Economics 326: Input Demands

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October 24, 2012

Outline

1. Terms

2. Input Demands

1 Terms

- Labor Productivity: Output per unit of labor.

$$\frac{Y(K, L)}{L}$$

What is the labor productivity of the US? Output is rouhgly US$14.7 trillion. The labor force is roughly 153 million people. Therefore, the aggregate labor productivity for the US is:

$$\$96,000$$

How is this different from GDP per capita. The main difference is that we are measuring output per working person not output per person living in the US.

- Marginal Product: The additional output from an addition unit of input:

$$\begin{aligned} \frac{\partial F}{\partial L} &= \text{Marginal Product of Labor} \\ \frac{\partial F}{\partial K} &= \text{Marginal Product of Capital} \\ \frac{\partial F}{\partial T} &= \text{Marginal Product of Land} \end{aligned}$$

- Declining Marginal Productivity: We usually assume that the marginal product of an input declines with input usage. Think about farm production. We have land, labor and capital. If we start with 2 people and 40 acres of land and zero capital and we then buy a tractor, output will increase a lot. If we then buy a second tractor, output might increase less than with one tractor because the second trator will probably not be used as often as the first. If we then buy a third tractor, the increase in output will be very low. (Show graph).

2 Input Demands

- The producer solves the profit maximization problem choosing the amount of capital and labor to employ. In doing so, the producer derives input demands. These are the analogues of Marshallian Demand in consumer theory. They are a function of prices of inputs and the price of output.

- We assume (for now) that firms act competitively. This is a behavioral assumption. We assume that firms act as if they have no impact on price. In many models, producers may actually have a small impact on price. However, they do not take into account their impact on price when they choose how labor and capital to employ and how much to produce.

- Producers maximize (choosing K,L):

$$\Pi(K, L) = pF(K, L) - wL - rK$$

- The competitive assumption is that p is not a function of K and L but rather an exogenous parameter.

- The three parameters are p, w, r. The two variables are K, L.

- There are two main aspects of a production function:

  – Returns to Scale: No Analogue on the Consumer Side

– Rate of Technical Substitution: Analogy with Marginal Rate of Substitution. Rate of technical substitution is the rate at which a firm trades off one input for another holding output constant.

$$F(K, L) = \bar{Y}$$

Totally differentiating, we get:

$$\begin{aligned} \frac{\partial F}{\partial K} dK + \frac{\partial F}{\partial L} dL &= 0 \\ &\implies \frac{dK}{dL} = -\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} \end{aligned}$$

Interpretation: if we increase labor by one unit, then we have to increase capital by $-\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}}$units (or increase capital by $\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}}$ units) in order to keep output constant. If $\frac{\partial F}{\partial L}$ (the marginal product of labor) is large, then you are gaining a lot of output when you gain one unit of labor. The rate of technical substitution is defined (similar to the marginal rate of substitutions) as a positive number:

$$RTS = -\frac{dK}{dL} = \frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}}$$

- Example with Cobb-Douglas:

$$F(K, L) = K^{\alpha} L^{\beta}$$

- What is the returns to scale. Lets increase capital and labor by a factor λ :

$$\begin{aligned} F(\lambda K, \lambda L) &= (\lambda K)^{\alpha} (\lambda L)^{\beta} \\ &= \lambda^{\alpha} K^{\alpha} \lambda^{\beta} L^{\beta} \\ &= \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} \end{aligned}$$

  In other words if we increase capital and labor by a factor λ, output goes up by same factor $\lambda^{\alpha+\beta}$. This production function exhibits constant returns to scale. Notice that when $\beta = 1 - \alpha$, then output goes up by proportionally with inputs - if we increase all factors by λ, output also goes up by λ :

$$F(\lambda K, \lambda L) = \lambda K^{\alpha} L^{1-\alpha}$$

- Lets try $\lambda = 2$. What happens if we double purchases of labor and capital:

$$\begin{aligned} F(2K, 2L) &= (2K)^{\alpha} (2L)^{1-\alpha} \\ &= 2^{\alpha} K^{\alpha} 2^{1-\alpha} L^{1-\alpha} \\ &= 2K^{\alpha} L^{1-\alpha} \end{aligned}$$

- Now we solve for labor and capital demand. It is very similar to the consumer side.

- First note that we should check second order conditions to make sure we have a global maximum. Essentially, the second derivative of the profit function (and thus the production function) should be negative. We will show this using a simple example with only one factor of production. The second order condition being satisfied basically is the same as assuming a declining marginal product. Then:

$$\begin{aligned}
\Pi(L) &= pF(L) - wL \\
\frac{\partial \Pi(L)}{\partial L} &= p\frac{\partial F}{\partial L} - w \\
\frac{\partial^2 \Pi(L)}{\partial L^2} &= \frac{\partial \frac{\partial \Pi(L)}{\partial L}}{\partial L} = p\frac{\partial^2 F}{\partial L^2} < 0 \text{ if } \frac{\partial^2 F}{\partial L^2} < 0
\end{aligned}$$

- Returning to two factors., we have two first order conditions - one for each input (remember, no Lagrange multiplier here!):

$$\begin{aligned}
& \max_{K,L} pF(K,L) - wL - rK \\
p\frac{\partial F}{\partial L} - w &= 0 \\
&\Longrightarrow \frac{\partial F}{\partial L} = \frac{w}{p}
\end{aligned}$$

- Intuition: the marginal product of labor is equal to the real wage (show graph) or the marginal revenue product $\left(p\frac{\partial F}{\partial L}\right)$ is equal to the wage (w).

- Similarly, we can derive the analogous formula for capital:

$$\begin{aligned} & \max_{K,L} pF(K,L) - wL - rK \\ p\frac{\partial F}{\partial K} - r &= 0 \\ &\Longrightarrow \frac{\partial F}{\partial K} = \frac{r}{p} \end{aligned}$$

- Intuition: the marginal product of labor is equal to the real wage (show graph) or the marginal revenue product $\left(p\frac{\partial F}{\partial K}\right)$ is equal to the wage (r).

- From this, we can compute a formula for the RTS (rate of technical substitution):

$$\frac{dK}{dL} = \frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} = \frac{\frac{w}{p}}{\frac{r}{p}} = \frac{w}{r}$$

- In the particular case of Cobb-Douglas, we get:

$$\max_{K,L} pK^{\alpha}L^{\beta} - wL - rK$$

$$\frac{\partial \Pi}{\partial L} = p\beta K^{\alpha}L^{\beta-1} - w = 0 \tag{1}$$

$$\frac{\partial \Pi}{\partial K} = p\alpha K^{\alpha-1}L^{\beta} - r = 0 \tag{2}$$

- Now we solve simultaneously for K and L. First, we re-express the first order condtions from equations (1) and (2):

$$p\beta K^{\alpha}L^{\beta-1} = w \tag{3}$$

$$p\alpha K^{\alpha-1}L^{\beta} = r \tag{4}$$

- Now we divide equation (3) by equation (4):

$$\frac{K\beta}{L\alpha} = \frac{w}{r}$$

which we can re-express as:

$$K = \frac{\alpha w}{\beta r}L \tag{5}$$

which we then plug in to (3) :

$$p\beta K^{\alpha} L^{\beta-1} = p\beta \left(\frac{\alpha w}{\beta r} L\right)^{\alpha} L^{\beta-1} = w \qquad (6)$$

- Re-expressing equation (6), we get:

$$\begin{aligned} L^{\alpha+\beta-1} &= \frac{w}{p\beta}\left(\frac{\beta r}{\alpha w}\right)^{\alpha} \\ &= \frac{1}{p}\left(\frac{w}{\beta}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha} \end{aligned}$$

- This leads to a solution for the input demand function for labor $L^*(p, r, w)$:

$$\begin{aligned} L^*(p, r, w) &= \left[\frac{1}{p}\left(\frac{w}{\beta}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}\right]^{\frac{1}{\alpha+\beta-1}} \\ &= p^{\frac{1}{1-\alpha-\beta}}\left(\frac{w}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}}\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} \end{aligned}$$

- Now we can plug our solution for L^* into (5) :

$$\begin{aligned}
K^* &= \frac{\alpha w}{\beta r} L^* \\
&= \frac{\alpha w}{\beta r} \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \left(\frac{w}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}} \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} \\
&= \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \left(\frac{w}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}+1} \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}-1} \\
&= p^{\frac{1}{1-\alpha-\beta}} \left(\frac{w}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}} \left(\frac{r}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}}
\end{aligned}$$

- Comparative statics:

$$\frac{\partial K^*}{\partial p} = \frac{1}{1-\alpha-\beta} p^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left(\frac{w}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}} \left(\frac{r}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}} > 0$$

greater than zero if $\alpha + \beta > 1$

$\Longrightarrow$ decreasing returns to scale

- $$\frac{\partial K^*}{\partial w} = \frac{\beta}{\alpha+\beta-1} p^{-\frac{1}{\alpha+\beta-1}} w^{\frac{1-\alpha}{\alpha+\beta-1}} \left(\frac{r}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}} > 0$$

  greater than zero if if $\alpha + \beta - 1 > 0$

  $\implies$ decreasing returns to scale

- We can also compute $\frac{\partial K^*}{\partial r}, \frac{\partial L^*}{\partial w}, \frac{\partial L^*}{\partial r}, \frac{\partial L^*}{\partial p}$.

- Suppose we plug K^* and L^* back into $Y(K, L)$? What do we get? Optimal output as a function of output prices and input prices.

  $$\begin{aligned} & Y\left(K^*(p, w, r), L^*(p, w, r)\right) \\ = \; & Y^*(p, w, r) \end{aligned}$$

  This function has a famous name in economics. What is it? If we graph $Y^*(p, w, r)$ on p (holding fixed w and r), what is it called?

- In our particular Cobb-Douglas example, we get:

$$
\begin{aligned}
Y^*(p,w,r) &= \\
&\left[p^{\frac{1}{1-\alpha-\beta}}\left(\frac{w}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}}\left(\frac{r}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}}\right]^{\alpha} * \\
&\left[p^{\frac{1}{1-\alpha-\beta}}\left(\frac{w}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}}\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}}\right]^{\beta} \\
&= p^{\frac{\alpha\beta}{1-\alpha-\beta}}\left(\frac{w}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}}\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}}
\end{aligned}
$$

- What is the sign of $\frac{\partial Y^*}{\partial p}$? Does that make intuitive sense?

- What happens with increasing and constant returns to scale?

  - For increasing returns, the production function is convex not concave. A solution to the calculus problem gives a global minimum.

- For constant returns, it is slightly more complicated

- We will cover both of these cases soon.