ECON 747 – LECTURE 4:
FROM COMPLETE TO INCOMPLETE MARKETS

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OVERVIEW OF THIS LECTURE

1. Why start with a complete markets model?
2. Consumption and risk sharing with complete markets
3. Asset pricing with complete markets
4. The Lucas Tree model
5. Asset pricing applications

+ Extra material on no-Ponzi vs. transversality conditions

PURPOSE OF THIS LECTURE

- Acquire tools that we need throughout the course (e.g. how to price assets)
- Build a benchmark model that models with financial frictions will deviate from
- Remember the two deviations from a complete markets representative agent model that are needed to meaningfully introduce financial frictions
  1. Incomplete markets
  2. Heterogeneous agents
- We will see today that one of them is not enough: (certain) heterogeneity between agents can be insured away if markets are complete

- If you understand the complete markets benchmark, you will realize some important aspects of many macro models
- Some examples ...

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- A representative agent model actually relies on **lots of** trade in financial assets!
- Why?
- The representative agent model is isomorphic to a model in which a continuum of heterogeneous agents trade insurance (state-contingent claims) to eliminate all idiosyncratic risk!

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- Models with financial frictions often feature several layers of market incompleteness
- Example of Kiyotaki-Moore style collateral constraint

$$b_t \leq \theta k_t$$

- Usually <u>two</u> types of market incompleteness!
  1. Bond is not state contingent
  2. Bond trade is limited by physical asset

  (great explanation in Cao and Nie, 2017)

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- In many models, some asset markets are complete, others are incomplete
- Example 1: friction in asset trade between two countries, complete markets (full insurance) within each country
- Example 2: friction in asset trade between households and firms, complete markets (full insurance) within household sector and within firm sector

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- You may occasionally see in a seminar that someone has actually made some asset markets incomplete without noticing
- Someone in the audience will say something like: "you must be assuming that agents cannot insure away this problem"
- Lesson: always remind yourself about the market structure you are operating in, be conscious about everything you are (implicitly) assuming about asset trade

References for lectures 4 and 5

- This lecture and the next one are pretty close to some chapters in the Ljungqvist and Sargent text book *Recursive Macroeconomic Theory* (2nd edition)
  - Lecture 4: Chapters 8, 12, 13
  - Lecture 5: Chapters 16, 17
- The notation will be similar to theirs
- I will provide a few additional references throughout

CONSUMPTION AND RISK SHARING WITH COMPLETE MARKETS

ENVIRONMENT

- Households indexed by $i = 1, ..., I$
- Preferences:

$$U_t^i = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c_\tau^i)$$

- Technology: exogenous endowment y_t^i in period t
- Denote state of the world s_t
  - State is vector of realizations $\{y_t^1, ..., y_t^I\}$
  - $\pi(s^t)$ is the probability of a history of states s^t
  - $\pi(s_{t+1}|s^t)$ is the probability of state s_{t+1} given history s^t

COMMENT ON NOTATION

- We make the state s_t explicit in the notation
- This is because we will work with state-contingent contracts
- But this model is not different from what we have seen so far: in the model of the previous two lectures, a state is a realization of (Z_t, V_t)

TWO WAYS TO FORMULATE COMPLETE MARKERS

1. **Arrow-Debreu securities**
   - Contingent claims traded in period-0, exercised every period
   - $q^0(s^t)$: time-0 price of asset that pays 1 unit of consumption if history s^t realizes
   - $a^i(s^t)$: agent i's holdings of this asset

2. **Arrow securities**
   - One-period ahead contingent claims
   - $Q(s_{t+1}|s^t)$: history-s^t price of asset that pays 1 unit of consumption if state s_{t+1} realizes next period
   - $A^i(s_{t+1}|s^t)$: agent i's holdings of this asset

contingent = every state can be contracted on

ARROW-DEBREU SECURITIES, MARKET OUTCOME

- The time-0 problem of the agent i is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$\begin{aligned} c^i(s^t) &= y^i(s^t) + a^i(s^t) \\ \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) a^i(s^t) &= 0 \end{aligned}$$

- Interpretation of last constraint: buy and sell claims to a clearing house in period 0, remain with zero balance

- Substitute out $a^i(s^t)$ and denote μ^i the Lagrange multiplier on the combined constraint

- The FOC w.r.t $c^i(s^t)$ is

$$\beta^t \pi(s^t) u'(c^i(s^t)) = q^0(s^t)\mu^i$$

- This condition holds for all i and all s^t

COMPETITIVE EQUILIBRIUM DEFINITION

- Define $c(s^t) = \sum_i c^i(s^t)$ and $y(s^t) = \sum_i y^i(s^t)$
- A **competitive equilibrium** in this economy is defined as a sequence of allocations $\{c(s^t)\}_{t=0}^{\infty}$ and prices $\{q^0(s^t)\}_{t=0}^{\infty}$ such that for all s^t:
  1. Markets clear: $c(s^t) = y(s^t)$
  2. Given the price system, the optimality conditions are satisfied

IMPLICATIONS OF COMPLETE MARKETS

1. Perfect risk sharing
2. Consumption smoothing

- Combine the optimality condition for two agents i and j

$$\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\mu^i}{\mu^j} \quad \forall i, j, s^t$$

PERFECT RISK SHARING

- Solve for $c^i(s^t)$ and sum over i

$$c^i(s^t) = u'^{-1}\left\{\frac{\mu^i}{\mu^j}u'(c^j(s^t))\right\}$$

$$\sum_i c^i(s^t) = \sum_i u'^{-1}\left\{\frac{\mu^i}{\mu^j}u'(c^j(s^t))\right\}$$

- From market clearing, $\sum_i c^i(s^t) = y(s^t)$
- Therefore, we get an optimality condition for j's consumption that only depends on the aggregate endowment!
- Idiosyncratic risk is insured away

CONSUMPTION SMOOTHING

- Combine the optimality condition for two states s^t and s^τ

$$\beta^{t-\tau}\frac{\pi(s^t)u'(c^i(s^t))}{\pi(s^\tau)u'(c^i(s^\tau))} = \frac{q^0(s^t)}{q^0(s^\tau)} \quad \forall i, s^t, s^\tau$$

- Desire to equate marginal utility of consumption across time and states, attenuated by probabilities and prices ($MRS = MRT$)

PLANNER SOLUTION

- Planner problem in this economy (no prices!)

$$\max \sum_i \omega^i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$\sum_i c^i(s^t) = \sum_i y^i(s^t) \quad \forall s^t$$

- ω^i is the welfare weight on agent i

OPTIMALITY

- Planner FOC w.r.t. $c^i(s^t)$ is

$$\beta^t \omega^i \pi(s^t) u'(c^i(s^t)) = \lambda(s^t) \quad \forall s^t$$

- $\lambda(s^t)$, the Lagrange multiplier on the resource constraint, is a function of s^t

MARKET VS. PLANNER OUTCOME

- The above condition is the same as the market FOC if

$$\begin{aligned} \omega^i &= (\mu^i)^{-1} \\ \lambda(s^t) &= q^0(s^t) \end{aligned}$$

- The competitive equilibrium is *a particular* Pareto optimal allocation of resources
    - Welfare weight inversely related to shadow price on individual constraint
- Market prices reflect shadow prices on resources

ARROW SECURITIES, MARKET OUTCOME

- Agent i's problem is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$\begin{aligned} c^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s^t) A^i(s_{t+1}|s^t) &= y^i(s^t) + A^i(s^t) \\ A^i(s^{t+1}) &\geq -\bar{A}^i(s^{t+1}) \end{aligned}$$

- $A^i(s_{t+1}|s^t) > 0 \rightarrow$ saving, $A^i(s_{t+1}|s^t) < 0 \rightarrow$ borrowing
- $\bar{A}^i(s^{t+1})$ is agent i's natural debt limit

THE NATURAL DEBT LIMIT

- In the formulation with Arrow securities, we need to impose some restriction on asset trade to prevent Ponzi schemes
- The *natural debt limit* is the weakest of such restrictions
- It is the value of the maximum amount that agent i can repay starting form the period, assuming consumption is zero forever
- We do not need such a restriction with Arrow-Debreu securities. It is only needed when budget constraints are formulated in a sequential way.

THE NATURAL DEBT LIMIT

- In the setting above, we impose

$$\bar{A}^i(s^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}(s^\tau) y^i(s^\tau)$$

- $q^{t+1}(s^\tau)$: Arrow-Debreu price in units of the state s^{t+1} consumption good

- RHS of above equation: maximum value that agent i can repay, assuming that her consumption is zero from $t+1$ onwards

- In the setting above, each agent i faces one no-Ponzi constraints for each state that can occur next period

  $\rightarrow$ More detailed remarks on no-Ponzi conditions at the end of the lecture

OPTIMALITY

- The FOC w.r.t $c^i(s^t)$ is

$$\pi(s^t)u'(c^i(s^t)) = \gamma^i(s^t)$$

- The FOC w.r.t $A^i(s_{t+1}|s^t)$ is

$$\beta\gamma^i(s^t) = Q(s_{t+1}|s^t)\gamma^i(s^{t+1})$$

- These condition hold for all i and all s^t and s^{t+1}
- i has a Lagrange multiplier for each s^t

OPTIMALITY

- We can combine the two conditions above and show that the outcome is the same as with Arrow-Debreu securities (see slide 18) if

$$Q(s_{t+1}|s^t) = \frac{q^0(s^{t+1})}{q^0(s^t)}$$

- We can iterate on this relation to get

$$q^0(s^{t+1}) = q^0(s^0) \prod_{j=1}^{t} Q(s_j|s^{j-1})$$

SOME NOTES

- With Arrow securities, decisions can still be thought of as made in period-0
    - Re-optimizing would not change allocation because optimal plan is time consistent in this setting
- Defining a competitive equilibrium with Arrow securities requires specifying an initial asset distribution
    - For example: $A^i(0) = 0 \; \forall i$

RECURSIVENESS

- So far we did not require a recursive structure
- We can additionally assume that s_t follows a Markov process
- In that case we can formulate a Bellmann equation with a time-invariant value function
- Above we were able to characterize the full sequence of allocations in a setting that is not necessarily recursive

ASSET PRICING WITH COMPLETE MARKETS

PRICING ASSETS WITH COMPLETE MARKETS

- Take framework from above and normalize $q^0(s^0) = 1$
- Use Euler equation (consumption smoothing relation) to solve for the price

$$q^0(s^t) = \beta^t \frac{\pi(s^t)u'(c^i(s^t))}{u'(c^i(s^0))}$$

- This is the time-0 price of an asset that pays one unit of consumption in period t if history s^t occurs

PRICING SYNTHETIC ASSETS

- After we have priced $q^0(s^t)$, we can price any asset
- For example, a risk-free bond (non-state contingent) in period t after history s^t

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- For example, a risk-free bond (non-state contingent) in period t after history s^t

$$P_{B,t} = R_t^{-1} = \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) = \sum_{s_{t+1}} \frac{q^0(s_{t+1})}{q^0(s_t)}$$

MORE ASSETS

- Risk-free consol in period 0

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$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

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- Risk-free consol in period 0

$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

- Stock with dividend stream $d(s^t)$

MORE ASSETS

- Risk-free consol in period 0

$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

- Stock with dividend stream $d(s^t)$

$$P_0^{stock} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) d(s^t)$$

IMPORTANT NOTE

- The consumption plan remains the same whether or not we introduce these additional assets
- Trading in Arrow-Debreu / Arrow securities already allows agents to generate the respective payment streams

THE LUCAS TREE MODEL

- Based on Lucas (1978) : *Asset pricing in an exchange economy*
- Purpose of this model is to price risky assets and to understand the relation to the marginal utility of consumption
- The setting features a representative agent
  - Idiosyncratic risk has been insured away via complete markets

ENVIRONMENT

- One consumer (or a large number of identical ones)
- Preferences: $U_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c_\tau)$
- Technology: endowment of one tree that pays fruit y_t (with Markov property)
- Price the following assets:
  - Shares in the tree, denoted a_t, at price p_t
  - Risk-free bonds b_t, pays gross return R_t at the beginning of the period

STATE VARIABLAES

- What are the state variables in this economy?

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  b_t, a_t, y_t

BELLMAN EQUATION

$$V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1})\}$$

subject to

$$c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t$$

BELLMAN EQUATION

$$V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1})\}$$

subject to

$$c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t$$

or

$$c_t + b_{t+1} + \boldsymbol{p_t(a_{t+1} - a_t)} = a_t y_t + R_t b_t$$

OPTIMALITY

- Combine FOCs w.r.t c_t and b_{t+1} to get Euler equation

$$u'(c_t) = \beta R_t \mathbb{E}_t \left[u'(c_{t+1}) \right]$$

OPTIMALITY

- Combine FOCs w.r.t c_t and a_{t+1} to get asset pricing equation for the tree

$$p_t = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + y_{t+1}) \right]$$

- This is a recursive equation in p_t
- This is basic the (consumption) CAPM equation

RISK PREMIUM

- Define the expected asset return as the price appreciation plus the payoff (dividend) relative to the current price

$$\mathbb{E}_t R_t^A = \mathbb{E}_t \left[\frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

RISK PREMIUM

- Using the optimality conditions and the fact that $E(XY) = E(X)E(Y) + COV(X,Y)$ to obtain

$$\mathbb{E}_t R_t^A = R_t - \frac{cov(R_t^A, \frac{u'(c_{t+1})}{u'(c_t)})}{\mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)}\right]}$$

- The consumer likes assets that give a relatively high return when the relative marginal utility of consumption is high
  - Consumer is happy with a lower expected return as long as the return has the "right" correlation with her expected consumption patterns

EQUILIBRIUM

- A **competitive equilibrium** in this economy is defined as a sequence of allocations $\{c_t, a_t, b_t\}_{t=0}^{\infty}$ and prices $\{p_t, R_t\}_{t=0}^{\infty}$ such that for all s^t:

  1. Markets clear:

$$\begin{aligned} c_t &= y_t \\ b_t &= 0 \\ a_t &= 1 \end{aligned}$$

  2. Given the price system, the optimality conditions are satisfied

GE ASSET PRICING

- When we impose equilibrium we can obtain asset prices (or relationships between asset prices) as a function of the model primitives

- The general equilibrium asset pricing relationship in the setting above is then

$$\mathbb{E}_t R_t^A = R_t - \frac{cov(R_t^A, \frac{u'(y_{t+1})}{u(y_t)})}{\mathbb{E}_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right]}$$

THE ILLUSION OF CHOICE

- Price mechanism provides agent with the illusion of choice: prices adjust so that agent consumes own fruit and does not save/borrow
- In this economy, quantities are trivial but prices are not
- Share and bond trade “do not matter” for consumption plan

ASSET PRICING APPLICATIONS

HOW TO PRICE ASSETS?

1. Define environment: preferences, technology, market structure
2. Solve agents' maximization problems (illusion of choice)
3. Only then apply market clearing to get general equilibrium asset pricing relationships

APPLICATION 1: THE TERM STRUCTURE

- From Euler equation

$$R_{1,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

- Two period bond

APPLICATION 1: THE TERM STRUCTURE

- From Euler equation

$$R_{1,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

- Two period bond

$$\begin{aligned}
R_{2,t}^{-1} &= \beta^2 \mathbb{E}_t \left[\frac{u'(c_{t+2})}{u'(c_t)} \right] \\
R_{2,t}^{-1} &= \beta^2 \mathbb{E}_t \left[\frac{u'(c_{t+2})}{u'(c_t)} \frac{u'(c_{t+1})}{u'(c_{t+1})} \right] \\
R_{2,t}^{-1} &= \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right] \\
R_{2,t}^{-1} &= \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} R_{1,t+1}^{-1} \right]
\end{aligned}$$

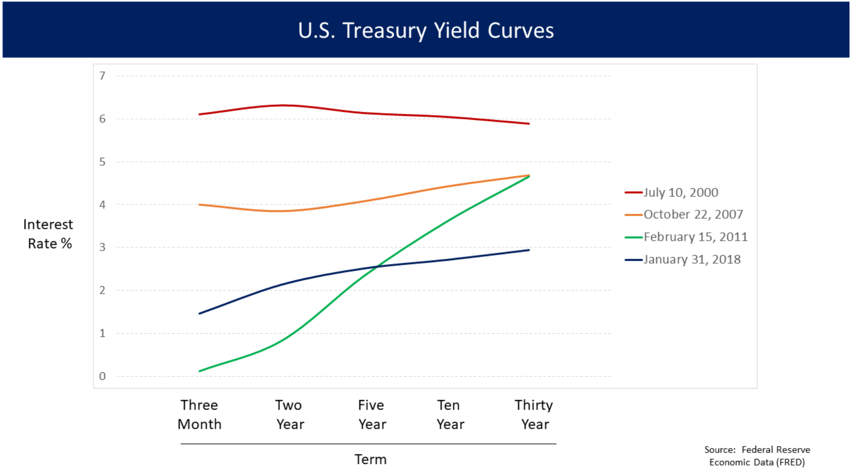
APPLICATION 1: THE TERM STRUCTURE

- If we assume risk neutrality $u(c) = c$, it follows that

$$R_{2,t} = R_{1,t}\mathbb{E}_t R_{1,t+1}$$

- The return on a two-period bond is the expected cumulative return on two one period bonds
- "Expectations hypothesis" result in finance
- Does not hold in practice, but useful to think about empirically observed yield curve in deviations from this benchmark (term premium, risk premium)

APPLICATION 1: THE TERM STRUCTURE



APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- Use Lucas' model asset pricing relationship with CRRA preferences

$$\mathbb{E}_t R_t^A = R_t - \frac{cov\left(R_t^A, \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)}{\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right]}$$

- If we think of R_t^A as the return on US equity, we can compute from the data whether this relation makes sense in practice

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- In the data

$$\begin{aligned} R_t^A &\approx 1.07 \\ R_t &\approx 1.01 \\ \frac{c_{t+1}}{c_t} &\approx 1.02 \end{aligned}$$

- Pick ex-ante admissible values for σ and use the data to compute LHS and RHS of the above relation. Does the equation hold?

- Not at all: this is the insight of Mehra and Prescott (1985)

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

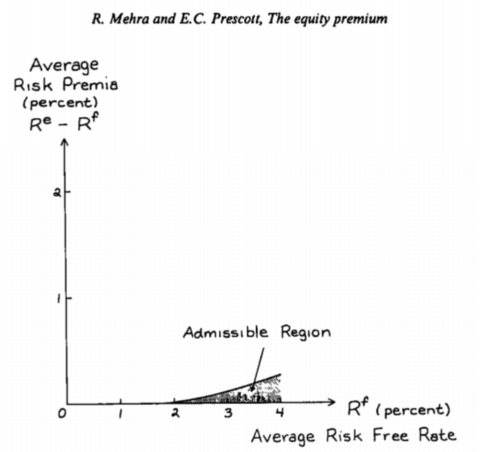


Fig. 4. Set of admissible average equity risk premia and real returns.

- Proposed solutions to the equity premium puzzle
  - Separate IES from risk aversion (e.g. Epstein-Zin preferences)
  - Habits in consumption $\rightarrow$ Campbell and Cochrane (1999)
  - Rare disasters $\rightarrow$ Barro (2006)
  - Behavioural finance (Thaler and others)
  - Long-run risk $\rightarrow$ Bansal and Yaron (2004)
- See also Mehra (2007) for a summary paper

APPLICATION 3: PRICING PRODUCTIVE CAPITAL

- The risky assets we have priced above were endowment technologies
- We can also do asset pricing in the presence of production technology
- In particular, we can price productive capital that is installed in a firm
- This is the "Q" theory of investment

APPLICATION 3: PRICING PRODUCTIVE CAPITAL

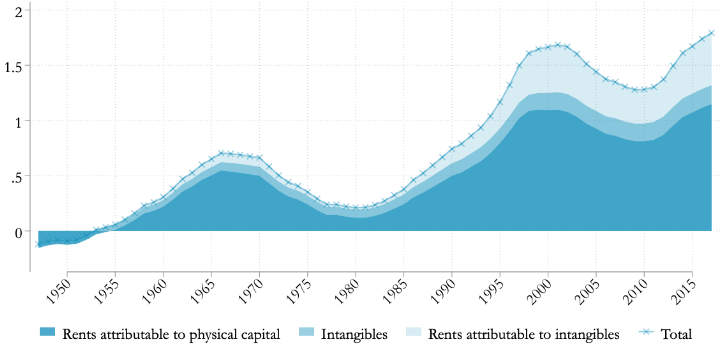
- Q is a relative price, the market value of capital (the value "inside the firm") relative to the replacement value
- In a neoclassical model, Q is the shadow price on the investment accumulation equation (see e.g. Hayashi, 1982)
- Important: *marginal Q* and *average Q* are different
- **Deriving and interpreting Q formally will be part of your second assignment**

Application 3: Pricing Productive Capital

- In theory, a firms' marginal Q should be the sole predictor of investment
- In the data, marginal Q is hard to measure
- In the data, other determinants matter a lot for firm investment
  - For example financial constraints!
  - Plenty of empirical research, going back to Fazzari, Hubbard, and Petersen (1988)
- See also the literature on *dynamic corporate finance*, as surveyed by Strebulaev and Whited (2011)
  - Neoclassical models of various firm decision margins

A RECENT APPLICATION

- Crouzet and Eberly (2020) construct a decomposition of the gap between valuation ($\approx$ average Q) and investment ($\approx$ marginal Q) in the US



TAKING STOCK

TAKING STOCK

- We have build a benchmark model with complete markets
- We have priced assets
- The next lecture will start putting our necessary ingredients together: Incomplete markets, heterogeneous agents and precautionary savings

EXTRA MATERIAL ON NO-PONZI VS. TRANSVERSALITY CONDITIONS

NO-PONZI CONDITIONS

- Ruling out Ponzi schemes is
  1. A commonsense requirement economically
  2. In many settings a formal restriction that ensures the existence of a solution, as it *bounds utility*
- Above, we have formalized the no-Ponzi condition as a state-by-state inequality, using Arrow-Debreu prices
- Typically, no-Ponzi conditions are expressed as limits
- No-Ponzi conditions are used to consolidate budget constraints
- No-Ponzi conditions are conceptually different from transversality conditions

NO-PONZI CONDITIONS

- Suppose the following:
  - Agent starts with b_0
  - Has sequential budget constraint $c_t + b_{t+1} = Rb_t$
  - A candidate solution to her problem is $\{c_t^*\}_{t=0}^{\infty}$
- Without additional constraints, the agent could
  1. Choose $\tilde{c}_0 = c_0^* + 1$ and $\tilde{b}_1 = b_1^* - 1$
  2. For $t \geq 1$ choose $\tilde{c}_t = c_t^*$ and $\tilde{b}_{t+1} = b_{t+1}^* - R^t$
- This strategy satisfies the period-by-period constraint
- It is possible for *any* c_0^*, so there is no finite solution!

NO-PONZI CONDITIONS

- This situation is ruled out by adding the condition

$$\lim_{t\to\infty} \frac{b_t}{R^t} \geq 0$$

- This means that "terminal" asset holdings cannot be negative
- In the presence of this condition the agent cannot choose a solution that implies unbounded consumption/utility
- There can of course be stronger restrictions that make the no-Ponzi condition redundant, for example

$$b_t \geq 0 \;\; \forall t$$

CONSOLIDATING THE BUDGET CONSTRAINT

- Start with constraint in period 0 and iterate to period T

$$\sum_{t=0}^{T}\left(c_t \frac{1}{R^t}\right) + \frac{b_{T+1}}{R^T} = Rb_0$$

- Doing this until infinity gives

$$\lim_{T\to\infty}\left\{\sum_{t=0}^{T}\left(c_t \frac{1}{R^t}\right) + \frac{b_{T+1}}{R^T}\right\} = Rb_0$$

$$\sum_{t=0}^{\infty}\left(c_t \frac{1}{R^t}\right) + \lim_{T\to\infty}\frac{b_{T+1}}{R^T} = Rb_0$$

- Together with no-Ponzi condition, this gives

$$\sum_{t=0}^{\infty}\left(c_t \frac{1}{R^t}\right) \leq Rb_0$$

TRANSVERSALITY CONDITIONS

- Transversality conditions ensure the *sufficiency* of a solution
- We do not impose the transversality condition on the agent, but the agent will require this condition as part of her solution
- It is a prescription how to behave optimally, given a choice set
- It is needed because the solution to second-order difference require an initial and a terminal condition
  - Recall the remarks in Lecture 2

TRANSVERSALITY CONDITIONS

- Think of an Euler equation with substituted budget constraint:

$$u'(Rb_t - b_{t+1}) = \beta R u'(Rb_{t+1} - b_{t+2})$$

- This is a second order difference equation in b_t
- The full solution to the agent's problem requires an initial condition (b_0 given) and the transversality condition

$$\lim_{t \to \infty} \frac{b_t}{R^t} \leq 0$$

- This says that the agent does not want to have savings in the limit
- It turns out this the same equation as the no-Ponzi condition, but with the weak inequality going the other way

NO-PONZI VS. TRANSVERSALITY

- **No-Ponzi condition**
  - Ensures existence (bounds utility)
  - Imposed on agent's program
  - Only needed in competitive solution
- **Transversality condition**
  - Ensures sufficiency (optimality)
  - Comes out as part of agent's program's solution
  - Part of competitive and planner solution

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