into this specific structure. The example considers the case where states have equal probability and that two outcomes are equally likely. Let \( p_i = \frac{r_i}{m_i} \) (where \( r_i \) and \( m_i \) are positive integers) denote the probability that signal \( i \)
is correct. We assume that $2r_i > m_i > r_i$ so that $p_i$ lies in the open interval $(0.5, 1)$. $p_i$ will also equal the probability that signal $i$ is correct conditional on each outcome.

To construct $K$ ($K \geq 2$, integer) interpreted signals that are conditionally independent on the outcome, we set $N$, the number of states of equal probability, equal to $2m_1 \times \ldots \times m_K$.

We denote a state as a vector of $K + 1$ attributes, $(\theta, x_1, x_2, \ldots, x_K)$, in which the first attribute $\theta$ takes one of two values, for convenience, 0 or 1, and each of the remaining $K$ variables take values in the set $\{1, 2, \ldots, m_i\}$. We construct the outcome function so that if an even number of the last $K$ attributes are greater than $r_i$, the value of the function equals $\theta$. Otherwise, the value equals $(1 - \theta)$.

$$f(\theta, \vec{x}) = \theta \text{ if } \left| \{ i : x_i > r_i \} \right| = 2j \text{ for some non negative integer } j$$

$$= 1 - \theta \text{ else}$$

We define the interpretations and the interpreted signals as follows. Interpretation $i$ considers every attribute except attribute $x_i$, i.e. $(\theta, \vec{x}_{-i})$. The interpreted signal, $s_i$, based on this interpretation equals $\theta$ if an even number of the attributes other than $x_i$ take values greater than $r_j$ and equals $1 - \theta$ otherwise.

$$s_i(\theta, \vec{x}_{-i}) = \theta \text{ if } \left| \{ j \neq i : x_j > r_j \} \right| = 2j \text{ for some non negative integer } j$$

$$= 1 - \theta \text{ else}$$

Let’s now consider the properties of the signals in this example keeping in mind that all attribute vectors are equally likely. First, $s_i$ makes a mistake if and only if $x_i$ has a value larger than $r_i$. This is true what ever the true outcome is. So, $s_i$ is correct with probability $\frac{r_i}{m_i}$. Also, $s_i$ is equal to the true outcome plus an error term (independent of the true outcome) which is determined entirely by $x_i$. By construction, $x_i$ and $x_j$ are independent. Therefore $s_i$ and $s_j$ are independent conditional on either outcome value.

Unlike generated signals, in which an agent gets a signal that is correct with some probability, these interpreted signals correspond to models that leave out one piece of information – the value of an attribute. Most of the time, the realization of that value will not change the value of the outcome, but sometimes it will. Thus, it is possible for conditionally independent signals to occur provided each agent constructs a predictive model that ignores a different attribute and if the functional form has the property that each attribute has a proportionally similar effect on the outcome regardless of the outcome value.

As we now show, this last restriction rules out all functions over attributes except those isomorphic to the class of examples we just described. We provide the formal statement of this result below. Its proof along with all proofs of subsequent claims is contained in the appendix.

**Claim 1** Let $F : \Omega \rightarrow \{0, 1\}$ be an outcome function. Each outcome is equally likely. For any $i \in \{1, \ldots, K\}$, let $\phi_i$ be agent $i$’s interpreted signal associated with the outcome.

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9For notational convenience, we denote the binary outcomes in this section as $\{0, 1\}$. The rest of the paper will use $\{G, B\}$. 

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function $F$. Assume that each signal $\phi_i$ is correct with probability $p_i = \frac{r_i}{m_i}$ given either outcome. If the interpreted signals $\phi_1, \ldots, \phi_K$ are independent conditional on each outcome, then the original state space $\Omega$ can be represented by the space of attributes in the above example such that the outcome function $F$ and the signals $\phi_1, \ldots, \phi_K$ are represented by the outcome function $f$ and the interpreted signals $s_1, \ldots, s_K$ in the above example.

This claim implies that while it is possible to create conditionally independent interpreted signals, doing so imposes a unique outcome function and requires that each agent neglects one piece of information. Thus given a specific function mapping attributes to outcomes, paradoxically, conditionally independent signals are not consistent with agents looking at different attributes but they are only consistent with agents neglecting distinct pieces of information. This assumption seems untenable in cases with endogenous information acquisition. Why would an agent choose not to look at or include a piece of information that everyone else possesses when having that single piece would give the agent full information? The assumptions would only seem to hold for a situation in which agents have marks on their foreheads that they cannot see but which are visible to everyone else, hardly a situation of broad relevance.

5 Independent Interpreted Signals

A natural next question to ask is what property of signals are consistent with agents who use interpreted signals? In this section, we consider interpreted signals based on independent interpretations. We restrict our focus to independent interpretations for three reasons. First, as Claim 2 below shows, independent interpretations provide a natural benchmark: they formally define what might be loosely described as people seeing the world differently. Second, in a binary classification problem, they provide the most diverse predictive models possible. This follows because interpretations cannot be negatively correlated once we tack on predictions. Finally, in the case where interpretations are chosen strategically, maximizing the value of information would entail choosing an interpretation as different from those of others. That would mean an independent interpretation.$^{10}$

The main result in this section establishes that independently correct signals or conditionally independent signals, both commonly assumed in the literature, are inconsistent with interpreted signals based on independent interpretations. To get to this result, it is useful to first have a discussion of various types of independence that relate to interpretations and predictions.

$^{10}$Here we take the attributes as given. If one player chooses a single factor with the highest predictive value, a second player could then choose the factor with the highest residual predictive value, which would by construction be orthogonal.
5.1 Definitions and Relationships of Types of Independence

We first provide definitions and characterize known relationships of various types of independence. We include this discussion to put them in the context of interpreted signals. We consider only pairwise independence. Extensions to include any finite number of interpretations can be constructed. Here we also show the result that independent interpretations can be viewed as formally defining people seeing the world differently.

**Definition 1** Interpretations, $\Pi_1$ and $\Pi_2$, are independent interpretations if

$$\text{Prob}(\pi_1^i \cap \pi_2^j) = \text{Prob}(\pi_1^i) \times \text{Prob}(\pi_2^j)$$

for all $i \in \{1, ..., n_1\}$ and $j \in \{1, ..., n_2\}$

Two interpretations are independent implies that knowing how one agent interprets a state provides no information about how the other agent interprets that same state. Note that interpretations are not predictions, they are the sets within which agents place states.

We now state what may seem to be a surprising result: independent interpretations imply that the state space can be written as a product space and the interpretations can be written as projections onto variables. To make the logic that drives this result as transparent as possible, we first assume that each state in $\Omega$ is equally likely. Consider the following trivial observation: if $\Omega$ can be represented as a product of attribute spaces and if agents make partitions by looking at disjoint subsets of attributes, then the agents have independent interpretations. For example, if the state space is written as a two by two lattice and one agent considers the row and the other considers the column, then these interpretations are independent. The logic here is straightforward: knowledge of a state’s row, tells us nothing about its column.

The counterintuitive finding is that the converse also holds. If two interpretations are independent, then the state space can be mapped into a coordinate system (a two attribute, $x$ and $y$, model) where each state is represented by $(x, y)$, and one interpretation considers only the $x$ attribute and the other is along the $y$ attribute. This result implies that any two independent interpretations can be rewritten as projections onto different attributes of the same perspective (Hong and Page 2001, Page 2007).\(^{11}\)

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\(^{11}\)Recall that a perspective is a representation of the entire space of possibilities. Two agents use different perspectives if they represent the set of the possible alternatives with different languages. These languages can be basis. For example, one agent may identify a point in the plane using Cartesian coordinates $(x, y)$. Another agent may use polar coordinates $(r, \theta)$. The natural interpretations differ for these two perspectives. In the former, someone might partition the space into points in which $x \leq 5$ and points in which $x > 5$. In the latter, an agent might partition the space into points in which $r \leq 10$ and points in which $r > 10$. 

Claim 2 Assume that each state in \( \Omega \) is equally likely. Let \( \Pi^1, \cdots, \Pi^n \) \( (n \geq 2) \) be non-trivial interpretations of \( \Omega \). If they are independent, then \( \Omega \) can be represented by an \( n \)-attribute rectangle such that \( \Pi^i \) is along the \( i \)th attribute. Thus \( N = \prod_{h=1}^{n} a_h \) for some larger-than-1 integers \( a_h \), \( h = 1, \ldots, n \).  

Intuitively, Claim 2 implies a bound on the number of independent interpretations. It cannot exceed the number of primes in the factorization of \( N \). As stated in the Corollary below, a finite set of states admits few independent interpretations.

Corollary 1 Assume events are equally likely. Let \( \prod_{i=1}^{k} p_i \) be the unique prime factorization of \( N \), that is, 
\[
N = \prod_{i=1}^{k} p_i
\]
where each \( p_i \) is a prime. Then, the maximum number of independent non-trivial interpretations is \( k \).

The implications of this corollary sink in when applied to a specific example such as the set of possible independent interpretations of all of the 300 million people in the United States. Such interpretations, the parsing of people into categories like soccer moms or NASCAR dads, are used to construct predictive models for economic, political, and social outcomes. The corollary implies that there exist fewer than thirty independent interpretations for the entire US population.\(^{13}\)

Independent interpretations are distinct from independent predictions. Saying two agents’ predictions are independent means that knowing one agent’s prediction about the outcome of a state provides no information about the other agent’s prediction.

**Definition 2** Predictions, \( \phi_1 \) and \( \phi_2 \), are **independent predictions** if they are independent random variables.

\(^{12}\)This result is established with the assumption that all states are equally likely. This assumption is not essential. We can show that if states in the original space \( \Omega \) do not have equal probability, there exists another state space \( \Omega' \) that has greater cardinality (the least common denominator of probabilities of original events expressed in fractions) such that all states in \( \Omega' \) is equally likely and \( \Omega' \) can be represented by an \( n \)-attribute rectangle and the independent interpretations of the original state space \( \Omega \) correspond to interpretations of the new event space \( \Omega' \) along different attributes. The original \( \Omega \) consists of lumping of some states in \( \Omega' \). The key is that for independent interpretations, probabilities have the rectangle property, i.e., 
\[
\text{Prob}(\pi_1^1 \cap \pi_2^2) = \text{Prob}(\pi_1^1) \times \text{Prob}(\pi_2^2)
\]

\(^{13}\)This result assumes that each attribute is binary such as \{ male, female \}. To be precise, \( 2^{28} \) is slightly less than 300 million and \( 2^{29} \) exceeds it by a substantial margin.
Relatedly, we say that two predictions are independently correct, if knowing that one agent’s prediction about the outcome of a state is correct gives no information about whether the other’s prediction of the same state is correct.

**Definition 3** Predictions, $\phi_1$ and $\phi_2$, are independently correct predictions if $\delta_1$ and $\delta_2$ are independent random variables.

We now list as observations the known relationship between these various types of independence but put them in the context of our interpreted framework.

**Observation 1** (i) Independent interpretations imply independent predictions. (ii) Independent predictions may not imply independent interpretations. (iii) Independent predictions need not be independently correct. (iv) Independent interpretations may not lead to independently correct predictions.

Independence of interpretations is a stronger property than independence of predictions because of the measurability requirement on predictions. So predictions can be independent without agents having independent interpretations. Observation (iii) above reveals the absence of a causal linkage between independent predictions and independently correct predictions. Consider the following example:

<table>
<thead>
<tr>
<th>Predictions</th>
<th>$g$</th>
<th>$g$</th>
<th>$b$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>G</td>
<td>G</td>
<td>G</td>
<td>B</td>
</tr>
<tr>
<td>$g$</td>
<td>G</td>
<td>G</td>
<td>B</td>
<td>G</td>
</tr>
<tr>
<td>$b$</td>
<td>G</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>$b$</td>
<td>B</td>
<td>G</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

In this example, assume that each of the sixteen states is equally likely. The upper case letters represent outcomes of states. The lower case letters in the first column and in the first row are predictions of the row agent and the column agent respectively. By construction, the two predictions are independent. However, they are not independently correct. The joint probability of both agents making correct predictions is $\frac{1}{2}$ while the multiplication of the probabilities of each agent making correct predictions equals $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$. They are not equal. In this example, not only the predictions of the two agents but also their interpretations are independent. Therefore, the correctness of predictions need not be independent even if the interpretations are.

### 5.2 Properties of Independent Interpreted Signals

The above casual observations indicate that independent interpreted signals can not be independently correct. Here we prove that their correctness is negatively correlated. We also show that conditional on at least one outcome, the signals themselves are negatively correlated. Since independent interpretations imply independent predictions, these results also apply to independent interpretations. Together these results imply a fundamental inconsistency between standard assumptions of incomplete
information models and interpreted signals based on independent interpretations. We establish these results for the case of binary outcomes in which agents’ predictions have identical probability distributions and equal accuracy.\footnote{14}{We have derived a set of results that do not assume that agents’s predictions have identical distributions.}

As before, let upper case letters, $G$ and $B$, refer to outcomes and lower case letters, $g$ and $b$ refer to predictions. $P(G)$ and $P(B)$ denote priors, assumed to be common among all agents. $P(g)$ and $P(b)$ denote the probabilities of predicting $g$ and $b$ respectively. $P(g,g)$ denotes the probability of both predicting $g$. $P(b,b)$, $P(g,b)$ and $P(b,g)$ are similarly defined. $P(c)$ and $P(i)$ denote the probabilities of making correct and incorrect predictions, which are also assumed to be the same for both agents. $P(c,c)$, $P(i,i)$, $P(c,i)$ and $P(i,c)$ denote joint probabilities of both correct, both incorrect, agent 1 correct but agent 2 incorrect and agent 1 incorrect but agent 2 correct respectively. We impose the following symmetry assumptions: $P(g,b) = P(b,g)$ and $P(c,i) = P(i,c)$. Conditional probabilities of a given event are denoted by $P(\cdot | G)$ and $P(\cdot | B)$.

5.2.1 Reasonable and Informative Predictions

Given an interpretation, an agent need not make the best possible predictions. For example, an agent who categorized agents by gender could mistakenly predict that women are, on average, taller than men. To impose a minimal degree of competence, we assume that predictions are either reasonable or informative.

\textbf{definition 4} A prediction is \textbf{reasonable} if it is correct at least half of the time, i.e., $P(c) \geq \frac{1}{2}$.

\textbf{definition 5} A prediction is \textbf{informative} if it is correct more than half of the time, i.e., $P(c) > \frac{1}{2}$.

In the binary outcome case, an experience generated prediction must be reasonable. Further if at least one prediction is correct more than half of the time, then an experience generated prediction is also informative. However, as we observe next, an informative prediction need not predict correctly half of the time conditional on every outcome.

\textbf{Observation 2} An informative prediction need not be reasonable conditional on every outcome.

This observation may be obvious to some. Nevertheless, we provide an example because the underlying logic is central to our analysis.
An Informative Prediction

<table>
<thead>
<tr>
<th>Prediction</th>
<th>Outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>G G B</td>
</tr>
<tr>
<td>g</td>
<td>G G B</td>
</tr>
<tr>
<td>g</td>
<td>G G B</td>
</tr>
<tr>
<td>g</td>
<td>G G B</td>
</tr>
<tr>
<td>b</td>
<td>B B G</td>
</tr>
</tbody>
</table>

Assuming each outcome to be equally likely, $P(c) = \frac{2}{3}$, implying that the prediction is informative. However, conditional on outcome $B$, the probability of making correct prediction equals $P(c \mid B) = P(b \mid B) = \frac{1}{3}$, implying that the prediction is not reasonable conditional on the bad outcome. This example provides insight into why when predicting rare events, agents may not make reasonable predictions.

5.2.2 Negative Correlation in Correctness

We now state a lemma that leads to the result of negative correlation in the correctness of predictions presented in Claim 3 below. Even though Claim 3 can be established directly, the lemma is interesting in its own right. It reveals a tension between the accuracy of predictions and the correlation of their correctness in a more general environment where the probabilities of a good prediction and a bad prediction are not required to be equal. When predictions are independent, the higher the accuracy, the less correlated their correctness. Without loss of generality, we assume $P(g) \geq \frac{1}{2}$. Also note that unless otherwise specified, we do not require that the prior probabilities over $G$ and $B$ to be equal.

**Lemma 1** The correctness of independent and reasonable predictions exhibit positive (zero, negative) correlation if and only if $P(g) > [=, <] P(c)$.

The intuition that drives this result is straightforward. If the probability of predicting the good outcome is large relative to the probability of being correct, then both agents often predict good outcomes at the same time whether or not the prediction is correct. Thus, the correctness of their predictions must also be positively correlated.

Our claim follows from Lemma 1.

**Claim 3** Independent informative predictions that predict good and bad outcomes with equal probability must be negatively correlated in their correctness. Furthermore, the negative correlation of their correctness is defined by the following expression

$$\rho = 1 - \frac{1}{4(p - p^2)}$$
where \( p \) is the accuracy of the individual predictions.\(^{15}\)

The first part of this claim follows directly from Lemma 1. The second part of this claim means that the accuracy of the individual predictions is a sufficient statistic for the correctness correlation between agents. Thus, changes in the outcome function do not lead to any change in the correctness correlation beyond the change in the accuracy. The following corollaries follow immediately.

**Corollary 2** Any independent and independently correct predictions that predict good and bad outcomes with equal probability cannot be informative.

In other words, they are as good as flipping a coin. Since independent interpretations imply independent predictions, we also have the following:

**Corollary 3** If interpretations are independent and if their associated predictions are informative and predict good and bad outcomes with equal probability, then the correctness of their predictions must be negatively correlated.

The claim together with the corollaries reveal a fundamental conflict between an assumption that agents see the world independently (independent interpretations) and an assumption that their associated predictions are independently correct.

The claim has a third, unexpected corollary, namely that the outcome function has no effect on correlation other than through the accuracy of the interpreted signals. If we think of the complexity of the outcome function as being some measure of the number of interaction terms, this corollary states that complexity influences correlation only in so far that it effects signal accuracy.

**Corollary 4** The accuracy of two independent interpreted signals uniquely determines their correlation. Therefore, any increase or decrease in the complexity of the inference task has no direct influence on correlation other than its effect on the accuracy of the interpreted signals.

### 5.2.3 Negative Correlation of Interpreted Signals

The negative correlation in signal correctness suggests a parallel result of negative correlation of the conditional distributions of the signals if signals are based on independent interpretations. We state this result below. Note that this contradicts the independent conditional distributions typically assumed for signals.

**Claim 4** Assume that predictions are experience generated and are informative. If predictions are independent, then for at least one outcome, predictions conditional on that outcome are negatively correlated.

\(^{15}\)Given that the correlation coefficient can not be less than -1, it can be easily computed based on the expression here that the accuracy of the individual predictions can not exceed \( \frac{1}{2} + \frac{\sqrt{2}}{4} \) which is approximately equal to 0.8535.
Again, since independent interpretations imply independent predictions, we have the following corollary.

**Corollary 5** If interpretations are independent and if their associated predictions are experience generated and informative, then for at least one outcome, predictions conditional on that outcome are negatively correlated.

This final result together with Corollary 3 reveals a fundamental incompatibility between standard signaling assumptions and independent interpretations: Seeing the world independently, looking at different attributes, not only does not imply, it is inconsistent with, both conditional independence of signals and independently correct signals.

### 6 Interpreted Signals and Information Aggregation

To show how to apply the interpreted signal framework, we describe a toy model of information aggregation. We leave a more general analysis for future work. Through this example, we show that the key to whether private information represented by interpreted signals aggregate through a mechanism, specifically the majority rule voting mechanism, lies in the structure of the outcome function relative to agents’ interpretations. In other words, in the interpreted signal framework, the information being aggregated is the functional relation between the outcome and the interpreted signals. Thus if this functional relationship is ”complex”, then a ”simple” mechanism may not be able to aggregate information perfectly.

In our toy model, agents vote on candidates for a political office. Whether a candidate is a good or a bad choice depends on five attributes. Specifically, we assume that it is represented by the following outcome function: $F : \{0,1\}^5 \rightarrow \{0,1\}$ such that $F(a_1, ..., a_5) = 1$ iff $\sum_{i=1}^{5} a_i \geq 3$. Since we want to focus purely on information aggregation as opposed to preference aggregation, we assume that agents agree on what makes a good or a bad choice. Assume also that a priori, each attribute vector is equally likely.

We consider five agents, each observing the value of a single, distinct attribute prior to casting their votes. We can think of these attributes as interpreted signals. We now consider using the majority rule voting mechanism to aggregate information. Straightforward calculations show that conditional on the true outcome, (i) each attribute is correct with probability $\frac{11}{16}$ (ii) the probability of a pair of attributes being correct at the same time is $\frac{7}{16}$ and (iii) since $\frac{7}{16} < \frac{11}{16} \times \frac{11}{16}$, attributes are pairwise negatively correlated, which by our previous results must be true.

All agents voting their information constitutes a Nash equilibrium (NE) of this model. To see this, note that given all other agents vote their information, agent $i$ is pivotal if and only if other agents’ attributes split, two 1’s and two 0’s. In those
situations, agent $i$’s vote alone determines the outcome of the game. Since the true outcome in such a situation always matches with agent $i$’s attribute value, it is in the interest of agent $i$ to vote her attribute value. Therefore all agents voting their information is a NE. It is obvious that the outcome of this majority rule voting game when agents vote their information always matches with the true outcome. Therefore, the majority rule voting mechanism perfectly aggregates information in this example.

We now consider a slightly different outcome function and our conclusion about the performance of the majority rule voting mechanism changes. Here, the outcome function is the same as before except that when all attributes agree, the outcome switches from either 0 to 1 or 1 to 0. Let

$$F'(a_1, ..., a_5) = F(a_1, ..., a_5)$$

for all $(a_1, ..., a_5)$ such that $\sum_{i=1}^{5} a_i \neq 0$ or 5; otherwise,

$$F'(a_1, ..., a_5) = 1 - F(a_1, ..., a_5).$$

Again, it is easy to compute that conditional on the true outcome, (i) each attribute is correct with probability $\frac{10}{16}$, (ii) the probability of a pair of attributes being correct at the same time is $\frac{6}{16}$ and (iii) since $\frac{6}{16} < \frac{10}{16} \times \frac{10}{16}$, the attributes are pairwise negatively correlated. The analysis to establish that informative voting by all agents still constitutes a NE is the same as that in the previous example because the true outcome switch happens when no agent is pivotal. Since informative voting combined with the majority rule voting mechanism leads to outcomes that match with the previous outcome function, they do not match with the outcome function in this example. When $\sum_{i=1}^{5} a_i = 0$ or 5, information does not aggregate.

These two examples demonstrate that whether a particular mechanism aggregates information perfectly depends on the structure of the outcome function. In both examples, agents have informative signals, conditional on the true outcome, signals are pairwise negatively correlated, voting informatively is a NE, but in one case information is perfectly aggregated and in the other case, information fails to always aggregate. The structure of the two outcome functions relative to their respective interpreted signals differs. $F$ as a function of the interpreted signals (attributes) is monotonic and no externality is present. On the other hand, $F'$ as a function of the interpreted signals is not monotonic and includes interaction terms among the attributes. The fact that information fails to aggregate when interactions are present means that the majority rule voting mechanism can not resolve externalities in private information.

These examples suggest that a different type of analysis is needed when we consider aggregating information represented by interpreted signals as opposed to generated signals. The functional relationship as opposed to the statistical relationship between outcomes and signals should be the focus. A companion paper (Hong & Page 2008) develops this approach in the context of information aggregation properties of general voting mechanisms.
7 Discussion

The results of this paper demonstrate how the standard statistical assumptions about signals may not apply for a broad class of economically relevant cases. Our distinction between generated signals and interpreted signals clarifies why this is the case. The standard assumptions of signals in incomplete information literature, which are reasonable for generated signals, are not appropriate for interpreted signals due to the substantive difference in how interpreted signals are obtained. Therefore, to the extent that interpreted signals capture a large class of incomplete information environments in economics and other social sciences, existing analyses of incomplete information models may be less pertinent than thought.

The point of this paper was not to criticize existing models but to suggest new avenues of research based on interpreted signals. For example, the distinction between the two types of signals might enable us to better understand differences between experimental and real world results. In experiments, private information is often generated using the standard conditionally independent signal model. Our results suggest that in the real world, that may not be the case. Therefore, testing theory using experiments may not be testing one of the most important assumptions: the assumption of conditionally independent signals.

A similar insight applies to attempts to calibrate computational models with standard models of signals. These efforts may also run up against this fundamental inconsistency. In an agent based model (Tesfatsion 1997, Holland and Miller 1991), the signals are often lower dimensional projections of a larger reality. In rich, fine detailed computer models, such as the trading agent competition (Wellman et al 2003), agents do not take into account all of the information in the environment. Instead, they monitor a lower dimensional world than the one within which they interact. In spatial models and network models, something close to dimensional reduction also occurs. Agents only see what happens in a local region, thereby creating interpreted signals.

Owing to its close connections to computer science and to psychology, the interpreted signal framework can be seen as a more cognitively based, computational approach to incomplete information. Ideally, computational and analytical models inform and complement one another (Judd 1997, Judd and Page 2004). However, our ability to align psychological, computational, and analytical models is hindered if the assumptions about signals that we make in our analytical models are not consistent with the information that agents realize in the computational implementations of those models.

Though interpreted signals may be more cumbersome to analyze, they provide greater flexibility in the modeling of endogenous signals. If agents are concerned with collective performance, such as in the case of voting to aggregate information, agents have an incentive to look at different attributes. By looking at different attributes, the correctness of agents’ predictions are negatively correlated (by a specific amount), so information aggregates better than standard models assume. Unfortunately, the
number of different attributes that can be considered depends upon the dimensionality of the problem. Therefore, large groups of agents may do worse than generated signal theory predicts because they necessarily have lots of overlap. Small groups, in contrast, may do better than the generated signal theory predicts.

The choice over which attributes to include in interpretations in competitive situations, such as auctions, may be among the most interesting questions to consider. Competitive situations create incentives to be correct and an incentive to be different. These two counterbalancing effects may allow us to model the structure of incomplete information as an equilibrium phenomenon as opposed to assuming it as exogenously given.

As demonstrated in our analysis of negative correlation, the interpretation framework also permits more fine grained analysis of the link between complexity and uncertainty. The complexity-uncertainty link is also the focus of a paper by Al-Najjar, Casadesus-Masanell, and Ozdenoren (2003). They consider the continual addition of more and more attributes. As the number of attributes considered increases, the signals should improve. In their model, a problem is complex if no matter how many attributes are considered, the uncertainty never goes away. Relatedly, in our framework, within some sets in a partition, both good and bad outcomes can exist. Our formulation differs in that it emphasizes the nonlinearity and interaction terms in the mapping from attributes to outcomes. As this mapping becomes more complex, the inference problem becomes more difficult unless interaction terms are absorbed by an interpretation. This suggests the need for deeper investigations into the relationship between the complexity of mappings from attributes to outcomes and the uncertainty of interpreted signals based on those mappings.

Finally, the interpretation framework suggests a normative rethinking of signal heterogeneity. With generated signals, more variation is generally worse. That is because we see variation as noise. However, in the interpreted signal framework, signal variation reflects interpretive model variation. Variation in interpretive models need not be seen negatively. It may well imply greater robustness and accuracy.

8 Appendix: Proofs

Proof of Claim 1. Let \( \delta \in \{0, 1\} \). Let \( J \) denote a subset of \( \{1, ..., K\} \). For a given \( \delta \) and a given \( J \subseteq \{1, ..., K\} \), consider the event

\[
\left\{ \omega \in \Omega : F(\omega) = \delta, \; \phi_j(\omega) = \delta \text{ for } j \in J, \; \phi_j(\omega) = 1 - \delta \text{ for } j \in J^C \right\}.
\]

If \( |J^C| \) is even, identify the above event with the following event in the example:

\[
\left\{ (\theta; x_1, ..., x_K) \in \{0, 1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. \\
\left. \theta = \delta, \; x_j \leq r_j \text{ for } j \in J, \; x_j > r_j \text{ for } j \in J^C \right\}
\]
If $|J^C|$ is odd, identify it with the following event in the example:

$$\left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
\theta = 1 - \delta, \ x_j \leq r_j \text{ for } j \in J, \ x_j > r_j \text{ for } j \in J^C \\
\end{array} \right\}$$

This identification is valid for two reasons. First, the probabilities of these events are equal. By the assumption of conditional independence and the assumption that each signal $i$ is correct with probability $\frac{r_i}{m_i}$ conditional on each outcome,

$$P \left( \{ \omega \in \Omega : F(\omega) = \delta, \phi_j(\omega) = \delta \text{ for } j \in J, \phi_j(\omega) = 1 - \delta \text{ for } j \in J^C \} \right)$$

$$= \prod_{j \in J} P(\phi_j(\omega) = \delta \mid F(\omega) = \delta) \prod_{j \in J^C} P(\phi_j(\omega) = 1 - \delta \mid F(\omega) = \delta) \cdot P(F(\omega) = \delta)$$

$$= \frac{1}{2} \prod_{j \in J} \frac{r_j}{m_j} \prod_{j \in J^C} \left(1 - \frac{r_j}{m_j}\right)$$

By the structure of our example,

$$P \left( \left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
\theta = 1 - \delta, \ x_j \leq r_j \text{ for } j \in J, \ x_j > r_j \text{ for } j \in J^C \\
\end{array} \right\} \right)$$

$$= P \left( \left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
\theta = 1 - \delta, \ x_j \leq r_j \text{ for } j \in J, \ x_j > r_j \text{ for } j \in J^C \\
\end{array} \right\} \right)$$

$$= \frac{1}{2} \prod_{j \in J} \frac{r_j}{m_j} \prod_{j \in J^C} \left(1 - \frac{r_j}{m_j}\right)$$

Second, the events of the original $\Omega$ we considered are the only events relevant to our interpreted signals. So from our proof so far, these events are represented by combining events in the attribute vector space of our example. By the definition of $f$ and $s_1, ..., s_K$ in our example, if $|J^C|$ is even,

$$\left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
\theta = \delta, \ x_j \leq r_j \text{ for } j \in J, \ x_j > r_j \text{ for } j \in J^C \\
\end{array} \right\}$$

$$= \left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
f = \delta, \ s_j = \delta \text{ for } j \in J, \ s_j = 1 - \delta \text{ for } j \in J^C \\
\end{array} \right\}$$

If $|J^C|$ is odd,

$$\left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
\theta = 1 - \delta, \ x_j \leq r_j \text{ for } j \in J, \ x_j > r_j \text{ for } j \in J^C \\
\end{array} \right\}$$

$$= \left\{ (\theta; x_1, ..., x_K) \in \{0,1\} \times \{1, ..., m_1\} \times ... \times \{1, ..., m_K\} : \right. $$

$$\left. \begin{array}{l}
f = \delta, \ s_j = \delta \text{ for } j \in J, \ s_j = 1 - \delta \text{ for } j \in J^C \\
\end{array} \right\}$$

In other words, the outcome function $F$ and its associated interpreted signals $\phi_1, ..., \phi_K$ are identical to the outcome function $f$ and its associated interpreted signals $s_1, ..., s_K$ once $\Omega$ is appropriately represented by the attribute vector space of our example.

**Proof of Claim 2.** We prove the claim for $n = 2$. The proof for more general cases follows the same procedure.
Without loss of generality, assume that $\Pi^i = \{\pi^i_1, \ldots, \pi^i_{n_i}\}$ where for each $i = 1, 2, n_i \geq 2$. We write the state space, $\Omega$, in the following form which helps to visualize the proof.

$$
\begin{array}{cccc}
\pi^1_1 \cap \pi^2_1 & \pi^1_1 \cap \pi^2_2 & \cdots & \pi^1_1 \cap \pi^2_{n_2} \\
\pi^1_2 \cap \pi^2_1 & \pi^1_2 \cap \pi^2_2 & \cdots & \pi^1_2 \cap \pi^2_{n_2} \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\pi^1_{n_1} \cap \pi^2_1 & \pi^1_{n_1} \cap \pi^2_2 & \cdots & \pi^1_{n_1} \cap \pi^2_{n_2}
\end{array}
$$

(1)

Each cell above can be represented by a 2-dimensional rectangle with the property that cells so represented in the same row have the same height and cells in the same column have the same width.

To show this, we first show that for each $j = 2, \ldots, n_2$, the number of states contained in each cell in any given column is proportional to the number of states in each cell in the first column:

$$
\frac{|\pi^1_i \cap \pi^2_j|}{|\pi^1_i \cap \pi^2_1|} = \frac{|\pi^1_2 \cap \pi^2_j|}{|\pi^1_2 \cap \pi^2_1|} = \cdots = \frac{|\pi^1_{n_1} \cap \pi^2_j|}{|\pi^1_{n_1} \cap \pi^2_1|}
$$

(2)

where $|\cdot|$ denotes the cardinality of a set. By independence (recall that each state in $\Omega$ is equally likely), for all $i = 1, \ldots, n_1$ and all $j = 2, \ldots, n_2$,

$$
\frac{|\pi^1_i \cap \pi^2_j|}{N} = \frac{|\pi^1_i|}{N} \cdot \frac{|\pi^2_j|}{N}
$$

and

$$
\frac{|\pi^1_i \cap \pi^2_1|}{N} = \frac{|\pi^1_i|}{N} \cdot \frac{|\pi^2_1|}{N}
$$

Therefore,

$$
\frac{|\pi^1_i \cap \pi^2_j|}{|\pi^1_i \cap \pi^2_1|} = \frac{|\pi^2_j|}{|\pi^2_1|}
$$

This proves (2) above.

For each $j = 2, \ldots, n_2$, let the ratio in (2) be equal to $\frac{u_j}{d_j}$ where both $u_j$ and $d_j$ are positive integers and $\frac{u_j}{d_j}$ can not be further simplified. That is, for each $i = 1, \ldots, n_1$, we can write the number of events in the $i$th row and $j$th column as $\frac{u_j}{d_j}$ times the number of events in the first column of the $i$th row.

$$
|\pi^1_i \cap \pi^2_j| = \frac{u_j}{d_j} \cdot |\pi^1_i \cap \pi^2_1|
$$

This implies that for each $i = 1, \ldots, n_1$, $|\pi^1_i \cap \pi^2_1|$ is divisible by all $d_j$'s, $j = 2, \ldots, n_2$. Let $d$ be the smallest positive integer that is divisible by all $d_j$'s. Then for each $i = 1, \ldots, n_1$, there exists a unique positive integer $k_i$ such that

$$
|\pi^1_i \cap \pi^2_1| = k_i \cdot d.
$$
Thus,
\[ |\pi_1^i \cap \pi_2^j| = k_i \cdot \left( u_j \cdot \frac{d}{d_j} \right) \]
for all \( i = 1, ..., n_1 \) and \( j = 2, ..., n_2 \). Notice that \( \frac{d}{d_j} \) is a positive integer in the above expression.

The above argument proves that for any \( i = 1, ..., n_1 \) and \( j = 2, ..., n_2 \), \( \pi_1^i \cap \pi_2^j \) can be represented by a 2-dimensional rectangle of \( k_i \) rows (height) and \( u_j \cdot \frac{d}{d_j} \) columns (width). Here we have implicitly defined \( u_1 = d_1 = 1 \). Therefore, each cell in (2) can be represented by a 2-dimensional rectangle such that cells in row \( i \) all have the same height of \( k_i \) and cells in column \( j \) all have the same width of \( u_j \cdot \frac{d}{d_j} \). Therefore, (2) can be represented by a 2-dimensional rectangle with \( \sum_{i=1}^{n_1} k_i \) rows and \( \sum_{j=1}^{n_2} u_j \cdot \frac{d}{d_j} \) columns. That means, \( N = (\sum_{i=1}^{n_1} k_i) \cdot \left( \sum_{j=1}^{n_2} u_j \cdot \frac{d}{d_j} \right) \). It is obvious that the number in each parenthesis is larger than 1.

**Proof of Corollary 1.** By Claim 2, a necessary condition for \( n \) non-trivial interpretations to be independent is that \( N \) can be written as the multiplications of \( n \) larger-than-1 integers. Thus, the largest number of independent non-trivial interpretations is bounded by the number of prime factors which is \( k \). Now we only need to show that there exist \( k \) many non-trivial interpretations that are independent. When \( N = \prod_{i=1}^{k} p_i \), \( \Omega \) can be represented by a \( k \)-dimensional rectangle where the \( i \)-th dimension has a length of \( p_i \). Let \( \Pi^i \) be the interpretation that can only identify events along the \( i \)-th dimension. Showing that these \( k \) interpretations are independent is a straightforward exercise.

**Proof of Lemma 1.** First, observe the following identity:
\[ P(g, b) + P(b, g) = P(c, i) + P(i, c) \]
Each side of this equation expresses the probability that agents disagree. Then by symmetry,
\[ P(g, b) = P(c, i) \]
Second, notice that the function \( x(1 - x) \) is a decreasing function of \( x \) for \( x \geq \frac{1}{2} \). Therefore,
\[ P(g)(1 - P(g)) < [=, >] P(c)(1 - P(c)) \]
if and only if
\[ P(g) > [=, <] P(c) \]
That is,
\[ P(g)P(b) < [=, >] P(c)P(i) \]
if and only if
\[ P(g) > [=, <] P(c) \]

24
Now we prove Lemma 1. If predictions are independent, then
\[ P(g, b) = P(g)P(b) \]

Also, by definition, the correctness of predictions are positively correlated (independent or negatively correlated) iff
\[ P(c, i) < [\leq, \geq] P(c)P(i) \]

Combine the above two equations with the identity at the beginning of the proof, we have the correctness of predictions are positively correlated (independent or negatively correlated) iff \( P(g)P(b) < [\leq, \geq] P(c)P(i) \). The result then follows.

**Proof of Claim 3.** The first part of Claim 3 is a special case of Lemma 1. Now we establish the exact expression for the negative correlation coefficient. Define variable \( \chi_i \) as the indicator of the correctness of prediction \( \phi_i \). In other words, \( \chi_i = 1 \) if \( \phi_i \) predicts correctly and otherwise \( \chi_i = 0 \). Then the correlation coefficient of the correctness of the predictions, \( \rho \), is defined as
\[ \rho = \frac{Cov(\chi_1, \chi_2)}{\sqrt{Var(\chi_1)Var(\chi_2)}} \]

Consider the following relationship between events and their probabilities. Let \( p_i \) denote the probability that \( \phi_i \) is correct. Then
\[ p_1 = Pr(1 \text{ agrees with } 2 \text{ and } 1 \text{ is correct}) + Pr(1 \text{ disagree with } 2 \text{ and } 1 \text{ is correct}) \]
\[ = Pr(1 \text{ and } 2 \text{ agree and both correct}) + Pr(1 \text{ and } 2 \text{ disagree and } 2 \text{ is incorrect}) \]

And, similarly,
\[ p_2 = Pr(1 \text{ and } 2 \text{ agree and both correct}) + Pr(1 \text{ and } 2 \text{ disagree and } 2 \text{ is correct}) \]

Add the two equations, we have
\[ p_1 + p_2 = 2Pr(\text{both agree and correct}) + Pr(1 \text{ and } 2 \text{ disagree}) \]

The above equation holds without any specific assumptions. Now we assume that \( p_i = p_j = p \) and that predictions \( \phi_1 \) and \( \phi_2 \) are independent. Then
\[ Pr(1 \text{ and } 2 \text{ disagree}) = 2P(g)P(b) = 2P(g)(1 - P(g)) \]

So,
\[ Pr(\text{both agree and correct}) = p - P(g)(1 - P(g)) \]

We can now compute the correlation coefficient of the correctness of predictions, \( \rho \).
\[ Var(\chi_i) = E\chi_i^2 - (E\chi_i)^2 = p - p^2 \]
\[ Cov(\chi_1, \chi_2) := E(\chi_1 \chi_2) - (E\chi_1)(E\chi_2) := p - P(g)(1 - P(g)) - p^2 \]

Therefore,
\[ \rho = \frac{p - P(g)(1 - P(g)) - p^2}{p - p^2} = 1 - \frac{P(g)(1 - P(g))}{p - p^2} \]

To complete the proof, note that \( P(g) = \frac{1}{2} \), then
\[ \rho = 1 - \frac{1}{4(p - p^2)} \]

**Proof of Claim 4.** For notational convenience, let \( p = P(g \mid G) \) and \( q = P(b \mid B) \). We prove this claim by way of contradiction. Suppose neither conditional distribution of predictions is negatively correlated. Then
\[ P(g, g \mid G) \geq p^2 \]
and
\[ P(g, g \mid B) \geq (1 - q)^2 \]

Since predictions are independent,
\[ P(g)^2 = P(g, g) \]

By definition,
\[ P(g) = P(G)p + P(B)(1 - q) \]
and
\[ P(g, g) = P(G)P(g, g \mid G) + P(B)P(g, g \mid B) \]

Thus,
\[ [P(G)p + P(B)(1 - q)]^2 \geq P(G)p^2 + P(B)(1 - q)^2 \]
which simplifies to
\[ P(G)P(B) (p + q - 1)^2 \leq 0 \]

Since
\[ p + q > 1 \]
a contradiction.
References


