ECON 747 – LECTURE 4: FROM COMPLETE TO INCOMPLETE MARKETS

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Spring 2021
OVERVIEW OF THIS LECTURE

1. Why start with a complete markets model?
2. Consumption and risk sharing with complete markets
3. Asset pricing with complete markets
4. Asset pricing with incomplete markets
5. Asset pricing applications
   + Extra material on no-Ponzi vs. transversality conditions
PURPOSE OF THIS LECTURE

- Acquire tools that we need throughout the course (e.g. how to price assets)

- Build a benchmark model that models with financial frictions will deviate from

- Remember the two deviations from a complete markets representative agent model that are needed to meaningfully introduce financial frictions
  1. Incomplete markets
  2. Heterogeneous agents

- We will see today that one of them is not enough: (certain) heterogeneity between agents can be insured away if markets are complete
INSIGHTS FROM THE COMPLETE MARKETS MODEL

- If you understand the complete markets benchmark, you will realize some important aspects of many macro models

- Some examples ...
A representative agent model actually relies on lots of trade in financial assets!

Why?

The representative agent model is isomorphic to a model in which a continuum of heterogeneous agents trade insurance (state-contingent claims) to eliminate all idiosyncratic risk!
INSIGHTS FROM THE COMPLETE MARKETS MODEL

- Models with financial frictions often feature several layers of market incompleteness

- Example of Kiyotaki-Moore style collateral constraint

\[ b_t \leq \theta k_t \]

- Usually two types of market incompleteness!
  1. Bond is not state contingent
  2. Bond trade is limited by physical asset

(great explanation in Cao and Nie, 2017)
INSIGHTS FROM THE COMPLETE MARKETS MODEL

- In many models, some asset markets are complete, others are incomplete

- Example 1: friction in asset trade between two countries, complete markets (full insurance) within each country

- Example 2: friction in asset trade between households and firms, complete markets (full insurance) within household sector and within firm sector
You may occasionally see in a seminar that someone has actually made some asset markets incomplete without noticing.

Someone in the audience will say something like: “you must be assuming that agents cannot insure away this problem”.

Lesson: always remind yourself about the market structure you are operating in, be conscious about everything you are (implicitly) assuming about asset trade.
REFERENCES FOR LECTURES 4 AND 5

- This lecture and the next one are pretty close to some chapters in the Ljungqvist and Sargent text book *Recursive Macroeconomic Theory* (2nd edition)
  - Lecture 4: Chapters 8, 12, 13
  - Lecture 5: Chapters 16, 17
- The notation will be similar to theirs
- I will provide a few additional references throughout
CONSUMPTION AND RISK SHARING WITH COMPLETE MARKETS
Households indexed by $i = 1, \ldots, I$

Preferences:

$$U^i_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c^i_\tau)$$

Technology: exogenous endowment $y^i_t$ in period $t$

Denote state of the world $s^t$

- State is vector of realizations $\{y^1_t, \ldots, y^I_t\}$
- $\pi(s^t)$ is the probability of a history of states $s^t$
- $\pi(s_{t+1}|s^t)$ is the probability of state $s_{t+1}$ given history $s^t$
COMMENT ON NOTATION

- We make the state $s_t$ explicit in the notation

- This is because we will work with state-contingent contracts

- But this model is not different from what we have seen so far: in the model of the previous two lectures, a state is a realization of $(Z_t, V_t)$
TWO WAYS TO FORMULATE COMPLETE MARKERS

1. **Arrow-Debreu securities**
   - Contingent claims traded in period-0, exercised every period
   - $q^0(s^t)$: time-0 price of asset that pays 1 unit of consumption if history $s^t$ realizes
   - $a^i(s^t)$: agent $i$’s holdings of this asset

2. **Arrow securities**
   - One-period ahead contingent claims
   - $Q(s_{t+1}|s^t)$: history-$s^t$ price of asset that pays 1 unit of consumption if state $s_{t+1}$ realizes next period
   - $A^i(s_{t+1}|s^t)$: agent $i$’s holdings of this asset

*contingent* = every state can be contracted on
The time-0 problem of the agent $i$ is

$$\max_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$c^i(s^t) = y^i(s^t) + a^i(s^t)$$

$$\sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)a^i(s^t) = 0$$

Interpretation of last constraint: buy and sell claims to a clearing house in period 0, remain with zero balance
Substitute out $a^i(s^t)$ and denote $\mu^i$ the Lagrange multiplier on the combined constraint.

The FOC w.r.t $c^i(s^t)$ is

$$\beta^t \pi(s^t) u'(c^i(s^t)) = q^0(s^t) \mu^i$$

This condition holds for all $i$ and all $s^t$. 
Define \( c(s^t) = \sum_i c^i(s^t) \) and \( y(s^t) = \sum_i y^i(s^t) \).

A **competitive equilibrium** in this economy is defined as a sequence of allocations \( \{c(s^t)\}_{t=0}^{\infty} \) and prices \( \{q^0(s^t)\}_{t=0}^{\infty} \) such that for all \( s^t \):

1. Markets clear: \( c(s^t) = y(s^t) \)
2. Given the price system, the optimality conditions are satisfied
IMPLICATIONS OF COMPLETE MARKETS

1. Perfect risk sharing

2. Consumption smoothing
Perfect Risk Sharing

Combine the optimality condition for two agents $i$ and $j$

\[
\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\mu^i}{\mu^j} \quad \forall i, j, s^t
\]
PERFECT RISK SHARING

> Solve for $c^i(s^t)$ and sum over $i$

$$c^i(s^t) = u'^{-1} \left\{ \frac{\mu^i}{\mu^j} u'(c^j(s^t)) \right\}$$

$$\sum_i c^i(s^t) = \sum_i u'^{-1} \left\{ \frac{\mu^i}{\mu^j} u'(c^j(s^t)) \right\}$$

> From market clearing, $\sum_i c^i(s^t) = y_t$

> Therefore, we get an optimality condition for $j$’s consumption that only depends on the aggregate!

> Idiosyncratic risk is insured away
CONSUMPTION SMOOTHING

- Combine the optimality condition for two states $s^t$ and $s^\tau$

$$\beta^{t-\tau} \frac{\pi(s^t)u'(c^i(s^t))}{\pi(s^\tau)u'(c^i(s^\tau))} = \frac{q^0(s^t)}{q^0(s^\tau)} \quad \forall i, s^t, s^\tau$$

- Desire to equate marginal utility of consumption across time and states, attenuated by probabilities and prices ($MRS = MRT$)
PLANNER SOLUTION

- Planner problem in this economy (no prices!)

\[
\max \sum_i \omega^i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))
\]

subject to

\[
\sum_i c^i(s^t) = \sum_i y^i(s^t) \quad \forall s^t
\]

- \( \omega^i \) is the welfare weight on agent \( i \)
OPTIMALITY

- Planner FOC w.r.t. $c^i(s^t)$ is

$$\beta^t \omega^i \pi(s^t) u'(c^i(s^t)) = \lambda(s^t) \quad \forall s^t$$

- $\lambda(s^t)$, the Lagrange multiplier on the resource constraint, is a function of $s^t$
The above condition is the same as the market FOC if

\[ \omega^i = (\mu^i)^{-1} \]
\[ \lambda(s^t) = q^0(s^t) \]

The competitive equilibrium is a particular Pareto optimal allocation of resources

- Welfare weight inversely related to shadow price on individual constraint

- Market prices reflect shadow prices on resources
Agent $i$’s problem is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$c^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s^t) A^i(s_{t+1}|s^t) = y^i(s^t) + A^i(s^t)$$

$$A^i(s^{t+1}) \geq -\bar{A}^i(s^{t+1})$$

- $A^i(s_{t+1}|s^t) > 0 \rightarrow$ saving, $A^i(s_{t+1}|s^t) < 0 \rightarrow$ borrowing
- $\bar{A}^i(s^{t+1})$ is agent $i$’s natural debt limit
In the formulation with Arrow securities, we need to impose some restriction on asset trade to prevent Ponzi schemes.

The *natural debt limit* is the weakest of such restrictions.

It is the value of the maximum amount that agent $i$ can repay starting from the period, assuming consumption is zero forever.

We do not need such a restriction with Arrow-Debreu securities. It is only needed when budget constraints are formulated in a sequential way.
In the setting above, we impose

\[ A_i(s^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}(s^\tau) y^i(s^{\tau+1}) \]

- \( q^{t+1}(s^\tau) \) is the Arrow-Debreu price in units of the time \( t + 1 \) history \( s^{t+1} \) consumption good

- In the setting above, each agent \( i \) faces one no-Ponzi constraints for each state that can occur next period

\[ \rightarrow \text{More detailed remarks on no-Ponzi conditions at the end of the lecture} \]
OPTIMALITY

- The FOC w.r.t $c^i(s^t)$ is

$$\pi(s^t)u'(c^i(s^t)) = \gamma^i(s^t)$$

- The FOC w.r.t $A^i(s^t+1|s^t)$ is

$$\beta \gamma^i(s^{t+1}) = Q(s^t+1|s^t)\gamma^i(s^{t+1})$$

- These condition hold for all $i$ and all $s^t$ and $s^{t+1}$

- $i$ has a Lagrange multiplier for each $s^t$
We can combine the two conditions above and show that the outcome is the same as with Arrow-Debreu securities if

\[ Q^i(s_{t+1} | s^t) = \frac{q^0(s_{t+1})}{q^0(s^t)} \]

We can iterate on this relation to get

\[ q^0(s_{t+1}) = q^0(s^0) \prod_{j=1}^{t} Q(s_j | s_{j-1}) \]
With Arrow securities, decisions can still be thought of as made in period-0

- Re-optimizing would not change allocation because optimal plan is time consistent in this setting

Defining a competitive equilibrium with Arrow securities requires specifying an initial asset distribution

- For example: $A^i(0) = 0 \forall i$
So far we did not require a recursive structure.

We can additionally assume that $s_t$ follows a Markov process.

In that case we can formulate a Bellmann equation with a time-invariant value function.

Above we were able to characterize the full sequence of allocations in a setting that is not necessarily recursive.
ASSET PRICING WITH COMPLETE MARKETS
Take framework from above and normalize $q^0(s^0) = 1$

Use Euler equation (consumption smoothing relation) to solve for the price

$$q^0(s^t) = \beta^t \frac{\pi(s^t)u'(c^i(s^t))}{u'(c^i(s^0))}$$

This is the time-0 price of an asset that pays one unit of consumption in period $t$ if history $s^t$ occurs
After we have priced $q^0(s^t)$, we can price any asset.

For example, a risk-free bond (non-state contingent) in period $t$ after history $s^t$. 

\[
P_{B,t} = R_t - 1 = \sum_{s^t} q^0(s^t + 1|s^t) q^0(s^t)\]
After we have priced $q^0(s^t)$, we can price any asset.

For example, a risk-free bond (non-state contingent) in period $t$ after history $s^t$

$$P_{B,t} = R_t^{-1} = \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) = \sum_{s_{t+1}} \frac{q^0(s_{t+1})}{q^0(s_t)}$$
MORE ASSETS

- Risk-free consol in period 0
MORE ASSETS

- Risk-free consol in period 0

\[ P_0^{\text{consol}} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) \]
MORE ASSETS

- Risk-free consol in period 0

\[ P_{0}^{\text{consol}} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) \]

- Stock with dividend stream \( d(s^t) \)
MORE ASSETS

- Risk-free consol in period 0

\[ P_{0}^{\text{consol}} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) \]

- Stock with dividend stream \( d(s^t) \)

\[ P_{0}^{\text{stock}} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)d(s^t) \]
The consumption plan remains the same whether or not we introduce these additional assets.

Trading in Arrow-Debreu / Arrow securities already allows agents to generate the respective payment streams.
Let us now move to incomplete markets

“Incomplete markets” means that not all state-contingent claims (Arrow-Debreu or Arrow securities) can be traded

Need to specify what are the assets that can be traded
THE LUCAS TREE MODEL

- Based on Lucas (1978): *Asset pricing in an exchange economy*

- Purpose of this model is to price risky assets and to understand the relation to the marginal utility of consumption

- The setting is incomplete markets without heterogeneity

- So now we focus on the other of our two necessary deviations from the representative agent complete markets model
One consumer (or a large number of identical ones)

Preferences: \[ U_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c_\tau) \]

Technology: endowment of one tree that pays fruit \( y_t \) (with Markov property)

Incomplete market structure:

- Can trade shares in the tree, denoted \( a_t \), at price \( p_t \)
- Can trade risk-free bonds \( b_t \), pays gross return \( R_t \) at the beginning of the period
- Cannot trade any other assets: for example, cannot write a contract that makes the payoff related to the fruit of the tree
STATE VARIABLES

► What are the state variables in this economy?
What are the state variables in this economy?

$b_t, a_t, y_t$
BELLMAN EQUATION

\[ V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \left\{ u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1}) \right\} \]

subject to

\[ c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t \]
**BELLMAN EQUATION**

\[ V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \{ u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1}) \} \]

subject to

\[ c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t \]

or

\[ c_t + b_{t+1} + p_t (a_{t+1} - a_t) = a_t y_t + R_t b_t \]
Combine FOCs w.r.t $c_t$ and $b_{t+1}$ to get Euler equation

$$u'(c_t) = \beta R_t \mathbb{E}_t [u'(c_{t+1})]$$
OPTIMALITY

- Combine FOCs w.r.t $c_t$ and $a_{t+1}$ to get asset pricing equation for the tree

$$p_t = \beta \mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + y_{t+1}) \right]$$

- This is a recursive equation in $p_t$

- This is basic the (consumption) CAPM equation
Define the expected asset return as the price appreciation plus the payoff (dividend) relative to the current price:

\[ \mathbb{E}_t R_t^A = \mathbb{E}_t \left[ \frac{p_{t+1} + y_{t+1}}{p_t} \right] \]
Using the optimality conditions and the fact that 
\[ E(XY) = E(X)E(Y) + \text{COV}(X, Y) \] 
to obtain

\[
\mathbb{E}_t R_t^A = R_t - \frac{\text{COV}(R_t^A, \frac{u'(c_{t+1})}{u'(c_t)})}{\mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right]}
\]

The consumer likes assets that give a relatively high return when the relative marginal utility of consumption is high

- Consumer is happy with a lower expected return as long as the return has the “right” correlation with her expected consumption patterns
A competitive equilibrium in this economy is defined as a sequence of allocations \( \{c_t, a_t, b_t\}_{t=0}^{\infty} \) and prices \( \{p_t, R_t\}_{t=0}^{\infty} \) such that for all \( s^t \):

1. Markets clear:

\[
\begin{align*}
    c_t &= y_t \\
    b_t &= 0 \\
    a_t &= 1
\end{align*}
\]

2. Given the price system, the optimality conditions are satisfied
When we impose equilibrium we can obtain asset prices (or relationships between asset prices) as a function of the model primitives.

The general equilibrium asset pricing relationship in the setting above is then:

\[
E_t R_t^A = R_t - \frac{\text{cov}(R_t^A, u'(y_{t+1})/u(y_t))}{E_t \left[ \frac{u'(y_{t+1})}{u'(y_t)} \right]}
\]
Price mechanism provides agent with the illusion of choice: prices adjust so that agent consumes own fruit and does not save/borrow

In this economy, quantities are trivial but prices are not

Share and bond trade “do not matter” for consumption plan
ASSET PRICING APPLICATIONS
HOW TO PRICE ASSETS?

1. Define environment: preferences, technology, market structure

2. Solve agents’ maximization problems (illusion of choice)

3. Only then apply market clearing to get general equilibrium asset pricing relationships
APPLICATION 1: THE TERM STRUCTURE

- From Euler equation

\[
R_{1,t}^{-1} = \beta E_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right]
\]

- Two period bond
APPLICATION 1: THE TERM STRUCTURE

- From Euler equation

\[ R_{1,t}^{-1} = \beta \mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \right] \]

- Two period bond

\[ R_{2,t}^{-1} = \beta^2 \mathbb{E}_t \left[ \frac{u'(c_{t+2})}{u'(c_t)} \right] \]

\[ R_{2,t}^{-1} = \beta^2 \mathbb{E}_t \left[ \frac{u'(c_{t+2})}{u'(c_t)} \frac{u'(c_{t+1})}{u'(c_{t+1})} \right] \]

\[ R_{2,t}^{-1} = \beta \mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right] \]

\[ R_{2,t}^{-1} = \beta \mathbb{E}_t \left[ \frac{u'(c_{t+1})}{u'(c_t)} R_{1,t+1}^{-1} \right] \]
APPLICATION 1: THE TERM STRUCTURE

- If we assume risk neutrality $u(c) = c$, it follows that

$$R_{2,t} = R_{1,t} \mathbb{E}_t R_{1,t+1}$$

- The return on a two-period bond is the expected cumulative return on two one period bonds

- “Expectations hypothesis” result in finance

- Does not hold in practice, but useful to think about empirically observed yield curve in deviations from this benchmark (term premium, risk premium)
APPLICATION 1: THE TERM STRUCTURE

U.S. Treasury Yield Curves

<table>
<thead>
<tr>
<th>Term</th>
<th>Three Month</th>
<th>Two Year</th>
<th>Five Year</th>
<th>Ten Year</th>
<th>Thirty Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest Rate %</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Red: July 10, 2000
- Orange: October 22, 2007
- Green: February 15, 2011
- Blue: January 31, 2018

Source: Federal Reserve Economic Data (FRED)
APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- Use Lucas’ model asset pricing relationship with CRRA preferences

\[
E_t R_t^A = R_t - \frac{cov \left( R_t^A, \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \right)}{E_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \right]}
\]

- If we think of \( R_t^A \) as the return on US equity, we can compute from the data whether this relation makes sense in practice
APPLICATION 2: THE EQUITY PREMIUM PUZZLE

▶ In the data

\[
\begin{align*}
R_t^A & \approx 1.07 \\
R_t & \approx 1.01 \\
\frac{c_{t+1}}{c_t} & \approx 1.02
\end{align*}
\]

▶ Pick ex-ante admissible values for \( \sigma \) and use the data to compute LHS and RHS of the above relation. Does the equation hold?

▶ Not at all: this is the insight of Mehra and Prescott (1985)
APPLICATION 2: THE EQUITY PREMIUM PUZZLE

R. Mehra and E.C. Prescott, The equity premium

![Diagram](image)

Fig. 4. Set of admissible average equity risk premia and real returns.
APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- Proposed solutions to the equity premium puzzle
  - Separate IES from risk aversion (e.g. Epstein-Zin preferences)
  - Habits in consumption → Campbell and Cochrane (1999)
  - Rare disasters → Barro (2006)
  - Behavioural finance (Thaler and others)
  - Long-run risk → Bansal and Yaron (2004)

- See also Mehra (2007) for a summary paper
The risky assets we have priced above were endowment technologies.

We can also do asset pricing in the presence of production technology.

In particular, we can price productive capital that is installed in a firm.

This is the “Q” theory of investment.
Q is a relative price, the market value of capital (the value “inside the firm”) relative to the replacement value

In a neoclassical model, Q is the shadow price on the investment accumulation equation (see e.g. Hayashi, 1982)

Important: marginal Q and average Q are different

Deriving and interpreting Q formally will be part of your second assignment
APPLICATION 3: PRICING PRODUCTIVE CAPITAL

- In theory, a firms’ marginal Q should be the sole predictor of investment
- In the data, marginal Q is hard to measure
- In the data, other determinants matter a lot for firm investment
  - For example financial constraints!
  - Plenty of empirical research, going back to Fazzari, Hubbard, and Petersen (1988)
- See also the literature on dynamic corporate finance, as surveyed by Strebulaev and Whited (2011)
  - Neoclassical models of various firm decision margins
A RECENT APPLICATION

- Crouzet and Eberly (2020) construct a decomposition of the gap between valuation ($\approx$ average Q) and investment ($\approx$ marginal Q) in the US.
TAKING STOCK
We have built a benchmark model with complete markets.

We have priced assets and started introducing market incompleteness.

The next lecture will start putting the ingredients together: Incomplete markets, heterogeneous agents and precautionary savings.
EXTRA MATERIAL ON NO-PONZI VS. TRANSVERSALITY CONDITIONS
Ruling out Ponzi schemes is

1. A commonsense requirement economically

2. In many settings a formal restriction that ensures the existence of a solution, as it bounds utility

Above, we have formalized the no-Ponzi condition as a state-by-state inequality, using Arrow-Debreu prices

Typically, no-Ponzi conditions are expressed as limits

No-Ponzi conditions are used to consolidate budget constraints

No-Ponzi conditions are conceptually different from transversality conditions
NO-PONZI CONDITIONS

- Suppose the following:
  - Agent starts with $b_0$
  - Has sequential budget constraint $c_t + b_{t+1} = Rb_t$
  - A candidate solution to her problem is $\{c_t^*\}_{t=0}^{\infty}$

- Without additional constraints, the agent could
  1. Choose $\tilde{c}_0 = c_0^* + 1$ and $\tilde{b}_1 = b_1^* - 1$
  2. For $t \geq 1$ choose $\tilde{c}_t = c_t^*$ and $\tilde{b}_{t+1} = b_{t+1}^* - R^t$

- This strategy satisfies the period-by-period constraint

- It is possible for any $c_0^*$, so there is no finite solution!
NO-PONZI CONDITIONS

- This situation is ruled out by adding the condition

\[
\lim_{t \to \infty} \frac{b_t}{R^t} \geq 0
\]

- This means that “terminal” asset holdings cannot be negative

- In the presence of this condition the agent cannot choose a solution that implies unbounded consumption/utility

- There can of course be stronger restrictions that make the no-Ponzi condition redundant, for example

\[
b_t \geq 0 \ \forall t
\]
CONSOLIDATING THE BUDGET CONSTRAINT

- Start with constraint in period 0 and iterate to period $T$

$$\sum_{t=0}^{T} \left( c_t \frac{1}{R^t} \right) + \frac{b_{T+1}}{R^T} = Rb_0$$

- Doing this until infinity gives

$$\lim_{T \to \infty} \left\{ \sum_{t=0}^{T} \left( c_t \frac{1}{R^t} \right) + \frac{b_{T+1}}{R^T} \right\} = Rb_0$$

$$\sum_{t=0}^{\infty} \left( c_t \frac{1}{R^t} \right) + \lim_{T \to \infty} \frac{b_{T+1}}{R^T} = Rb_0$$

- Together with no-Ponzi condition, this gives

$$\sum_{t=0}^{\infty} \left( c_t \frac{1}{R^t} \right) \leq Rb_0$$
Transversality conditions ensure the *sufficiency* of a solution.

We do not impose the transversality condition on the agent, but the agent will require this condition as part of her solution.

It is a prescription how to behave optimally, given a choice set.

It is needed because the solution to second-order difference require an initial and a terminal condition.

Recall the remarks in Lecture 2.
Think of an Euler equation with substituted budget constraint:

\[ u'(Rb_t - b_{t+1}) = \beta Ru'(Rb_{t+1} - b_{t+2}) \]

This is a second order difference equation in \( b_t \).

The full solution to the agent’s problem requires an initial condition (\( b_0 \) given) and the transversality condition:

\[ \lim_{t \to \infty} b_t R^t \leq 0 \]

This says that the agent does not want to have savings in the limit.

It turns out this the same equation as the no-Ponzi condition, but with the weak inequality going the other way.
No-Ponzi vs. Transversality

- **No-Ponzi condition**
  - Ensures existence (bounds utility)
  - Imposed on agent's program
  - Only needed in competitive solution

- **Transversality condition**
  - Ensures sufficiency (optimality)
  - Comes out as part of agent's program's solution
  - Part of competitive and planner solution
Bibliography


