Imperfect Common Knowledge and Exchange Rate Determination

Ethan Kaplan*

The University of California at Berkeley

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Abstract

We look at a simple financial model of exchange rate determination with investors who have heterogeneous information. The exchange rate can be expressed as a sum of averages of higher order expectations of future interest rates. Lack of common knowledge of information about the interest rate process leads investors to rationally confuse persistent with transitory

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shocks. Average higher order expectations are biased towards commonly known information and what is commonly known is that on average shocks do not occur. Future higher order expectations of a given period's interest rate are less biased merely because there is less higher order averaging involved. Therefore, exchange rates will slowly move in the direction of a shock even if investors are on average well informed. This framework is capable of explaining attenuation of the Fama coefficient as well as excess volatility of the exchange rate relative to interest rates.

1. INTRODUCTION

Exchange rate prediction has long been mostly unsuccessful. Meese and Rogoff's (1983) essential conclusion that, for countries with moderate to low inflation, random walk models outpredict all other exchange rate models at low and medium frequencies seems to have remained true over time and econometric technique. Only at time horizons of two to three years do macroeconomic models begin to have predictive power. Moreover, exchange rates often times even more blatantly contradict theory. The best known example of this is the famous forward premium bias (otherwise known as the Fama puzzle). A large empirical literature, exemplified by Fama (1984), has run the following regressions: $S_{t+1} - S_t = \alpha + \beta\,(f_t - S_t) + \epsilon_t$

where S_t is the log of a country's bilateral exchange rate at date t and f_t is the log of the one period forward rate at date t. Whereas traditional models with homogeneous information, no learning and risk premia that are constant over time predict that interest rate increases should lead, one-to-one, to instantaneous appreciations and subsequent exchange rate depreciations and thus that β should equal 1, regressions have shown that, in fact, the following depreciations are always less than one-to-one and often exchange rates actually appreciate ($\beta < 0$). Most of Fama's estimates were between 0 and -1. Subsequent research has found estimates that are often below -1 though sometimes positive but always below 1. Engel (1996) has a nice review of empirical results. Lewis (1996) has a review of attempted theoretical resolutions of the puzzle.

Early theories of exchange rate determination suggested that bilateral exchange rates for two countries with stable macroeconomic policies should follow a random walk because all information about future exchange rate events should be capitalized into the current exchange rate. However, such theories also predict that there should be no Fama puzzle.

The current paper is an attempt to simultaneously explain both the Fama puzzle and the excess volatility puzzle by looking at the role of information heterogeneity of investors. In the absence of informational heterogeneity, there is no reason for investors to speculate against eachother. However, with informational heterogeneity, investors are convinced that their private information will be better reflected in future exchange rates. On the one hand, this speculative component to exchange rate determination leads exchange rates to be more volatile than interest rates. On the other hand, however, private information gets incorporated into exchange rates slowly and this causes exchange rates to not react to persistent shocks immediately. This slow release of information into the exchange rate causes attenuation in the Fama coefficient.

In part 2 of this part, we review the relevant literature. In part 3, we set up the basic model. In part 4, we solve the model for the case of "Walrasian Investors" who have no behavioral biases but do not use the information embedded in the exchange rate or the information embedded in the interest rate. This case is analytically tractable and provides intuition for the case with "Rational Expectations" investors considered in part 5. Lastly, in part 6, we conclude and discuss future research possibilities.

2. LITERATURE REVIEW

Traditionally, the forward premium bias puzzle has been explained either by time-varying risk premia or by investor expectational errors. However, Froot and

Frankel (1989), using survey data on trader forecasts, show that most of the puzzle must be explained by expectational errors. Lewis (1989) constructs and estimates a learning model where investors learn about whether or not the central bank changed its monetary policy. This generates average expectational errors of investors. However, Lewis' model does not predict that the Fama coefficient should necessarily be less than one. In Lewis' model, it is equally likely that the Fama coefficient be greater than one.

A burgeoning literature has focussed upon incomplete information theories of asset price determination. In international finance, the focus has been upon speculative attacks under fixed exchange rates where agents receive private information about the state of the economy. In such models, agents are incompletely informed about both the economy as well as other agents' beliefs about the economy. The current paper extends this literature to look at flexible exchange rates.

This paper is most similar to Bacchetta and Van Wincoop (2004), who also build a model of heterogeneous investors and flexible exchange rate determination. However, they do not focus upon the forward premium bias. Second, their results are not analytically tractable. Third, they have a somewhat non-standard information structure, where information about the future arrives T periods in advance and all information about the past is commonly known.

Bacchetta and Van Wincoop (2005) explain the forward premium bias using rational inattention: investors with high costs of information acquisition do not update often and investors with low costs of information acquisition update every period.

In a financial asset pricing paper by Allen, Morris and Shin (2004), the authors consider a model of asset price determination with heterogeneous information. In particular, they demonstrate how the law of iterated expectations does not hold for averages beliefs of groups of people, an analysis which is very useful when we apply it to exchange rates.

Jeanne and Rose (2002) generate excess volatility in flexible exchange rates but in a reduced form one-period model and only given the presence of irrational traders. Moreover, they do not explain the forward premium bias puzzle. Gourinchas and Tornell (2004) explain the forward premium bias and the excess volatility of exchange rates relative to interest rates. However, they do so by assuming a behavioral bias where agents irrationally believe that shocks are more transitory than they actually are. The current paper is similar in spirit except that agents rationally misperceive.

3. THE MODEL

Investors live for two periods, investing the wealth they were born with in the first period and consuming in the second period. Wealth endowments are constant across people and over time. There is a continuum of investors of measure 1 from the home country who can choose between a one period risk-free domestic bond and a one period foreign bond. Whereas the interest rate on the domestic bond is fixed at i, the supply of the foreign bond is fixed at B. The foreign bond's interest rate is subject to both persistent and temporary shocks so that future foreign interest rates are uncertain[1]. The movements in the foreign interest rate impact the demand for foreign bonds and thus cause fluctuations in the nominal exchange rate. In addition, the exchange rate is subject to additional random noise, independent of the shocks to the interest rate. The motivation for this is that the foreign central bank targets interest rates, using a moving target with both persistent and transitory components; in addition, there are liquidity traders who randomly purchase foreign bonds, impacting the exchange rate without affecting the foreign interest rate.

Agent j's utility is given by a constant absolute risk aversion utility function

[1] Alternatively, one can think of the random shock to the foreign interest rate as a shock to the forward premium. This interpretation doesn´t involve asymmetries in shocks across countries.

(CARA). She maximizes the expected utility of next period's consumption choosing the share of her endowed wealth allocated towards domestic versus foreign bonds. Since the investor consumes in the domestic country, any wealth put into foreign bonds must be converted back into domestic currency at the next period's exchange rate in the second period of the investor's life.

$$\max_{\alpha_{jt}} -E_t e^{-\frac{C_{t+1}}{\tau}} \quad s.t. \; C_{t+1} = \alpha_{jt}(1+i_t^*)\frac{s_{t+1}}{s_t} + (W - \alpha_{jt})(1+i) \tag{3.1}$$

We use α_{jt} to denote the amount of wealth invested by investor j in the foreign bond at time t, C_t to be equal to consumption, i_t^* to be the period t foreign interest rate, i to be the domestic interest rate, W to be investor wealth, s_t to be the period t exchange rate, and τ to be the inverse of the coefficient of absolute risk aversion[2]. We log-approximate the exchange rate and rewrite the problem as:

$$\max_{\alpha_{jt}} -E_t e^{-\frac{C_{t+1}}{\tau}} \quad s.t. \; C = W(1+i) + \alpha_{jt}\left[i_t^* + S_{t+1} - S_t - i\right] \tag{3.2}$$

Here S_t refers to the log of the exchange rate. For the moment, we assume that the log of the period $t+1$ exchange rate is normally distributed; therefore

[2] We assume here an interior solution; namely that parameters are such that $W \geq \alpha_{jt} \geq 0$.

the exchange rate is log-normally distributed and we can derive the formula for the investor's expected utility and then maximize expected utility, deriving the investor's demand for foreign bonds:

$$\alpha_{jt} = \frac{E_t S_{t+1} + i_t^* - S_t - i}{\tau\sigma^2} \tag{3.3}$$

For the moment, we assume that investor j's conditional variance of the next period's exchange rate is constant across individuals and across time. The intuition for this assumption is that there is no heterogeneity across investors of beliefs about second moments. The only heterogeneity of beliefs will be about first moments of the interest rate and the exchange rate. We later solve for the that conditional subjective exchange rate variances.

Now we turn to a discussion of the interest rate process and the information structure. We assume that time begins at date $-\infty$. Deviations at time t of the interest from its long term mean, $\bar{i}$, are distributed randomly with a purely transitory shock, v_t, and a persistent shock, ϵ_t :

$$i_t^* = \bar{i} + \epsilon_t + v_t \tag{3.4}$$

The persistent shock is an AR(1) shock whose innovation is distributed normally with mean zero and constant variance (precision):

$$\begin{aligned} \epsilon_t &= \lambda \epsilon_{t-1} + \mu_t \\ \mu_t &\sim N\left(0, \sigma_\mu^2\right) \end{aligned} \tag{3.5}$$

The transitory shock is distributed i.i.d. normal: $v_t \sim N\left(0, \frac{1}{v}\right)$. Since $V(\epsilon_t)$ will be used often throughout the rest of the paper, we define $\gamma = \frac{1}{V(\epsilon_t)} = \frac{1-\lambda^2}{\sigma_\mu^2}$. In addition to these two interest rate shocks, there is also a shock to domestic demand for foreign bonds. This shock will impact the equilibrium exchange rate without impacting the equilibrium interest rate. As a motivation for this shock, we have in mind a central bank which sterilizes capital inflows in order to maintain a target interest rate but allows inflows to effect the exchange rate. The shock to domestic demand for foreign bonds is given by a normally distributed random variable with constant variance: $N_t \sim N\left(0, \sigma_N^2\right)$. With this shock, domestic demand for foreign bonds becomes:

$$\int_0^1 \alpha_{jt} dj + N_t = \int_0^1 \frac{E_t S_{t+1} + i_t^* - S_t - i}{\tau \sigma^2} dj + N_t$$

Agents are assumed to know the distributions of the transitory and persistent interest rate shocks as well as the distribution of the exchange rate shock. Moreover, investors know that others investors know this information. In other words, there is common knowledge of the distributions of the three shocks. In addition to this public information, the investors have private information. Each investor, upon birth, receives a private signal about the current state of the persistent shock, ϵ_t. The signal is a garbled signal of the actual realization of the persistent component of the shock where the garbling is distributed normally with constant variance:

$$\begin{aligned} \epsilon_t &: \quad x_{jt} = \epsilon_t + \delta_{jt} \\ \delta_{jt} &\sim N\left(0, \frac{1}{\omega}\right) \end{aligned} \tag{3.6}$$

We have now fully specified the model except for the equilibrium concept. We will not discuss the solution concept in detail in this section because we will solve the model in two different ways. Both of the ways in which we solve the model are market clearing models, however, so that our fundamental equation of equilibrium in the foreign bond market is that the supply of bonds equal the demand for bonds. We do not need to look at the domestic bond market since the domestic interest rate is fixed by the home central bank. The equation for equilibrium in the foreign bond market is:

$$\int_0^1 \frac{E_t S_{t+1} + i_t^* - S_t - i}{\tau\sigma^2} dj + N_t = B \tag{3.7}$$

From now on, we will denote the date t first order average belief operator of investors by $\overline{E}_t$ and the k^{th} order average belief operator of investors by $\overline{E}_t^k$. In other words, $\overline{E}_t \epsilon = \int\limits_0^1 E_{jt}\epsilon$ and $\overline{E}_t^k \epsilon = \int\limits_0^1 E_{jt} \int\limits_0^1 E_{jt+1...} \int\limits_0^1 E_{jt+k}\epsilon dj_1...dj_k$. One important note is that whereas an individual's expectation today of their expectation tomorrow of a random variable is equal to their current expectation from the law of iterated expectations, the law of iterated expectations does not hold for averages. In other words, whereas $E_t E_{t+1}\epsilon = E_t\epsilon$, in general, $\int\limits_0^1 E_{jt} \int\limits_0^1 E_{jt+1}\epsilon dj_1 dj_2 \neq \int\limits_0^1 E_{jt}\epsilon dj_1 dj_2$. In particular, following Allen, Morris and Shin (2004), higher order expectations are biased towards public signals. Suppose that agents have two sources of information on ϵ: (1.) they know that the distribution has mean m and precision π_1 and they also receive a private signal, x_{jt}, which has mean ϵ and precision π_2. Then any individual's expectation of ϵ is a precision-weighted average of the two signals: $\frac{\pi_2 x_{jk} + \pi_1 m}{\pi_1 + \pi_2}$. The average belief in the economy is then:

$\int_0^1 E_{jt} \frac{\pi_2 x_{jk} + \pi_1 m}{\pi_1 + \pi_2} dj = \frac{\pi_2 \epsilon + \pi_1 m}{\pi_1 + \pi_2}$. This itself is a random variable over which agents can have beliefs. We can calculate the belief of a given agent j of the average belief: $\int_0^1 E_{jt+1} \frac{\pi_2 i_{t+k} + \pi_1 m}{\pi_1 + \pi_2} \, dj = \frac{\pi_2 \frac{\pi_2 x_{jk} + \pi_1 m}{\pi_1 + \pi_2} + \pi_1 m}{\pi_1 + \pi_2} = \frac{\pi_2^2 x_{jt} + (\pi_1 + \pi_2)^2 - m}{(\pi_1 + \pi_2)^2} = \frac{\pi_2^2 x_{jt} + \left[(x_1 + \pi_2)^2 - \pi_2^2 \right] m}{(\pi_1 + \pi_2)^2}$. In general, we have $\overline{E}_t^k \epsilon = \frac{\pi_2^k}{(\pi_1 + \pi_2)^k} x_{jt} + \left[1 - \frac{\pi_2^k}{(\pi_1 + \pi_2)^k} \right] m$. In fact, $\lim_{k \to \infty} \overline{E}_t^k \epsilon = m$.This example shows that higher order expectations are biased towards public information (commonly known information) and that this bias increases as does the order of the expectation. In simpler terms, suppose that my private signal is 1 and my public signal is 0. Moreover, suppose that the variances are equal. Then, my estimate of ϵ is .5. This means that I guess that on average everyone gets a signal of .5. Therefore, they average .5 with zero (at least on average) to come up with an average estimate of .25. So my belief about the average belief about ϵ is .25 (closer to zero than my belief itself). In addition, I think that the average person's estimate is .25. Therefore, I think that they think that the average person comes up with an estimate for ϵ by averaging .25 with zero. Thus, I believe that the average belief about the average belief of the value of ϵ is .125. This process continues until, in the limit, the infinite order belief is equal to the public signal: 0.

We can now use our bond market equilibrium equation to derive a recursive equation for the exchange rate:

$$s_t = \overline{E_t} s_{t+1} + i_t^* - i + N_t - B\tau\sigma^2 \tag{3.8}$$

The recursive formula for our exchange rate is almost the same as we normally obtain in a homogeneous information rational expectations environment. Today's exchange rate is equal to the average expectation of the next period's rate plus the forward premium and the liquidity trader demand minus the risk premium. The only difference between our equation and standard uncovered interest parity equations is that the exchange rate depends upon the average expectation of the next period's exchange rate rather than the unconditional expectation. This is due to the presence of heterogeneous information. Note that in a homogeneous information model, the two formulations would be exactly the same. Also, the above equation makes it look like there is a one-to-one positive relationship between the foreign interest rate and the price of foreign currency. However, the average market expectation of the exchange rate depends upon the current foreign interest rate. If a rise (fall) in the foreign interest rate today leads to an increase (decrease) in the price of foreign currency tomorrow, then the Fama coefficient will be attenuated.

We can use the recursive formula for the exchange rate to re-express the exchange rate as a sum of average higher order beliefs about future interest rates. Currently, we have that the exchange rate today is equal to the expected exchange rate tomorrow plus the forward premium minus the risk premium plus the current liquidity demand shock (henceforth noise). Substituting in for the next period's exchange rate, we get that today's exchange rate is equal to the average expectation of the average expectation of the exchange rate in two periods plus the average expectation of the forward premium next period plus the forward premium today minus the risk premium minus the average expectation of the risk premium plus today's noise:

$$s_t = \overline{E_t E}_{t+1} s_{t+2} + \overline{E}_t \left(i^*_{t+1} - i \right) + i^*_t - i + N_t - B\tau\sigma^2 - E_t B\tau\sigma^2$$

Solving recursively forward, we finally obtain the exchange rate as a sum of higher order average expectations of future interest rates:

$$s_t = \sum_{k=t}^{\infty} \left(\overline{E}_t^{k-t} i^*_k - i - \tau B\sigma^2 \right) + N_t \tag{3.9}$$

Using this formula, we can look at the change in the exchange rate between date t and date $t+1$:

$$s_{t+1} - s_t = \sum_{k=t}^{\infty} \left(\overline{E}_{t+1}^{k-t-1} i_{k+1}^* - \overline{E}_t^{k-t} i_k^* \right) + N_{t+1} - N_t \tag{3.10}$$

This is just the sum over interest rates of the difference between the k^{th} order average belief at date $t+1$ and the $k+1^{th}$ order average belief at date t of all higher order interest rates minus the current forward premium. In the case where investors have common knowledge of heterogeneous information, then the law of iterated expectations holds and therefore, on average, the k^{th} order average belief at date $t+1$ and the $k+1^{th}$ order average belief at date t are the same, implying that the exchange rate change is just equal to the forward premium. However, when there is not informational homogeneity, then there is not common knowledge of information and in general higher order beliefs can also impact the exchange rate.

We can rewrite equation (3.10) as the sum over future interest rates of the difference in expectations between the current and the following period:

$$s_{t+1} - s_t = \sum_{k=t+1}^{\infty} \left(\overline{E}_{t+1}^{k-t-1} i_k^* - \overline{E}_t^{k-t} i_k^* \right) - i_t^* + N_{t+1} - N_t \tag{3.11}$$

This approach generalizes previous theories of the dynamics of exchange rate determination. In the next section, we will show why we higher order beliefs about a shock are more reflective of the shock in later periods.

We now return to a discussion of our two equilibrium concepts. Though our two solution concepts will both entail market clearing in the bond market, they will differ in assumptions about how investors use relevant information. In our first solution method, we will assume that investors use the private information as well as the commonly known distribution of shocks in order to forecast future exchange rates and thus make their decisions about their demands for current period foreign bonds. They will explicitly ignore the information in the interest rate and more importantly in the exchange rate. In other words, they will not use either the exchange rate or the interest to estimate the persistent component of the interest rate shock (ϵ_t). However, they will not suffer from any behavioral biases and thus will use the information they use correctly.

We will then turn to a model with rational expectations investors who not only use the private information and commonly known distribution of signals, but in addition use the information in the exchange rate and the interest rate to infer the current state of the persistent shock. In both models, we find, for certain parameter values, that the exchange rate overreacts to the current level of the persistent shock and that the Fama coefficient is attenuated strictly below one.

4. WALRASIAN INVESTORS

We now look at a variant of the model we specified in the previous section: namely the variant with Walrasian investors. Walrasian investors use all relevant information except for the information embedded in prices when they make their decisions about how to allocate their wealth. The Walrasian investor case is interesting to look at because it contains the essential ingredients necessary in order to have higher order beliefs matter and yet it is analytically tractable. We will show numerically that our results in the rational expectations case are qualitatively very similar to those in the Walrasian case.

We will build much of our intuition with the Walrasian model and then check to see if our intuition is robust to allowing investors to use information embedded in the interest rate and the exchange rate in order to forecast the future value of the exchange and makes decisions about demand for foreign bonds.

In this variant of the model, we assume that the investors do not use either the exchange rate or the interest rate as a signal. We could of course make the intermediate assumption that they use the interest rate as a signal but not the exchange rate. We have done this and results are essentially the same as in the general Walrasian case. The only information agents use, then, is the known mean and variance of the distribution of the shocks to the interest rate and the private information.

We will assume that the exchange rate follows an equilibrium equation of:

$$s_t = \beta_0 + \beta_1 i_t^* + \beta_2 \epsilon_t + \beta_3 N_t \tag{4.1}$$

We can now plug this into the recursive equation (3.8) and verify that the equilibrium exchange rate equation is indeed of the form we guessed.

$$s_t = \overline{E_t}\left[\beta_0 + \beta_1 i_{t+1}^* + \beta_2 \epsilon_{t+1} + \beta_3 N_{t+1}\right] + i_t^* - i + N_t - B\tau\sigma^2 \tag{4.2}$$

Using the method of undetermined coefficients, we compute:

$$\begin{aligned} \beta_1 &= 1; \beta_2 = \frac{\lambda R}{1-\lambda R}; \beta_3 = 1 \\ \text{where } R &= \frac{\omega}{\gamma + \omega} \end{aligned}$$

This shows immediately that exchange rates under-react to the persistent shock. The degree of under-reaction is positively related to the noise to total variance ratio. The under-reaction is due to higher-order attenuation. From equation (3.9), we see that the exchange rate is a sum of higher order averages of future interest rates: the farther in the future the interest rate, the higher the order of the expectation involved. Moreover, whereas higher order individual expectations should be equal to lower order expectations for an individual due to the law of iterated expectations (I should believe that I will believe tomorrow on average what I currently believe today), this in general will not hold for groups of individuals with heterogeneous beliefs. Since the exchange rate is the sum of higher order averages of future interest rates and since higher order averages weight public over private information, the exchange rate underweights private information. In other words, instead of the coefficient on the current shock being $\beta_1 = \frac{1}{1-\lambda}$ as would occur in a model of homogeneous investors, the coefficient is attenuated, reflecting less private information due to higher order averaging.

We can write the variance of the interest rate as:

$$V\left(i_t^*\right) = \frac{1}{\gamma} + \frac{1}{\pi}$$

We have built in to the exchange rate an extra source of noise from the fact that the central bank is accommodating noise traders with respect to the interest rate though not with respect to the exchange rate. This would tend to bias upwards our calculation of the variance of the exchange rate in comparison with our calculation of the variance of the interest rate. Therefore we calculate the variance of the exchange rate ignoring the part of the variance due to the exchange rate noise[3]:

$$\left(\frac{1}{1-\lambda R}\right)^2 \frac{1}{\gamma} + \frac{1}{\pi} \tag{4.3}$$

The variance of the exchange rate is greater than that of the interest rate due to over-reaction of investors to the persistent component's signal. We can also use the exchange rate equation to determine the Fama coefficient. To do this, we calculate:

$$cov\left(s_{t+1} - s_t, i - i_t^*\right)$$

The change in the exchange rate equals:

$$0 < \frac{1-\lambda}{1-\lambda\frac{\omega}{\gamma+\omega}} = \frac{1-\lambda}{1-\lambda R} < 1 \tag{4.4}$$

[3]We include noise traders in the exchange rate market to prevent the exchange rate plus the interest rate from fully revealing the transitory and persistent shocks to the interest rate. In the appendix, we include the proof that with rational expectations investors, two informational instruments and two dimensions of unceraint y, we obtain full revelation.

Additionally, the equilibrium with Walrasian investors is unique. Solving for the excess demand function, we obtain:

$$\begin{aligned} Z\left(s_t\right) &= \int_0^1 \frac{E_{jt}s_{t+1}dj + i_t^* - s_t - i}{\tau\sigma^2} dj + N_t - B \\ \Rightarrow \quad \frac{dZ\left(s_t\right)}{ds_t} &= -\frac{1}{\tau\left[\left(\frac{1}{1-\lambda R}\right)^2 \frac{1}{\gamma} + \frac{1}{\pi} + \sigma_N^1\right]} < 0 \end{aligned}$$

Since, for $0 < \lambda < 1$ and $\tau > 0$, the excess demand curve is everywhere downward sloping, we have a unique equilibrium.

Note, first of all, that the Fama coefficient is strictly below one as long as persistence of the shock is between zero and one, and private information contains at least some signal component. Also, notice that the Fama coefficient is decreasing in λ. Consider the case where λ is zero; then private information plays no role in the prediction of future interest rates and thus future exchange rates. In this case, private information does not matter for exchange rate determination and thus changes in the accuracy of private information also do not impact exchange rate determination. Additionally, the Fama coefficient is increasing in the noisiness of private information. In the extreme case where private information is purely noise, the Fama coefficient is one as in the standard case. This is because private information is not informative and thus is not reflected in current or future exchange rates; consequently, it is not released slowly over time into the exchange rate.

The reason for attenuation in the Fama coefficient is due to the role of heterogeneous information. As discussed before, private information about shocks is incorporated into the price of foreign currency slowly over time. Therefore, counteracting the depreciation of an exchange rate due to the country having a higher interest rate (after an a positive interest rate shock), there will be an offsetting expectations increase effect. The average belief about the price of the country's currency will increase because private information that there was a persistent shock to the interest rate will be more reflected in future exchange rates in comparison with the current one.

Note, however, that the increase in average expectation of the following period's exchange rate is not due to learning. It is due to less higher order averaging (Keynesian Beauty Contests) in the future compared with the present. Let us perform a thought experiment. Suppose that (1.) each agent receives as their private signal the exact value of the current innovation to the persistent shock of the interest rate, (2.) each agent is aware that they are receiving the true value as a signal, and (3.) each agent believes that all other investors in the economy are receiving mean zero draws of a non-degenerate random distribution around the true value of the shock. Then the average investor expectation and all higher order expectations will be the same as in the rational expectations case where all investors receive mean zero shocks. Since all higher order expectations will be the same, (3.10) will be the same and thus the Fama coefficient will be the same as in the rational expectations case. However, in this thought experiment, agents do not learn about the interest rate slowly over time. They are perfectly well informed at any point in time.

We have just shown that if we consider a non "rational expectations" variant of the model where agents know with infinite precision the state of the persistent shock to the interest rate but their higher order knowledge is the same in the model just presented, then our results will be exactly the same. In other words, if agents do not learn but do not know what others know, we retain the same results.

5. RATIONAL EXPECTATIONS INVESTORS

In the rational expectations model, we solve for an equation for exchange rate determination where investors use the equation to extract information about the persistent component of the interest rate shock, formulate a guess of the level of the shock, use that to forecast the following period's exchange rate and then figure out their demand for bonds. In equilibrium, it turns out that the equation for the exchange rate that investors use to extract information out of the exchange rate and make forecasts of the following period's exchange rate is exactly the actual equilibrium equation which balances supply and demand for the bond and exchange rate markets.

We solve for the rational expectations equilibrium using the guess and verify method. Since currency demand is linear in the exchange rate, we can equate aggregate demand and aggregate supply to obtain an equation of the exchange rate which is linear in the shocks[4]. Our guess for the law of motion for the exchange rate, therefore is that it depends linearly upon the foreign interest rate, the persistent shock, and the transitory shock to the exchange rate:

$$s_t = \beta_0 + \beta_1 i_t^* + \beta_2 \epsilon_t + \beta_3 N_t \tag{5.1}$$

Given this equilibrium formula for the exchange rate, each investor can form an estimate of the current state of the persistent shock by solving for the persistent shock plus noise using only publicly available information. The estimate which

[4]Note that if we are correct, then the exchange rate is just a sum of normally distributed variables in which case it is a normally distributed random variable.

each investor forms turns out to be normally distributed with mean ϵ_t and variance $\left(\frac{\beta_3}{\beta_2}\sigma_N\right)^2$ (which we will call $\frac{1}{\eta}$) :

$$\frac{s_t - \beta_0 - \beta_1 i_t^*}{\beta_2} = \epsilon_t + \frac{\beta_3}{\beta_2} N_t \sim N\left(\epsilon_t, \left(\frac{\beta_3}{\beta_2}\sigma_N\right)^2\right) \tag{5.2}$$

In addition to using the exchange rate as a source of information about the persistent component of the shock, each investor has recourse to three other sources: (1.) the private information they receive, (2.) the interest rate, and (3.) the publicly known distribution of the persistent shock. The investor weighs each of these pieces of information by its relative precision in forming an overall estimate of ϵ_t. The interest rate can be used to form an estimate of the persistent shock by differencing it with its long-term mean:

$$i_t^* - \bar{i} = \epsilon_t + v_t \sim N\left(\epsilon_t, \frac{1}{\pi}\right) \tag{5.3}$$

Since each investor is born the same date that they invest, we assume that their prior distribution of ϵ_t is just the unconditional distribution[5]. Since the unconditional distribution of ϵ_t is stationary, the variance does not depend on

[5] In the future, we will assume that the entire history of prices is available to the newly born investor. Hopefully, it will turn out that the prior period's price will be a sufficient statistic for all prior information.

time or person. The expectation of epsilon is then given by:

$$E_{jt}\epsilon_t = \frac{\omega x_{jt} + v(i_t^* - \bar{i}) + \eta \frac{s_t - \beta_0 - \beta_1 i_t^*}{\beta_2}}{\omega + \pi + \gamma + \eta} \tag{5.4}$$

Under the assumption that we have correctly guessed the exchange rate equation, we now use our expectations formation equation and plug it into the equation for the exchange rate which we obtained from bond market equilibrium. Investors will use the guessed linear form of the exchange rate when formulating an expectation of the following period's exchange, which will rely upon their estimate of the current state of the persistent shock. When we plug our guess into the bond equation we get:

$$\begin{aligned}
\eta &= \left(\frac{\beta_3}{\beta_2}\sigma_N\right)^2 \qquad (5.5)\\
\beta_1 &= 1 + \frac{\lambda\pi}{(1-\lambda)(\omega+\eta+\pi)+\gamma}\\
\beta_2 &= \frac{\lambda(\omega+\eta)}{\gamma+(1-\lambda)(\omega+\pi+\eta)}\\
\beta_3 &= 1 + \frac{\eta}{\omega}\\
\beta_0 &= \frac{\left(\beta_0 + i_t^* - i - \frac{B\sigma^2}{\tau} - \frac{\beta_0\eta\lambda}{\lambda(\omega+\eta)}\right)\frac{[\omega+\pi+\gamma+\eta]}{(\beta_1+\beta_2)} - \lambda\pi\bar{i}}{\beta_2\lambda(\omega+\eta)-\eta\lambda}\\
\sigma^2 &= \beta_1^2\frac{1}{\pi} + (\beta_1+\beta_2)^2\frac{1}{\gamma} + \beta_3^2\sigma_N^2
\end{aligned}$$

Note that both β_1 and β_2 are positive. This means that there are no sufficiently perverse signal extraction problems leading the price of foreign currency to decrease with a positive shock to the foreign interest rate. It will also imply that the Fama coefficient is bounded below by zero. Also, notice that both β_1 and β_3 are greater than one. Since both coefficients tell how the exchange rate reacts to temporary shocks, one might expect the coefficients to be equal to one. In fact, in our Walrasian version of the model, the coefficients were both equal to one. Now that investors extract information from the exchange rate and the interest rate, they over-react to noise, rationally confusing persistent shocks with temporary ones.

One problem with the solution to the model is that it is sufficiently non-linear (remember that η is a function of both β_2^{-2} and β_3^2) that for some parameter values we have multiple equilibria. In particular, if the exchange rate is believed to have high variance, then there will not be much signal extraction from marginal increases in the exchange rate, leading demand to be relatively inelastic. This inelasticity of demand then causes the exchange rate to be volatile. Similarly, a hypothesized low variance exchange rate leads to high demand elasticity, making the hypothesized low variance exchange rate self-fulfilling. This is similar to the general results about belief-driven multiplicity of rational expectations equilibria discussed by Driskill and McCafferty (1980) and Driskill and McCafferty (1982). In the Walrasian case, investors do not extract information about the persistent component of the shock to the foreign interest rate from the exchange rate itself or in fact from any endogenous variable. Therefore, changes in subjective variances of the exchange rate don't change the elasticity of foreign exchange demand; they only change the level of demand for foreign currency. The variance term for the exchange rate enters only into the constant term of the exchange rate. On the other hand, the variance of the exchange rate, encapsulated by η, does enter into the elasticity parameter for the persistent shock to the interest rate in the rational expectations investor case[6].

One potential problem with multiple equilibria is stability of the equilibrium. An equilibrium is less believable as an actual outcome if small shocks or deviations from the equilibrium price lead to large shifts away from the equilibrium. So, we check the conditions for the equilibrium to be "tatonnement" stable. We derive the aggregate excess demand function and then check to see if, at the equilibrium, a rise in the price of foreign currency (the exchange rate) leads to excess supply of foreign currency and a drop in the price of foreign currency leads to excess demand

[6] Technically, η is not the variance of the exchange rate but rather the variance of the signal that investors use, incorporating the exchange rate, in order to infer the level of ϵ_t.

for foreign currency. In other words, we check to see whether an appreciation of the foreign currency will lead to excess supply of the foreign currency and thus a return in the price to equilibrium or instead whether an appreciation of the currency will lead to an increase in the demand for the foreign currency followed by a further appreciation away from equilibrium.

We start by deriving the aggregate excess demand function, which is just the sum over the individual demand functions; we also replace the period $t+1$ exchange rate with the equilibrium formula for the exchange rate:

$$Z(s_t) = \frac{\overline{E_t}\left[\beta_0 + \beta_1 i^*_{t+1} + \beta_2 \epsilon_{t+1} + \beta_3 N_{t+1}\right] + i^*_t - s_t - i}{\tau\sigma^2} - B \qquad (5.6)$$

The expectation of the future interest rate and the future shock reduce to expectations at date t of the date $t+1$ level of the persistent shock. Thus we get:

$$Z(s_t) = \frac{\beta_0 + (\beta_1 + \beta_2)\,\lambda \frac{\omega\epsilon_t + \pi(i^*_t - \bar{i}) + \eta \frac{s_t - \beta_0 - \beta_1 i^*_t}{\beta_2}}{\omega + \pi + \gamma + \eta} + i^*_t - s_t - i}{\tau\sigma^2} - B$$

Since we have only one price ratio to consider (the price of foreign currency in terms of domestic currency), it is sufficient to look at how aggregate demand changes when the price changes. The stability condition is that aggregate excess demand for foreign currency goes down with an increase in the price of foreign exchange:

$$\begin{aligned}
\frac{dZ(s_t)}{ds_t} &= -\frac{1}{\tau\sigma^2} + \frac{(\beta_1+\beta_2)\,\lambda\eta}{\beta_2\,(\omega+\pi+\gamma+\eta)\,\tau\sigma^2} < 0 \qquad (5.7)\\
&\Leftrightarrow \frac{\beta_1+\beta_2}{\beta_2} < \frac{\omega+\pi+\gamma+\eta}{\lambda\eta}
\end{aligned}$$

We now calculate the Fama coefficient. The coefficient is just the covariance of the change in the exchange rate and the forward premium divided by the variance of the forward premium. Since the variance of the forward premium is always positive, the sign of the Fama coefficient is given by the sign of the covariance between the exchange rate change and the interest rate differential. We can calculate it directly:

$$cov(s_{t+1}-s_t, i-i_t^*) = -cov(s_{t+1}-s_t, i_t^*) \qquad (5.8)$$

Substituting in the equation for the exchange rate at dates t and $t+1$, and replacing the interest rate differential with the random variables which determine the interest rate, we compute the coefficient to be:

$$\frac{\gamma + \pi (\beta_1 + \beta_2)(1 - \lambda)}{\pi + \gamma} \tag{5.9}$$

Results from simulations are presented in two tables (see appendix). In both tables, all results come from stable equilibria. The first table shows comparative statics on the persistence of the persistent shock to the interest rate. The Fama coefficient is decreasing in the degree of persistence of the persistent component of the interest rate shock. The more persistent the shock (higher λ), the greater the importance of private information and thus the less the exchange rate will fully reflect all available information in the economy immediately. In this case, the average expectation of the price of foreign currency will move in the direction of the foreign interest rate shock after the shock has occurred, offsetting the standard interest rate differential effect (with constant average expectations). Thus, average beliefs will appreciate for a country with a higher interest rate, causing the exchange rate to depreciate less than one-to-one with the interest rate differential.

Next we turn to the second table of simulation results where we consider comparative statics on the variance of the liquidity shock to foreign currency demand, the variance of private information, the variance of the transitory shock to the interest rate, and the variance of the persistent shock to the interest rate. A decrease in the transitory shock to foreign currency demand raises the coefficient on foreign currency demand shocks, β_3. The lower the variance of the transitory shock to currency demand, the greater a currency demand shock of a given size will be misperceived as a persistent shock to the interest rate and thus the larger the effect on the exchange rate. This is somewhat different from the Walrasian case where the coefficient on the exchange rate shock was exactly one. As mentioned before, this comes about because investors now extract information from exchange rate movements and thus over-react to a temporary demand shock for foreign currency. However, the impact of a change in the variance of the currency demand shock upon the Fama Coefficient is not large. An increase in variance of the private signal attenuates the Fama coefficient. Greater variance leads to less weight being placed on private information; thus, private information will enter into the exchange rate more slowly, attenuating the Fama coefficient. A decrease in the transitory interest rate shock variance increases the response of the exchange rate to an interest rate shock. Investors are more likely to think that a given shock to the interest rate is driven by a persistent rather than transitory shock and this gets capitalized into the exchange rate. Also, this increases the quality of the interest rate as well as the exchange rate as signals of the persistent component of the interest rate shock. As a result of the increase in quality of publicly observed information, the lack of initial incorporation of private information into the price of foreign currency does not have as large an impact; the exchange rate more quickly reflects the actual shock and the Fama coefficient is closer to one. Lastly, an increase in the variance of the persistent shock to the exchange rate has a non-monotonic effect on the Fama coefficient. This is due to the dual role of persistent shock variance: it guides both the informativeness of signals as well as the importance of information in prediction. On the one hand, an increased variance lowers the quality of the exchange rate and interest rate signals, leading to slower information release and attenuation of the Fama coefficient. On the other hand, a higher variance also means that persistent shocks play a less important role in exchange rate determination. In the limit, with infinite variance of the persistence component of the interest rate shock, knowledge of the shock is not at all useful for prediction. In this case, the Fama coefficient returns to one because private information is not slowly released into the exchange rate in an exchange rate which never reflects private information. The simulation results are presented below:

6. CONCLUSION

In this paper, we present an international financial model of investors with heterogeneous information. It has a more intuitive information structure than Bacchetta and Van Wincoop (2004) and is also more tractable. In addition, we use the structure to explain the forward premium bias whereas they use their structure to explain exchange rate disconnect. The current version of the paper is able to show generic attenuation of the Fama coefficient, improving upon the learning and time-varying risk premia models which have come before. Short-comings of the paper include (1.) current inability to generate negative coefficients for reasonable parameter values, and (2.) exogeneity of the interest rate. Future research could modify the current setup to endogenize the interest rate with a monetary model and also allow for an exchange rate with a unit root where an exchange rate with a unit root. Heterogeneous information may also be useful for explaining other puzzles in international macroeconomics like the prevalence of domestic consumer-price stickiness of foreign country imports and the existence of j-curves in international trade.

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7. APPENDIX

7.1. Tables

Simulation Results I

Persistence (λ)	0.5	0.9	0.99	0.9999
Dem. Var. (σ_N^2)	5	5	5	5
Private Signal Var. ($\frac{1}{\omega}$)	10	10	10	10
Trans. Interest Rate Var. ($\frac{1}{\pi}$)	5	5	5	5
Persistent. Shock Var. (σ_μ^2)	5	5	5	5
β_1	1.3217	2.2311	4.389	13.7708
β_2	0.0906	0.6784	5.6964	165.9154
β_3	1.0309	1.6645	5.4194	48.2661
Fama Coefficient	0.8321	0.4041	0.1184	0.0182

Simulation Results II

Persistence (λ)	0.5	0.5	0.5	0.5	0.5
Dem. Var. (σ_N^2)	1	5	5	5	5
Private Signal Var. ($\frac{1}{\omega}$)	10	1	10	10	10
Trans. Interest Rate Var. ($\frac{1}{\pi}$)	5	5	1	5	5
Persistent. Shock Var. (σ_μ^2)	5	5	5	10	1
β_1	0.9682	1.0057	1.9339	1.4897	1.0258
β_2	0.0878	0.8853	0.0093	0.152	0.0064
β_3	1.3955	1.0296	1.0003	1.0793	1.0002
Fama Coefficient	0.7303	0.9688	0.9724	0.8492	0.9755

7.2. Method of Undetermined Coefficients: Walrasian Case

$$s_t = \beta_0 + \beta_1 i_t^* + \beta_2 \epsilon_t + \beta_3 N_t$$

$$s_t = \overline{E_t}\left[\beta_0 + \beta_1 i_{t+1}^* + \beta_2 \epsilon_{t+1} + \beta_3 N_{t+1}\right] + i_t^* - i + N_t - B\tau\sigma^2$$

$$= \beta_0 + \beta_1 \lambda R \epsilon_t + \beta_2 \lambda R \epsilon_t + i_t^* - i + N_t - B\tau\sigma^2$$

$$\Rightarrow \beta_1 = 1$$

$$\beta_2 = \beta_1 \lambda R + \beta_2 \lambda R = \lambda R + \beta_2 \lambda R \Rightarrow \beta_2 = \frac{\lambda R}{1-\lambda R}$$

$$\beta_3 = 1$$

$$\Rightarrow cov(s_{t+1} - s_t, i_t^* - i_t) =$$

$$cov\left[\begin{array}{c}\beta_0+\lambda\epsilon_t+\frac{\lambda^2R}{1-\lambda R}\epsilon_t+v_{t+1}+N_{t+1}-\beta_0-\epsilon_t-\frac{\lambda R\epsilon_t}{1-\lambda R}-v_t-N_t,\\ i-\epsilon_t-v_t-N_t\end{array}\right]$$

$\Rightarrow cov(s_{t+1}-s_t,i_t^*-i_t)=\frac{1-\lambda}{1-\lambda R}\sigma_\epsilon^2+\frac{1}{\pi}=\frac{1-\lambda}{1-\lambda R}\frac{\sigma_\mu^2}{1-\lambda^2}+\frac{1}{\pi}$

where $R=\frac{\omega}{\gamma+\omega}$

7.3. Method of Undetermined Coefficients: Rational Expectations Case

$E\epsilon_t=\frac{\omega x_{jt}+\pi(i_t^*-\bar{i})+\eta\frac{s_t-\beta_0-\beta_1 i_t^*}{\beta_2}}{\omega+\pi+\gamma+\eta}$

But $s_t=\overline{E_t}s_{t+1}+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t=$

$\Rightarrow s_t=\overline{E_t}\left[\beta_0+\beta_1 i_{t+1}^*+\beta_2\epsilon_{t+1}+\beta_3N_{t+1}\right]+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

$\Rightarrow s_t=\beta_0+\beta_1\overline{E_t}\epsilon_{t+1}+\beta_2\overline{E_t}\epsilon_{t+1}+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

$\Rightarrow s_t=\beta_0+(\beta_1+\beta_2)\overline{E_t}\epsilon_{t+1}+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

$\Rightarrow s_t=\beta_0+(\beta_1+\beta_2)\overline{E_t}\left(\lambda\epsilon_t+\mu_{t+1}\right)+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

$\Rightarrow s_t=\beta_0+(\beta_1+\beta_2)\lambda\overline{E_t}\epsilon_t+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

which equals $s_t=\beta_0+(\beta_1+\beta_2)\lambda\frac{\omega\epsilon_t+\pi(i_t^*-\bar{i})+\eta\frac{s_t-\beta_0-\beta_1 i_t^*}{\beta_2}}{\omega+\pi+\gamma+\eta}+i_t^*-i-\frac{B\sigma^2}{\tau}+N_t$

or

$$\left\{ \begin{array}{c} \left(1 - \frac{\eta(\beta_1+\beta_2)\lambda}{\beta_2[\omega+\pi+\gamma+\eta]}\right) s_t = \left[\beta_0 - i - \frac{B\sigma^2}{\tau} - \frac{\beta_0\eta(\beta_1+\beta_2)\lambda}{\beta_2[\omega+\pi+\gamma+\eta]} - \frac{(\beta_1+\beta_2)\lambda\pi\bar{i}}{\omega+\pi+\gamma+\eta}\right] + \\ \left[\frac{(\beta_1+\beta_2)\lambda\left(\pi - \frac{\eta\beta_1}{\beta_2}\right)+\omega+\pi+\gamma+\eta}{\omega+\pi+\gamma+\eta}\right] i_t^* + \left[\frac{(\beta_1+\beta_2)\lambda\omega}{\omega+\pi+\gamma+\eta}\right]\epsilon_t + N_t \end{array} \right\}$$

Thus

$$\left\{ \begin{array}{c} \frac{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda}{\beta_2[\omega+\pi+\gamma+\eta]} s_t = \left[\beta_0 - i - \frac{B\sigma^2}{\tau} - \frac{\beta_0\eta(\beta_1+\beta_2)\lambda}{\beta_2[\omega+\pi+\gamma+\eta]} - \frac{(\beta_1+\beta_2)\lambda\pi\bar{i}}{\omega+\pi+\gamma+\eta}\right] + \\ \left[\frac{(\beta_1+\beta_2)\lambda\left(\pi - \frac{\eta\beta_1}{\beta_2}\right)+\omega+\pi+\gamma+\eta}{\omega+\pi+\gamma+\eta}\right] i_t^* + \left[\frac{(\beta_1+\beta_2)\lambda\omega}{\omega+\pi+\gamma+\eta}\right]\epsilon_t + N_t \end{array} \right\}$$

and:

$$\left\{ \begin{array}{c} s_t = \frac{\left[\beta_0 - i - \frac{B\sigma^2}{\tau} - \frac{\beta_0\eta(\beta_1+\beta_2)\lambda}{\beta_2[\omega+\pi+\gamma+\eta]} - \frac{(\beta_1+\beta_2)\lambda\pi\bar{i}}{\omega+\pi+\gamma+\eta}\right]}{\frac{\beta_2[\omega+\pi+\gamma+\eta]-(\beta_1+\beta_2)\lambda\eta}{\beta_2[\omega+\pi+\gamma+\eta]}} + \beta_2\left[\frac{(\beta_1+\beta_2)\lambda\left(\pi - \frac{\eta\beta_1}{\beta_2}\right)+\omega+\pi+\gamma+\eta}{\beta_2(\omega+\pi+\gamma+\eta)-\eta(\beta_1+\beta_2)\lambda}\right] i_t^* + \\ \left[\frac{\beta_2(\beta_1+\beta_2)\lambda\omega}{(\omega+\pi+\gamma+\eta)\beta_2-\eta(\beta_1+\beta_2)\lambda}\right]\epsilon_t + \frac{\beta_2[\omega+\pi+\gamma+\eta]}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda} N_t \end{array} \right\}$$

So:(1.) $\beta_1 = \beta_2 \frac{(\beta_1+\beta_2)\lambda\left(\pi - \frac{\eta\beta_1}{\beta_2}\right)+\omega+\pi+\gamma+\eta}{\beta_2(\omega+\pi+\gamma+\eta)-\eta(\beta_1+\beta_2)\lambda}$

(2.) $\beta_2 = \frac{\beta_2(\beta_1+\beta_2)\lambda\omega}{(\omega+\pi+\gamma+\eta)\beta_2-\eta(\beta_1+\beta_2)\lambda}$

(3.) $\beta_3 = \frac{\beta_2[\omega+\pi+\gamma+\eta]}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda}$

(4.) $\beta_0 = \frac{\left[\beta_0[\omega+\pi+\gamma+\eta]+(i_t^*-i)[\omega+\pi+\gamma+\eta]-\frac{B\sigma^2}{\tau}[\omega+\pi+\gamma+\eta]-\frac{\beta_0\eta(\beta_1+\beta_2)\lambda}{\beta_2}-(\beta_1+\beta_2)\lambda\pi\bar{i}\right]}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda}$

$\Rightarrow (\omega+\pi+\gamma+\eta)\beta_2 - \eta(\beta_1+\beta_2)\lambda = (\beta_1+\beta_2)\lambda\omega$ (from (2.))

$\Rightarrow (\omega+\pi+\gamma+\eta)\beta_2 = \eta(\beta_1+\beta_2)\lambda + (\beta_1+\beta_2)\lambda\omega$

$\Rightarrow \beta_2 = \frac{(\beta_1+\beta_2)\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}$

$\Rightarrow \frac{\beta_2}{\beta_1+\beta_2} = \frac{\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}$

$\Rightarrow \beta_1 = \frac{\lambda(\beta_2\pi-\eta\beta_1)+(\omega+\pi+\gamma+\eta)\frac{\beta_2}{\beta_1+\beta_2}}{\frac{\beta_2}{\beta_1+\beta_2}(\omega+\pi+\gamma+\eta)-\eta\lambda}$ (dividing (1.) by $\frac{\beta_1+\beta_2}{\beta_1+\beta_2}$)

$\Rightarrow \beta_1 = \frac{\lambda(\beta_2\pi-\eta\beta_1)+(\omega+\pi+\gamma+\eta)\frac{\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}}{\frac{\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}(\omega+\pi+\gamma+\eta)-\eta\lambda} = \frac{\lambda(\beta_2\pi-\eta\beta_1)+\lambda(\omega+\eta)}{\lambda(\omega+\eta)-\eta\lambda}$

$\Rightarrow \beta_1 = \frac{\beta_2\pi-\eta\beta_1+\omega+\eta}{\omega} = 1 + \frac{\beta_2\pi-\eta\beta_1+\eta}{\omega}$

$\Rightarrow \beta_1\left(\frac{\omega+\eta}{\omega}\right) = 1 + \frac{\beta_2\pi+\eta}{\omega}$

$\Rightarrow \beta_1 = \frac{\beta_2\pi+\eta+\omega}{\omega+\eta} = 1 + \frac{\beta_2\pi}{\omega+\eta}$

$\Rightarrow \beta_3 = \frac{\beta_2[\omega+\pi+\gamma+\eta]}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda} = \frac{\frac{\beta_2}{\beta_1+\beta_2}[\omega+\pi+\gamma+\eta]}{\frac{\beta_2}{\beta_1+\beta_2}[\omega+\pi+\gamma+\eta]-\eta\lambda}$

$\Rightarrow \beta_3 = \frac{\lambda(\omega+\eta)}{\lambda(\omega+\eta)-\eta\lambda} = \frac{\omega+\eta}{\omega} = 1 + \frac{\eta}{\omega}$

$\beta_0 = \frac{\left(\beta_0+i_t^*-i-\frac{B\sigma^2}{\tau}\right)[\omega+\pi+\gamma+\eta]-\frac{\beta_0\eta(\beta_1+\beta_2)\lambda}{\beta_2}-(\beta_1+\beta_2)\lambda\pi\bar{i}}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda}$

$\Rightarrow \beta_0 = \frac{\left(\beta_0+i_t^*-i-\frac{B\sigma^2}{\tau}-\frac{\beta_0\eta\lambda}{\lambda(\omega+\eta)}\right)[\omega+\pi+\gamma+\eta]-(\beta_1+\beta_2)\lambda\pi\bar{i}}{\beta_2[\omega+\pi+\gamma+\eta]-\eta(\beta_1+\beta_2)\lambda}$

$\Longrightarrow \beta_0 = \frac{\left(\beta_0+i_t^*-i-\frac{B\sigma^2}{\tau}-\frac{\beta_0\eta\lambda}{\lambda(\omega+\eta)}\right)\frac{[\omega+\pi+\gamma+\eta]}{(\beta_1+\beta_2)}-\lambda\pi\bar{i}}{\beta_2\lambda(\omega+\eta)-\eta\lambda}$

(1.) $\beta_1 = 1 + \frac{\beta_2\pi}{\omega+\eta}$

(2.) $\beta_2 = \frac{(\beta_1+\beta_2)\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}$

(3.) $\beta_3 = 1 + \frac{\eta}{\omega}$

(4.) $\beta_0 = \frac{\left(\beta_0+i_t^*-i-\frac{B\sigma^2}{\tau}-\frac{\beta_0\eta\lambda}{\lambda(\omega+\eta)}\right)\frac{[\omega+\pi+\gamma+\eta]}{(\beta_1+\beta_2)}-\lambda\pi\bar{i}}{\beta_2\lambda(\omega+\eta)-\eta\lambda}$

$\beta_1 = \frac{\omega+\eta+\beta_2\pi}{\omega+\eta}$

and $\beta_2 = \frac{(\beta_1+\beta_2)\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta}$

$\Rightarrow \beta_2 = \frac{\left(\frac{\omega+\eta+\beta_2\pi}{\omega+\eta}+\beta_2\right)\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta} = \frac{(\omega+\eta+\beta_2(\omega+\eta+\pi))\lambda}{\omega+\pi+\gamma+\eta}$

$$\Rightarrow \beta_2 (\omega + \pi + \gamma + \eta) = (\omega + \eta + \beta_2 (\omega + \eta + \pi)) \lambda$$

$$\Rightarrow \beta_2 [\gamma + (1 - \lambda)(\omega + \pi + \eta)] = \lambda (\omega + \eta)$$

$$\Rightarrow \beta_2 = \frac{\lambda(\omega+\eta)}{\gamma+(1-\lambda)(\omega+\pi+\eta)}$$

$$\beta_1 = \frac{\omega+\eta+\beta_2\pi}{\omega+\eta} = \frac{\omega+\eta+\frac{\lambda(\omega+\eta)}{\gamma+(1-\lambda)(\omega+\pi+\eta)}\pi}{\omega+\eta} = 1 + \frac{\lambda\pi}{\gamma+(1-\lambda)(\omega+\pi+\eta)}$$

$$\begin{aligned}
\eta &= \left(\frac{\beta_3}{\beta_2}\sigma_N\right)^2 \\
\beta_1 &= 1 + \frac{\lambda\pi}{(1-\lambda)(\omega+\eta+\pi)+\gamma} \\
\beta_2 &= \frac{\lambda(\omega+\eta)}{\gamma+(1-\lambda)(\omega+\pi+\eta)} \\
\beta_3 &= 1 + \frac{\eta}{\omega} \\
\beta_0 &= \frac{\left(\beta_0 + i_t^* - i - \frac{B\sigma^2}{\tau} - \frac{\beta_0\eta\lambda}{\lambda(\omega+\eta)}\right)\frac{[\omega+\pi+\gamma+\eta]}{(\beta_1+\beta_2)} - \lambda\pi\bar{i}}{\beta_2\lambda(\omega+\eta) - \eta\lambda} \\
\sigma^2 &= \beta_1^2\frac{1}{\pi} + (\beta_1+\beta_2)^2\frac{1}{\gamma} + \beta_3^2\sigma_N^2
\end{aligned}$$

7.4. Fama Calculation

$$cov(s_{t+1} - s_t, i - i_t^*) = cov(s_{t+1} - s_t, -i_t^*) = cov(s_{t+1} - s_t, -i_t^*)$$

$$=cov(\beta_0 + \beta_1 i_{t+1}^* + \beta_2\epsilon_{t+1} + \beta_3 N_{t+1} - \beta_0 - \beta_1 i_t^* - \beta_2\epsilon_t - \beta_3 N_t, -i_t^*)$$

$$=cov\left(\beta_1\left(i_{t+1}^* - i_t^*\right) + \beta_2(\epsilon_{t+1} - \epsilon_t) + \beta_3(N_{t+1} - N_t), -i_t^*\right)$$

$$=cov\left(\beta_1\left(i_{t+1}^* - i_t^*\right) + \beta_2(\epsilon_{t+1} - \epsilon_t), i_t^*\right)$$

$$=\text{-}cov\left(\beta_1\left(\lambda\epsilon_t+\mu_{t+1}+v_{t+1}-\epsilon_t-v_t\right)+\beta_2\left(\lambda\epsilon_t+\mu_{t+1}-\epsilon_t\right),\epsilon_t+v_t\right)$$

$$=\text{-}cov\left(\beta_1\left(\lambda-1\right)\epsilon_t+\beta_2\left(\lambda-1\right),\epsilon_t\right)+cov(v_t,v_t)$$

$$=\tfrac{(\beta_1+\beta_2)(1-\lambda)}{\gamma}+\tfrac{1}{\pi}$$

$$var(i-i_t^*)=\tfrac{1}{\pi}+\tfrac{1}{\gamma}$$

Fama Coefficient$=\frac{cov(S_{t+1}-S_t,i-i_t^*)}{V(i-i_t^*)}=\left[\frac{(\beta_1+\beta_2)(1-\lambda)}{\gamma}+\frac{1}{\pi}\right]\left(\frac{\pi\gamma}{\pi+\gamma}\right)=\frac{\gamma+\pi(\beta_1+\beta_2)(1-\lambda)}{\pi+\gamma}$

7.5. Full Revelation Rational Expectations

Demand for bonds (note will never demand just currency... get higher rate of return conditional upon currency by being invested in bonds):

$$\frac{\int\int\tau[i_t^*+s_{t+1}-s_t-i]d\mu_{t+1}dv_t}{\sigma_{jt}^2}$$

Summing over individual bond demands, we have supply equals demand:

$$\int_0^1\frac{\int\int\tau[i_t^*+s_{t+1}-s_t-i]d\mu_{t+1}dv_{t+1}}{\sigma_{jt}^2}dj=B$$

or

$$\int_0^1\frac{\tau\left[i_t^*+\int s_{t+1}d\mu_{t+1}-s_t-i\right]}{\sigma_{jt}^2}dj=B$$

or (assuming constant subjective variances across agents and over time)

$$\frac{\tau\left[i_t^*+\int\int s_{t+1}d\mu_{t+1}dj-s_t-i\right]}{\sigma^2}=B$$

$$s_t=\int_0^1\int s_{t+1}d\mu_{t+1}dj+i_t^*-i-\tfrac{B\sigma^2}{\tau}$$

$$s_t=\overline{E_t}s_{t+1}+i_t^*-i-\tfrac{B\sigma^2}{\tau}+N_t$$

Guess#1

$s_t = \beta_1 \epsilon_t + \beta_2 i_t^* + \beta_3$

In this case, we can fully solve for $\epsilon_t = \frac{s_1 - \beta_2 i_t^* - \beta_3}{\beta_1}$

Plugging this into the exchange rate equation (which we got from individual bond demand), we get:

$s_t = \int_0^1 \int s_{t+1} d\mu_{t+1} dj + i_t^* - i - \frac{B\sigma^2}{\tau} = \overline{E}_t \left(\beta_1 \epsilon_{t+1} + \beta_2 i_{t+1}^* + \beta_3\right) + i_t^* - i - \frac{B\sigma^2}{\tau}$

$= \beta_1 \lambda \epsilon_t + \beta_2 \lambda \epsilon_t + \beta_3 + i_t^* - i - \frac{B\sigma^2}{\tau}$

So, $\beta_2 = 1$

This means that $(\beta_1 + \beta_2)\lambda = \beta_1 \Rightarrow \lambda = (1-\lambda)\beta_1$ (using that $\beta_2 = 1$) $\Rightarrow \beta_1 = \frac{\lambda}{1-\lambda}$

Lastly, $\beta_3 = \beta_3 - i - \frac{B\sigma^2}{\tau}$

No solutions? Is this the wrong guess? It works with $B = 0$ and $i = 0$

7.6. Solving for the Variance of the Prior

$\frac{1}{\gamma} = \sigma^2 = V(\epsilon_t) = V(\lambda \epsilon_{t-1} + \mu_t) = \lambda^2 \sigma^2 + \sigma_\mu^2 \Rightarrow$

$\sigma^2 = \lambda^2 \sigma^2 + \sigma_\mu^2$

or $\sigma^2 = \frac{\sigma_\mu^2}{1-\lambda^2} = \frac{1}{\gamma}$

7.7. Proof of Bounds on Fama Coefficient in Rational Expectations Equilibria

Since the formula for the Fama coefficient is: $\frac{\beta_1+\pi(\beta_1+\beta_2)(1-\lambda)}{\pi+\gamma}$

Sufficient to show that (1.) $\beta_1, \beta_2 > 0$ and (2.) $(1-\lambda)(\beta_1+\beta_2) < 1$.

7.7.1. $\beta_1, \beta_2 > 0$

See calculation of coefficients and note that $\gamma, \lambda, \omega, \pi, \eta > 0$ and $\lambda < 1$. Also, note that both β_1 and $\beta_2 > 0 \Rightarrow \beta_1 + \beta_2 > 0 \Rightarrow \frac{\beta_1+\pi(\beta_1+\beta_2)(1-\lambda)}{\pi+\gamma} > 0$

7.7.2. $(1-\lambda)(\beta_1+\beta_2) < 1$

$\beta_2 = \frac{(\beta_1+\beta_2)\lambda(\omega+\eta)}{\omega+\pi+\gamma+\eta} \Rightarrow \beta_1 + \beta_2 = \frac{\omega+\pi+\gamma+\eta}{\lambda(\omega+\eta)}\beta_2$

This means that $\beta_1 + \beta_2 = \frac{\omega+\pi+\gamma+\eta}{\lambda(\omega+\eta)} \frac{\lambda(\omega+\eta)}{\gamma+(1-\lambda)(\omega+\pi+\eta)} = \frac{\omega+\pi+\gamma+\eta}{\gamma+(1-\lambda)(\omega+\pi+\eta)}$

Thus, $(1-\lambda)(\beta_1+\beta_2) = \frac{[\omega+\pi+\gamma+\eta](1-\lambda)}{\gamma+(1-\lambda)(\omega+\pi+\eta)}$

but $\frac{[\omega+\pi+\gamma+\eta](1-\lambda)}{\gamma+(1-\lambda)(\omega+\pi+\eta)} < 1$

$\Rightarrow (1-\lambda)(\beta_1+\beta_2) < 1$

$\Rightarrow \frac{\beta_1+\pi(\beta_1+\beta_2)(1-\lambda)}{\pi+\gamma} < 1$