# ECON 747 – LECTURE 6: MODELS WITH CONSTRAINTS ON (RISK-FREE) DEBT

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# OVERVIEW

- We now turn to one of the two canonical classes of financial frictions that will be at the core of this course
- The basic idea is that because of *limited enforcement*, the amount of debt/savings flows between agents is limited
- This leads to feedback effects between debt and real activity:
  - Financial variables "matter"
- The other class of financial frictions that we will study in a few weeks time are *costly state verification* (CSV) frictions, which give rise to risky debt contracts

# OVERVIEW

- The models we are about to get to know feature the two ingredients that are required for financial frictions to affect allocations
- Typically, the **market incompleteness** will be that
  1. Only non-state contingent bonds are available to the agents
  2. The amount of debt is limited by an endogenous variable (e.g. an asset which serves as collateral)
- Typically, **the heterogeneity between agents** will be relatively simple, for example two agents out of which one is patient and one is impatient

## PLAN FOR NEXT FEW WEEKS

- We will look at this class of models in a big topical block, from a variety of angles
  - Original paper of Kiyotaki and Moore (1997)
  - Microfoundation for borrowing constraints:
    Limited contract enforcement and Hart and Moore's (1994) theory of debt
  - Issues and limitations of the Kiyotaki-Moore environment
  - Plenty of recent research centering around borrowing constraints:
    - Applications to households, firms, financial institutions, countries, ...
    - Variations of the constraint: multi-period debt, occasionally binding, earnings-based vs. asset-based, ...

# 1. KIYOTAKI AND MOORE (1997)

# THE MOTIVATION

- The paper is a "*theoretical study into how credit constraints interact with aggregate economic activity over the business cycle*"
- Main idea: make **credit limit endogenous** to real variables, so that production and credit flows are linked and can amplify one another
- Main insight: very strong endogenous amplification $\rightarrow$ small shock, but large and persistent response of the real activity

# HOW THEIR PAPER IS STRUCTURED

- First a baseline model
  - Lots of stark assumption, e.g. fixed aggregate capital
  - Equilibrium can be characterized very well analytically (and graphically), dynamics can be traced out with a simple log-linearization
- Then a more detailed model version in which various assumptions are relaxed
- Today's lecture will cover the baseline model version in detail and provide a concise summary of the full version

# SETTING

- Discrete time
- Continuum of agents, live infinitely
  - Farmers (population of 1)
  - Gatherers (population of $m$)
- Two goods
  - Durable asset ("land"): fixed supply of $\bar{K}$
  - Nondurable good ("fruit"): cannot be stored
- Both types of agents produce and eat fruit
- Both types of agents are risk neutral

# PREFERENCES

- Farmers

$$\mathbb{E}_t \left( \sum_{s=0}^{\infty} \beta^s x_{t+s} \right)$$

- Gatherers

$$\mathbb{E}_t \left( \sum_{s=0}^{\infty} \beta'^s x'_{t+s} \right)$$

- $x_{t+s}$ and $x'_{t+s}$ denote consumption

# ASSUMPTION 1

$$\beta < \beta'$$

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$$\beta < \beta'$$

- This means that gatherers are more patient than farmers (they place a higher weight on utility from consumption in the future relative to the present)
- In equilibrium, farmers will borrow and gatherers will save
- This is a simple (ex-ante) heterogeneity
- Quite typical in this class of models
- There are related assumption with the similar equilibrium consequences: e.g. one agent dies every period with an exogenous probability, which also renders the agent less patient

# TECHNOLOGY: FARMERS

- Farmers produce fruit using land with CRS technology

$$y_{t+1} = F(k_t) = (a+c)k_t$$

- $ak_t$ can be consumed or traded
- $ck_t$ is bruised and can only be consumed by the farmer
- Therefore the maximum savings rate of farmers is $\frac{a}{a+c}$
- Note also the timing assumption on capital here (would be important for Dynare)

# ASSUMPTION 2

$$
\begin{aligned}
\frac{a}{a+c} &< \beta \\
\Leftrightarrow \quad c &> \left(\frac{1}{\beta} - 1\right) a
\end{aligned}
$$

- Maximum savings rate smaller than discount factor
- Ensures that in equilibrium the farmers will not want to consume more than the bruised fruit, and will want to use all the tradable fruit for investment

# FURTHER ASSUMPTIONS

- Two further critical assumptions about farmers

  1. Once production has started, only the farmer has the skill for the land to bear a fruit. This means that if between $t$ and $t+1$ she decides to withdraw her labor, there is no fruit $(y_{t+1})$ at $t+1$, only land $(k_t)$ remains
  2. The farmers have freedom to withdraw their labor, cannot pre-commit to work $\rightarrow$ *Inalienability of human capital* (Hart and Moore, 1994)

# INALIENABLE HUMAN CAPITAL

- The assumptions above will be the reason that there is a borrowing constraint
- We will see this further below, when we combine the asset market structure with these assumptions
- The second part of this lecture will elaborate in more detail on the **microfoundation** of borrowing constraints

# TECHNOLOGY: GATHERERS

- Gatherers have production function

$$y'_{t+1} = G(k'_t)$$

- It is assumed that $G' > 0, G'' < 0, G'(0) > aR > G'(\bar{K}/m)$
- Gatherers' production does not require any specific skill and the fruit they produced is fully tradable
- The last inequality ensures that both farmers and gatherers produce in equilibrium

# MARKETS

- Land market: at date $t$, land can be exchanged for fruit at price $q_t$ (the numérarie in this economy is the fruit)
- Financial market: one-period bonds, not state contingent, interest rate $R_t$
  - We will see that in equilibrium, interest rate will equal the gatherers' time preference: $R_t = 1/\beta' = R$
  - You should already be able to see why

# ASSET MARKETS: BORROWING CONSTRAINT

- Farmers' human capital is inalienable $\Rightarrow$ if a farmer has a lot of debt, she can threaten creditors to withdraw labor and repudiate the debt contract
- Creditors protect themselves: collateralize farmer's land
- Important: without farmer's labor, land does not yield the fruit, its *liquidation value* is therefore below the *inside value*
  - Remember Tobin's "Q"
- Creditors will only lend up to the liquidation value

(More details on this reasoning later in this lecture)

# ASSET MARKETS: BORROWING CONSTRAINT

- Since creditors know of the possibility of debt repudiation by the farmer, they do not allow the size of debt (gross of interest) to be above the value of collateral

$$Rb_t \leq q_{t+1}k_t$$

# PUTTING THINGS TOGETHER

- Farmer's program

$$\max \mathbb{E}_t \left( \sum_{s=0}^{\infty} \beta^s x_{t+s} \right)$$

subject to

$$q_t(k_t - k_{t-1}) + Rb_{t-1} + x_t - ck_{t-1} = ak_{t-1} + b_t$$

$$Rb_t \leq q_{t+1} k_t$$

# PUTTING THINGS TOGETHER

- Gatherer's program

$$\max \mathbb{E}_t \left( \sum_{s=0}^{\infty} \beta'^s x'_{t+s} \right)$$

subject to

$$q_t(k'_t - k'_{t-1}) + Rb'_{t-1} + x'_t = G(k'_{t-1}) + b'_t$$

# EQUILIBRIUM

- A sequence of prices $\{q_t, R_t\}_{t=0}^{\infty}$ and allocations $\{x_t, x'_t, y_t, y'_t, k_t, k'_t, b_t, b'_t\}_{t=0}^{\infty}$, such that:

  1. Farmers solve their maximization program
  2. Gatherers solve their maximization program
  3. Land, fruit and debt markets clear

  (Details on market clearing further below)

# EQUILIBRIUM CHARACTERIZATION

- To solve for the equilibrium, we make a claim and then verify the claim
- Claim:
  - Farmers borrow up to limit of constraint
  - Farmers consume no more than their nontradable fruit
- Formally:

$$b_t = q_{t+1} k_t / R$$

$$x_t = c k_{t-1}$$

# EQUILIBRIUM CHARACTERIZATION

- Combine this with budget constraint to get the condition

$$k_t = \frac{1}{q_t - \frac{1}{R}q_{t+1}} \left[(a + q_t)k_{t-1} - Rb_{t-1}\right]$$

## EQUILIBRIUM CHARACTERIZATION

- Combine this with budget constraint to get the condition

$$k_t = \frac{1}{q_t - \frac{1}{R}q_{t+1}} \left[ (a + q_t)k_{t-1} - Rb_{t-1} \right]$$

- $(a + q_t)k_{t-1} - Rb_{t-1}$ is the farmer's *net worth*:
(tradable output and assets net of debt)

- $u_t \equiv q_t - \frac{1}{R}q_{t+1}$ is the *downpayment per unit of land purchased*
(the difference between the price of a unit of land and how much can be borrowed against a unit of land)

# EQUILIBRIUM CHARACTERIZATION

- Proof of claim: for each unit of tradable fruit in $t$, the farmers could choose one of the following consumption paths from $t+1$ onwards
  - Invest the unit in land:
    $0, \frac{c}{u_t}, \frac{a}{u_t}\frac{c}{u_{t+1}}, \frac{a}{u_t}\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, \dots$
  - Invest the unit in bond, then in land:
    $0, 0, R\frac{c}{u_{t+1}}, R\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, \dots$
  - Consume the unit immediately:
    $1, 0, 0, 0, \dots$
- Will show that first path leads to highest utility around the steady state (proof will be concluded further below)

# EQUILIBRIUM CHARACTERIZATION

- A farmer's optimality conditions are linear, so we can aggregate across farmers:

$$K_t = \frac{1}{u_t}\left[(a + q_t)K_{t-1} - RB_{t-1}\right]$$

$$B_t = \frac{1}{R} q_{t+1} K_t$$

- With $u_t \equiv q_t - \frac{1}{R} q_{t+1}$

(upper case letters denote quantities in aggregate farming sector)

## FARMER DEMAND FOR LAND

- Rearrange first equation to

$$K_t = \frac{q_t K_{t-1}}{u_t} + \frac{aK_{t-1} - RB_{t-1}}{u_t}$$

- Suppose $q_t$ and $q_{t+1}$ rise by 1 percent
- If $RB_{t-1} > aK_{t-1}$ (high leverage), then $K_t$ increases
- This means that a higher price increases demand!
- This is how the financial accelerator operates

# FARMER DEMAND FOR LAND

- Another way to see this

$$q_t K_t - \frac{1}{R} q_{t+1} K_t = q_t K_{t-1} + a K_{t-1} - R B_{t-1}$$

$q_t \uparrow$ gives usual demand effect ($K_t \downarrow$)

$q_{t+1} \uparrow$ allows for more borrowing ($K_t \uparrow$)

$q_t \uparrow$ increases net worth ($K_t \uparrow$)

# GATHERERS DEMAND FOR LAND

- The gatherer is financially unconstrained, so the demand for land comes from a standard Euler equation

$$q_t = \beta' \left( q_{t+1} + G'(k_t') \right)$$

- The gatherers bond Euler equation (linear utility function!) will give us

$$\beta' = 1/R$$

- Can combine this to

$$\frac{1}{R} G'(k_t') = u_t$$

# GATHERERS DEMAND FOR LAND

$$\frac{1}{R}G'(k_t') = u_t$$

- This says that the discounted marginal product of land is equal to the gatherer's opportunity cost of land
- Note that the gatherer's opportunity cost of land is equal to the required downpayment for each unit of land for the farmer

# MARKET CLEARING

- Gatherers are identical, so aggregate demand for land from gathering sector is equal to $m$ times $k'_t$
- Market clearing in the land market is thus

$$K_t + mk'_t = \bar{K}$$

- Combine this with the gatherers Euler equation to obtain

$$u_t = q_t - \frac{1}{R}q_{t+1} = \frac{1}{R}G'\left[\frac{1}{m}(\bar{K} - K_t)\right]$$

- The right hand side is increasing in $K_t$, so $u_t = u(K_t)$

# MARKET CLEARING (CONTINUED)

- Since gatherers have linear preferences and are not credit constrained, they are indifferent between any amount of consumption and debt
- Equilibrium in debt market fully described by $B_t$
- Fruit market clears by Walras' law
  (Market clearing condition is $X_t + mx'_t = Y_t + my'_t$)

# EQUILIBRIUM CHARACTERIZATION

- Consider perfect foresight equilibria
- Given $K_{t-1}, B_{t-1}$, an equilibrium from date $t$ onwards is is given by $\{q_{t+s}, K_{t+s}, B_{t+s}\}_{s>0}^{\infty}$ satisfying

$$K_t = \frac{1}{q_t - \frac{1}{R}q_{t+1}} \left[(a + q_t)K_{t-1} - RB_{t-1}\right]$$

$$B_t = \frac{1}{R}q_{t+1}K_t$$

$$q_t - \frac{1}{R}q_{t+1} = \frac{1}{R}G'\left[\frac{1}{m}(\bar{K} - K_t)\right]$$

- Add assumption 3

$$\lim_{s\to\infty} E_t(R^{-s}q_{t+s}) = 0$$

- There is a unique steady state $(q^*, K^*, B^*)$ such that

$$q^* = \frac{R}{R-1}a$$

$$\frac{1}{R}G'\left[\frac{1}{m}(\bar{K} - K^*)\right] = a$$

$$B^* = \frac{a}{R-1}K^*$$

- Also note that $u^* = a$

# VERIFY CLAIM

- Remember the earlier claim: farmers invest all tradable units and borrow up to constraint
- To verify claim, need to show that first strategy out of the following three gives highest utility around the steady state
  - Invest the unit in land:
    $0, \frac{c}{u_t}, \frac{a}{u_t}\frac{c}{u_{t+1}}, \frac{a}{u_t}\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, \ldots$
  - Invest the unit in bond, then in land:
    $0, 0, R\frac{c}{u_{t+1}}, R\frac{a}{u_{t+1}}\frac{c}{u_{t+2}}, \ldots$
  - Consume the unit immediately:
    $1, 0, 0, 0, \ldots$

# VERIFY CLAIM

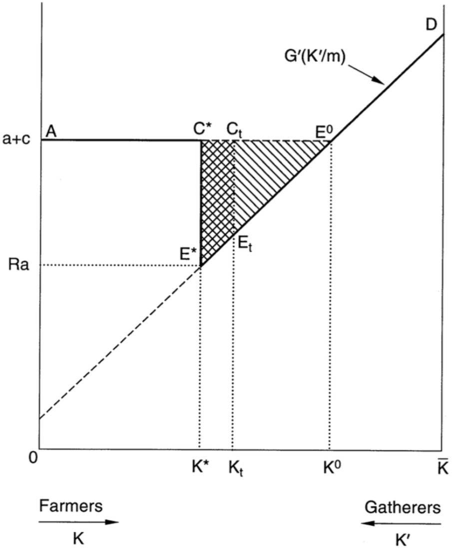
- In steady state, we get the following utility for the farmer
  - $\frac{\beta}{1-\beta}\frac{c}{a}$
  - $\frac{R\beta^2}{1-\beta}\frac{c}{a} = \frac{\beta}{\beta'}\frac{\beta}{1-\beta}\frac{c}{a}$
  - 1
- By assumption 1: $\beta < \beta'$, so first strategy better than second
- By assumption 2: $\beta c > (1-\beta)a$, so first better than third

$\Rightarrow$ proof is completed

# EQUILIBRIUM CHARACTERIZATION

- Back to characterizing the equilibrium

# EQUILIBRIUM CHARACTERIZATION



# EQUILIBRIUM CHARACTERIZATION

- Horizontal line plots farmer's MPK: $a + c$
- Diagonal line plots gatherer's decreasing MPK: $G'$
- Point $E^0$ is the efficient equilibrium without credit constraints: MPKs are equal
- Point $E^*$ is the constrained steady state equilibrium: MPK of gatherers is $G' = Ra$ (by assumptions 1 and 2 $Ra < a + c$)
- In both cases, the area under the upper MPK lines is aggregate output
- Giving more $K$ to farmers (looseing the constraint) increases aggregate output

# DYNAMICS

- We now turn to studying an *unanticipated* shock
- Suppose we start at steady state equilibrium in $t-1$, but in period $t$ the fruit harvest of both farmers and gatherers is multiplied by $1+\Delta$ for one period
- This is a completely transitory shock, so any persistence in the responses will come from the model's endogenous dynamics

# DYNAMICS

- Remember the equilibrium conditions

$$K_t = \frac{1}{q_t - \frac{1}{R}q_{t+1}}\left[(a+q_t)K_{t-1} - RB_{t-1}\right]$$

$$B_t = \frac{1}{R}q_{t+1}K_t$$

$$q_t - \frac{1}{R}q_{t+1} = \frac{1}{R}G'\left[\frac{1}{m}(\bar{K} - K_t)\right]$$

with

$$u(K_t) = q_t - \frac{1}{R}q_{t+1}$$

- The transmission of the shock from period $t$ onwards is characterized by combining the first and the second equation on the previous slide:

- In period $t$

$$u(K_t)K_t = (a + \Delta a + q_t - q*)K^*$$

- In period $t+1, t+2, t+3, ...$

$$u(K_{t+s})K_{t+s} = aK_{t+s-1} \quad \text{for } s \geq 1$$

# DYNAMICS

- Comparing the equation shows that in period $t$ the unanticipated shock gives a rise in the farmers net worth
- Value of land is now $q_t K^*$ while debt repayment is still $RB^* = q^* K^*$
- In the following periods, gross debt repayment and the value of land are equal since the constraint binds

# DYNAMICS

- The dynamics in response to the shock can be summarized by a system of difference equations in $K_t$ and $q_t$
- As usual, we can add equations to compute other variables (e.g. output and debt)
- The system of difference equations in nonlinear: $u(K)$ is nonlinear, since it depends on $G(K)$
- The system can be linearized around the steady state (for the derivations, see the paper)

# LINEARIZED SYSTEM

- Denote variables in percentage deviations from steady state: $\hat{X} = \frac{X_t - X^*}{X^*}$
- The linearized system is given by

$$\left(1+\frac{1}{\eta}\right)\hat{K}_t = \Delta + \frac{R}{R-1}\hat{q}_t \quad \text{at date } t$$

$$\left(1+\frac{1}{\eta}\right)\hat{K}_{t+s} = \hat{K}_{t+s-1} \quad \text{for } s \geq 1$$

and

$$\hat{q}_t = \frac{1}{\eta}\frac{R-1}{R}\frac{1}{1-\frac{1}{R}\frac{\eta}{1+\eta}}\hat{K}_t$$

where $1/\eta$ is the elasticity of $u(K_t)$ w.r.t. $K_t$ evaluated at the steady state (more on this below)

# DYNAMICS

- How to inperpret the equations above?
- At date $t$, $\Delta$ is the direct effect of the shock on $\hat{K}_t$, while the indirect effect through prices in scaled up by $\frac{R}{R-1}$ because of leverage
- The factor $\left(1+\frac{1}{\eta}\right)$ reflects that user cost must rise in response to growing demand from farmers, for markets to clear
- The effect of the shock persists into the future via the second equation: the farmers' ability to invest in $t+s$ depends on their net worth, which in turn depends on production in date $t+s-1$

- For period $t$ can solve the system explicitly for $\hat{K}_t$ and $\hat{q}_t$

$$\hat{q}_t = \frac{1}{\eta}\Delta$$

$$\hat{K}_t = \frac{1}{1+\frac{1}{\eta}}\left(1 + \frac{R}{R-1}\frac{1}{\eta}\right)\Delta$$

# AMPLIFICATION

- The response of the land price is scaled by $1/\eta$ the elasticity of $u(K_t)$ w.r.t. $K_t$ evaluated at the steady state
- This means the size of change in the marginal product of capital in the gathering sector is key to the amplification
- The response of capital is scaled up by a big margin, due to the presence of the term $R/(R-1)$
- Overall there is large and persistent internal propagation on impact

# AMPLIFICATION

- Can derive the dynamics of output as

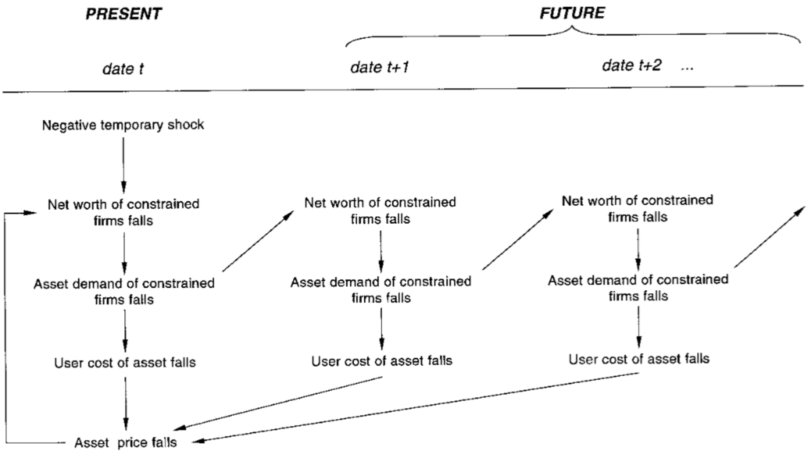
$$\hat{Y}_{t+s} = \frac{a + c - Ra}{a + c} \frac{(a + c)K^*}{Y^*} \hat{K}_{t+s-1} \qquad \text{for } s \geq 1$$

- Again the steady state relative marginal products ($a + c$ vs. $aR$) amplify the magnitude of the response

# COMPARISON TO FIRST-BEST ECONOMY

- In the first-best economy, without a borrowing constraint, output will rise by $\Delta$
- There would be no effect on land prices, no effect on capital, and therefore no effect on future output
- The comparison between the dynamics characterized above and the case in which output rises by $\Delta$ provides us with the magnitude of the internal propagation via the collateral constraint

# DYNAMIC VS. STATIC MULTIPLIERS



# DYNAMIC VS. STATIC MULTIPLIERS

- Can decompose amplification into static and dynamic multipliers
- The dynamic multiplier comes from future price changes
- We can ask: what are counterfactual dynamics when keeping $q_{t+s}$, $s \geq 1$ at $q^*$?
- This counterfactual isolates the static multiplier

- Fixing $q_{t+s} = q^*$ we obtain

$$\hat{q}_t|_{q_{t+s}=q^*} = \frac{R-1}{R}\frac{1}{\eta}\Delta$$

$$\hat{K}_t|_{q_{t+s}=q^*} = \Delta$$

# DYNAMIC VS. STATIC MULTIPLIERS

- Can check the difference between these equations to their original counterparts to understand the contribution of dynamic multipliers
- Dynamic multiplier scales up static effect on $\hat{q}_t$ by $\frac{R}{R-1}$
- Dynamic multiplier scales up static effect on $\hat{K}_t$ by $\frac{1}{1+\frac{1}{\eta}}\left(1+\frac{R}{R-1}\frac{1}{\eta}\right)$
- Overall, large additional effects due to dynamic multiplier
  $\rightarrow$ forward-looking prices give amplification

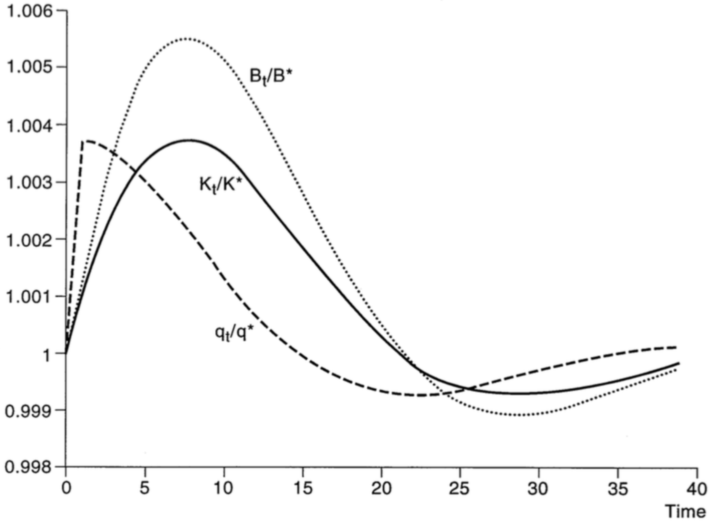
# SUMMING UP

- Model delivers a great deal of amplification
- The financial constraint matters a lot for the dynamics of activity in this economy
- Neat and intuitive characterization of the model

# FULL MODEL: BRIEF OVERVIEW

- Kiyotaki and Moore also write down a more general version of the model
- Two substantive changes
  1. Reproducable capital: farmers can plant fruits
  2. Only a fraction of farmers get investment opportunities

# FULL MODEL: BRIEF OVERVIEW



# OUTLOOK

# MICROFOUNDATION: OVERVIEW

- Next step: turn to *rationalizing* the presence of a debt limit of the type that Kiyotaki and Moore explore

# Bibliography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