Analytic Bias Correction for Dynamic Panel Models with Fixed Effects: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

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1 Bias Correction using Higher Order Expansions

In Section 2 we explain the bias of the GMM estimator as a result of the biases of the 2SLS estimators. In this appendix, we attempt eliminating the bias (i) by adopting the second order Taylor type approximation as by Nagar (1959), and Rothenberg (1983); and (ii) by adopting a "long difference" perspective.

For this purpose we first analyze the second order bias of a general minimization estimator b of a single parameter $\beta \in \mathbb{R}$ defined by

$$b = \underset{c \in C}{\operatorname{argmin}}\, Q_n(c) \tag{1}$$

for some $C \subset \mathbb{R}$. The score for the minimization problem is denoted by $S_n(c) \equiv \partial Q_n(c) / \partial c$. The criterion function is assumed to be of the form $Q_n(c) = g(c)' G(c)^{-1} g(c)$ where $g(c)$ and $G(c)$ are defined in Condition 4. The criterion function depends on primitive functions $\delta(w_i, c)$ and $\psi(w_i, c)$ mapping $\mathbb{R}^k \times \mathbb{R}$ into $\mathbb{R}^d$ for $d \geq 1$, where w_i are i.i.d. observations. We assume that $E[\delta(w_i, \beta)] = 0$. We impose the following additional conditions on w_i, δ, and ψ.

Condition 1 *The random variables w_i are i.i.d.*

Condition 2 *The functions $\delta(w, c)$ and $\psi(w, c)$ are three times differentiable in c for $c \in C$ where $C \subset \mathbb{R}$ is a compact set such that $\beta \in \operatorname{int} C$. Assume that $\delta(w_i, c)$ and $\psi(w_i, c)$ satisfy a Lipschitz condition $\|\delta(w_i, c_1) - \delta(w_i, c_2)\| \leq M_\delta(w_i)|c_1 - c_2|$ for some function $M_\delta(.) : \mathbb{R}^k \to \mathbb{R}$ and $c_1, c_2 \in C$ with the same statement holding for ψ. The functions $M_\delta(.)$ and $M_\psi(.)$ satisfy $E[M_\delta(w_i)] < \infty$ and $E\left[|M_\psi(w_i)|^2\right] < \infty$.*

Condition 3 *Let $\delta_j(w_i, c) \equiv \partial^j \delta(w_i, c) / \partial c^j$, $\Psi(w_i, c) \equiv \psi(w_i, c)\psi(w_i, c)'$ and $\Psi_j(w_i, c) \equiv \partial^j \Psi(w_i, c) / \partial c^j$. Then, $\lambda_j(c) \equiv E[\delta_j(w_i, c)]$, and $\Lambda_j(c) \equiv E[\Psi_j(w_i, c)]$ all exist and are finite for $j = 0, ..., 3$. For simplicity, we use the notation $\lambda_j \equiv \lambda_j(\beta)$, $\Lambda_j \equiv \Lambda_j(\beta)$, $\lambda(c) \equiv \lambda_0(c)$ and $\Lambda(c) \equiv \Lambda_0(c)$.*

Condition 4 *Let $g(c) \equiv \frac{1}{n}\sum_{i=1}^n \delta(w_i, c)$, $g_j(c) \equiv \frac{1}{n}\sum_{i=1}^n \delta_j(w_i, c)$, $G(c) \equiv \frac{1}{n}\sum_{i=1}^n \psi(w_i, c)\psi(w_i, c)'$ and $G_j(c) \equiv \frac{1}{n}\sum_{i=1}^n \Psi_j(w_i, c)$. Then $g(c) \xrightarrow{p} E[\delta(w_i, c)]$, $g_j(c) \to E[\delta_j(w_i, c)]$, $G(c) \xrightarrow{p} E[\Psi(w_i, c)]$, and $G_j(c) \xrightarrow{p} E[\Psi_j(w_i, c)]$ for all $c \in C$.*

Our asymptotic approximation of the second order bias of b is based on an approximate estimator $\tilde{b}$ such that $b - \tilde{b} = o_p\left(n^{-1}\right)$. The approximate bias of b is then defined as $E\left[\tilde{b}\right] - \beta$ while the original estimator b need not necessarily possess moments of any order. In order to justify our approximation we need to establish that b is $\sqrt{n}$-consistent and that $S_n(b) = 0$ with probability tending to one. For this purpose we introduce the following additional conditions.

Condition 5 *(i) There exists some finite* $0 < \mathbf{M} < \infty$ *such that the eigenvalues of* $E\left[\Psi\left(w_i, c\right)\right]$ *are contained in the compact interval* $\left[\mathbf{M}^{-1}, \mathbf{M}\right]$ *for all* $c \in C$*; (ii) the vector* $E\left[\delta\left(w_i, c\right)\right] = 0$ *if and only if* $c = \beta$*; (iii)* $\lambda_1 \neq 0$.

Condition 6 *There exists some* $\eta > 0$ *such that* $E\left[M_\delta\left(w_i\right)^{2+\eta}\right] < \infty$, $E\left[M_\psi\left(w_i\right)^{2+\eta}\right] < \infty$, $E\left[\sup_{c \in C}\left\|\delta\left(w_i, c\right)\right\|^{2+\eta}\right] < \infty$*, and* $E\left[\sup_{c \in C}\left\|\psi\left(w_i, c\right)\right\|^{4+\eta}\right] < \infty$.

Condition 5 is an identification condition that guarantees the existence of a unique interior minimum of the limiting criterion function. Condition 6 corresponds to Assumption B of Andrews (1994) and is used to establish a stochastic equicontinuity property of the criterion function.

Theorem 1 *Under conditions 1 - 6, b defined in (1) satisfies* $\sqrt{n}(b - \beta) = O_p(1)$ *and* $S_n(b) = 0$ *with probability tending to 1.*

Proof. See Section 2. ■

Based on Theorem 1 the first order condition for (1) can be characterized by

$$0 = 2 g_1(b)' G(b)^{-1} g(b) - g(b)' G(b)^{-1} G_1(b) G(b)^{-1} g(b) \;\; \text{wp} \;\to\; 1. \tag{2}$$

A second order Taylor expansion of (2) around β leads to a representation of $b - \beta$ up to terms of order $o_p\left(n^{-1}\right)$:

Definition 1

$$\Psi \equiv 3\lambda_1' \Lambda^{-1} \lambda_2 - 3\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1,$$

$$\Upsilon \equiv 2\lambda_1 \Lambda^{-1} \lambda_1, \quad \frac{1}{\sqrt{n}}\Phi \equiv 2\lambda_1' \Lambda^{-1} g,$$

$$\frac{1}{\sqrt{n}}\Xi \equiv 4\left(g_1 - \lambda_1\right)' \Lambda^{-1} \lambda_1 - 2\lambda_1' \Lambda^{-1}\left(G - \Lambda\right) \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g + 2\lambda_2' \Lambda^{-1} g,$$

$$\frac{1}{n}\Gamma \equiv 2\left(g_1 - \lambda_1\right)' \Lambda^{-1} g - 2\lambda_1' \Lambda^{-1}\left(G - \Lambda\right) \Lambda^{-1} g - g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g.$$

In Section 3, it is shown that

$$\sqrt{n}\left(b - \beta\right) = -\frac{1}{\Upsilon}\Phi + \frac{1}{\sqrt{n}}\left(-\frac{1}{\Upsilon}\Gamma + \frac{1}{\Upsilon^2}\Phi\Xi - \frac{\Psi}{\Upsilon^3}\Phi^2\right) + o_p\left(\frac{1}{\sqrt{n}}\right). \tag{3}$$

Ignoring the $o_p\left(\frac{1}{\sqrt{n}}\right)$ term in (3), and taking expectations, we obtain the "approximate mean" of $\sqrt{n}\left(b - \beta\right)$. We present the second order bias of b in the next Theorem.

Theorem 2 *Under Conditions 1-6, the second order bias of b is equal to*

$$-\frac{1}{\sqrt{n}}\frac{1}{\Upsilon}E\left[\Phi\right] - \frac{1}{n}\frac{1}{\Upsilon}E\left[\Gamma\right] + \frac{1}{n}\frac{1}{\Upsilon^2}E\left[\Phi\Xi\right] - \frac{\Psi}{n\Upsilon^3}E\left[\Phi^2\right]. \tag{4}$$

where

$$E[\Phi] = 0,$$

$$E[\Gamma] = 2\,\text{trace}\left(\Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta}\right]\right) - 2\lambda_1' \Lambda^{-1} E\left[\psi_i \psi_i' \Lambda^{-1} \delta_i\right] - \text{trace}\left(\Lambda^{-1} \Lambda_1 \Lambda^{-1} E\left[\delta_i \delta_i'\right]\right),$$

and

$$\begin{aligned} E[\Phi\Xi] &= 8\lambda_1' \Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta}\right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E\left[\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i'\right] \Lambda^{-1} \lambda_1 \\ &\quad -8\lambda_1' \Lambda^{-1} E\left[\delta_i \delta_i'\right] \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 + 4\lambda_1' \Lambda^{-1} E\left[\delta_i \delta_i'\right] \Lambda^{-1} \lambda_2, \end{aligned}$$

and

$$E\left[\Phi^2\right] = 4\lambda_1' \Lambda^{-1} E\left[\delta_i \delta_i'\right] \Lambda^{-1} \lambda_1,$$

Proof. See Section 3. ■

Remark 1 *For the particular case where $\psi_i = \delta_i$, i.e. when b is a CUE, the bias formula (4) exactly coincides with Newey and Smith's (2000).*

We now apply these general results to the GMM estimator of the dynamic panel model. The GMM estimator $\widehat{b}_{GMM}$ can be understood to be a solution to the minimization problem

$$\min_c \left(\frac{1}{n}\sum_{i=1}^n m_i(c)\right)' V_n^{-1} \left(\frac{1}{n}\sum_{i=1}^n m_i(c)\right)$$

for $c \in C$ where C is some closed interval on the real line containing the true parameter value and

$$m_i(c) = \begin{pmatrix} z_{i1}\left(y_{i1}^* - c \cdot x_{i1}^*\right) \\ \vdots \\ z_{i,T-1}\left(y_{i,T-1}^* - c \cdot x_{i,T-1}^*\right) \end{pmatrix}, \qquad V_n = \frac{1}{n}\sum_{i=1}^n \begin{bmatrix} z_{i1} z_{i1}' & & 0 \\ & \ddots & \\ 0 & & z_{i,T-1} z_{i,T-1}' \end{bmatrix}.$$

We now characterize the finite sample bias of the GMM estimator $\widehat{b}_{GMM}$ of the dynamic panel model using Theorem 2. Recall that we assume

Condition 7 $\varepsilon_{it} \overset{i.i.d.}{\sim} \mathcal{N}\left(0, \sigma^2\right)$ *over i and t.*

Condition 8 $y_{i0}|\,\alpha_i \sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta}, \frac{\sigma^2}{1-\beta^2}\right)$ *and $\alpha_i \sim N\left(0, \sigma_\alpha^2\right)$.*

It can be shown that:

Theorem 3 *Under Conditions 7-8 the second order bias of $\widehat{b}_{GMM}$ is equal to*

$$\frac{B_1 + B_2 + B_3}{n} + o\left(\frac{1}{n}\right), \tag{5}$$

where

$$\begin{aligned} B_1 &\equiv \Upsilon_1^{-1} \textstyle\sum_{t=1}^{T-1} \text{trace}\left(\left(\Gamma_t^{zz}\right)^{-1} \Gamma_{t,t}^{\varepsilon xzz}\right) \\ B_2 &\equiv -2\Upsilon_1^{-2} \textstyle\sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{zx\prime} \left(\Gamma_t^{zz}\right)^{-1} \Gamma_{t,s}^{\varepsilon xzz} \left(\Gamma_s^{zz}\right)^{-1} \Gamma_s^{zx} \\ B_3 &\equiv \Upsilon_1^{-2} \textstyle\sum_{t=1}^{T-1} \sum_{s=1}^{T-1} \Gamma_t^{zx\prime} \left(\Gamma_t^{zz}\right)^{-1} B_{3,1}(t,s) \left(\Gamma_s^{zz}\right)^{-1} \Gamma_s^{zx}. \end{aligned}$$

where $\Gamma_t^{zz} \equiv E\left[z_{it} z_{it}'\right], \Gamma_t^{zx} \equiv E\left[z_{it} x_{it}^*\right], \Gamma_{t,s}^{\varepsilon xzz} \equiv E\left[\varepsilon_{it}^* x_{is}^* z_{it} z_{is}'\right],\ B_{3,1}(t,s) \equiv E\left[\varepsilon_{it}^* z_{it} \Gamma_s^{zx\prime} \left(\Gamma_s^{zz}\right)^{-1} z_{is} z_{is}'\right]$ *and* $\Upsilon_1 \equiv \sum_{t=1}^{T-1} \Gamma_t^{zx\prime} \left(\Gamma_t^{zz}\right)^{-1} \Gamma_t^{zx}$.

Proof. See Section 4. ■

Because the summand

$$E\left[E\left[z_{it}x_{it}^{*}\right]' E\left[z_{it}z_{it}'\right]^{-1}\varepsilon_{it}^{*}z_{it}\lambda_s'\Lambda_s^{-1}z_{is}z_{is}'E\left[z_{is}z_{is}'\right]^{-1}E\left[z_{is}x_{is}^{*}\right]\right]$$

in the numerator of B_3 is equal to zero for $s<t$, we may also write

$$\begin{aligned}
B_1 &\equiv \Upsilon_1^{-1}\textstyle\sum_{t=1}^{T-1}\operatorname{trace}\left(\left(\Gamma_t^{zz}\right)^{-1}\Gamma_{t,t}^{\varepsilon xzz}\right)\\
B_2 &\equiv -2\Upsilon_1^{-2}\textstyle\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}\Gamma_t^{zx\prime}\left(\Gamma_t^{zz}\right)^{-1}\Gamma_{t,s}^{\varepsilon xzz}\left(\Gamma_s^{zz}\right)^{-1}\Gamma_s^{zx}\\
B_3 &\equiv \Upsilon_1^{-2}\textstyle\sum_{t=1}^{T-1}\sum_{s=t}^{T-1}\Gamma_t^{zx\prime}\left(\Gamma_t^{zz}\right)^{-1}B_{3,1}(t,s)\left(\Gamma_s^{zz}\right)^{-1}\Gamma_s^{zx}.
\end{aligned}$$

2 Proof of Theorem 1

By Lemma 2 (a) and Theorem 1(a) of Andrews (1992) and Conditions 1-4 it follows that $\sup_{c\in C}\left|Q_n(c)-Q(c)\right|=o_p(1)$, where $Q(c)\equiv\lambda(c)'\Lambda(c)^{-1}\lambda(c)$. Let $B(\beta,\epsilon)$ be an open interval of length ϵ centered at β. By Condition 5 it follows that $\inf_{c\notin B(\beta,\epsilon)}Q(c)>Q(\beta)=0$ for all $\epsilon>0$. It then follows from standard arguments that $b-\beta=o_p(1)$. It therefore follows that $\Pr\left(S_n(b)\neq 0\right)\leq\Pr\left(b\in\partial C\right)=1-\Pr\left(b\in\operatorname{int}C\right)\leq 1-\Pr\left(b\in B(\beta,\epsilon)\right)\to 0$ for any $\epsilon>0$ where ∂C denotes the boundary of C.

Using similar arguments as in Pakes and Pollard (1989) we write

$$\begin{aligned}
\left|Q(b)\right| &= \left|\lambda(b)'\Lambda(b)^{-1}\lambda(b)\right|\\
&\leq \left|g(b)'G(b)^{-1}g(b)-\lambda(b)'\Lambda(b)^{-1}\lambda(b)-g(\beta)'G(b)^{-1}g(\beta)\right|\\
&\quad+\left|g(b)'G(b)^{-1}g(b)\right|+\left|g(\beta)'G(b)^{-1}g(\beta)\right|\\
&\leq \left|g(b)'G(b)^{-1}g(b)-\lambda(b)'G(b)^{-1}\lambda(b)-g(\beta)'G(b)^{-1}g(\beta)\right|\\
&\quad+\left|\lambda(b)'G(b)^{-1}\lambda(b)-\lambda(b)'\Lambda(b)^{-1}\lambda(b)\right|\\
&\quad+\left|g(b)'G(b)^{-1}g(b)\right|+\left|g(\beta)'G(b)^{-1}g(\beta)\right|,
\end{aligned}$$

where $\left|g(b)'G(b)^{-1}g(b)\right|\leq\left|g(\beta)'G(\beta)^{-1}g(\beta)\right|=O_p\left(n^{-1}\right)$ by the definition of b and Condition 6. We have $G(b)^{-1}=O_p(1)$ by consistency of b and the uniform law of large numbers, from which we obtain $\left|g(\beta)'G(b)^{-1}g(\beta)\right|=O_p\left(n^{-1}\right)$. We also have

$$\begin{aligned}
&g(b)'G(b)^{-1}g(b)-\lambda(b)'G(b)^{-1}\lambda(b)-g(\beta)'G(b)^{-1}g(\beta)\\
=\;&g(b)'G(b)^{-1}\left(g(b)-g(\beta)-\lambda(b)\right)+\left(g(b)-g(\beta)-\lambda(b)\right)'G(b)^{-1}\lambda(b)\\
&+2g(\beta)'G(b)^{-1}\lambda(b)+\left(g(b)-g(\beta)-\lambda(b)\right)'G(b)^{-1}g(\beta).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left|Q(b)\right| &\leq \left|\lambda(b)'G(b)^{-1}\lambda(b)-\lambda(b)'\Lambda(b)^{-1}\lambda(b)\right|\\
&\quad+\left|g(b)'G(b)^{-1}\left(g(b)-g(\beta)-\lambda(b)\right)\right|\\
&\quad+\left|\left(g(b)-g(\beta)-\lambda(b)\right)'G(b)^{-1}\lambda(b)\right|\\
&\quad+2\left|g(\beta)'G(b)^{-1}\lambda(b)\right|+\left|\left(g(b)-g(\beta)-\lambda(b)\right)'G(b)^{-1}g(\beta)\right|\\
&\quad+O_p\left(n^{-1}\right).
\end{aligned}$$

The terms $G(\beta)^{-1}$, and $\Lambda(b)^{-1}$ are $O_p(1)$ by consistency of b and the uniform law of large numbers. Also, the terms $g(b)$ and $\lambda(b)$ are $o_p(1)$ by consistency of b and the uniform law of large numbers. From Theorems 1 and 2 in Andrews (1994) and Conditions 1-6 it follows that $g(b) - \lambda(b) - g(\beta) = o_p\left(n^{-1/2}\right)$. From a standard CLT and consistency of b it follows that $\left(g(b) - g(\beta) - \lambda(b)\right)' G(b)^{-1} g(\beta) = o_p\left(n^{-1}\right)$, and $g(\beta)' G(b)^{-1} = O_p\left(n^{-1/2}\right)$. These results show that

$$
\begin{aligned}
|Q(b)| &\leq \left\| G(b)^{-1} - \Lambda(b)^{-1} \right\| \|\lambda(b)\|^2 + O_p\left(n^{-1/2}\right) \|\lambda(b)\| + o_p\left(n^{-1}\right) \\
&= o_p(1) \|\lambda(b)\|^2 + O_p\left(n^{-1/2}\right) \|\lambda(b)\| + o_p\left(n^{-1}\right).
\end{aligned}
$$

Because $|Q(b)| = \left|\lambda(b)' \Lambda(b)^{-1} \lambda(b)\right| \geq \frac{1}{\mathbf{M}} \|\lambda(b)\|^2$, we conclude that

$$
\left(\frac{1}{\mathbf{M}} - o_p(1)\right) \|\lambda(b)\|^2 - O_p\left(n^{-1/2}\right) \|\lambda(b)\| \leq o_p\left(n^{-1}\right)
$$

or

$$
\|\lambda(b)\| = O_p\left(n^{-1/2}\right),
$$

which implies that $b - \beta = O_p\left(n^{-1/2}\right)$.

3 Proof of Theorem 2

Note that we have

$$
\begin{aligned}
g_1(b) &= g_1 + \frac{1}{\sqrt{n}} g_2 \cdot \sqrt{n}(b-\beta) + \frac{1}{2n} g_3 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right), \\
g(b) &= g + \frac{1}{\sqrt{n}} g_1 \cdot \sqrt{n}(b-\beta) + \frac{1}{2n} g_2 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
G(b)^{-1} &= G^{-1} - \frac{1}{\sqrt{n}} G^{-1} G_1 G^{-1} \cdot \sqrt{n}(b-\beta) \\
&\quad + \frac{1}{2}\frac{1}{n}\left(2G^{-1}G_1G^{-1}G_1G^{-1} - G^{-1}G_2G^{-1}\right)\left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right), \\
G_1(b) &= G_1 + \frac{1}{\sqrt{n}} G_2 \cdot \sqrt{n}(b-\beta) + \frac{1}{2}\frac{1}{n} G_3 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right)
\end{aligned}
$$

where g, g_j, G, G_j and $\sqrt{n}(b-\beta)$ are $O_p(1)$ by Conditions 3 and 4 and Lemma 1**.** Therefore, we have

$$
g_1(b)' G(b)^{-1} g(b) = g_1' G^{-1} g + \frac{1}{\sqrt{n}} h_1 \cdot \sqrt{n}(b-\beta) + \frac{1}{n} h_2 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right),
$$

and

$$
\begin{aligned}
g(b)' G(b)^{-1} G_1(b) G(b)^{-1} g(b) &= g' G^{-1} G_1 G^{-1} g + \frac{1}{\sqrt{n}} h_3 \cdot \sqrt{n}(b-\beta) \\
&\quad + \frac{1}{n} h_4 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
h_1 &= g_2' G^{-1} g - g_1' G^{-1} G_1 G^{-1} g + g_1' G^{-1} g_1, \\
h_2 &= \frac{1}{2} g_3' G^{-1} g + \frac{1}{2} g_1' \left(2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}\right) g + \frac{1}{2} g_1' G^{-1} g_2 \\
&\quad - g_2' G^{-1} G_1 G^{-1} g - g_1' G^{-1} G_1 G^{-1} g_1 + g_2' G^{-1} g_1 \\
&= \frac{1}{2} g_3' G^{-1} g + g_1' G^{-1} G_1 G^{-1} G_1 G^{-1} g - \frac{1}{2} g_1' G^{-1} G_2 G^{-1} g + \frac{3}{2} g_1' G^{-1} g_2 \\
&\quad - g_2' G^{-1} G_1 G^{-1} g - g_1' G^{-1} G_1 G^{-1} g_1,
\end{aligned}
$$

and

$$
\begin{aligned}
h_3 &= g_1' G^{-1} G_1 G^{-1} g - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_2 G^{-1} g \\
&\quad - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_1 G^{-1} g_1 \\
&= 2 g_1' G^{-1} G_1 G^{-1} g - 2 g' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g' G^{-1} G_2 G^{-1} g,
\end{aligned}
$$

$$
\begin{aligned}
h_4 &= \frac{1}{2} g_2' G^{-1} G_1 G^{-1} g + \frac{1}{2} g' \left(2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}\right) G_1 G^{-1} g \\
&\quad + \frac{1}{2} g' G^{-1} G_3 G^{-1} g + \frac{1}{2} g' G^{-1} G_1 \left(2G^{-1} G_1 G^{-1} G_1 G^{-1} - G^{-1} G_2 G^{-1}\right) g \\
&\quad + \frac{1}{2} g' G^{-1} G_1 G^{-1} g_2 \\
&\quad - g_1' G^{-1} G_1 G^{-1} G_1 G^{-1} g + g_1' G^{-1} G_2 G^{-1} g - g_1' G^{-1} G_1 G^{-1} G_1 G^{-1} g \\
&\quad + g_1' G^{-1} G_1 G^{-1} g_1 - g' G^{-1} G_1 G^{-1} G_2 G^{-1} g \\
&\quad + g' G^{-1} G_1 G^{-1} G_1 G^{-1} G_1 G^{-1} g - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g_1 - g' G^{-1} G_2 G^{-1} G_1 G^{-1} g \\
&\quad + g' G^{-1} G_2 G^{-1} g_1 - g' G^{-1} G_1 G^{-1} G_1 G^{-1} g_1
\end{aligned}
$$

We may therefore rewrite the score $S_n(b)$ as

$$
\begin{aligned}
S_n(b) &= \left(2 g_1' G^{-1} g - g' G^{-1} G_1 G^{-1} g\right) + \frac{1}{\sqrt{n}} (2h_1 - h_3) \sqrt{n} (b - \beta) \\
&\quad + \frac{1}{n} (2h_2 - h_4) \left(\sqrt{n} (b - \beta)\right)^2 + o_p\left(\frac{1}{n}\right). \qquad (6)
\end{aligned}
$$

Next note that $P(|S_n(b)| > \epsilon) = P(|S_n(b)| > \epsilon/n^{-1})$ for any $\epsilon > 0$ because of Lemma 1. Thus $S_n(b) = o_p(n^{-1})$ and we can subsume this error into the $o_p(n^{-1})$ term of 6. Using these arguments and Lemmas 1, 2, and 3 below, we may rewrite the first order condition (6) as

$$
0 = \frac{1}{\sqrt{n}} \Phi + \frac{1}{n} \Gamma + \frac{1}{\sqrt{n}} \left(\Upsilon + \frac{1}{\sqrt{n}} \Xi\right) \sqrt{n} (b - \beta) + \frac{1}{n} \Psi \left(\sqrt{n} (b - \beta)\right)^2 + o_p\left(\frac{1}{n}\right)
$$

or

$$
0 = \Phi + \frac{1}{\sqrt{n}} \Gamma + \left(\Upsilon + \frac{1}{\sqrt{n}} \Xi\right) \sqrt{n} (b - \beta) + \frac{1}{\sqrt{n}} \Psi \left(\sqrt{n} (b - \beta)\right)^2 + o_p\left(\frac{1}{\sqrt{n}}\right),
$$

based on which we obtain (3). Noting that

$$
E[\Phi] = 2\sqrt{n} \lambda_1' \Lambda^{-1} E[g] = 0, \qquad (7)
$$

$$
\begin{aligned}
E[\Gamma] &= 2nE\left[(g_1 - \lambda_1)' \Lambda^{-1} g\right] - 2n\lambda_1' \Lambda^{-1} E\left[(G - \Lambda) \Lambda^{-1} g\right] - nE\left[g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g\right] \\
&= 2 \operatorname{trace}\left(\Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta}\right]\right) - 2\lambda_1' \Lambda^{-1} E\left[\psi_i \psi_i' \Lambda^{-1} \delta_i\right] - \operatorname{trace}\left(\Lambda^{-1} \Lambda_1 \Lambda^{-1} E\left[\delta_i \delta_i'\right]\right), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
E[\Phi\Xi] &= 8n\lambda_1'\Lambda^{-1}E\left[g(g_1-\lambda_1)'\right]\Lambda^{-1}\lambda_1 - 4nE\left[\lambda_1'\Lambda^{-1}g\lambda_1'\Lambda^{-1}(G-\Lambda)\Lambda^{-1}\lambda_1\right] \\
&\quad -8n\lambda_1'\Lambda^{-1}E[gg']\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + 4n\lambda_1'\Lambda^{-1}E[gg']\Lambda^{-1}\lambda_2 \\
&= 8\lambda_1'\Lambda^{-1}E\left[\delta_i\frac{\partial\delta_i'}{\partial\beta}\right]\Lambda^{-1}\lambda_1 - 4\lambda_1'\Lambda^{-1}E\left[\delta_i\lambda_1'\Lambda^{-1}\psi_i\psi_i'\right]\Lambda^{-1}\lambda_1 \\
&\quad -8\lambda_1'\Lambda^{-1}E\left[\delta_i\delta_i'\right]\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + 4\lambda_1'\Lambda^{-1}E\left[\delta_i\delta_i'\right]\Lambda^{-1}\lambda_2, \qquad (9)
\end{aligned}
$$

and

$$
E\left[\Phi^2\right] = 4\lambda_1'\Lambda^{-1}E\left[\delta_i\delta_i'\right]\Lambda^{-1}\lambda_1, \qquad (10)
$$

we obtain the desired conclusion.

Lemma 1 *Under Conditions 3 and 4*

$$
\begin{aligned}
h_2 &= \frac{3}{2}\lambda_1'\Lambda^{-1}\lambda_2 - \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + o_p(1), \\
h_4 &= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 + o_p(1).
\end{aligned}
$$

Proof. Follows from $\operatorname{plim} g = 0$. ■

Lemma 2 *Under Conditions 3 and 4*

$$
2h_1 - h_3 = \Upsilon + \frac{1}{\sqrt{n}}\Xi + o_p\left(\frac{1}{\sqrt{n}}\right).
$$

Proof. Because

$$
\begin{aligned}
g_2'G^{-1}g &= \lambda_2'\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right), \\
g_1'G^{-1}G_1G^{-1}g &= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
g_1'G^{-1}g_1 &= (\lambda_1 + (g_1-\lambda_1))'\left(\Lambda^{-1} - \Lambda^{-1}(G-\Lambda)\Lambda^{-1} + o_p\left(\frac{1}{\sqrt{n}}\right)\right)(\lambda_1 + (g_1-\lambda_1)) \\
&= \lambda_1\Lambda^{-1}\lambda_1 + 2(g_1-\lambda_1)'\Lambda^{-1}\lambda_1 - \lambda_1\Lambda^{-1}(G-\Lambda)\Lambda^{-1}\lambda_1 + o_p\left(\frac{1}{\sqrt{n}}\right), \qquad (11)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
h_1 &= \lambda_2'\Lambda^{-1}g - \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g \\
&\quad + \lambda_1\Lambda^{-1}\lambda_1 + 2(g_1-\lambda_1)'\Lambda^{-1}\lambda_1 - \lambda_1\Lambda^{-1}(G-\Lambda)\Lambda^{-1}\lambda_1 + o_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

Similarly, we obtain

$$
h_3 = 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}g + o_p\left(\frac{1}{\sqrt{n}}\right).
$$

The conclusion follows. ■

Lemma 3 *Under Conditions 3 and 4*

$$
2g_1'G^{-1}g - g'G^{-1}G_1G^{-1}g = \frac{1}{\sqrt{n}}\Phi + \frac{1}{n}\Gamma + o_p\left(\frac{1}{n}\right).
$$

Proof. We have

$$g_1' G^{-1} g = (\lambda_1 + (g_1 - \lambda_1))' \left(\Lambda^{-1} - \Lambda^{-1} (G - \Lambda) \Lambda^{-1} + o_p \left(\frac{1}{\sqrt{n}} \right) \right) g$$
$$= \lambda_1' \Lambda^{-1} g + (g_1 - \lambda_1)' \Lambda^{-1} g - \lambda_1' \Lambda^{-1} (G - \Lambda) \Lambda^{-1} g + o_p \left(\frac{1}{n} \right)$$

and

$$g' G^{-1} G_1 G^{-1} g = g' \Lambda^{-1} \Lambda_1 \Lambda^{-1} g + o_p \left(\frac{1}{n} \right).$$

from which the conclusion follows. ■

4 Proof of Theorem 3

The second order bias is computed using Theorem 3. Because the "weight matrix" here does not involve the parameter of interest, we have $\Lambda_1 = 0$. Also, because the moment restriction is linear in the parameter of interest, we have $\lambda_2 = 0$. It therefore follows that

$$\Psi = 3\lambda_1' \Lambda^{-1} \lambda_2 - 3\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 = 0,$$

and the second order bias is equal to

$$-\frac{1}{n} \frac{1}{\Upsilon} E[\Gamma] + \frac{1}{n} \frac{1}{\Upsilon^2} E[\Phi \Xi],$$

where

$$\begin{aligned} E[\Gamma] &= 2 \operatorname{trace} \left(\Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) - 2\lambda_1' \Lambda^{-1} E\left[\psi_i \psi_i' \Lambda^{-1} \delta_i \right] \\ E[\Phi \Xi] &= 8\lambda_1' \Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E\left[\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i' \right] \Lambda^{-1} \lambda_1 \end{aligned}$$

Furthermore, because $E\left[z_{it} z_{it}' E\left[z_{it} z_{it}' \right]^{-1} z_{it} \varepsilon_{it}^* \right] = 0$ under conditional symmetry of ε_{it}^*,

$$\lambda_1' \Lambda^{-1} E\left[\psi_i \psi_i' \Lambda^{-1} \delta_i \right] = -\sum_{t=1}^{T-1} E\left[z_{it} x_{it}^* \right]' E\left[z_{it} z_{it}' \right]^{-1} E\left[z_{it} z_{it}' E\left[z_{it} z_{it}' \right]^{-1} z_{it} \varepsilon_{it}^* \right] = 0$$

and therefore, we have further simplification

$$\begin{aligned} E[\Gamma] &= 2 \operatorname{trace} \left(\Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) \\ E[\Phi \Xi] &= 8\lambda_1' \Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 - 4\lambda_1' \Lambda^{-1} E\left[\delta_i \lambda_1' \Lambda^{-1} \psi_i \psi_i' \right] \Lambda^{-1} \lambda_1 \end{aligned}$$

the second term should be equal to zero. Because

$$\lambda_1 \Lambda^{-1} \lambda_1 = \sum_{t=1}^{T-1} E\left[z_{it} x_{it}^* \right]' E\left[z_{it} z_{it}' \right]^{-1} E\left[z_{it} x_{it}^* \right],$$

$$\operatorname{trace} \left(\Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \right) = -\sum_{t=1}^{T-1} \operatorname{trace} \left(E\left[z_{it} z_{it}' \right]^{-1} E\left[\varepsilon_{it}^* x_{it}^* z_{it} z_{it}' \right] \right),$$

$$\lambda_1' \Lambda^{-1} E\left[\delta_i \frac{\partial \delta_i'}{\partial \beta} \right] \Lambda^{-1} \lambda_1 = -\sum_{t=1}^{T-1} \sum_{s=1}^{T-1} E\left[z_{it} x_{it}^* \right]' E\left[z_{it} z_{it}' \right]^{-1} E\left[\varepsilon_{it}^* x_{is}^* z_{it} z_{is}' \right] E\left[z_{is} z_{is}' \right]^{-1} E\left[z_{is} x_{is}^* \right],$$

and

$$\lambda_1'\Lambda^{-1}E\left[\delta_i\lambda_1'\Lambda^{-1}\psi_i\psi_i'\right]\Lambda^{-1}\lambda_1$$
$$= -\sum_{t=1}^{T-1}\sum_{s=1}^{T-1} E\left[z_{it}x_{it}^*\right]' E\left[z_{it}z_{it}'\right]^{-1} E\left[\varepsilon_{it}^* z_{it} E\left[z_{is}x_{is}^*\right]' E\left[z_{is}z_{is}'\right]^{-1} z_{is}z_{is}'\right] E\left[z_{is}z_{is}'\right]^{-1} E\left[z_{is}x_{is}^*\right],$$

we now obtain

$$\begin{aligned}\Upsilon &= 2\lambda_1\Lambda^{-1}\lambda_1\\ &= 2\sum_{t=1}^{T-1} E\left[z_{it}x_{it}^*\right]' E\left[z_{it}z_{it}'\right]^{-1} E\left[z_{it}x_{it}^*\right]\\ &= 2\Upsilon_1,\end{aligned}$$

$$\begin{aligned}E\left[\Gamma\right] &= 2\,\mathrm{trace}\left(\Lambda^{-1}E\left[\delta_i\frac{\partial\delta_i'}{\partial\beta}\right]\right)\\ &= -2\sum_{t=1}^{T-1}\mathrm{trace}\left(E\left[z_{it}z_{it}'\right]^{-1}E\left[\varepsilon_{it}^*x_{it}^*z_{it}z_{it}'\right]\right)\\ &= -2\textstyle\sum_{t=1}^{T-1}\mathrm{trace}\left(\left(\Gamma_t^{zz}\right)^{-1}\Gamma_{t,t}^{\varepsilon xzz}\right),\end{aligned}$$

$$\begin{aligned}E\left[\Phi\Xi\right] &= 8\lambda_1'\Lambda^{-1}E\left[\delta_i\frac{\partial\delta_i'}{\partial\beta}\right]\Lambda^{-1}\lambda_1 - 4\lambda_1'\Lambda^{-1}E\left[\delta_i\lambda_1'\Lambda^{-1}\psi_i\psi_i'\right]\Lambda^{-1}\lambda_1\\ &= -8\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}E\left[z_{it}x_{it}^*\right]'E\left[z_{it}z_{it}'\right]^{-1}E\left[\varepsilon_{it}^*x_{is}^*z_{it}z_{is}'\right]E\left[z_{is}z_{is}'\right]^{-1}E\left[z_{is}x_{is}^*\right]\\ &\quad +4\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}E\left[z_{it}x_{it}^*\right]'E\left[z_{it}z_{it}'\right]^{-1}E\left[\varepsilon_{it}^*z_{it}E\left[z_{is}x_{is}^*\right]'E\left[z_{is}z_{is}'\right]^{-1}z_{is}z_{is}'\right]E\left[z_{is}z_{is}'\right]^{-1}E\left[z_{is}x_{is}^*\right]\\ &= -8\textstyle\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}\Gamma_t^{zx\prime}\left(\Gamma_t^{zz}\right)^{-1}\Gamma_{t,s}^{\varepsilon xzz}\left(\Gamma_s^{zz}\right)^{-1}\Gamma_s^{zx}\\ &\quad +4\textstyle\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}\Gamma_t^{zx\prime}\left(\Gamma_t^{zz}\right)^{-1}B_{3,1}(t,s)\left(\Gamma_s^{zz}\right)^{-1}\Gamma_s^{zx}\end{aligned}$$

from which we obtain the desired conclusion.

Higher Order Bias and MSE of GMM: Supplementary Appendix for “Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects”

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1 Higher Order Expansion of GMM

Consider a general minimization estimator b of a single parameter $\beta \in \mathbb{R}$ defined by

$$b = \operatorname*{argmin}_{c \in C} g_n(c)' \, G_n\left(\widetilde{\beta}\right)^{-1} g_n(c) \tag{1}$$

where

$$g_n(c) \equiv \frac{1}{n} \sum_{i=1}^{n} \psi(z_i, c)$$

and

$$G_n\left(\widetilde{\beta}\right) \equiv \frac{1}{n} \sum_{i=1}^{n} \psi\left(z_i, \widetilde{\beta}\right) \psi\left(z_i, \widetilde{\beta}\right)'$$

Here, $\widetilde{\beta}$ is an initial $\sqrt{n}$-consistent estimator for β such that

$$\sqrt{n}\left(\widetilde{\beta} - \beta\right) = L_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right)$$

where

$$L_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i$$

f_i is i.i.d. mean zero, and $Q_n = O_p(1)$, $S_n = O_p(1)$.

Definition 1

$$\begin{aligned} \psi_j(z_i, c) &\equiv \partial^j \psi(z_i, c) / \partial c^j \\ \lambda_j(c) &\equiv E\left[\psi_j(z_i, c)\right] \end{aligned}$$

$$\begin{aligned} \Psi(z_i, c) &\equiv \psi(z_i, c) \, \psi(z_i, c)' \\ \Psi_j(z_i, c) &\equiv \partial^j \Psi(z_i, c) / \partial c^j \\ \Lambda_j(c) &\equiv E\left[\Psi_j(z_i, c)\right] \end{aligned}$$

For simplicity, we use the notation

$$\begin{aligned}\lambda_j &\equiv \lambda_j(\beta)\\ \Lambda_j &\equiv \Lambda_j(\beta)\\ \lambda(c) &\equiv \lambda_0(c)\\ \Lambda(c) &\equiv \Lambda_0(c)\end{aligned}$$

Definition 2

$$\begin{aligned}g_n(c) &\equiv \frac{1}{n}\sum_{i=1}^{n}\psi(z_i,c)\\ g_{j,n}(c) &\equiv \frac{1}{n}\sum_{i=1}^{n}\psi_j(z_i,c)\\ G_n(c) &\equiv \frac{1}{n}\sum_{i=1}^{n}\Psi(z_i,c)=\frac{1}{n}\sum_{i=1}^{n}\psi(z_i,c)\,\psi(z_i,c)'\\ G_{j,n}(c) &\equiv \frac{1}{n}\sum_{i=1}^{n}\Psi_j(z_i,c)\end{aligned}$$

Definition 3

$$\begin{aligned}w_n &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\psi(z_i,\beta)-\lambda)=\sqrt{n}g_n(\beta)\\ w_{1,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\psi_1(z_i,\beta)-\lambda_1)\\ w_{2,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\psi_2(z_i,\beta)-\lambda_2)\\ w_{3,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\psi_3(z_i,\beta)-\lambda_3)\\ w_{4,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\psi_4(z_i,\beta)-\lambda_4)\end{aligned}$$

$$\begin{aligned}W_n &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Psi(z_i,\beta)-\Lambda)\\ W_{1,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Psi_1(z_i,\beta)-\Lambda_1)\\ W_{2,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Psi_2(z_i,\beta)-\Lambda_2)\\ W_{3,n} &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\Psi_3(z_i,\beta)-\Lambda_3)\end{aligned}$$

Definition 4

$$\begin{aligned}\widetilde{\Lambda}_1 &\equiv W_n+\Lambda_1L_n\\ \widetilde{\Lambda}_2 &\equiv 2\Lambda_1Q_n+2W_{1,n}L_n+\Lambda_2L_n^2\\ \widetilde{\Lambda}_3 &\equiv 6\Lambda_1S_n+6W_{1,n}Q_n+3W_{2,n}L_n^2+6\Lambda_2L_nQ_n+\Lambda_3L_n^3\end{aligned}$$

Lemma 1

$$G_n\left(\widetilde{\beta}\right)=\Lambda+\frac{1}{\sqrt{n}}\widetilde{\Lambda}_1+\frac{1}{2n}\widetilde{\Lambda}_2+\frac{1}{6n\sqrt{n}}\widetilde{\Lambda}_3+o_p\left(\frac{1}{n\sqrt{n}}\right).$$

Proof. Write

$$\begin{aligned}
G_n\left(\widetilde{\beta}\right) &= \frac{1}{n}\sum_{i=1}^{n}\Psi\left(z_i,\widetilde{\beta}\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\Psi\left(z_i,\beta\right)\\
&\quad +\frac{1}{\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_1\left(z_i,\beta\right)\right)\cdot\sqrt{n}\left(\widetilde{\beta}-\beta\right)\\
&\quad +\frac{1}{2n}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_2\left(z_i,\beta\right)\right)\cdot\left(\sqrt{n}\left(\widetilde{\beta}-\beta\right)\right)^2\\
&\quad +\frac{1}{6n\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_3\left(z_i,\beta\right)\right)\cdot\left(\sqrt{n}\left(\widetilde{\beta}-\beta\right)\right)^3+o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}$$

Note that

$$\frac{1}{n}\sum_{i=1}^{n}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'=\Lambda+\frac{1}{\sqrt{n}}W_n,$$

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_1\left(z_i,\beta\right)\right)\cdot\sqrt{n}\left(\widetilde{\beta}-\beta\right)\\
=\ &\frac{1}{\sqrt{n}}\left(\Lambda_1+\frac{1}{\sqrt{n}}W_{1,n}\right)\cdot\left(L_n+\frac{1}{\sqrt{n}}Q_n+\frac{1}{n}S_n+o_p\left(\frac{1}{n}\right)\right)\\
=\ &\frac{1}{\sqrt{n}}\Lambda_1L_n\\
&+\frac{1}{n}\left(\Lambda_1Q_n+W_{1,n}L_n\right)\\
&+\frac{1}{n\sqrt{n}}\left(\Lambda_1S_n+W_{1,n}Q_n\right)+o_p\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2n}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_2\left(z_i,\beta\right)\right)\cdot\left(\sqrt{n}\left(\widetilde{\beta}-\beta\right)\right)^2\\
=\ &\frac{1}{2n}\left(\Lambda_2+\frac{1}{\sqrt{n}}W_{2,n}\right)\cdot\left(L_n+\frac{1}{\sqrt{n}}Q_n+\frac{1}{n}S_n+o_p\left(\frac{1}{n}\right)\right)^2\\
=\ &\frac{1}{2n}\Lambda_2L_n^2+\frac{1}{2n\sqrt{n}}\left(W_{2,n}L_n^2+2\Lambda_2L_nQ_n\right)+o_p\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{6n\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^{n}\Psi_3\left(z_i,\beta\right)\right)\cdot\left(\sqrt{n}\left(\widetilde{\beta}-\beta\right)\right)^3\\
=\ &\frac{1}{6n\sqrt{n}}\left(\Lambda_3+\frac{1}{\sqrt{n}}W_{3,n}\right)\left(L_n+\frac{1}{\sqrt{n}}Q_n+\frac{1}{n}S_n+o_p\left(\frac{1}{n}\right)\right)^3\\
=\ &\frac{1}{6n\sqrt{n}}\Lambda_3L_n^3+o_p\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

It follows that

$$
\begin{aligned}
G_n\left(\widetilde{\beta}\right) &= \Lambda + \frac{1}{\sqrt{n}}W_n + \frac{1}{\sqrt{n}}\Lambda_1 L_n \\
&\quad + \frac{1}{n}\left(\Lambda_1 Q_n + W_{1,n}L_n\right) + \frac{1}{2n}\Lambda_2 L_n^2 \\
&\quad + \frac{1}{n\sqrt{n}}\left(\Lambda_1 S_n + W_{1,n}Q_n\right) + \frac{1}{2n\sqrt{n}}\left(W_{2,n}L_n^2 + 2\Lambda_2 L_n Q_n\right) + \frac{1}{6n\sqrt{n}}\Lambda_3 L_n^3 \\
&\quad + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \Lambda + \frac{1}{\sqrt{n}}\widetilde{\Lambda}_1 + \frac{1}{2n}\widetilde{\Lambda}_2 + \frac{1}{6n\sqrt{n}}\widetilde{\Lambda}_3 + o_p\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}
$$

■

Definition 5

$$
\begin{aligned}
H_1 &\equiv -\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} \\
H_2 &\equiv 2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1} \\
H_3 &\equiv -6\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1} - \Lambda^{-1}\widetilde{\Lambda}_3\Lambda^{-1}
\end{aligned}
$$

Lemma 2

$$
G_n\left(\widetilde{\beta}\right)^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}}H_1 + \frac{1}{2n}H_2 + \frac{1}{6n\sqrt{n}}H_3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
$$

Proof. Using Lemma 1, we obtain

$$
\begin{aligned}
G_n\left(\widetilde{\beta}\right)^{-1} &= \Lambda^{-1} - \frac{1}{\sqrt{n}}\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} \\
&\quad + \frac{1}{2n}\left(2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1}\right) \\
&\quad - \frac{1}{6n\sqrt{n}}\left(6\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} - 3\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} - 3\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1} + \Lambda^{-1}\widetilde{\Lambda}_3\Lambda^{-1}\right) \\
&\quad + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}
$$

■

Lemma 3

$$
H_1 = H_{1,0} + L_n \cdot H_{1,1}
$$

where

$$
\begin{aligned}
H_{1,0} &\equiv -\Lambda^{-1}W_n\Lambda^{-1} \\
H_{1,1} &\equiv -\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
H_1 &= -\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} = -\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} \\
&= -\Lambda^{-1}W_n\Lambda^{-1} - L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}
$$

■

Lemma 4

$$H_2 = H_{2,0} + L_n \cdot H_{2,1} + L_n^2 \cdot H_{2,2} + Q_n \cdot H_{2,3}$$

where

$$\begin{aligned}
H_{2,0} &\equiv 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\
H_{2,1} &\equiv 2\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 2\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} - 2\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
H_{2,2} &\equiv 2\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_2\Lambda^{-1} \\
H_{2,3} &\equiv -2\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}$$

Proof.

$$\begin{aligned}
H_2 &= 2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} - \Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1} \\
&= 2\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} - \Lambda^{-1}\left(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2\right)\Lambda^{-1} \\
&= 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} + 2L_n \cdot \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} + 2L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 2L_n^2 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&\quad -2Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1} - 2L_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1} - L_n^2 \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1} \\
&= 2\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\
&\quad +L_n \cdot \left(2\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} + 2\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 2\Lambda^{-1}W_{1,n}\Lambda^{-1}\right) \\
&\quad +L_n^2 \cdot \left(2\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_2\Lambda^{-1}\right) \\
&\quad -2Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}$$

■

Lemma 5

$$H_3 = H_{3,0} + L_n \cdot H_{3,1} + L_n^2 \cdot H_{3,2} + L_n^3 \cdot H_{3,3} + Q_n \cdot H_{3,4} + Q_n L_n \cdot H_{3,5} + S_n \cdot H_{3,6}$$

where

$$\begin{aligned}
H_{3,0} &\equiv -6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\
H_{3,1} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&\quad +6\Lambda^{-1}W_{1,n}\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_n\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
H_{3,2} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} - 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
&\quad +3\Lambda^{-1}\Lambda_2\Lambda^{-1}W_n\Lambda^{-1} + 3\Lambda^{-1}W_n\Lambda^{-1}\Lambda_2\Lambda^{-1} \\
&\quad +6\Lambda^{-1}W_{1,n}\Lambda^{-1}\Lambda_1\Lambda^{-1} + 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
&\quad -3\Lambda^{-1}W_{2,n}\Lambda^{-1} \\
H_{3,3} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} + 3\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_2\Lambda^{-1} + 3\Lambda^{-1}\Lambda_2\Lambda^{-1}\Lambda_1\Lambda^{-1} - \Lambda^{-1}\Lambda_3\Lambda^{-1} \\
H_{3,4} &\equiv 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
H_{3,5} &\equiv 12\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} - 6\Lambda^{-1}\Lambda_2\Lambda^{-1} \\
H_{3,6} &\equiv -6\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
H_3 &= -6\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1} + 3\Lambda^{-1}\widetilde{\Lambda}_1\Lambda^{-1}\widetilde{\Lambda}_2\Lambda^{-1} - \Lambda^{-1}\widetilde{\Lambda}_3\Lambda^{-1} \\
&= -6\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} \\
&\quad +3\Lambda^{-1}\left(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} \\
&\quad +3\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2\right)\Lambda^{-1} \\
&\quad -\Lambda^{-1}\left(6\Lambda_1 S_n + 6W_{1,n}Q_n + 3W_{2,n}L_n^2 + 6\Lambda_2 L_n Q_n + \Lambda_3 L_n^3\right)\Lambda^{-1}
\end{aligned}
$$

Note that the terms on the RHS can be expanded as follows:

$$
\begin{aligned}
& -6\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} \\
=\ & -6\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} \\
& -6L_n \cdot \left(\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\right) \\
& -6L_n^2 \cdot \left(\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} + \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1} + \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\right) \\
& -6L_n^3 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& 3\Lambda^{-1}\left(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2\right)\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1} \\
=\ & 6L_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1}W_n\Lambda^{-1} \\
& +L_n^2 \cdot \left(3\Lambda^{-1}\Lambda_2\Lambda^{-1}W_n\Lambda^{-1} + 6\Lambda^{-1}W_{1,n}\Lambda^{-1}\Lambda_1\Lambda^{-1}\right) \\
& +3L_n^3 \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
& +6Q_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1} \\
& +6Q_n L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& 3\Lambda^{-1}\left(W_n + \Lambda_1 L_n\right)\Lambda^{-1}\left(2\Lambda_1 Q_n + 2W_{1,n}L_n + \Lambda_2 L_n^2\right)\Lambda^{-1} \\
=\ & 6L_n \cdot \Lambda^{-1}W_n\Lambda^{-1}W_{1,n}\Lambda^{-1} \\
& +L_n^2 \cdot \left(3\Lambda^{-1}W_n\Lambda^{-1}\Lambda_2\Lambda^{-1} + 6\Lambda^{-1}\Lambda_1\Lambda^{-1}W_{1,n}\Lambda^{-1}\right) \\
& +3L_n^3 \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_2\Lambda^{-1} \\
& +6Q_n \cdot \Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1} \\
& +6Q_n L_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& -\Lambda^{-1}\left(6\Lambda_1 S_n + 6W_{1,n}Q_n + 3W_{2,n}L_n^2 + 6\Lambda_2 L_n Q_n + \Lambda_3 L_n^3\right)\Lambda^{-1} \\
=\ & -6S_n \cdot \Lambda^{-1}\Lambda_1\Lambda^{-1} - 6Q_n \cdot \Lambda^{-1}W_{1,n}\Lambda^{-1} - 3L_n^2 \cdot \Lambda^{-1}W_{2,n}\Lambda^{-1} - 6Q_n L_n \cdot \Lambda^{-1}\Lambda_2\Lambda^{-1} - L_n^3 \cdot \Lambda^{-1}\Lambda_3\Lambda^{-1}
\end{aligned}
$$

■

Definition 6

$$
\begin{aligned}
h_{0,1} &\equiv \lambda_1'\Lambda^{-1}w_n \\
h_{0,2} &\equiv w_{1,n}'\Lambda^{-1}w_n + \lambda_1' H_1 w_n \\
h_{0,3} &\equiv w_{1,n}' H_1 w_n + \frac{1}{2}\lambda_1' H_2 w_n
\end{aligned}
$$

$$
\begin{aligned}
h_{1,1} &\equiv \lambda_1' \Lambda^{-1} \lambda_1 \\
h_{1,2} &\equiv 2\lambda_1' \Lambda^{-1} w_{1,n} + \lambda_2' \Lambda^{-1} w_n + \lambda_1' H_1 \lambda_1 \\
h_{1,3} &\equiv w_{1,n}' \Lambda^{-1} w_{1,n} + w_{2,n}' \Lambda^{-1} w_n + 2\lambda_1' H_1 w_{1,n} + \lambda_2' H_1 w_n + \frac{1}{2} \lambda_1' H_2 \lambda_1
\end{aligned}
$$

$$
\begin{aligned}
h_{2,2} &\equiv 3\lambda_1' \Lambda^{-1} \lambda_2 \\
h_{2,3} &\equiv 3\lambda_2' \Lambda^{-1} w_{1,n} + 3\lambda_1' \Lambda^{-1} w_{2,n} + \lambda_3' \Lambda^{-1} w_n + 3\lambda_1' H_1 \lambda_2
\end{aligned}
$$

$$
h_{3,3} \equiv 3\lambda_2' \Lambda^{-1} \lambda_2 + 4\lambda_1' \Lambda^{-1} \lambda_3
$$

Lemma 6

$$
g_1(b)' G\left(\widetilde{\beta}\right)^{-1} g(b) = h + h_1 \cdot \sqrt{n}(b-\beta) + \frac{1}{2} h_2 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + \frac{1}{6} h_3 \cdot \left(\sqrt{n}(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right),
$$

where

$$
\begin{aligned}
h &\equiv \frac{1}{\sqrt{n}} h_{0,1} + \frac{1}{n} h_{0,2} + \frac{1}{n\sqrt{n}} h_{0,3} \\
h_1 &\equiv \frac{1}{\sqrt{n}} h_{1,1} + \frac{1}{n} h_{1,2} + \frac{1}{n\sqrt{n}} h_{1,3} \\
h_2 &\equiv \frac{1}{n} h_{2,2} + \frac{1}{n\sqrt{n}} h_{2,3} \\
h_3 &\equiv \frac{1}{n\sqrt{n}} h_{3,3}
\end{aligned}
$$

Proof. Note that we have

$$
\begin{aligned}
g_1(b) &= g_1 + \frac{1}{\sqrt{n}} g_2 \cdot \sqrt{n}(b-\beta) + \frac{1}{2n} g_3 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + \frac{1}{6n\sqrt{n}} g_4 \cdot \left(\sqrt{n}(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \lambda_1 + \frac{1}{\sqrt{n}} w_{1,n} + \frac{1}{\sqrt{n}}\left(\lambda_2 + \frac{1}{\sqrt{n}} w_{2,n}\right)\sqrt{n}(b-\beta) + \frac{1}{2n}\left(\lambda_3 + \frac{1}{\sqrt{n}} w_{3,n}\right) \cdot \left(\sqrt{n}(b-\beta)\right)^2 \\
&+ \frac{1}{6n\sqrt{n}}\left(\lambda_4 + \frac{1}{\sqrt{n}} w_{4,n}\right) \cdot \left(\sqrt{n}(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
g(b) &= \frac{1}{\sqrt{n}} w_n + \frac{1}{\sqrt{n}} g_1 \cdot \sqrt{n}(b-\beta) + \frac{1}{2n} g_2 \cdot \left(\sqrt{n}(b-\beta)\right)^2 + \frac{1}{6n\sqrt{n}} g_3 \cdot \left(\sqrt{n}(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{n}} w_n + \frac{1}{\sqrt{n}}\left(\lambda_1 + \frac{1}{\sqrt{n}} w_{1,n}\right) \cdot \sqrt{n}(b-\beta) + \frac{1}{2n}\left(\lambda_2 + \frac{1}{\sqrt{n}} w_{2,n}\right) \cdot \left(\sqrt{n}(b-\beta)\right)^2 \\
&+ \frac{1}{6n\sqrt{n}}\left(\lambda_3 + \frac{1}{\sqrt{n}} w_{3,n}\right) \cdot \left(\sqrt{n}(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
\end{aligned}
$$

By Lemma 2, we also have

$$
G\left(\widetilde{\beta}\right)^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}} H_1 + \frac{1}{2n} H_2 + \frac{1}{6n\sqrt{n}} H_3 + o_p\left(\frac{1}{n\sqrt{n}}\right)
$$

■

Lemma 7

$$\begin{aligned}
h_{0,1} &= \lambda_1' \Lambda^{-1} w_n \\
h_{0,2} &= w_{1,n}' \Lambda^{-1} w_n + \lambda_1' H_{1,0} w_n + L_n \cdot \lambda_1' H_{1,1} w_n \\
h_{0,3} &= w_{1,n}' H_{1,0} w_n + \frac{1}{2} \lambda_1' H_{2,0} w_n \\
&\quad + L_n \cdot w_{1,n}' H_{1,1} w_n + \frac{1}{2} L_n \cdot \lambda_1' H_{2,1} w_n + \frac{1}{2} L_n^2 \cdot \lambda_1' H_{2,2} w_n + \frac{1}{2} Q_n \cdot \lambda_1' H_{2,3} w_n
\end{aligned}$$

$$\begin{aligned}
h_{1,1} &= \lambda_1' \Lambda^{-1} \lambda_1 \\
h_{1,2} &= 2\lambda_1' \Lambda^{-1} w_{1,n} + \lambda_2' \Lambda^{-1} w_n + \lambda_1' H_{1,0} \lambda_1 + L_n \cdot \lambda_1' H_{1,1} \lambda_1 \\
h_{1,3} &= w_{1,n}' \Lambda^{-1} w_{1,n} + w_{2,n}' \Lambda^{-1} w_n + 2\lambda_1' H_{1,0} w_{1,n} + 2 L_n \cdot \lambda_1' H_{1,1} w_{1,n} \\
&\quad + \lambda_2' H_{1,0} w_n + L_n \cdot \lambda_2' H_{1,1} w_n \\
&\quad + \frac{1}{2} \lambda_1' H_{2,0} \lambda_1 + \frac{1}{2} L_n \cdot \lambda_1' H_{2,1} \lambda_1 + \frac{1}{2} L_n^2 \cdot \lambda_1' H_{2,2} \lambda_1 + \frac{1}{2} Q_n \cdot \lambda_1' H_{2,3} \lambda_1
\end{aligned}$$

$$\begin{aligned}
h_{2,2} &= 3\lambda_1' \Lambda^{-1} \lambda_2 \\
h_{2,3} &= 3\lambda_2' \Lambda^{-1} w_{1,n} + 3\lambda_1' \Lambda^{-1} w_{2,n} + \lambda_3' \Lambda^{-1} w_n + 3\lambda_1' H_{1,0} \lambda_2 + 3 L_n \cdot \lambda_1' H_{1,1} \lambda_2
\end{aligned}$$

$$h_{3,3} = 3\lambda_2' \Lambda^{-1} \lambda_2 + 4\lambda_1' \Lambda^{-1} \lambda_3$$

Proof. Follows from Lemmas 3, 4, and 5. ■

Lemma 8 *Let*

$$b_1 \equiv \underset{c \in C}{\operatorname{argmin}}\, g(c)' G\left(\widetilde{\beta}\right)^{-1} g(c)$$

Then,

$$\sqrt{n}\,(b - \beta) = T_1 + \frac{1}{2\sqrt{n}} T_2 + \frac{1}{6n} T_3 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
T_1 &\equiv -\frac{1}{h_{1,1}} h_{0,1} \\
T_2 &\equiv -\frac{2}{h_{1,1}} h_{0,2} + \frac{2}{h_{1,1}^2} h_{1,2} h_{0,1} - \frac{1}{h_{1,1}^3} h_{2,2} h_{0,1}^2 \\
T_3 &\equiv -\frac{6}{h_{1,1}} h_{0,3} + \frac{6 h_{1,2} h_{0,2} + 6 h_{1,3} h_{0,1}}{h_{1,1}^2} + \frac{-6 h_{1,2}^2 h_{0,1} - 3 h_{2,3} h_{0,1}^2 - 6 h_{2,2} h_{0,1} h_{0,2}}{h_{1,1}^3} \\
&\quad + \frac{9 h_{1,2} h_{2,2} h_{0,1}^2 + h_{3,3} h_{0,1}^3}{h_{1,1}^4} - \frac{3}{h_{1,1}^5} h_{2,2}^2 h_{0,1}^3
\end{aligned}$$

Proof. We now solve for $\sqrt{n}\,(b - \beta)$ by iterative substitution. Using Lemma 6, we first write the FOC as

$$0 = \frac{1}{\sqrt{n}} h_{0,1} + \left(\frac{1}{\sqrt{n}} h_{1,1}\right) \cdot \sqrt{n}\,(b - \beta) + o_p\left(\frac{1}{\sqrt{n}}\right),$$

from which we obtain

$$\sqrt{n}\,(b-\beta) = -\frac{1}{h_{1,1}}h_{0,1} + o_p(1) = T_1 + o_p(1)$$

We now write the FOC as

$$0 = \frac{1}{\sqrt{n}}h_{0,1} + \frac{1}{n}h_{0,2} + \left(\frac{1}{\sqrt{n}}h_{1,1} + \frac{1}{n}h_{1,2}\right)\cdot\left(\sqrt{n}\,(b-\beta)\right) + \frac{1}{2}\left(\frac{1}{n}h_{2,2}\right)\cdot\left(\sqrt{n}\,(b-\beta)\right)^2 + o_p\left(\frac{1}{n}\right).$$

Writing

$$\sqrt{n}\,(b-\beta) = -\frac{1}{h_{1,1}}h_{0,1} + \frac{x}{\sqrt{n}}$$

we can rewrite the FOC as

$$0 = \frac{1}{n}\left(h_{0,2} + \frac{1}{2}\frac{h_{2,2}}{h_{1,1}^2}h_{0,1}^2 + h_{1,1}x - \frac{h_{1,2}}{h_{1,1}}h_{0,1}\right) + o_p\left(\frac{1}{n}\right).$$

Solving for x, we obtain

$$\begin{aligned}\sqrt{n}\,(b-\beta) &= T_1 + \frac{1}{2\sqrt{n}}\left(-\frac{2h_{0,2}h_{1,1}^2 + h_{2,2}h_{0,1}^2 - 2h_{1,2}h_{0,1}h_{1,1}}{h_{1,1}^3}\right) + o_p\left(\frac{1}{\sqrt{n}}\right)\\ &= T_1 + \frac{1}{2\sqrt{n}}T_2 + o_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

We now write the FOC as

$$\begin{aligned}0 &= \frac{1}{\sqrt{n}}h_{0,1} + \frac{1}{n}h_{0,2} + \frac{1}{n\sqrt{n}}h_{0,3} + \left(\frac{1}{\sqrt{n}}h_{1,1} + \frac{1}{n}h_{1,2} + \frac{1}{n\sqrt{n}}h_{1,3}\right)\cdot\sqrt{n}\,(b-\beta)\\ &\quad + \frac{1}{2}\left(\frac{1}{n}h_{2,2} + \frac{1}{n\sqrt{n}}h_{2,3}\right)\cdot\left(\sqrt{n}\,(b-\beta)\right)^2 + \frac{1}{6}\frac{1}{n\sqrt{n}}h_{3,3}\cdot\left(\sqrt{n}\,(b-\beta)\right)^3 + o_p\left(\frac{1}{n\sqrt{n}}\right).\end{aligned}$$

Writing

$$\sqrt{n}\,(b-\beta) = -\frac{1}{h_{1,1}}h_{0,1} - \frac{1}{2\sqrt{n}}\frac{2h_{0,2}h_{1,1}^2 + h_{2,2}h_{0,1}^2 - 2h_{1,2}h_{0,1}h_{1,1}}{h_{1,1}^3} + \frac{x}{n}$$

we can rewrite the FOC as

$$\begin{aligned}0 &= \frac{h_{1,1}x - \frac{h_{1,2}}{h_{1,1}}h_{0,2} + \frac{1}{2}\frac{h_{2,2}^2}{h_{1,1}^4}h_{0,1}^3 + \frac{h_{1,2}^2}{h_{1,1}^2}h_{0,1} + \frac{h_{2,2}}{h_{1,1}^2}h_{0,1}h_{0,2} - \frac{h_{1,3}}{h_{1,1}}h_{0,1} + h_{0,3} + \frac{1}{2}\frac{h_{2,3}}{h_{1,1}^2}h_{0,1}^2 - \frac{3}{2}\frac{h_{1,2}}{h_{1,1}^3}h_{2,2}h_{0,1}^2 - \frac{1}{6}\frac{h_{3,3}}{h_{1,1}^3}h_{0,1}^3}{n^{\frac{3}{2}}}\\ &\quad + o_p\left(\frac{1}{n\sqrt{n}}\right)\end{aligned}$$

Solving for x, we obtain

$$\sqrt{n}\,(b-\beta) = T_1 + \frac{1}{2\sqrt{n}}T_2 + \frac{1}{6n}T_3 + o_p\left(\frac{1}{n}\right)$$

■

Lemma 9

$$\begin{aligned}T_1 &= \overline{T}_1\\ T_2 &= \overline{T}_2 + L_n\cdot\widetilde{T}_2\\ T_3 &= \overline{T}_3 + L_n\cdot\widetilde{T}_{3,1} + L_n^2\cdot\widetilde{T}_{3,2} + Q_n\cdot\widetilde{T}_{3,3}\end{aligned}$$

where

$$\overline{T}_1 \equiv -\frac{\lambda_1' \Lambda^{-1} w_n}{\lambda_1' \Lambda^{-1} \lambda_1}$$

$$\begin{aligned}
\overline{T}_2 \quad \equiv \quad & -\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \left(w_{1,n}' \Lambda^{-1} w_n \right) \\
& -\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \left(\lambda_1' H_{1,0} w_n \right) \\
& +\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1 \right)^2} \left(2\lambda_1' \Lambda^{-1} w_{1,n} \right) \left(\lambda_1' \Lambda^{-1} w_n \right) \\
& +\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1 \right)^2} \left(\lambda_2' \Lambda^{-1} w_n \right) \left(\lambda_1' \Lambda^{-1} w_n \right) \\
& +\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1 \right)^2} \left(\lambda_1' H_{1,0} \lambda_1 \right) \left(\lambda_1' \Lambda^{-1} w_n \right) \\
& -\frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1 \right)^3} \left(3\lambda_1' \Lambda^{-1} \lambda_2 \right) \left(\lambda_1' \Lambda^{-1} w_n \right)^2
\end{aligned}$$

$$\widetilde{T}_2 \equiv -\frac{2}{\lambda_1' \Lambda^{-1} \lambda_1} \lambda_1' H_{1,1} w_n + \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1 \right)^2} \left(\lambda_1' H_{1,1} \lambda_1 \right) \left(\lambda_1' \Lambda^{-1} w_n \right)$$

$$
\begin{aligned}
\overline{T}_3 \quad \equiv \quad & -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(w_{1,n}'H_{1,0}w_n\right) - \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(\lambda_1'H_{2,0}w_n\right) \\
& + \frac{12\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{12\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'H_{1,0}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{6\left(\lambda_2'\Lambda^{-1}w_n\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,0}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{6\left(\lambda_1'H_{1,0}\lambda_1\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'H_{1,0}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{6\left(w_{1,n}'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(w_{2,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{12\left(\lambda_1'H_{1,0}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_2'H_{1,0}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{3\left(\lambda_1'H_{2,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& - \frac{24\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{6\left(\lambda_2'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{6\left(\lambda_1'H_{1,0}\lambda_1\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{24\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{12\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{24\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{9\left(\lambda_2'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{9\left(\lambda_1'\Lambda^{-1}w_{2,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{3\left(\lambda_3'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{9\left(\lambda_1'H_{1,0}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{18\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{18\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,0}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& + \frac{54\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} + \frac{27\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} \\
& + \frac{27\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} \\
& + \frac{3\left(\lambda_2'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} + \frac{4\left(\lambda_1'\Lambda^{-1}\lambda_3\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} \\
& - \frac{27}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)^3
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{T}_{3,1} \equiv & -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(w_{1,n}'H_{1,1}w_n\right) - \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(\lambda_1'H_{2,1}w_n\right) \\
& + \frac{6\left(\lambda_1'H_{1,1}\lambda_1\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'H_{1,0}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{12\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'H_{1,1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'H_{1,1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{12\left(\lambda_1'H_{1,1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{6\left(\lambda_2'H_{1,1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} + \frac{3\left(\lambda_1'H_{2,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& - \frac{24\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} - \frac{12\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{12\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{9\left(\lambda_1'H_{1,1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& - \frac{18\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \\
& + \frac{27\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{T}_{3,2} \equiv & -\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(\lambda_1'H_{2,2}w_n\right) \\
& + \frac{6\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'H_{1,1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{3\left(\lambda_1'H_{2,2}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
& + \frac{-6\left(\lambda_1'H_{1,1}\lambda_1\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}
\end{aligned}
$$

$$
\widetilde{T}_{3,3} \equiv -\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)}\left(\lambda_1'H_{2,3}w_n\right) + \frac{3\left(\lambda_1'H_{2,3}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}
$$

Proof. Follows from Lemmas 7 and 8. ■

Theorem 1 *Let*

$$
b_2 \equiv \underset{c\in C}{\operatorname{argmin}}\, g(c)'\, G(b_1)^{-1}\, g(c)
$$

and

$$
b_3 \equiv \underset{c\in C}{\operatorname{argmin}}\, g(c)'\, G(b_2)^{-1}\, g(c)
$$

Then,

$$
\sqrt{n}\left(b_3-\beta\right) = \overline{T}_1 + \frac{1}{2\sqrt{n}}\left(\overline{T}_2 + \overline{T}_1\cdot\widetilde{T}_2\right) + \frac{1}{6n}\left(\overline{T}_3 + \overline{T}_1\cdot\widetilde{T}_{3,1} + \overline{T}_1^2\cdot\widetilde{T}_{3,2} + \left(\overline{T}_2 + \overline{T}_1\cdot\widetilde{T}_2\right)\cdot\widetilde{T}_{3,3}\right) + o_p\left(\frac{1}{n}\right)
$$

Proof. We use Lemmas 8 and 9. With induction, we can obtain

$$\sqrt{n}\left(b_2-\beta\right)=\overline{T}_1+\frac{1}{2\sqrt{n}}\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)+\frac{1}{6n}\left(\overline{T}_3+\overline{T}_1\cdot\widetilde{T}_{3,1}+\overline{T}_1^2\cdot\widetilde{T}_{3,2}+\left(\overline{T}_2+L_1\cdot\widetilde{T}_2\right)\cdot\widetilde{T}_{3,3}\right)+o_p\left(\frac{1}{n}\right)$$

and

$$\sqrt{n}\left(b_3-\beta\right)=\overline{T}_1+\frac{1}{2\sqrt{n}}\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)+\frac{1}{6n}\left(\overline{T}_3+\overline{T}_1\cdot\widetilde{T}_{3,1}+\overline{T}_1^2\cdot\widetilde{T}_{3,2}+\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)\cdot\widetilde{T}_{3,3}\right)+o_p\left(\frac{1}{n}\right)$$

■

2 Higher Order Moments

For Maple/Mathematica programming purpose, it is convenient to introduce a new set of symbols:

$$\begin{aligned}
u_0 &\equiv \psi\left(z_i,\beta\right)-\lambda=\psi\left(z_i,\beta\right)\\
u_1 &\equiv \psi_1\left(z_i,\beta\right)-\lambda_1\\
u_2 &\equiv \psi_2\left(z_i,\beta\right)-\lambda_2\\
u_3 &\equiv \psi_3\left(z_i,\beta\right)-\lambda_3\\
u_4 &\equiv \psi_4\left(z_i,\beta\right)-\lambda_4
\end{aligned}$$

$$\begin{aligned}
U_0 &\equiv \Psi\left(z_i,\beta\right)-\Lambda\\
U_1 &\equiv \Psi_1\left(z_i,\beta\right)-\Lambda_1\\
U_2 &\equiv \Psi_2\left(z_i,\beta\right)-\Lambda_2\\
U_3 &\equiv \Psi_3\left(z_i,\beta\right)-\Lambda_3
\end{aligned}$$

$$\begin{aligned}
v_0 &\equiv \psi\left(z_j,\beta\right)-\lambda=\psi\left(z_j,\beta\right)\qquad i\neq j\\
v_1 &\equiv \psi_1\left(z_j,\beta\right)-\lambda_1\\
v_2 &\equiv \psi_2\left(z_j,\beta\right)-\lambda_2\\
v_3 &\equiv \psi_3\left(z_j,\beta\right)-\lambda_3\\
v_4 &\equiv \psi_4\left(z_j,\beta\right)-\lambda_4
\end{aligned}$$

$$\begin{aligned}
V_0 &\equiv \Psi\left(z_j,\beta\right)-\Lambda\\
V_1 &\equiv \Psi_1\left(z_j,\beta\right)-\Lambda_1\\
V_2 &\equiv \Psi_2\left(z_j,\beta\right)-\Lambda_2\\
V_3 &\equiv \Psi_3\left(z_j,\beta\right)-\Lambda_3
\end{aligned}$$

2.1 Higher Order Bias

We focus on the version of GMM b_3 iterated 3 times.[1] This is convenient because its third order properties are invariant to the choice of initial consistent estimator. With Theorem 1, we can see that

Theorem 2

$$
\begin{aligned}
E\left[b_2-\beta\right] &\approx \frac{1}{\sqrt{n}}E\left[\overline{T}_1\right]+\frac{1}{2n}E\left[\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right]\\
&= \frac{1}{2n}E\left[\overline{T}_2\right]+\frac{1}{2n}E\left[\overline{T}_1\cdot\widetilde{T}_2\right]\\
&\equiv \frac{B}{n}
\end{aligned}
$$

Definition 7

$$
\begin{aligned}
\Gamma_{0,1} &\equiv E\left[u_0u_1'\right]=E\left[\psi\left(z_i,\beta\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\right]\\
&= E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right]\\
\Gamma_{0,2} &\equiv E\left[u_0u_2'\right]=E\left[\psi\left(z_i,\beta\right)\left(\psi_2\left(z_i,\beta\right)-\lambda_2\right)'\right]\\
&= E\left[\psi\left(z_i,\beta\right)\psi_2\left(z_i,\beta\right)'\right]\\
\Gamma_{1,1} &\equiv E\left[u_1u_1'\right]=E\left[\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\right]\\
&= E\left[\psi_1\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right]-\lambda_1\lambda_1'
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{0,0} &\equiv E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\\
&= E\left[\psi\left(z_i,\beta\right)\lambda_1'\Lambda^{-1}\Psi\left(z_i,\beta\right)\right]\Lambda^{-1}\\
\Upsilon_{1,0} &\equiv E\left[u_1\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\\
&= E\left[\psi_1\left(z_i,\beta\right)\lambda_1'\Lambda^{-1}\Psi\left(z_i,\beta\right)\right]\Lambda^{-1}-\lambda_1\lambda_1'\Lambda^{-1}\\
\Upsilon_{0,1} &\equiv E\left[u_0\lambda_1'\Lambda^{-1}U_1\right]\Lambda^{-1}\\
&= E\left[\psi_0\left(z_i,\beta\right)\lambda_1'\Lambda^{-1}\Psi_1\left(z_i,\beta\right)\right]\Lambda^{-1}\\
\Upsilon_{0,0}^{(2)} &\equiv E\left[u_0\lambda_2'\Lambda^{-1}U_0\right]\Lambda^{-1}\\
&= E\left[\psi\left(z_i,\beta\right)\lambda_2'\Lambda^{-1}\Psi\left(z_i,\beta\right)\right]\Lambda^{-1}\\
\Upsilon_{0,0}^{(3)} &\equiv E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\cdot U_0\Lambda^{-1}\right]\\
&= E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\cdot\Psi\left(z_i,\beta\right)\Lambda^{-1}\right]\\
\Upsilon_{0,0}^{(4)} &\equiv E\left[U_0\Lambda^{-1}u_0\right]\\
&= E\left[\Psi\left(z_i,\beta\right)\Lambda^{-1}\psi\left(z_i,\beta\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{0,0} &\equiv E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right]\\
\Xi_{0,2} &\equiv E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right]
\end{aligned}
$$

[1] This means that b_3 is a four step estimator. Recall that

$$
\begin{aligned}
b_1 &= \operatorname*{argmin}_{c\in C} g_n\left(c\right)'G_n\left(\widetilde{\beta}\right)^{-1}g_n\left(c\right)\\
b_2 &\equiv \operatorname*{argmin}_{c\in C} g\left(c\right)'G\left(b_1\right)^{-1}g\left(c\right)\\
b_3 &\equiv \operatorname*{argmin}_{c\in C} g\left(c\right)'G\left(b_2\right)^{-1}g\left(c\right)
\end{aligned}
$$

Note that

$$B \equiv \sum_{k=1}^{8} \widetilde{B}_k$$

and

$$\begin{aligned}
\widetilde{B}_1 &\equiv \frac{1}{2}E\left[-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}w_{1,n}'\Lambda^{-1}w_n\right] \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\Lambda^{-1}\psi\left(z_i,\beta\right)\right] \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\operatorname{trace}\left(\Lambda^{-1}E\left[\psi\left(z_i,\beta\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\right]\right) \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\operatorname{trace}\left(\Lambda^{-1}E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right]\right) \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\frac{\operatorname{trace}\left(\Lambda^{-1}\Lambda_1\right)}{2} \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_2 &\equiv \frac{1}{2}E\left[-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}\lambda_1' H_{1,0}w_n\right] \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\left(-\Lambda^{-1}W_n\Lambda^{-1}\right)w_n\right] \\
&= \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\left(\Psi\left(z_i,\beta\right)-\Lambda\right)\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\right] \\
&= \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'-\Lambda\right)\Lambda^{-1}\psi\left(z_i,\beta\right)\right] \\
&= \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\psi\left(z_i,\beta\right)\right]
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_3 &\equiv \frac{1}{2}E\left[\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(2\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)\right)\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\right] \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\left(\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\Lambda^{-1}\lambda_1\right)\right] \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}E\left[\psi\left(z_i,\beta\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\right]\Lambda^{-1}\lambda_1 \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right]\Lambda^{-1}\lambda_1 \\
&= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\frac{1}{2}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \\
&= \frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{B}_4 &\equiv \frac{1}{2}E\left[\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_2'\Lambda^{-1}E\left[w_n w_n'\right]\Lambda^{-1}\lambda_1 \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_2'\Lambda^{-1}\Lambda\Lambda^{-1}\lambda_1 \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_2'\Lambda^{-1}\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_5 &\equiv \frac{1}{2}E\left[\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\left(-\Lambda^{-1}W_n\Lambda^{-1}\right)\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'-\Lambda\right)\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_6 &\equiv \frac{1}{2}E\left[-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\right] \\
&= -\frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}w_n\right)^2\right] \\
&= -\frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_1'\Lambda^{-1}\Lambda\Lambda^{-1}\lambda_1 \\
&= -\frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_1'\Lambda^{-1}\lambda_1 \\
&= -\frac{3\lambda_2'\Lambda^{-1}\lambda_1}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_7 &\equiv \frac{1}{2}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}\lambda_1'H_{1,1}w_n\right)\right] \\
&= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\left(-\Lambda^{-1}\Lambda_1\Lambda^{-1}\right)w_n\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}\Lambda\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

$$\begin{aligned}
\widetilde{B}_8 &\equiv \frac{1}{2}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\left(-\Lambda^{-1}\Lambda_1\Lambda^{-1}\right)\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\right] \\
&= \frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_1'\Lambda^{-1}\Lambda\Lambda^{-1}\lambda_1 \\
&= \frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
B &= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\frac{\operatorname{trace}\left(\Lambda^{-1}\Lambda_1\right)}{2}+\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\psi\left(z_i,\beta\right)\right] \\
&\quad+\frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_2'\Lambda^{-1}\lambda_1 \\
&\quad-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\right]-\frac{3\lambda_2'\Lambda^{-1}\lambda_1}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
&\quad-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1+\frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
&= -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\frac{\operatorname{trace}\left(\Lambda^{-1}\Lambda_1\right)}{2}+\frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}-\frac{\lambda_2'\Lambda^{-1}\lambda_1}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
&\quad+\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\psi\left(z_i,\beta\right)\right] \\
&\quad-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\Lambda^{-1}\lambda_1\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right]
\end{aligned}$$

or

Theorem 3

$$B \equiv \sum_{k=1}^{5} B_k$$

where

$$\begin{aligned}
B_1 &\equiv -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\frac{\text{trace}\left(\Lambda^{-1}\Lambda_1\right)}{2} \\
B_2 &\equiv \frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
B_3 &\equiv -\frac{\lambda_2'\Lambda^{-1}\lambda_1}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \\
B_4 &\equiv \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}u_0\right] = \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}\left(U_0+\Lambda\right)\Lambda^{-1}u_0\right] = \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right] \\
&= \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\text{trace}\left(\Upsilon_{0,0}\right) \\
B_5 &\equiv -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\lambda_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\lambda_1'\Lambda^{-1}u_0\right] = -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\lambda_1'\Lambda^{-1}\left(U_0+\Lambda\right)\Lambda^{-1}\lambda_1\lambda_1'\Lambda^{-1}u_0\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

Remark 1 *The last equality in (2) follows from*

$$\begin{aligned}
\Lambda_1 &= E\left[\partial\Psi\left(z_i,\beta\right)/\partial c\right] = E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)' + \psi_1\left(z_i,\beta\right)\psi\left(z_i,\beta\right)'\right] \\
&= E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right] + \left(E\left[\psi\left(z_i,\beta\right)\psi_1\left(z_i,\beta\right)'\right]\right)'
\end{aligned}$$

and the lemma below.

Lemma 10 *For an arbitrary $k\times k$ matrice A, and an arbitrary symmetric $k\times k$ matrice B, we have* $\text{trace}\left(B^{-1}A\right) = \frac{1}{2}\text{trace}\left(B^{-1}\left(A+A'\right)\right)$ *and* $\text{trace}\left(\left(A+A'\right)^{-1}A\right) = \frac{k}{2}$

Proof. The first claim follows from

$$\text{trace}\left(B^{-1}A'\right) = \text{trace}\left(A'B^{-1}\right) = \text{trace}\left(\left(A'B^{-1}\right)'\right) = \text{trace}\left(B^{-1}A\right)$$

The second claim follows from

$$\text{trace}\left(\left(A+A'\right)^{-1}A'\right) = \text{trace}\left(A'\left(A+A'\right)^{-1}\right) = \text{trace}\left(\left(A'\left(A+A'\right)^{-1}\right)'\right) = \text{trace}\left(\left(A+A'\right)^{-1}A\right)$$

and

$$\text{trace}\left(\left(A+A'\right)^{-1}A\right) + \text{trace}\left(\left(A+A'\right)^{-1}A'\right) = \text{trace}\left(\left(A+A'\right)^{-1}\left(A+A'\right)\right) = \text{trace}\left(I_k\right) = k$$

■

2.2 Higher Order MSE

Again, we focus on the version of GMM iterated 3 times, i.e., b_3. With Theorem 1, we can see that

$$\begin{aligned} E\left[\left(\sqrt{n}\left(b_2-\beta\right)\right)^2\right] &\approx E\left[\overline{T}_1^2\right] \\ &+\frac{1}{\sqrt{n}}E\left[\overline{T}_1\cdot\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)\right] \quad (3)\\ &+\frac{1}{n}E\left[\left(\frac{1}{2}\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)\right)^2\right] \quad (4)\\ &+\frac{1}{3n}E\left[\overline{T}_1\cdot\left(\overline{T}_3+\overline{T}_1\cdot\widetilde{T}_{3,1}+\overline{T}_1^2\cdot\widetilde{T}_{3,2}+\overline{T}_2\cdot\widetilde{T}_{3,3}+\overline{T}_1\cdot\widetilde{T}_2\cdot\widetilde{T}_{3,3}\right)\right] \quad (5)\end{aligned}$$

Theorem 4

$$E\left[\overline{T}_1^2\right]=E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)^2\right]=\frac{\lambda_1'\Lambda^{-1}\Lambda\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}=\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}$$

2.3 Term (3)

We now provide a detailed analysis of (3).

Theorem 5

$$\frac{1}{\sqrt{n}}E\left[\overline{T}_1\cdot\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)\right]=\frac{1}{n}\sum_{k=1}^{8}M_k$$

where

$$\begin{aligned} M_1 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}w_{1,n}'\Lambda^{-1}w_n\right)\right]\\ M_2 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}\lambda_1'H_{1,0}w_n\right)\right]\\ M_3 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(2\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right)\right]\\ M_4 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right)\right]\\ M_5 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'H_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right)\right]\\ M_6 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\left(-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\right)\right]\\ M_7 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)^2\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}\lambda_1'H_{1,1}w_n\right)\right]\\ M_8 &\equiv \sqrt{n}E\left[\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)^2\left(\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'H_{1,1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\right)\right]\end{aligned}$$

Lemma 11

$$M_1=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\operatorname{trace}\left(\Psi_{1,0}\right)$$

Proof. Note that

$$
\begin{aligned}
& \sqrt{n}\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\cdot\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}w_{1,n}'\Lambda^{-1}w_n\right) \\
= & \sqrt{n}\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\psi_1\left(z_j,\beta\right)-\lambda_1\right)'\Lambda^{-1}\left(\psi\left(z_k,\beta\right)-\lambda\right)\right) \\
= & \frac{1}{n}\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\left(\psi_1\left(z_j,\beta\right)-\lambda_1\right)'\Lambda^{-1}\left(\psi\left(z_k,\beta\right)-\lambda\right)
\end{aligned}
$$

Because $E\left[\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\left(\psi_1\left(z_j,\beta\right)-\lambda_1\right)'\Lambda^{-1}\left(\psi\left(z_k,\beta\right)-\lambda\right)\right]=0$ unless $i=j=k$, we have

$$
\begin{aligned}
& E\left[\sqrt{n}\left(-\frac{\lambda_1'\Lambda^{-1}w_n}{\lambda_1'\Lambda^{-1}\lambda_1}\right)\cdot\left(-\frac{2}{\lambda_1'\Lambda^{-1}\lambda_1}w_{1,n}'\Lambda^{-1}w_n\right)\right] \\
= & \frac{1}{n}\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\sum_{i=1}^{n}\lambda_1'\Lambda^{-1}E\left[\left(\psi\left(z_i,\beta\right)-\lambda\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\right] \\
= & \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\lambda_1'\Lambda^{-1}E\left[\left(\psi\left(z_i,\beta\right)-\lambda\right)\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)-\lambda\right)\right]
\end{aligned}
$$

We therefore have

$$
M_1=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}u_0\right)\right]
$$

Because

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[u_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\right] \\
&= E\left[u_1'\Lambda^{-1}\Psi\left(z_i,\beta\right)\Lambda^{-1}\lambda_1\right] \\
&= E\left[u_1'\Lambda^{-1}\left(U_0+\Lambda\right)\Lambda^{-1}\lambda_1\right] \\
&= E\left[u_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\
&= E\left[\operatorname{trace}\left(u_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\operatorname{trace}\left(\lambda_1u_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right] \\
&= E\left[\operatorname{trace}\left(\Lambda^{-1}U_0\Lambda^{-1}\lambda_1u_1'\right)\right] \\
&= E\left[\operatorname{trace}\left(\left(\lambda_1u_1'\right)'\left(\Lambda^{-1}U_0\Lambda^{-1}\right)'\right)\right] \\
&= E\left[\operatorname{trace}\left(u_1\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right] \\
&= \operatorname{trace}\left(\Psi_{1,0}\right)
\end{aligned}
$$

we have

$$
M_1=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\operatorname{trace}\left(\Psi_{1,0}\right)
$$

■

Lemma 12

$$
M_2=-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\Xi_{0,0}
$$

Proof. As in the proof of Lemma 11, we can obtain

$$M_2 = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)\lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)' - \Lambda\right)\Lambda^{-1}\psi\left(z_i,\beta\right)\right]$$

and hence

$$\begin{aligned}
M_2 &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\left(U_0+\Lambda\right)\Lambda^{-1}\lambda_1\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\
&= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} \Xi_{0,0}
\end{aligned}$$

■

Lemma 13

$$M_3 = -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned}
M_3 &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)^2\lambda_1'\Lambda^{-1}\left(\psi_1\left(z_i,\beta\right)-\lambda_1\right)\right] \\
&= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\left(\lambda_1'\Lambda^{-1}u_1\right)\right] \\
&= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right] \\
&= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\left(U_0+\Lambda\right)\Lambda^{-1}\lambda_1\right)\right] \\
&= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1
\end{aligned}$$

■

Lemma 14

$$M_4 = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_4 &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)^2 \lambda_2'\Lambda^{-1}\psi\left(z_i,\beta\right)\right] \\ &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2 \left(\lambda_2'\Lambda^{-1}u_0\right)\right] \\ &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\right] \\ &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} \lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

■

Lemma 15

$$M_5 = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\Xi_{0,2}$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_5 &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)^2 \lambda_1'\Lambda^{-1}\left(\psi\left(z_i,\beta\right)\psi\left(z_i,\beta\right)' - \Lambda\right)\Lambda^{-1}\lambda_1\right] \\ &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2 \left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_0u_0'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right] \\ &= \end{aligned}$$

■

Lemma 16

$$M_6 = 3\frac{\left(\lambda_1'\Lambda^{-1}\lambda_2\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_6 &= \frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}\psi\left(z_i,\beta\right)\right)^3\right] \\ &= \frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^3\right] \\ &= \frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= \frac{\left(3\lambda_1'\Lambda^{-1}\lambda_2\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right) \end{aligned}$$

■

Lemma 17

$$M_7 = \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} \lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_7 &= \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(\lambda_1' \Lambda^{-1} \psi\left(z_i, \beta\right)\right)^2 \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \psi\left(z_i, \beta\right)\right)\right] \\ &= \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(\lambda_1' \Lambda^{-1} u_0\right)^2 \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0\right)\right] \\ &= \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\right] \\ &= \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} \lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Upsilon_{0,0} \lambda_1 \end{aligned}$$

■

Lemma 18

$$M_8 = -\frac{2\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right)$$

Proof. As in the proof of Lemma 11, we can obtain

$$\begin{aligned} M_8 &= -\frac{2\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_1' \Lambda^{-1} \psi\left(z_i, \beta\right)\right)^3\right] \\ &= -\frac{2\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_1' \Lambda^{-1} u_0\right)^3\right] \\ &= -\frac{2\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\right] \\ &= -\frac{2\left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right) \end{aligned}$$

■

2.4 Term (4)

We now provide a detailed analysis of (4). Write

$$\frac{1}{2}\left(\overline{T}_2 + \overline{T}_1 \cdot \widetilde{T}_2\right) = \sum_{k=1}^{7} S_k$$

where

$$\begin{aligned}
S_1 &\equiv -\frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\left(w_{1,n}'\Lambda^{-1}w_n\right) \\
S_2 &\equiv \frac{1}{\lambda_1'\Lambda^{-1}\lambda_1}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right) \\
S_3 &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right) \\
S_4 &\equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right) \\
S_5 &\equiv -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right) \\
S_6 &\equiv \left(-\frac{3\lambda_1'\Lambda^{-1}\lambda_2}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}+\frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
S_7 &\equiv -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)
\end{aligned}$$

We then have

Theorem 6

$$E\left[\left(\frac{1}{2}\left(\overline{T}_2+\overline{T}_1\cdot\widetilde{T}_2\right)\right)^2\right]=\sum_{k=1}^{7}E\left[S_k^2\right]+2\sum_{k<k'}E\left[S_kS_{k'}\right]$$

Lemma 19

$$E\left[S_1^2\right]=M_9+O\left(n^{-1}\right),$$

where

$$M_9=\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\right)^2+\text{trace}\left(\Lambda^{-1}\Gamma_{1,1}\right)+\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\right)\right)$$

Proof. As in Donald and Newey (1998), we use the fact that for a pair of double arrays, say $\left\{A_{ij}^1\right\}$ and $\left\{A_{ij}^2\right\}$, we have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}^1\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}^2 &= \sum_{i=1}^{n}A_{ii}^1A_{ii}^2+\sum_{i=1}^{n}\sum_{j\neq i}A_{ii}^1A_{jj}^2 \\
&\quad+\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^1A_{ij}^2+\sum_{i=1}^{n}\sum_{j\neq i}A_{ij}^1A_{ji}^2+\cdots
\end{aligned}$$

where the omitted terms will not be needed for our calculations because they will have zero expectations in our context. It therefore follows that

$$\begin{aligned}
M_9 &\equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(u_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}u_0\right)\right]
\end{aligned}$$

Now note that

$$E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\right] = \left(E\left[u_1'\Lambda^{-1}u_0\right]\right)^2 = \left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\right)^2$$

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(u_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}u_1\right)\right] = E\left[u_1'\Lambda^{-1}v_0v_0'\Lambda^{-1}u_1\right] \\ &= E\left[u_1'\Lambda^{-1}u_1\right] = \text{trace}\left(\Lambda^{-1}\Gamma_{1,1}\right) \end{aligned}$$

and

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}u_0\right)\right] &= E\left[u_1'\Lambda^{-1}E\left[v_0v_1'\right]\Lambda^{-1}u_0\right] = \text{trace}\left(\Lambda^{-1}E\left[u_0u_1'\right]\Lambda^{-1}E\left[u_0u_1'\right]\right) \\ &= \text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\right) \end{aligned}$$

■

Lemma 20

$$E\left[S_2^2\right] = M_{10} + O\left(n^{-1}\right),$$

where

$$M_{10} \equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\left(\left[\text{trace}\left(\Upsilon_{0,0}\right)\right]\right)^2 + \Xi_{0,0} + \text{trace}\left(\Upsilon_{0,0}\Upsilon_{0,0}\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{10} \equiv\; & \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Note that

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] &= \left(E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]\right)^2 \\ &= \left(E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]\right)^2 \\ &= \left(E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\right)^2 \\ &= \left(\left[\text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\right)\right]\right)^2 \\ &= \left(\left[\text{trace}\left(\Upsilon_{0,0}\right)\right]\right)^2 \end{aligned}$$

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0v_0'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\ &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \end{aligned}$$

and

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right] &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}u_0\right] \\ &= E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}u_0\right)\right] \\ &= E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}\right)\right] \\ &= \text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}\right) \\ &= \text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\right) \\ &= \text{trace}\left(\Upsilon_{0,0}\Upsilon_{0,0}\right) \end{aligned}$$

■

Lemma 21

$$E\left[S_3^2\right]=M_{11}+O\left(n^{-1}\right),$$

where

$$M_{11}\equiv\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2+\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}M_{11}\quad\equiv\quad&\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\&+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\&+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\right)^2=\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2$$

$$\begin{aligned}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\quad&=\quad E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)^2\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)^2\right]\\&=\quad\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)^2\right]\\&=\quad\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1\right)\end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]=\left(E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\right)^2=\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 22

$$E\left[S_4^2\right]=M_{12}+O\left(n^{-1}\right),$$

where

$$M_{12}\equiv\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2+\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_2\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}M_{12}\quad\equiv\quad&\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\&+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\&+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\end{aligned}$$

Now note that

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\right)^2 = \left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2$$

$$\begin{aligned} & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)^2\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)^2\right] = \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)^2\right] \\ = & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_2\right) \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ = & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)^2 \end{aligned}$$

■

Lemma 23

$$E\left[S_5^2\right] = M_{13} + O\left(n^{-1}\right),$$

where

$$M_{13} \equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2 + \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\Xi_{0,2}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{13} \equiv & \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & \left(E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\right)^2 \\ = & \left(E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]\right)^2 \\ = & \left(\lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1\right)^2 \\ = & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2 \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right] \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ = & \left(E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\right)^2 \\ = & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2 \end{aligned}$$

■

Lemma 24

$$E\left[S_6^2\right] = M_{14} + O\left(n^{-1}\right),$$

where

$$M_{14} \equiv \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-\frac{3\lambda_1'\Lambda^{-1}\lambda_2}{2} + \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2$$

Proof. As in Lemma 19, we obtain

$$M_{14} \equiv 3\left(-\frac{3\lambda_1'\Lambda^{-1}\lambda_2}{2\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} + \frac{\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\right)^2 E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\left(\lambda_1'\Lambda^{-1}v_0\right)^2\right]$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\left(\lambda_1'\Lambda^{-1}v_0\right)^2\right] = \left(E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\right]\right)^2 = \left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 25

$$E\left[S_7^2\right] = M_{15} + O\left(n^{-1}\right),$$

where

$$M_{15} \equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2 + \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{15} \equiv\ & \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & \left(E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]\right)^2 \\ = & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2 \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)^2\right] \\ = & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\ = & \left(E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]\right)^2 \\ = & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2 \end{aligned}$$

■

Lemma 26

$$2\left[S_1S_2\right]=M_{16}+O\left(n^{-1}\right),$$

where

$$M_{16}\equiv-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\text{trace}\left(\Upsilon_{0,0}\right)+\text{trace}\left(\Upsilon_{1,0}\right)+\text{trace}\left(\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}M_{16}\quad\equiv\quad&-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right]\\&-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\right]\\&-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right]\end{aligned}$$

Now note that

$$\begin{aligned}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right]&=E\left[u_1'\Lambda^{-1}u_0\right]E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]\\&=\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]\\&=\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\\&=\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\right)\\&=\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\text{trace}\left(\Upsilon_{0,0}\right)\end{aligned}$$

$$\begin{aligned}&E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\right]\\=\;&E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]=E\left[u_1'\Lambda^{-1}v_0v_0'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right]\\=\;&E\left[u_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right]=E\left[\text{trace}\left(u_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]=E\left[\text{trace}\left(\lambda_1u_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\\=\;&E\left[\text{trace}\left(\left(\lambda_1u_1'\right)\left(\Lambda^{-1}U_0\Lambda^{-1}\right)\right)\right]=E\left[\text{trace}\left(\left(\Lambda^{-1}U_0\Lambda^{-1}\right)'\left(\lambda_1u_1'\right)'\right)\right]\\=\;&E\left[\text{trace}\left(\left(\lambda_1u_1'\right)'\left(\Lambda^{-1}U_0\Lambda^{-1}\right)'\right)\right]=E\left[\text{trace}\left(u_1\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\\=\;&\text{trace}\left(\Upsilon_{1,0}\right)\end{aligned}$$

and

$$\begin{aligned}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right]&=E\left[\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}E\left[u_0u_1'\right]\Lambda^{-1}v_0\right]\\&=E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}u_0\right]\\&=E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}u_0\right)\right]\\&=E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\right)\right]\\&=\text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\right)\\&=\text{trace}\left(\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\right)\end{aligned}$$

■

Lemma 27

$$2\left[S_1S_3\right]=M_{17}+O\left(n^{-1}\right),$$

where

$$M_{17} \equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right) + \lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1 + \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{17} \equiv & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= \text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right) \end{aligned}$$

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= E\left[\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}v_0v_0'\Lambda^{-1}\lambda_1\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}\lambda_1\right)\right] \\ &= E\left[\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_0v_1'\Lambda^{-1}\lambda_1\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

■

Lemma 28

$$2\left[S_1S_4\right] = M_{18} + O\left(n^{-1}\right),$$

where

$$M_{18} \equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right) + \lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 + \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{18} \equiv & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[u_1'\Lambda^{-1}u_0\right]E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= \text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right) \end{aligned}$$

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_0v_0'\Lambda^{-1}\lambda_1\right] \\
&= \lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_2\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2
\end{aligned}
$$

■

Lemma 29

$$2\left[S_1S_5\right] = M_{19} + O\left(n^{-1}\right),$$

where

$$M_{19} \equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
M_{19} \equiv\ & \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
=\ & \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\
=\ & \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & \lambda_1'\Lambda^{-1}E\left[u_1\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\
=\ & \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
=\ & \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\
=\ & \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

■

Lemma 30

$$2\left[S_1 S_6\right] = M_{20} + O\left(n^{-1}\right),$$

where

$$M_{20} \equiv \frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(3\lambda_1' \Lambda^{-1} \lambda_2 - 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) \left(\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right) \left(\lambda_1' \Lambda^{-1} \lambda_1\right) + 2\left(\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{20} \equiv\ & \frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(3\lambda_1' \Lambda^{-1} \lambda_2 - 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(u_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(3\lambda_1' \Lambda^{-1} \lambda_2 - 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & + \frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(3\lambda_1' \Lambda^{-1} \lambda_2 - 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} u_0\right)\right] \end{aligned}$$

Now note that

$$E\left[\left(u_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right)\right] = E\left[\left(u_1' \Lambda^{-1} u_0\right)\right] E\left[\left(\lambda_1' \Lambda^{-1} v_0\right)^2\right] = \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right) \left(\lambda_1' \Lambda^{-1} \lambda_1\right)$$

$$\begin{aligned} E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right)\right] &= E\left[\left(\lambda_1' \Lambda^{-1} u_0\right) \left(u_1' \Lambda^{-1} v_0\right) \left(v_0' \Lambda^{-1} \lambda_1\right)\right] \\ &= E\left[\left(\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0\right) \left(v_0' \Lambda^{-1} \lambda_1\right)\right] \\ &= \lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

and

$$\begin{aligned} E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} u_0\right)\right] &= E\left[\left(\lambda_1' \Lambda^{-1} u_0\right) \left(u_1' \Lambda^{-1} v_0\right) \left(v_0' \Lambda^{-1} \lambda_1\right)\right] \\ &= E\left[\left(\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} v_0\right) \left(v_0' \Lambda^{-1} \lambda_1\right)\right] \\ &= \lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1 \end{aligned}$$

■

Lemma 31

$$2\left[S_1 S_7\right] = M_{21} + O\left(n^{-1}\right),$$

where

$$M_{21} \equiv \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} \left(\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right) \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) + \lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1 + \lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{21} \equiv\ & \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(u_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0\right)\right] \\ & + \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} u_0\right) \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} v_0\right)\right] \\ & + \frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^3} E\left[\left(u_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} v_0\right) \left(\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} u_0\right)\right] \end{aligned}$$

Now note that

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
&= \text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_0v_0'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_0v_0'\Lambda^{-1}\lambda_1\right] \\
&= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

■

Lemma 32

$$2\left[S_2S_3\right] = M_{22} + O\left(n^{-1}\right),$$

where

$$M_{22} = \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\text{trace}\left(\Upsilon_{0,0}\right) + \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
M_{22} \quad = \quad & \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}\lambda_1\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\text{trace}\left(E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\right) \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\text{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}E\left[u_1\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

■

Lemma 33

$$2\left[S_2S_4\right] = M_{23} + O\left(n^{-1}\right),$$

where

$$M_{23} = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_2'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Upsilon_{0,0}\right) + \lambda_2'\Lambda^{-1}\Upsilon_{00}\lambda_1 + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2\right)$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
M_{23} = {} & \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}U_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_2'\Lambda^{-1}\lambda_1\right)E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right] \\
&= \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_2'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\
&= \lambda_2'\Lambda^{-1}\Upsilon_{00}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2
\end{aligned}
$$

■

Lemma 34

$$2\left[S_2S_5\right] = M_{24} + O\left(n^{-1}\right),$$

where

$$M_{24} = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\text{trace}\left(\Upsilon_{0,0}\right)\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 + \Xi_{0,2} + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{24} &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]E\left[\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ =\ & E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ =\ & E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\ =\ & \text{trace}\left(\Upsilon_{0,0}\right)\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right] \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}\lambda_1\right] \\ =\ & \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}\lambda_1 \\ =\ & \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\ =\ & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

■

Lemma 35

$$2\left[S_2S_6\right] = M_{25} + O\left(n^{-1}\right),$$

where

$$M_{25} = \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\text{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{25} &= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)^2\right] \\ =\ & E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ =\ & E\left[\operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ =\ & \operatorname{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\ =\ & \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\right]\Lambda^{-1}\lambda_1 \\ =\ & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

■

Lemma 36

$$2\left[S_2S_7\right] = M_{26} + O\left(n^{-1}\right),$$

where

$$M_{26} = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\operatorname{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1 + \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{26} &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ =\ & E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\ =\ & E\left[\operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \\ =\ & \operatorname{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \end{aligned}$$

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
= & E\left[\lambda_1'\Lambda^{-1}u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right] \\
= & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
= & \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

■

Lemma 37

$$
2\left[S_3S_4\right] = M_{27} + O\left(n^{-1}\right),
$$

where

$$
M_{27} = \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right) + \left(\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
M_{27} = & \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)
\end{aligned}
$$

■

Lemma 38

$$2\left[S_3S_5\right] = M_{28} + O\left(n^{-1}\right),$$

where

$$M_{28} = -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{28} &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} &E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ &E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\ =\ &\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\right]\Lambda^{-1}\lambda_1\right) \\ =\ &\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right) \end{aligned}$$

$$\begin{aligned} &E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ &E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\ &\left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

and

$$\begin{aligned} &E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ =\ &E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\ =\ &\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right) \end{aligned}$$

■

Lemma 39

$$2\left[S_3S_6\right] = M_{29} + O\left(n^{-1}\right),$$

where

$$M_{29} = \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{29} &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 40

$$2\left[S_3S_7\right] = M_{30} + O\left(n^{-1}\right),$$

where

$$M_{30} = -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{30} = & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 41

$$2\left[S_4 S_5\right] = M_{31} + O\left(n^{-1}\right),$$

where

$$M_{31} = -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(2\left(\lambda_1' \Lambda^{-1} \lambda_2\right)\left(\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right) + \left(\lambda_1' \Lambda^{-1} \lambda_1\right)\left(\lambda_2' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{31} \quad = \quad & -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ = \quad & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] E\left[\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\right] \\ = \quad & \left(\lambda_1' \Lambda^{-1} \lambda_2\right)\left(\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right) \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ = \quad & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\right] E\left[\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ = \quad & \left(\lambda_2' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} \lambda_1\right) \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] \\ = \quad & E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] E\left[\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} V_0 \Lambda^{-1} \lambda_1\right)\right] \\ = \quad & \left(\lambda_1' \Lambda^{-1} \lambda_2\right)\left(\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \lambda_1\right) \end{aligned}$$

■

Lemma 42

$$2\left[S_4 S_6\right] = M_{32} + O\left(n^{-1}\right),$$

where

$$M_{32} = \frac{3}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^4} \left(-3\lambda_1' \Lambda^{-1} \lambda_2 + 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right)\left(\lambda_1' \Lambda^{-1} \lambda_2\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} M_{32} \quad = \quad & \frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^5} \left(-3\lambda_1' \Lambda^{-1} \lambda_2 + 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & +\frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^5} \left(-3\lambda_1' \Lambda^{-1} \lambda_2 + 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\ & +\frac{1}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^5} \left(-3\lambda_1' \Lambda^{-1} \lambda_2 + 2\lambda_1' \Lambda^{-1} \Lambda_1 \Lambda^{-1} \lambda_1\right) E\left[\left(\lambda_2' \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] \end{aligned}$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 43

$$2\left[S_4S_7\right] = M_{33} + O\left(n^{-1}\right),$$

where

$$M_{33} = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
M_{33} \;=\; & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 44

$$2\left[S_5S_6\right]=M_{34}+O\left(n^{-1}\right),$$

where

$$M_{34}=-\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2+2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}M_{34} &= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2+2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2+2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2+2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\end{aligned}$$

Now note that

$$\begin{aligned}& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

$$\begin{aligned}& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

and

$$\begin{aligned}& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]\\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

■

Lemma 45

$$2\left[S_5S_7\right]=M_{35}+O\left(n^{-1}\right),$$

where

$$M_{35}=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}M_{35} &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]\\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]\\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 46

$$2\left[S_6S_7\right] = M_{36} + O\left(n^{-1}\right),$$

where

$$M_{36} = -\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
M_{36} \;=\; & -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
& -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(-3\lambda_1'\Lambda^{-1}\lambda_2 + 2\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)^2\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

2.5 Term (5)

We now provide a detailed analysis of (5). We need to calculate the expectation of

$$\frac{1}{3}\overline{T}_1\cdot\overline{T}_3+\frac{1}{3}\left(\overline{T}_1\right)^2\cdot\widetilde{T}_{3,1}+\frac{1}{3}\overline{T}_1^3\cdot\widetilde{T}_{3,2}+\frac{1}{3}\overline{T}_1\cdot\overline{T}_2\cdot\widetilde{T}_{3,3}+\frac{1}{3}\left(\overline{T}_1\right)^2\cdot\widetilde{T}_2\cdot\widetilde{T}_{3,3}$$

We first note that

$$\frac{1}{3}\overline{T}_1\cdot\overline{T}_3=\sum_{k=1}^{31}A_k$$

where

$$\begin{aligned}
A_1 &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(w_{1,n}'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_2 &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_3 &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_4 &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_5 &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_6 &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_7 &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_8 &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_9 &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(w_{1,n}'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{10} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(w_{2,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)
\end{aligned}$$

$$
\begin{aligned}
A_{11} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{12} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{13} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{14} &\equiv \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{15} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{16} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{17} &\equiv \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{18} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{19} &\equiv -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{20} &\equiv \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)
\end{aligned}
$$

$$
\begin{aligned}
A_{21} &\equiv \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_{2,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{22} &\equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_3'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{23} &\equiv -\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{24} &\equiv \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{25} &\equiv -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{26} &\equiv -\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{27} &\equiv -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{28} &\equiv \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{29} &\equiv -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{30} &\equiv -\frac{4}{3\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_3\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{31} &\equiv \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}w_n\right)
\end{aligned}
$$

Lemma 47

$$
\begin{aligned}
& E\left[A_1\right] \\
= & -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^2}\left(\operatorname{trace}\left(\Upsilon_{1,0}\right)+\operatorname{trace}\left(\Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(3)}\right)+\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(4)}\right) \\
& +O\left(n^{-1}\right)
\end{aligned}
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_1\right] = & -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^2} E\left[\left(u_1' \Lambda^{-1} U_0 \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^2} E\left[\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^2} E\left[\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(u_1' \Lambda^{-1} U_0 \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\
= & E\left[\left(u_1' \Lambda^{-1} U_0 \Lambda^{-1} v_0\right)\left(v_0' \Lambda^{-1} \lambda_1\right)\right]=E\left[u_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right] \\
= & E\left[\operatorname{trace}\left(u_1' \Lambda^{-1} U_0 \Lambda^{-1} \lambda_1\right)\right]=E\left[\operatorname{trace}\left(\lambda_1 u_1' \Lambda^{-1} U_0 \Lambda^{-1}\right)\right] \\
= & E\left[\operatorname{trace}\left(\left(\lambda_1 u_1'\right)\left(\Lambda^{-1} U_0 \Lambda^{-1}\right)\right)\right]=E\left[\operatorname{trace}\left(\left(\Lambda^{-1} U_0 \Lambda^{-1}\right)\left(\lambda_1 u_1'\right)\right)\right] \\
= & E\left[\operatorname{trace}\left(\left(\lambda_1 u_1'\right)'\left(\Lambda^{-1} U_0 \Lambda^{-1}\right)'\right)\right]=E\left[\operatorname{trace}\left(u_1 \lambda_1' \Lambda^{-1} U_0 \Lambda^{-1}\right)\right] \\
= & \operatorname{trace}\left(\Upsilon_{1,0}\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} u_0\right)\left(\lambda_1' \Lambda^{-1} v_0\right)\right] \\
= & E\left[\left(\lambda_1' \Lambda^{-1} v_0\right) \cdot \operatorname{trace}\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} u_0\right)\right] \\
= & E\left[\left(\lambda_1' \Lambda^{-1} v_0\right) \cdot \operatorname{trace}\left(u_0 u_1' \Lambda^{-1} V_0 \Lambda^{-1}\right)\right] \\
= & E\left[\left(\lambda_1' \Lambda^{-1} v_0\right) \cdot \operatorname{trace}\left(\Gamma_{0,1} \Lambda^{-1} V_0 \Lambda^{-1}\right)\right] \\
= & E\left[\left(\lambda_1' \Lambda^{-1} u_0\right) \cdot \operatorname{trace}\left(\Gamma_{0,1} \Lambda^{-1} U_0 \Lambda^{-1}\right)\right] \\
= & E\left[\operatorname{trace}\left(\Gamma_{0,1} \Lambda^{-1} \cdot\left(\lambda_1' \Lambda^{-1} u_0\right) \cdot U_0 \Lambda^{-1}\right)\right] \\
= & \operatorname{trace}\left(\Gamma_{0,1} \Lambda^{-1} \cdot \Upsilon_{0,0}^{(3)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} v_0\right)\left(\lambda_1' \Lambda^{-1} u_0\right)\right] & = E\left[\left(\lambda_1' \Lambda^{-1} u_0\right)\left(u_1' \Lambda^{-1} V_0 \Lambda^{-1} v_0\right)\right] \\
& = E\left[\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} V_0 \Lambda^{-1} v_0\right] \\
& = E\left[\lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} U_0 \Lambda^{-1} u_0\right] \\
& = \lambda_1' \Lambda^{-1} \Gamma_{0,1} \Lambda^{-1} \Upsilon_{0,0}^{(4)}
\end{aligned}
$$

■

Lemma 48

$$
E\left[A_2\right]=\frac{2}{\left(\lambda_1' \Lambda^{-1} \lambda_1\right)^2}\left(\Xi_{0,0}+\operatorname{trace}\left(\Upsilon_{0,0} \Upsilon_{0,0}^{(3)}\right)+\lambda_1' \Lambda^{-1} \Upsilon_{0,0} \Upsilon_{0,0}^{(4)}\right)+O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_2\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}v_0v_0'\Lambda^{-1}\lambda_1\right] \\
&= E\left[\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\
&= \Xi_{0,0}
\end{aligned}$$

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\cdot \operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\cdot \operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\cdot \operatorname{trace}\left(\Upsilon_{0,0}V_0\Lambda^{-1}\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\cdot \operatorname{trace}\left(\Upsilon_{0,0}U_0\Lambda^{-1}\right)\right] \\
&= \operatorname{trace}\left(\Upsilon_{0,0}\Upsilon_{0,0}^{(3)}\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)V_0\Lambda^{-1}v_0\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\Upsilon_{0,0}V_0\Lambda^{-1}v_0\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\Upsilon_{0,0}U_0\Lambda^{-1}u_0\right] \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Upsilon_{0,0}^{(4)}
\end{aligned}$$

■

Lemma 49

$$\begin{aligned}
& E\left[A_3\right] \\
= & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1 + \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\cdot \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\right) \\
& + O\left(n^{-1}\right)
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_3\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad - \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad - \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = E\left[\lambda_1'\Lambda^{-1}u_1u_1'\Lambda^{-1}\lambda_1\right] = \lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1$$

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(v_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= E\left[v_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\lambda_1'\Lambda^{-1}v_0\right] \\ &= \text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\lambda_1'\Lambda^{-1}\Gamma_{0,1}\right) \\ &= \text{trace}\left(\left(\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\right)\right) \\ &= \text{trace}\left(\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\right)\left(\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\right) \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}v_0\right)\right] \\ &= E\left[\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\left(v_1'\Lambda^{-1}v_0\right)\right] \\ &= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\cdot\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right) \end{aligned}$$

■

Lemma 50

$$E\left[A_4\right] = \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\lambda_1 + \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\text{trace}\left(\Upsilon_{0,0}\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_4\right] = \; & \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & + \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\ &= \lambda_1'\Lambda^{-1}E\left[u_1\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\ &= \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1 \end{aligned}$$

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\right] \\ &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)u_0\right)\right] \\ &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}u_0\right)\right] \\ &= E\left[\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right] \\ &= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)E\left[\operatorname{trace}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}
$$

■

Lemma 51

$$
E\left[A_5\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 + \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2 + \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_5\right] = & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
$$

$$
\begin{aligned}
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= E\left[\left(\lambda_2'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}\lambda_2\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}v_0\right)\right] \\
&= E\left[\left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)
\end{aligned}
$$

■

Lemma 52

$$
E\left[A_6\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2 + \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\operatorname{trace}\left(\Upsilon_{0,0}\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_6\right] = & \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_2'\Lambda^{-1}u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right] \\
&= \lambda_2'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\
&= \lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_2\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_2\right] \\
&= \lambda_1'\Lambda^{-1}E\left[v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right]\lambda_2 \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\mathrm{trace}\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\mathrm{trace}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_2\right)\mathrm{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}
$$

■

Lemma 53

$$
E\left[A_7\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1 + \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Upsilon_{0,0}\lambda_1 + \mathrm{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_7\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(v_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[v_1'\Lambda^{-1}v_0\right]E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)
\end{aligned}
$$

■

Lemma 54

$$
E\left[A_8\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\Xi_{0,2} + \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Upsilon_{0,0}\lambda_1 + \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_8\right] = & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)\Upsilon_{0,0}\lambda_1\right] \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Upsilon_{0,0}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\lambda_1\right]E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\cdot E\left[\operatorname{trace}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}
$$

■

Lemma 55

$$
E\left[A_9\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\operatorname{trace}\left(\Lambda^{-1}\Gamma_{1,1}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E[A_9] &= -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_1'\Lambda^{-1}u_1)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&\quad -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&\quad -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)\right] + O(n^{-1})
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[(u_1'\Lambda^{-1}u_1)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)\right] &= E\left[(u_1'\Lambda^{-1}u_1)\right] E\left[(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&= \text{trace}\left(\Lambda^{-1}\Gamma_{1,1}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left[(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)\right] &= E\left[(\lambda_1'\Lambda^{-1}u_0)(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&= E\left[(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}v_1)(v_0'\Lambda^{-1}\lambda_1)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)\right] &= E\left[(u_1'\Lambda^{-1}v_1)(\lambda_1'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1
\end{aligned}
$$

■

Lemma 56

$$
E[A_{10}] = -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3}\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,2}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,2}\Lambda^{-1}\lambda_1\right)\right) + O(n^{-1})
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E[A_{10}] &= -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_2'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&\quad -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_2'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)\right] \\
&\quad -\frac{2}{(\lambda_1'\Lambda^{-1}\lambda_1)^3} E\left[(u_2'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)\right] + O(n^{-1})
\end{aligned}
$$

Now note that

$$
E\left[(u_2'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)\right] = \text{trace}\left(\Lambda^{-1}\Gamma_{0,2}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
$$

$$
\begin{aligned}
E\left[(u_2'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)(\lambda_1'\Lambda^{-1}v_0)\right] &= E\left[(\lambda_1'\Lambda^{-1}u_0)(u_2'\Lambda^{-1}v_0)(v_0'\Lambda^{-1}\lambda_1)\right] \\
&= E\left[(\lambda_1'\Lambda^{-1}u_0)(u_2'\Lambda^{-1}\lambda_1)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,2}\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[(u_2'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}v_0)(\lambda_1'\Lambda^{-1}u_0)\right] &= E\left[(\lambda_1'\Lambda^{-1}u_0)(u_2'\Lambda^{-1}v_0)(v_0'\Lambda^{-1}\lambda_1)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,2}\Lambda^{-1}\lambda_1
\end{aligned}
$$

■

Lemma 57

$$E\left[A_{11}\right]=\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\operatorname{trace}\left(\Upsilon_{1,0}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}'\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{11}\right] &= \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$\begin{aligned}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &= \operatorname{trace}\left(\Upsilon_{1,0}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

$$\begin{aligned}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &= E\left[\left(\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &= E\left[\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &= E\left[\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}v_1\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right]\\ &= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}'\Lambda^{-1}\lambda_1\end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]=\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Gamma_{0,1}'\Lambda^{-1}\lambda_1$$

■

Lemma 58

$$E\left[A_{12}\right]=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}^{(2)}\right)+2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}^{(2)}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{12}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$\begin{aligned}E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot E\left[\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}u_0\right]\\ &= \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}^{(2)}\right)\end{aligned}$$

$$\begin{aligned} E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\ &= E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}\right)\lambda_1\right] \\ &= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}^{(2)}\lambda_1 \end{aligned}$$

and

$$E\left[\left(\lambda_2'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Upsilon_{0,0}^{(2)}\lambda_1$$

■

Lemma 59

$$E\left[A_{13}\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\Xi_{0,0}\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda\Upsilon_{0,0}'\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{13}\right] = & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ &= \Xi_{0,0}\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}V_0\Lambda^{-1}\lambda_1\right)'\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Upsilon_{0,0}'\Lambda^{-1}\lambda_1\right)\right] \\ = & E\left[\lambda_1'\Lambda^{-1}\left(v_0\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\right)\Lambda\Upsilon_{0,0}'\Lambda^{-1}\lambda_1\right] \\ = & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda\Upsilon_{0,0}'\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda\Upsilon_{0,0}'\Lambda^{-1}\lambda_1$$

■

Lemma 60

$$E\left[A_{14}\right] = \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{14}\right] &= \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Gamma_{1,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

$$\begin{aligned} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2 \end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 61

$$E\left[A_{15}\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_2'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_2'\Lambda^{-1}\lambda_1\right)^2\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{15}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_2'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

$$\begin{aligned} E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= \left(\lambda_2'\Lambda^{-1}\lambda_1\right)^2 \end{aligned}$$

and

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \left(\lambda_2'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 62

$$E\left[A_{16}\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\Xi_{0,2} + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{16}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)^2\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
=\; & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2
\end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)^2$$

■

Lemma 63

$$E\left[A_{17}\right] = \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{17}\right] &= \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_2'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_2'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 64

$$A_{18} \equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
A_{18} \quad \equiv \quad & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \quad & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \quad & \left(\lambda_2'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \quad & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
= \quad & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \quad & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
= \quad & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)
\end{aligned}$$

■

Lemma 65

$$E\left[A_{19}\right] = -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{19}\right] \quad = \quad & -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \quad & E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \quad & \left(\lambda_1'\Lambda^{-1}\Upsilon_{1,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_1\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 66

$$E\left[A_{20}\right] = \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{20}\right] = & \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_2'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 67

$$E\left[A_{21}\right] = \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Gamma_{0,2}\Lambda^{-1}\lambda_1\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{21}\right] &= \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&+\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_2\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&+\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}u_2\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_2\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,2}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 68

$$E\left[A_{22}\right] = \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\lambda_3\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{22}\right] &= \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_3'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_3'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&+\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_3'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
E\left[\left(\lambda_3'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_3'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_3\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 69

$$E\left[A_{23}\right] = -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{23}\right] &= -\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&-\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&-\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_2\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 70

$$
E\left[A_{24}\right]=\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\right)+O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{24}\right] = \; & \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\left(u_0u_1'\right)\Lambda^{-1}\left(v_0v_0'\right)\Lambda^{-1}\lambda_1\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}v_1\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\lambda_1'\Lambda^{-1}\left(u_0u_0'\right)\Lambda^{-1}\left(v_1v_0'\right)\Lambda^{-1}\lambda_1\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1 \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(v_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(v_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)
\end{aligned}
$$

■

Lemma 71

$$
E\left[A_{25}\right]=-\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\operatorname{trace}\left(\Upsilon_{0,0}\right)\right)+O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{25}\right] = \; & -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

and

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}V_0\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\text{trace}\left(\Upsilon_{0,0}\right)
\end{aligned}$$

■

Lemma 72

$$E\left[A_{26}\right] = -\frac{54}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{26}\right] &= -\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 73

$$E\left[A_{27}\right] = -\frac{27}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2 + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{27}\right] &= -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 74

$$E\left[A_{28}\right]=\frac{27}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{28}\right] &= \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$\begin{aligned}&E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ &E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ &\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

■

Lemma 75

$$E\left[A_{29}\right]=-\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}\lambda_2\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{29}\right] &= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 76

$$E\left[A_{30}\right]=-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\lambda_3\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{30}\right] &= -\frac{4}{3\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_3\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4}{3\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_3\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4}{3\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_3\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 77

$$E\left[A_{31}\right]=\frac{27}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{31}\right] &= \frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

We also note that

$$\frac{1}{3}\left(\overline{T}_1\right)^2\cdot\widetilde{T}_{3,1}=\sum_{k=32}^{51}A_k$$

where

$$
\begin{aligned}
A_{32} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(w_{1,n}'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{33} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{34} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{35} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_{1,n}\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{36} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{37} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{38} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{39} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{40} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2
\end{aligned}
$$

$$
\begin{aligned}
A_{41} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{42} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{43} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{44} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{45} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_{1,n}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{46} &\equiv \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{47} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{48} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{49} &\equiv \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{50} &\equiv \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2 \\
A_{51} &\equiv -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)^2
\end{aligned}
$$

Lemma 78

$$E[A_{32}] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot \text{trace}\left(\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{32}] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\text{trace}\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= E\left[\text{trace}\left(u_0u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\right)\right]\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ &= \text{trace}\left(\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\right)\cdot\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1 \end{aligned}$$

■

Lemma 79

$$E[A_{33}] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot \text{trace}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}^{(4)}\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E[A_{33}] &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\; & E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ =\; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot \text{trace}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}^{(4)}\right) \end{aligned}$$

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)'\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
$$

■

Lemma 80

$$
E\left[A_{34}\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{34}\right] \;=\; & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
= \; & E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\Lambda_1\Lambda^{-1}\lambda_1\right] \\
= \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
$$

■

Lemma 81

$$
E\left[A_{35}\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,1}\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,1}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{35}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ = \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,1}\right) \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ = \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\right] \\ = \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,1}\lambda_1 \end{aligned}$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Upsilon_{0,1}\lambda_1$$

■

Lemma 82

$$E\left[A_{36}\right] = -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Gamma_{0,1}\Lambda^{-1}\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{36}\right] &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\operatorname{trace}\left(u_1'\Lambda^{-1}u_0\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &= E\left[\operatorname{trace}\left(u_0u_1'\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ &= \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Gamma_{0,1}\Lambda^{-1}\right) \end{aligned}$$

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

and

$$\begin{aligned} E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ &= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1 \end{aligned}$$

■

Lemma 83

$$E\left[A_{37}\right] = \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{37}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\operatorname{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\operatorname{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\ = \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\operatorname{trace}\left(\Upsilon_{0,0}\right) \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ = \; & E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\left(v_0v_0'\right)\Lambda^{-1}\lambda_1\right] \\ = \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\ = \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\ = \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1 \end{aligned}$$

■

Lemma 84

$$E\left[A_{38}\right] = -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{38}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad - \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(v_0'\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 85

$$
E\left[A_{39}\right]=-\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{39}\right] \; = \; & -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 86

$$E\left[A_{40}\right]=\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{40}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\lambda_1\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
=\ & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 87

$$E\left[A_{41}\right]=-\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{41}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_1\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 88

$$E\left[A_{42}\right]=-\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{42}\right] &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$\begin{aligned}& E\left[\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & E\left[\left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(u_0'\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & \left(\lambda_2'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\end{aligned}$$

■

Lemma 89

$$E\left[A_{43}\right]=\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{43}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$\begin{aligned}& E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)'\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ =\ & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\\ =\ & E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\Lambda_1\Lambda^{-1}\lambda_1\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\\ =\ & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)\end{aligned}$$

■

Lemma 90

$$E\left[A_{44}\right]=\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E[A_{44}] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\Lambda_1\Lambda^{-1}\lambda_1\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 91

$$
E[A_{45}] = -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,1}\lambda_1\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E[A_{45}] &= -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad - \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\lambda_1\right)\right] E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\lambda_1'\Lambda^{-1}\left(u_0\lambda_1'\Lambda^{-1}U_1\Lambda^{-1}\right)\lambda_1\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
= \; & \left(\lambda_1'\Lambda^{-1}\lambda_1\right)\cdot\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,1}\lambda_1\right)
\end{aligned}
$$

■

Lemma 92

$$
E[A_{46}] = \frac{24}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E[A_{46}] &= \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad + \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 93

$$A_{47}=\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_2'\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
A_{47} = \; & \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_2'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 94

$$E\left[A_{48}\right]=-\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{48}\right] = \; & -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 95

$$E\left[A_{49}\right]=\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{49}\right] &= \frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 96

$$E\left[A_{50}\right]=\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{50}\right] &= \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad+\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
=\ & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 97

$$E\left[A_{51}\right]=-\frac{27}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{51}\right] &= -\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad-\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad-\frac{9}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

We also have

$$\frac{1}{3}\overline{T}_1^3\cdot\widetilde{T}_{3,2}=\sum_{k=52}^{57}A_k$$

where

$$\begin{aligned}
A_{52} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{53} &\equiv -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{54} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{55} &\equiv -\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{56} &\equiv \frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{57} &\equiv \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3
\end{aligned}$$

Lemma 98

$$E\left[A_{52}\right]=\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{52}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\
&\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\
&\quad+\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 99

$$E\left[A_{53}\right]=-\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{53}\right] &= -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{1}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 100

$$E\left[A_{54}\right] = -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{54}\right] &= -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 101

$$E\left[A_{55}\right] = -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) + O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{55}\right] &= -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5} E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 102

$$E\left[A_{56}\right]=\frac{3}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{56}\right] &= \frac{\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{\left(\lambda_1'\Lambda^{-1}\Lambda_2\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 103

$$E\left[A_{57}\right]=\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{57}\right] &= \frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad +\frac{2}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

We also have

$$\frac{1}{3}\overline{T}_1\cdot\overline{T}_2\cdot\widetilde{T}_{3,3}=\sum_{k=58}^{69}A_k$$

where

$$\begin{aligned}
A_{58} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{59} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(w_{1,n}'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\\
A_{60} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)\\
A_{61} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\\
A_{62} &\equiv -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\\
A_{63} &\equiv \frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}w_{1,n}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3
\end{aligned}$$

$$\begin{aligned}
A_{64} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\\
A_{65} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_2'\Lambda^{-1}w_n\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{66} &\equiv \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\\
A_{67} &\equiv -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}W_n\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\\
A_{68} &\equiv \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)\\
A_{69} &\equiv -\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

Lemma 104

$$\begin{aligned}
&E\left[A_{58}\right]\\
=\;&\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{58}\right] &= \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\
&\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\
&\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\text{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)$$

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
&= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(u_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
&= \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}
$$

■

Lemma 105

$$
E\left[A_{59}\right] = -\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{59}\right] &= -\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$
E\left[\left(u_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = \operatorname{trace}\left(\Lambda^{-1}\Gamma_{0,1}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
$$

$$
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] = E\left[\left(u_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
$$

and

$$
E\left[\left(u_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = E\left[\left(u_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] = \lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1
$$

■

Lemma 106

$$
E\left[A_{60}\right] = -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}\left(\operatorname{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right) + \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1\right)\right) + O\left(n^{-1}\right)
$$

Proof. As in Lemma 19, we obtain

$$
\begin{aligned}
E\left[A_{60}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^3}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}
$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \\
= \; & \text{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\
= \; & \lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\Lambda_1\Lambda^{-1}\lambda_1 \\
= \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\Lambda_1\Lambda^{-1}\lambda_1
\end{aligned}$$

■

Lemma 107

$$E\left[A_{61}\right] = \frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\text{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) + 2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{61}\right] = \; & \frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& + \frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] + O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\text{trace}\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}u_0\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
= \; & E\left[\text{trace}\left(u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right)\right]\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \\
= \; & \text{trace}\left(\Upsilon_{0,0}\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \lambda_1'\Lambda^{-1}E\left[u_0\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\right]\lambda_1 \\
= \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

and

$$\begin{aligned}
& E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] \\
= \; & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right] \\
= \; & \lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1
\end{aligned}$$

■

Lemma 108

$$\begin{aligned}
& E\left[A_{62}\right] \\
= \; & -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)
\end{aligned}$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{62}\right] = \; & -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{8}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right] &= E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
&= \left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)
\end{aligned}$$

■

Lemma 109

$$E\left[A_{63}\right]=\frac{24\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{63}\right] = \; & \frac{8\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{8\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{8\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Gamma_{0,1}\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 110

$$E\left[A_{64}\right]=-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{64}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)$$

and

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)$$

■

Lemma 111

$$E\left[A_{65}\right]=\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_2\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{65}\right] &= \frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$E\left[\left(\lambda_2'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 112

$$E\left[A_{66}\right]=\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(2\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{66}\right] &= \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ &+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ &+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \end{aligned}$$

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right) \end{aligned}$$

and

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

■

Lemma 113

$$E\left[A_{67}\right]=-\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{67}\right] &= -\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &-\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &-\frac{4\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$\begin{aligned} & E\left[\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}U_0\Lambda^{-1}\lambda_1\right)\right]E\left[\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ = & \left(\lambda_1'\Lambda^{-1}\Upsilon_{0,0}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right) \end{aligned}$$

■

Lemma 114

$$E\left[A_{68}\right]=\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{68}\right] \quad = \quad & \frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& +\frac{6}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 115

$$E\left[A_{69}\right]=-\frac{18}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}
E\left[A_{69}\right] \quad = \quad & -\frac{6\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{6\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\
& -\frac{6\left(\lambda_1'\Lambda^{-1}\lambda_2\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)
\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

We also have

$$\frac{1}{3}\left(\overline{T}_1\right)^2\cdot\widetilde{T}_2\cdot\widetilde{T}_{3,3}=\sum_{k=70}^{73}A_k$$

where

$$\begin{aligned}
A_{70} \quad & \equiv \quad \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}w_n\right)^2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right)^2 \\
A_{71} \quad & \equiv \quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right) \\
A_{72} \quad & \equiv \quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}w_n\right)^3\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}w_n\right) \\
A_{73} \quad & \equiv \quad \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2\left(\lambda_1'\Lambda^{-1}w_n\right)^4
\end{aligned}$$

Lemma 116

$$E\left[A_{70}\right]=\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)+2\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2\right)+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{70}\right] &= \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]\\ &\quad+\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)$$

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2$$

and

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2$$

■

Lemma 117

$$E\left[A_{71}\right]=-\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned}E\left[A_{71}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]\\ &\quad-\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right)\end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 118

$$E\left[A_{72}\right]=-\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{72}\right] &= -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad -\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^5}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)\left(\lambda_1'\Lambda^{-1}\lambda_1\right)$$

■

Lemma 119

$$E\left[A_{73}\right]=\frac{12}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^4}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2+O\left(n^{-1}\right)$$

Proof. As in Lemma 19, we obtain

$$\begin{aligned} E\left[A_{73}\right] &= \frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad +\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right] \\ &\quad +\frac{4}{\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^6}\left(\lambda_1'\Lambda^{-1}\Lambda_1\Lambda^{-1}\lambda_1\right)^2E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\right]+O\left(n^{-1}\right) \end{aligned}$$

Now note that

$$E\left[\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}u_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\left(\lambda_1'\Lambda^{-1}v_0\right)\right]=\left(\lambda_1'\Lambda^{-1}\lambda_1\right)^2$$

■

3 GMM Estimation and Dynamic Panel Model with Fixed Effects (T=4)

We now assume that our model is generated by

$$\begin{aligned} y_{it} &= \alpha_i+\beta y_{it-1}+\varepsilon_{it} \qquad t=1,\ldots,T \\ \varepsilon_{it} &\sim N\left(0,1\right) \\ y_{i0} &\sim N\left(\frac{\alpha_i}{1-\beta},\frac{1}{1-\beta^2}\right) \\ \alpha_i &\sim N\left(0,1\right) \end{aligned}$$

We will assume that

$$T=4$$

3.1 Ahn and Schmidt

Ahn and Schmidt propose following moment restrictions

$$\begin{aligned} E\left[y_{is}\left(\left(y_{it}-y_{it-1}\right)-\beta\left(y_{it-1}-y_{is-1}\right)\right)\right] &= 0 \qquad t=2,\ldots,T: s=0,\ldots,t-2 \\ E\left[\left(y_{iT}-\beta y_{iT-1}\right)\left(\left(y_{it}-y_{it-1}\right)-\beta\left(y_{it-1}-y_{it-2}\right)\right)\right] &= 0 \qquad t=2,\ldots,T-1 \end{aligned}$$

Therefore, we have

$$\frac{T\left(T-1\right)}{2}+T-2=\frac{1}{2}T^2+\frac{1}{2}T-2=\frac{1}{2}\left(4\right)^2+\frac{1}{2}\left(4\right)-2=8$$

moment restrictions:

$$\psi\left(z_i,c\right)=\begin{bmatrix} y_{i,0}\left(\left(y_{i,2}-y_{i,1}\right)-c\left(y_{i,1}-y_{i,0}\right)\right) \\ y_{i,0}\left(\left(y_{i,3}-y_{i,2}\right)-c\left(y_{i,2}-y_{i,1}\right)\right) \\ y_{i,1}\left(\left(y_{i,3}-y_{i,2}\right)-c\left(y_{i,2}-y_{i,1}\right)\right) \\ y_{i,0}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ y_{i,1}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ y_{i,2}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ \left(y_{i,4}-cy_{i,3}\right)\left(\left(y_{i,2}-y_{i,1}\right)-c\left(y_{i,1}-y_{i,0}\right)\right) \\ \left(y_{i,4}-cy_{i,3}\right)\left(\left(y_{i,3}-y_{i,2}\right)-c\left(y_{i,2}-y_{i,1}\right)\right) \end{bmatrix}$$

3.2 Blundell and Bond

Blundell and Bond proposefollowing moment restrictions

$$\begin{aligned} E\left[y_{is}\left(\left(y_{it}-y_{it-1}\right)-\beta\left(y_{it-1}-y_{is-1}\right)\right)\right] &= 0 \qquad t=2,\ldots,T: s=0,\ldots,t-2 \\ E\left[\left(y_{it}-\beta y_{i,t-1}\right)\left(y_{i,t-1}-y_{i,t-2}\right)\right] &= 0 \qquad t=2,\ldots,T \end{aligned}$$

Therefore, we have

$$\frac{T\left(T-1\right)}{2}+T-1=\frac{1}{2}T^2+\frac{1}{2}T-1=\frac{1}{2}\left(4\right)^2+\frac{1}{2}\left(4\right)-1=9$$

moment restrictions:

$$\psi\left(z_i,c\right)=\begin{bmatrix} y_{i,0}\left(\left(y_{i,2}-y_{i,1}\right)-c\left(y_{i,1}-y_{i,0}\right)\right) \\ y_{i,0}\left(\left(y_{i,3}-y_{i,2}\right)-c\left(y_{i,2}-y_{i,1}\right)\right) \\ y_{i,1}\left(\left(y_{i,3}-y_{i,2}\right)-c\left(y_{i,2}-y_{i,1}\right)\right) \\ y_{i,0}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ y_{i,1}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ y_{i,2}\left(\left(y_{i,4}-y_{i,3}\right)-c\left(y_{i,3}-y_{i,2}\right)\right) \\ \left(y_{i,2}-cy_{i,1}\right)\left(y_{i,1}-y_{i,0}\right) \\ \left(y_{i,3}-cy_{i,2}\right)\left(y_{i,2}-y_{i,1}\right) \\ \left(y_{i,4}-cy_{i,3}\right)\left(y_{i,3}-y_{i,2}\right) \end{bmatrix}$$

3.3 Long Difference

Long difference note that

$$\left(y_{iT}-y_{i,1}\right)-\beta\left(y_{iT-1}-y_{i,0}\right)=\varepsilon_{iT}-\varepsilon_{i1}$$

is orthogonal to

$$\begin{aligned}
& y_{i,0} \\
y_{i,2} - \beta y_{i,1} &= \alpha_i + \varepsilon_{i,2} \\
& \vdots \\
y_{i,T-1} - \beta y_{i,T-2} &= \alpha_i + \varepsilon_{i,T-1}
\end{aligned}$$

In other words, it considers the moment restrictions

$$\begin{aligned}
E\left[y_{i,0}\left(\left(y_{iT} - y_{i1}\right) - \beta\left(y_{iT-1} - y_{i0}\right)\right)\right] &= 0 \\
E\left[\left(y_{i,s} - \beta y_{i,s-1}\right)\left(\left(y_{iT} - y_{i1}\right) - \beta\left(y_{iT-1} - y_{i0}\right)\right)\right] &= 0, \qquad s = 2, \ldots, T-1
\end{aligned}$$

We therefore have

$$1 + T - 2 = T - 1 = 4 - 1 = 3$$

moment restrictions:

$$\psi\left(z_i, c\right) = \left[\begin{array}{c}
y_{i,0}\left(\left(y_{i,4} - y_{i,1}\right) - c\left(y_{i,3} - y_{i,0}\right)\right) \\
\left(y_{i,2} - c y_{i,1}\right)\left(\left(y_{i,4} - y_{i,1}\right) - c\left(y_{i,3} - y_{i,0}\right)\right) \\
\left(y_{i,3} - c y_{i,2}\right)\left(\left(y_{i,4} - y_{i,1}\right) - c\left(y_{i,3} - y_{i,0}\right)\right)
\end{array}\right]$$

Higher Order Expansion of Iterated IV Estimator: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

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1 Iterated 2SLS

We consider 2SLS

$$b = \left[\left(\sum_{i=1}^{n} x_i \widehat{z}_i' \right) \left(\sum_{i=1}^{n} \widehat{z}_i \widehat{z}_i' \right)^{-1} \left(\sum_{i=1}^{n} \widehat{z}_i x_i \right) \right]^{-1} \left(\sum_{i=1}^{n} x_i \widehat{z}_i' \right) \left(\sum_{i=1}^{n} \widehat{z}_i \widehat{z}_i' \right)^{-1} \left(\sum_{i=1}^{n} \widehat{z}_i y_i \right) \tag{1}$$

applied to the single equation

$$y_i = \beta x_i + \varepsilon_i \tag{2}$$

using instrument $\widehat{z}_i = z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i$, where $\widehat{\theta} = \sqrt{n} \left(\widehat{\beta} - \beta \right)$. Here, z_i is the "proper" instrument, and $\widehat{\beta}$ is a preliminary estimator for β. We assume that

$$\widehat{\theta} = F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p \left(\frac{1}{n} \right) \tag{3}$$

where

$$F_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i$$

for some f_i i.i.d. with mean zero, $Q_n = O_p(1)$, and $S_n = O_p(1)$. Below in Theorem 1, we derive an expansion of the form

$$\sqrt{n} (b - \beta) = R_0 + \frac{1}{\sqrt{n}} R_1 + \frac{1}{n} R_2 + o_p \left(\frac{1}{n} \right) \tag{4}$$

where

$$\begin{aligned} R_0 &\equiv \frac{N_0}{D_0} \\ R_1 &\equiv -\frac{D_1 N_0}{D_0^2} + \frac{N_1}{D_0} \\ R_2 &\equiv -\frac{D_2 N_0}{D_0^2} + \frac{D_1^2 N_0}{D_0^3} + \frac{N_2}{D_0} - \frac{D_1 N_1}{D_0^2} \end{aligned}$$

where Rs depend on (F_n, Q_n, S_n).

We therefore can see that the higher order bias of b is given by[1]

$$\frac{1}{n}E\left[R_1\right]$$

which can be stochastically approximated by

$$\frac{1}{n}\widehat{E}\left[R_1\right]$$

where

$$\widehat{E}\left[R_1\right]=\frac{1}{J}\sum_{j=1}^{J}R_{1j}$$

for J large. We can also see that the expected value of $(b-\beta)^2$ is approximately equal to

$$\frac{1}{n}E\left[R_0^2\right]+\frac{2}{n\sqrt{n}}E\left[R_0R_1\right]+\frac{1}{n^2}E\left[R_1^2\right]+\frac{2}{n^2}E\left[R_0R_2\right]$$

which can be stochastically approximated by

$$\frac{1}{n}\widehat{E}\left[R_0^2\right]+\frac{2}{n\sqrt{n}}\widehat{E}\left[R_0R_1\right]+\frac{1}{n^2}\widehat{E}\left[R_1^2\right]+\frac{2}{n^2}\widehat{E}\left[R_0R_2\right]$$

where

$$\begin{aligned}
\widehat{E}\left[R_0^2\right] &= \frac{1}{J}\sum_{j=1}^{J}R_{0j}^2\\
\widehat{E}\left[R_0R_1\right] &= \frac{1}{J}\sum_{j=1}^{J}R_{0j}R_{1j}\\
\widehat{E}\left[R_1^2\right] &= \frac{1}{J}\sum_{j=1}^{J}R_{1j}^2\\
\widehat{E}\left[R_0R_2\right] &= \frac{1}{J}\sum_{j=1}^{J}R_{0j}R_{2j}
\end{aligned}$$

for J large.

2 Expansion

We present an expansion for 2SLS using instrument $\widehat{z}_i=z_i-\frac{1}{\sqrt{n}}\widehat{\theta}w_i$. We have

$$\sqrt{n}\left(b-\beta\right)=\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_ix_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i\right)}{\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_ix_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_ix_i\right)} \tag{5}$$

Definition 1

$$\begin{aligned}
\lambda &\equiv E\left[z_ix_i\right]\\
\Lambda &\equiv E\left[z_iz_i'\right]\\
\phi &\equiv E\left[w_ix_i\right]\\
\Delta &\equiv E\left[w_iz_i'+z_iw_i'\right]\\
\Omega &\equiv E\left[w_iw_i'\right]\\
\varphi &\equiv E\left[w_i\varepsilon_i\right]
\end{aligned}$$

[1] It can be verified that $E\left[R_0\right]=0$.

Lemma 1

$$\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i = \lambda + \frac{1}{\sqrt{n}}A_1 + \frac{1}{n}A_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
A_1 &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i x_i - \lambda) - F_n\phi = \sqrt{n}\left(\frac{1}{n}Z'x - \lambda\right) - F_n\phi \\
A_2 &\equiv -F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i x_i - \phi)\right) - Q_n\phi \\
&= -\sqrt{n}F_n\left(\frac{1}{n}W'x - \phi\right) - Q_n\phi
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i &= \frac{1}{n}\sum_{i=1}^{n}\left(z_i - \frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)x_i \\
&= \frac{1}{n}\sum_{i=1}^{n}z_i x_i - \frac{1}{\sqrt{n}}\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}w_i x_i\right)
\end{aligned}$$

Because

$$\frac{1}{n}\sum_{i=1}^{n}z_i x_i = \lambda + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i x_i - \lambda)\right)$$

and

$$\begin{aligned}
&\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}w_i x_i\right) \\
=\ &\left(F_n + \frac{1}{\sqrt{n}}Q_n + \frac{1}{n}S_n + o_p\left(\frac{1}{n}\right)\right)\left(\phi + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i x_i - \phi)\right)\right) \\
=\ &F_n\phi \\
&+\frac{1}{\sqrt{n}}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i x_i - \phi)\right) + \frac{1}{\sqrt{n}}Q_n\phi \\
&+o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(z_i - \frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)x_i \\
=\ &\lambda \\
&+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_i x_i - \lambda)\right) - \frac{1}{\sqrt{n}}F_n\phi \\
&-\frac{1}{n}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i x_i - \phi)\right) - \frac{1}{n}Q_n\phi \\
&+o_p\left(\frac{1}{n}\right)
\end{aligned}$$

■

Lemma 2

$$\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i' = \Lambda + \frac{1}{\sqrt{n}}L_1 + \frac{1}{n}L_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
L_1 &\equiv \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_iz_i' - \Lambda)\right) - F_n\Delta \\
&= \sqrt{n}\left(\frac{1}{n}Z'Z - \Lambda\right) - F_n\Delta \\
L_2 &\equiv -Q_n\Delta - F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_iz_i' + z_iw_i' - \Delta)\right) + F_n^2\Omega \\
&= -Q_n\Delta - \sqrt{n}F_n\left(\frac{1}{n}Z'W + \frac{1}{n}W'Z - \Delta\right) + F_n^2\Omega
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i' &= \frac{1}{n}\sum_{i=1}^{n}\left(z_i - \frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)\left(z_i - \frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)' \\
&= \frac{1}{n}\sum_{i=1}^{n}z_iz_i' - \frac{1}{\sqrt{n}}\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}(w_iz_i' + z_iw_i')\right) + \frac{1}{n}\widehat{\theta}^2\left(\frac{1}{n}\sum_{i=1}^{n}w_iw_i'\right)
\end{aligned}$$

Because

$$\frac{1}{n}\sum_{i=1}^{n}z_iz_i' = \Lambda + \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(z_iz_i' - \Lambda)\right)$$

$$\begin{aligned}
&\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}(w_iz_i' + z_iw_i')\right) \\
= &\left(F_n + \frac{1}{\sqrt{n}}Q_n + \frac{1}{n}S_n + o_p\left(\frac{1}{n}\right)\right)\left(\Delta + \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_iz_i' + z_iw_i' - \Delta)\right) \\
= &F_n\Delta \\
&+\frac{1}{\sqrt{n}}Q_n\Delta + \frac{1}{\sqrt{n}}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_iz_i' + z_iw_i' - \Delta)\right) \\
&+o_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\theta}^2\left(\frac{1}{n}\sum_{i=1}^{n}w_iw_i'\right) \\
= &\left(F_n + \frac{1}{\sqrt{n}}Q_n + \frac{1}{n}S_n + o_p\left(\frac{1}{n}\right)\right)^2\left(\Omega + \frac{1}{n}\sum_{i=1}^{n}(w_iw_i' - \Omega)\right) \\
= &F_n^2\Omega + o_p(1),
\end{aligned}$$

we have

$$
\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left(z_i-\frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)\left(z_i-\frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)' \\
= \; & \Lambda \\
& +\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)-\frac{1}{\sqrt{n}}F_n\Delta \\
& -\frac{1}{n}Q_n\Delta-\frac{1}{n}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i z_i'+z_i w_i'-\Delta\right)\right) \\
& +\frac{1}{n}F_n^2\Omega \\
& +o_p\left(\frac{1}{n}\right)
\end{aligned}
$$

■

Lemma 3

$$
\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}=\Lambda^{-1}+\frac{1}{\sqrt{n}}M_1+\frac{1}{n}M_2+o_p\left(\frac{1}{n}\right)
$$

where

$$
\begin{aligned}
M_1 &\equiv -\Lambda^{-1}L_1\Lambda^{-1} \\
&= -\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)\Lambda^{-1}+F_n\Lambda^{-1}\Delta\Lambda^{-1}
\end{aligned}
$$

$$
\begin{aligned}
M_2 \equiv\; & \Lambda^{-1}L_1\Lambda^{-1}L_1\Lambda^{-1}-\Lambda^{-1}L_2\Lambda^{-1} \\
=\; & \Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)\Lambda^{-1} \\
& -F_n\Lambda^{-1}\Delta\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)\Lambda^{-1}-F_n\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i'-\Lambda\right)\right)\Lambda^{-1}\Delta\Lambda^{-1} \\
& +F_n^2\Lambda^{-1}\Delta\Lambda^{-1}\Delta\Lambda^{-1} \\
& +Q_n\Lambda^{-1}\Delta\Lambda^{-1} \\
& +F_n\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i z_i'+z_i w_i'-\Delta\right)\right)\Lambda^{-1} \\
& -F_n^2\Lambda^{-1}\Omega\Lambda^{-1}
\end{aligned}
$$

Note that

$$
M_1=-\sqrt{n}\Lambda^{-1}\left(\frac{1}{n}Z'Z-\Lambda\right)\Lambda^{-1}+F_n\Lambda^{-1}\Delta\Lambda^{-1}
$$

and

$$
\begin{aligned}
M_2 &= n\Lambda^{-1}\left(\frac{1}{n}Z'Z-\Lambda\right)\Lambda^{-1}\left(\frac{1}{n}Z'Z-\Lambda\right)\Lambda^{-1} \\
&\quad -\sqrt{n}F_n\Lambda^{-1}\Delta\Lambda^{-1}\left(\frac{1}{n}Z'Z-\Lambda\right)\Lambda^{-1} \\
&\quad -\sqrt{n}F_n\Lambda^{-1}\left(\frac{1}{n}Z'Z-\Lambda\right)\Lambda^{-1}\Delta\Lambda^{-1} \\
&\quad +F_n^2\Lambda^{-1}\Delta\Lambda^{-1}\Delta\Lambda^{-1} \\
&\quad +Q_n\Lambda^{-1}\Delta\Lambda^{-1} \\
&\quad +\sqrt{n}F_n\Lambda^{-1}\left(\frac{1}{n}Z'W+\frac{1}{n}W'Z-\Delta\right)\Lambda^{-1} \\
&\quad -F_n^2\Lambda^{-1}\Omega\Lambda^{-1}
\end{aligned}
$$

Proof. From Lemma 2, we obtain

$$
\begin{aligned}
\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1} &= \left(\Lambda+\frac{1}{\sqrt{n}}L_1+\frac{1}{n}L_2+o_p\left(\frac{1}{n}\right)\right)^{-1} \\
&= \Lambda^{-1}-\frac{1}{\sqrt{n}}\Lambda^{-1}L_1\Lambda^{-1}+\frac{1}{n}\left(\Lambda^{-1}L_1\Lambda^{-1}L_1\Lambda^{-1}-\Lambda^{-1}L_2\Lambda^{-1}\right)+o_p\left(\frac{1}{n}\right)
\end{aligned}
$$

■

Lemma 4

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i = C_0+\frac{1}{\sqrt{n}}C_1+\frac{1}{n}C_2+o_p\left(\frac{1}{n}\right)
$$

where

$$
\begin{aligned}
C_0 &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i\varepsilon_i-F_n\varphi \\
&= \frac{1}{\sqrt{n}}Z'\varepsilon-F_n\varphi \\
C_1 &\equiv -F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i\varepsilon_i-\varphi)\right)-Q_n\varphi \\
&= -\sqrt{n}F_n\left(\frac{1}{n}W'\varepsilon-\varphi\right)-Q_n\varphi \\
C_2 &\equiv -Q_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(w_i\varepsilon_i-\varphi)\right)-S_n\varphi \\
&= -\sqrt{n}Q_n\left(\frac{1}{n}W'\varepsilon-\varphi\right)-S_n\varphi
\end{aligned}
$$

Proof. We have

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i-\frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)\varepsilon_i = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i\varepsilon_i-\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}w_i\varepsilon_i\right)
$$

Because

$$\begin{aligned}\widehat{\theta}\left(\frac{1}{n}\sum_{i=1}^{n} w_i\varepsilon_i\right) &= \left(F_n+\frac{1}{\sqrt{n}}Q_n+\frac{1}{n}S_n+o_p\left(\frac{1}{n}\right)\right)\left(\varphi+\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i\varepsilon_i-\varphi\right)\right)\right)\\ &= F_n\varphi\\ &\quad+\frac{1}{\sqrt{n}}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i\varepsilon_i-\varphi\right)\right)+\frac{1}{\sqrt{n}}Q_n\varphi\\ &\quad+\frac{1}{n}Q_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i\varepsilon_i-\varphi\right)\right)+\frac{1}{n}S_n\varphi\\ &\quad+o_p\left(\frac{1}{n}\right)\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i-\frac{1}{\sqrt{n}}\widehat{\theta}w_i\right)\varepsilon_i\\ =\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\varepsilon_i-F_n\varphi\\ &-\frac{1}{\sqrt{n}}F_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i\varepsilon_i-\varphi\right)\right)-\frac{1}{\sqrt{n}}Q_n\varphi\\ &-\frac{1}{n}Q_n\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(w_i\varepsilon_i-\varphi\right)\right)-\frac{1}{n}S_n\varphi\\ &+o_p\left(\frac{1}{n}\right)\end{aligned}$$

■

Lemma 5

$$\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i\right)=N_0+\frac{1}{\sqrt{n}}N_1+\frac{1}{n}N_2+o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}N_0 &\equiv \lambda'\Lambda^{-1}C_0\\ N_1 &\equiv \lambda'\Lambda^{-1}C_1+\lambda' M_1C_0+A_1'\Lambda^{-1}C_0\\ N_2 &\equiv A_1'M_1C_0+\lambda'\Lambda^{-1}C_2+\lambda'M_2C_0+\lambda'M_1C_1+A_1'\Lambda^{-1}C_1+A_2'\Lambda^{-1}C_0\end{aligned}$$

Proof. From Lemmas 1, 3, and 4, we obtain

$$\begin{aligned}&\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i\right)\\ =\ &\left(\lambda+\frac{1}{\sqrt{n}}A_1+\frac{1}{n}A_2+o_p\left(\frac{1}{n}\right)\right)'\left(\Lambda^{-1}+\frac{1}{\sqrt{n}}M_1+\frac{1}{n}M_2+o_p\left(\frac{1}{n}\right)\right)\left(C_0+\frac{1}{\sqrt{n}}C_1+\frac{1}{n}C_2+o_p\left(\frac{1}{n}\right)\right)\\ =\ &\lambda'\Lambda C_0\\ &+\frac{1}{\sqrt{n}}\left(\lambda'\Lambda C_1+\lambda'M_1C_0+A_1'\Lambda C_0\right)\\ &+\frac{1}{n}\left(A_1'M_1C_0+\lambda'\Lambda C_2+\lambda'M_2C_0+\lambda'M_1C_1+A_1'\Lambda C_1+A_2'\Lambda C_0\right)\end{aligned}$$

■

Lemma 6

$$\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i x_i\right)=D_0+\frac{1}{\sqrt{n}}D_1+\frac{1}{n}D_2+o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} D_0 &\equiv \lambda'\Lambda^{-1}\lambda \\ D_1 &\equiv 2\lambda'\Lambda^{-1}A_1+\lambda' M_1\lambda \\ D_2 &\equiv 2\lambda'\Lambda^{-1}A_2+2\lambda' M_1 A_1+\lambda' M_2\lambda+A_1'\Lambda^{-1}A_1 \end{aligned}$$

Proof. From Lemmas 1, and 3, we obtain

$$\begin{aligned} &\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i x_i\right) \\ =&\left(\lambda+\frac{1}{\sqrt{n}}A_1+\frac{1}{n}A_2+o_p\left(\frac{1}{n}\right)\right)'\left(\Lambda^{-1}+\frac{1}{\sqrt{n}}M_1+\frac{1}{n}M_2+o_p\left(\frac{1}{n}\right)\right)\left(\lambda+\frac{1}{\sqrt{n}}A_1+\frac{1}{n}A_2+o_p\left(\frac{1}{n}\right)\right) \\ =&\ \lambda'\Lambda^{-1}\lambda \\ &+\frac{1}{\sqrt{n}}\left(\lambda'\Lambda^{-1}A_1+\lambda' M_1\lambda+A_1'\Lambda^{-1}\lambda\right) \\ &+\frac{1}{n}\left(A_1' M_1\lambda+\lambda'\Lambda^{-1}A_2+\lambda' M_2\lambda+\lambda' M_1 A_1+A_1'\Lambda^{-1}A_1+A_2'\Lambda^{-1}\lambda\right) \end{aligned}$$

■

Theorem 1

$$\sqrt{n}\,(b-\beta)=R_0+\frac{1}{\sqrt{n}}R_1+\frac{1}{n}R_2+o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} R_0 &\equiv \frac{N_0}{D_0} \\ R_1 &\equiv -\frac{D_1 N_0}{D_0^2}+\frac{N_1}{D_0} \\ R_2 &\equiv -\frac{D_2 N_0}{D_0^2}+\frac{D_1^2 N_0}{D_0^3}+\frac{N_2}{D_0}-\frac{D_1 N_1}{D_0^2} \end{aligned}$$

Proof. From Lemmas 5 and 6, we obtain

$$\begin{aligned} \sqrt{n}\,(b-\beta) &= \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i\varepsilon_i\right)}{\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{z}_i\widehat{z}_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{z}_i x_i\right)} \\ &= \frac{N_0+\frac{1}{\sqrt{n}}N_1+\frac{1}{n}N_2+o_p\left(\frac{1}{n}\right)}{D_0+\frac{1}{\sqrt{n}}D_1+\frac{1}{n}D_2+o_p\left(\frac{1}{n}\right)} \\ &= \frac{N_0}{D_0}+\frac{1}{\sqrt{n}}\left(-\frac{D_1 N_0}{D_0^2}+\frac{N_1}{D_0}\right)+\frac{1}{n}\left(-\frac{D_2 N_0}{D_0^2}+\frac{D_1^2 N_0}{D_0^3}+\frac{N_2}{D_0}-\frac{D_1 N_1}{D_0^2}\right)+o_p\left(\frac{1}{n}\right) \end{aligned}$$

■

3 Long Difference

$$y_{iT} - y_{i1} = \beta\left(y_{iT-1} - y_{i0}\right) + \left(\varepsilon_{iT} - \varepsilon_{i1}\right) \tag{6}$$

It is easy to see that the initial observation y_{i0} would serve as a valid instrument. Using similar intuition as in Hausman and Taylor (1983) or Ahn and Schmidt (1995), we can see that $y_{iT-1} - \beta y_{iT-2}$, $\ldots$, $y_{i2} - \beta y_{i1}$ would be valid instruments as well. If we use a preliminary estimator $\widehat{\beta}$, then we would use

$$\begin{aligned}
\widehat{z}_i &\equiv \begin{bmatrix} y_{i0} \\ y_{i2} - \widehat{\beta} y_{i1} \\ \vdots \\ y_{iT-1} - \widehat{\beta} y_{iT-2} \end{bmatrix} = \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} - \left(\widehat{\beta} - \beta\right) y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} - \left(\widehat{\beta} - \beta\right) \beta y_{iT-2} \end{bmatrix} \\
&= \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} \end{bmatrix} - \left(\widehat{\beta} - \beta\right) \begin{bmatrix} 0 \\ y_{i1} \\ \vdots \\ y_{iT-2} \end{bmatrix} = z_i - \frac{1}{\sqrt{n}} \widehat{\theta} w_i
\end{aligned}$$

as instrument, where

$$z_i \equiv \begin{bmatrix} y_{i0} \\ y_{i2} - \beta y_{i1} \\ \vdots \\ y_{iT-1} - \beta y_{iT-2} \end{bmatrix}, \qquad w_i \equiv \begin{bmatrix} 0 \\ y_{i1} \\ \vdots \\ y_{iT-2} \end{bmatrix}, \qquad \widehat{\theta} = \sqrt{n}\left(\widehat{\beta} - \beta\right)$$

In our context,

$$\begin{aligned}
y_i &= y_{iT} - y_{i1} \\
x_i &= y_{iT-1} - y_{i0} \\
\varepsilon_i &= \varepsilon_{iT} - \varepsilon_{i1}
\end{aligned}$$

$$\begin{aligned}
\lambda &= E\left[z_i x_i\right] \\
\Lambda &= E\left[z_i z_i'\right] \\
\phi &= E\left[w_i x_i\right] \\
\Delta &= E\left[w_i z_i' + z_i w_i'\right] \\
\Omega &= E\left[w_i w_i'\right] \\
\varphi &= E\left[w_i \varepsilon_i\right]
\end{aligned}$$

Now, suppose that

$$\begin{aligned}
y_{it} &= \alpha_i + \beta y_{it-1} + \varepsilon_{it} \qquad t = 1, \ldots, T \\
\varepsilon_{it} &\sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right), \qquad \alpha_i \sim \mathcal{N}\left(0, \sigma_\alpha^2\right) \\
y_{i0} &\sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta}, \frac{1}{1-\beta^2}\right) = \frac{\alpha_i}{1-\beta} + \frac{1}{\sqrt{1-\beta^2}} \mathcal{N}(0,1)
\end{aligned}$$

Further suppose that

$$\sigma_\varepsilon^2 = \sigma_\alpha^2 = 1, \quad T = 5$$

We then have the following results:

$$\lambda = \begin{bmatrix} -\beta^2 - 1 \\ \beta^2 \\ \beta \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \frac{2}{(1-\beta^2)(1-\beta)} & \frac{1}{1-\beta} & \frac{1}{1-\beta} & \frac{1}{1-\beta} \\ \frac{1}{1-\beta} & 2 & 1 & 1 \\ \frac{1}{1-\beta} & 1 & 2 & 1 \\ \frac{1}{1-\beta} & 1 & 1 & 2 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 0 \\ -\beta \\ 0 \\ \beta \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} & \frac{-2-\beta+\beta^2}{-1+\beta} \\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{-3+\beta}{-1+\beta} & -\frac{2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} \\ -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{-2-\beta+\beta^2}{-1+\beta} & \frac{-3+\beta}{-1+\beta} & -\frac{2}{-1+\beta} \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} \\ 0 & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} \\ 0 & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} \end{bmatrix}$$

$$\varphi = \begin{bmatrix} 0 \\ -1 \\ -\beta \\ -\beta^2 \end{bmatrix}$$

Higher Order Expansion of Arellano-Bover Estimator for Dynamic Panel Model with Fixed Effects: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

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1 Arellano-Bover Estimator for Dynamic Panel Model with Fixed Effects

Consider the usual dynamic panel model with fixed effects:

$$y_{it} = \alpha_i + \beta y_{i,t-1} + \varepsilon_{it}, \qquad i = 1, \ldots, n; \quad t = 1, \ldots, T \tag{1}$$

We consider a version of the GMM estimator developed by Arellano and Bover (1995), which simplifies the characterization of the "weight matrix" in GMM estimation. We define the innovation $u_{it} \equiv \alpha_i + \varepsilon_{it}$. Arellano and Bover (1995) eliminate the fixed effect α_i in (1) by applying Helmert's transformation

$$u_{it}^* \equiv \sqrt{\frac{T-t}{T-t+1}} \left[u_{it} - \frac{1}{T-t} \left(u_{i,t+1} + \cdots + u_{iT} \right) \right], \qquad t = 1, \ldots, T-1$$

instead of first differencing. The transformation produces

$$y_{it}^* = \beta x_{it}^* + \varepsilon_{it}^*, \qquad t = 1, \ldots, T-1$$

where $x_{i,t}^* \equiv y_{i,t-1}^*$. Let $z_{it}^* \equiv (y_{i0}, \ldots, y_{it-1})'$. Our moment restriction is summarized by

$$E\left[z_{i,t}^* \varepsilon_{i,t}^* \right] = 0 \qquad t = 1, \ldots, T-1$$

It can be shown that, with the homoscedasticity assumption on ε_{it}, the optimal "weight matrix" is proportional to a block-diagonal matrix, with typical diagonal block equal to $E\left[z_{i,t}^* z_{i,t}^{*\prime} \right]$. Therefore, the optimal GMM estimator by Arellano-Bover is equal to

$$\widehat{b}_{AB} \equiv \frac{\sum_{t=1}^{T-1} x_t^{*\prime} P_t y_t^*}{\sum_{t=1}^{T-1} x_t^{*\prime} P_t x_t^*}$$

so that

$$\sqrt{n} \left(\widehat{b}_{AB} - \beta \right) \equiv \frac{\sum_{t=1}^{T-1} \left(\frac{1}{\sqrt{n}} x_t^{*\prime} P_t^* \varepsilon_t^* \right)}{\sum_{t=1}^{T-1} \left(\frac{1}{n} x_t^{*\prime} P_t^* x_t^* \right)} \tag{2}$$

where $x_t^* \equiv \left(x_{1,t}^*, \cdots, x_{n,t}^*\right)'$, $y_t^* \equiv \left(y_{1,t}^*, \cdots, y_{n,t}^*\right)'$, $Z_t^* \equiv \left(z_{1,t}^*, \cdots, z_{n,t}^*\right)'$, and $P_t^* \equiv Z_t^* \left(Z_t^{*\prime} Z_t^*\right)^{-1} Z_t^{*\prime}$. In Theorem 1 below, we provide an approximation of the form

$$\sqrt{n}\left(\widehat{b}_{AB} - \beta\right) = F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right).$$

2 Higher Order Expansion of $\sqrt{n}\left(\widehat{b}_{AB} - \beta\right)$

We now adopt obvious notations, and make a second order analysis of the right side of (2). Recall that

$$\sqrt{n}\left(\widehat{b}_{AB} - \beta\right) \equiv \frac{\sum_{t=1}^{T-1}\left(\frac{1}{\sqrt{n}} x_t^{*\prime} P_t^* \varepsilon_t^*\right)}{\sum_{t=1}^{T-1}\left(\frac{1}{n} x_t^{*\prime} P_t^* x_t^*\right)}$$

where $z_{it}^* \equiv (y_{i0}, \ldots, y_{it-1})'$, $x_t^* \equiv (x_{1t}^*, \cdots, x_{nt}^*)'$, $y_t^* \equiv (y_{1t}^*, \cdots, y_{nt}^*)'$, $Z_t^* \equiv (z_{1t}^*, \cdots, z_{nt}^*)'$, and $P_t^* \equiv Z_t^* \left(Z_t^{*\prime} Z_t^*\right)^{-1} Z_t^{*\prime}$. The higher order expansion for $\sqrt{n}\left(\widehat{b}_{AB} - \beta\right)$ is contained in Theorem 1 below.

Definition 1

$$\begin{aligned} \lambda_t^* &\equiv E\left[z_{i,t}^* x_{i,t}^*\right] \\ \Lambda_t^* &\equiv E\left[z_{i,t}^* z_{i,t}^{*\prime}\right] \end{aligned}$$

Definition 2

$$\begin{aligned} A_{1,t}^* &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_{i,t}^* x_{i,t}^* - \lambda_t^*\right) \\ &= \sqrt{n}\left(\frac{1}{n} Z_t^{*\prime} x_t^* - \lambda_t^*\right) \qquad (= O_p(1)) \end{aligned}$$

Definition 3

$$\begin{aligned} M_{1,t}^* &\equiv -\Lambda_t^{*-1}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_{i,t}^* z_{i,t}^{*\prime} - \Lambda_t^*\right)\right) \Lambda_t^{*-1} \\ &= -\sqrt{n} \Lambda_t^{*-1}\left(\frac{1}{n} Z_t^{*\prime} Z_t^* - \Lambda_t^*\right) \Lambda_t^{*-1} \qquad (= O_p(1)) \end{aligned}$$

$$\begin{aligned} M_{2,t}^* &\equiv \Lambda_t^{*-1}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_{i,t}^* z_{i,t}^{*\prime} - \Lambda_t^*\right)\right) \Lambda_t^{*-1}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(z_{i,t}^* z_{i,t}^{*\prime} - \Lambda_t^*\right)\right) \Lambda_t^{*-1} \\ &= n \Lambda_t^{*-1}\left(\frac{1}{n} Z_t^{*\prime} Z_t^* - \Lambda_t^*\right) \Lambda_t^{*-1}\left(\frac{1}{n} Z_t^{*\prime} Z_t^* - \Lambda_t^*\right) \Lambda_t^{*-1} \qquad (= O_p(1)) \end{aligned}$$

Definition 4

$$\begin{aligned} C_{0,t}^* &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n z_{i,t}^* \varepsilon_{i,t}^* \\ &= \frac{1}{\sqrt{n}} Z_t^{*\prime} \varepsilon_t^* \qquad (= O_p(1)) \end{aligned}$$

Lemma 1

$$\frac{1}{\sqrt{n}} x_t^{*\prime} P_t^* \varepsilon_t^* = N_{0,t}^* + \frac{1}{\sqrt{n}} N_{1,t}^* + \frac{1}{n} N_{2,t}^* + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
N_{0,t}^* &\equiv \lambda_t^{*\prime} \Lambda_t^{*-1} C_{0,t}^* && (= O_p(1)) \\
N_{1,t}^* &\equiv \lambda_t^{*\prime} M_{1,t}^* C_{0,t}^* + A_{1,t}^{*\prime} \Lambda_t^{*-1} C_{0,t}^* && (= O_p(1)) \\
N_{2,t}^* &\equiv A_{1,t}^{*\prime} M_{1,t}^* C_{0,t}^* + \lambda_t^{*\prime} M_{2,t}^* C_{0,t}^* && (= O_p(1))
\end{aligned}$$

Proof. Note that

$$\frac{1}{\sqrt{n}} x_t^{*\prime} P_t^* \varepsilon_t^* = \left(\frac{1}{n}\sum_{i=1}^n z_{it}^* x_{it}^*\right)' \left(\frac{1}{n}\sum_{i=1}^n z_{it}^* z_{it}^{*\prime}\right)^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n z_{it}^* \varepsilon_{it}^*\right)$$

The conclusion then follows by an argument as in Lemma 7. ■

Lemma 2

$$\frac{1}{n} x_t^{*\prime} P_t^* x_t^* = D_{0,t}^* + \frac{1}{\sqrt{n}} D_{1,t}^* + \frac{1}{n} D_{2,t}^* + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
D_{0,t}^* &\equiv \lambda_t^{*\prime} \Lambda_t^{*-1} \lambda_t^* && (= O_p(1)) \\
D_{1,t}^* &\equiv 2\lambda_t^{*\prime} \Lambda_t^{*-1} A_{1,t}^* + \lambda_t^{*\prime} M_{1,t}^* \lambda_t^* && (= O_p(1)) \\
D_{2,t}^* &\equiv \lambda_t^{*\prime} M_{2,t}^* \lambda_t^* + 2\lambda_t^{*\prime} M_{1,t}^* A_{1,t}^* + A_{1,t}^{*\prime} \Lambda_t^{*-1} A_{1,t}^* && (= O_p(1))
\end{aligned}$$

Proof. Note that

$$\frac{1}{n} x_t^{*\prime} P_t^* x_t^* = \left(\frac{1}{n}\sum_{i=1}^n z_{it}^* x_{it}^*\right)' \left(\frac{1}{n}\sum_{i=1}^n z_{it}^* z_{it}^{*\prime}\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^n z_{it}^* x_{it}^*\right)$$

The conclusion then follows by an argument as in Lemma 8. ■

Theorem 1

$$\sqrt{n}\left(\widehat{b}_{AB} - \beta\right) = F_n + \frac{1}{\sqrt{n}} Q_n + \frac{1}{n} S_n + o_p\left(\frac{1}{n}\right),$$

where

$$\begin{aligned}
F_n &\equiv \frac{N_0^*}{D_0^*} \\
Q_n &\equiv \frac{N_1^*}{D_0^*} - \frac{N_0^* D_1^*}{D_0^{*2}} \\
S_n &\equiv \frac{N_2^*}{D_0^*} - \frac{N_0^* D_2^*}{D_0^{*2}} - \frac{D_1^* N_1^*}{D_0^{*2}} + \frac{D_1^{*2} N_0^*}{D_0^{*3}}
\end{aligned}$$

and

$$\begin{aligned}
N_0^* &\equiv \sum_{t=1}^{T-1} N_{0,t}^* = O_p(1) \qquad N_1^* \equiv \sum_{t=1}^{T-1} N_{1,t}^* = O_p(1) \qquad N_2^* \equiv \sum_{t=1}^{T-1} N_{2,t}^* = O_p(1) \\
D_0^* &\equiv \sum_{t=1}^{T-1} D_{0,t}^* = O_p(1) \qquad D_1^* \equiv \sum_{t=1}^{T-1} D_{1,t}^* = O_p(1) \qquad D_2^* \equiv \sum_{t=1}^{T-1} D_{2,t}^* = O_p(1)
\end{aligned}$$

Proof. From equation (2), and Lemmas 1 and 2, we obtain

$$\sqrt{n}\left(\widehat{b}_{AB}-\beta\right)=\frac{N_0^*+\frac{1}{\sqrt{n}}N_1^*+\frac{1}{n}N_2^*+o_p\left(\frac{1}{n}\right)}{D_0^*+\frac{1}{\sqrt{n}}D_1^*+\frac{1}{n}D_2^*+o_p\left(\frac{1}{n}\right)}$$

■

3 Some simplification for stochastic approximation

Suppose that

$$\begin{aligned}
y_{it} &= \alpha_i+\beta y_{it-1}+\varepsilon_{it} \qquad t=1,\ldots,T \\
\varepsilon_{it} &\sim \mathcal{N}\left(0,\sigma_\varepsilon^2\right), \qquad \alpha_i\sim\mathcal{N}\left(0,\sigma_\alpha^2\right) \\
y_{i0} &\sim \mathcal{N}\left(\frac{\alpha_i}{1-\beta},\frac{1}{1-\beta^2}\right)=\frac{\alpha_i}{1-\beta}+\frac{1}{\sqrt{1-\beta^2}}\mathcal{N}(0,1)
\end{aligned}$$

Further suppose that

$$\sigma_\varepsilon^2=\sigma_\alpha^2=1, \quad T=5$$

We then have the following results:

$$\lambda_1^*=\left[\begin{array}{c}\frac{1}{10}\left(\beta^3+2\beta^2+3\beta+4\right)\frac{\sqrt{5}}{\beta+1}\end{array}\right]$$

$$\lambda_2^*=\left[\begin{array}{c}\frac{1}{6}\left(\beta^2+2\beta+3\right)\sqrt{3}\frac{\beta}{\beta+1}\\ \frac{1}{6}\left(\beta^2+2\beta+3\right)\frac{\sqrt{3}}{\beta+1}\end{array}\right]$$

$$\lambda_3^*=\left[\begin{array}{c}\frac{1}{6}(\beta+2)\sqrt{6}\frac{\beta^2}{\beta+1}\\ \frac{1}{6}(\beta+2)\sqrt{6}\frac{\beta}{\beta+1}\\ \frac{1}{6}(\beta+2)\frac{\sqrt{6}}{\beta+1}\end{array}\right]$$

$$\lambda_4^*=\left[\begin{array}{c}\frac{1}{2}\sqrt{2}\frac{\beta^3}{\beta+1}\\ \frac{1}{2}\sqrt{2}\frac{\beta^2}{\beta+1}\\ \frac{1}{2}\sqrt{2}\frac{\beta}{\beta+1}\\ \frac{1}{2}\frac{\sqrt{2}}{\beta+1}\end{array}\right]$$

$$\Lambda_1^*=\left[\begin{array}{c}\frac{2}{(-1+\beta^2)(-1+\beta)}\end{array}\right]$$

$$\Lambda_2^*=\left[\begin{array}{cc}\frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)}\end{array}\right]$$

$$\Lambda_3^*=\left[\begin{array}{ccc}\frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)}\end{array}\right]$$

$$\Lambda_4^*=\left[\begin{array}{cccc}\frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)}\\ -\frac{\beta^4-\beta^3-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^3-\beta^2-\beta-1}{(-1+\beta^2)(-1+\beta)} & -\frac{\beta^2-2\beta-1}{(-1+\beta^2)(-1+\beta)} & \frac{2}{(-1+\beta^2)(-1+\beta)}\end{array}\right]$$

A Digression: Some Useful Results Related to Higher Order Properties of 2SLS

We first make a digression to the discussion of single equation model,

$$\begin{aligned} y_i &= \beta x_i + \varepsilon_i, \\ x_i &= z_i'\pi + u_i \end{aligned}$$

The only reason we consider such analysis is because all the analysis we found in the literature are conditional analysis given instruments: They all assume that the instruments are nonstochastic. Our purpose is to make a "marginal" higher order analysis, which is more natural in the dynamic panel model context.

Definition 5

$$\begin{aligned} \lambda &\equiv E\left[z_i x_i\right] \\ \Lambda &\equiv E\left[z_i z_i'\right] \end{aligned}$$

Lemma 3

$$\frac{1}{n}\sum_{i=1}^{n} z_i x_i = \lambda + \frac{1}{\sqrt{n}} A_1$$

where

$$A_1 \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i x_i - \lambda\right) = O_p\left(1\right)$$

Lemma 4

$$\frac{1}{n}\sum_{i=1}^{n} z_i z_i' = \Lambda + \frac{1}{\sqrt{n}} L_1$$

where

$$L_1 \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i' - \Lambda\right) = O_p\left(1\right)$$

Lemma 5

$$\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1} = \Lambda^{-1} + \frac{1}{\sqrt{n}} M_1 + \frac{1}{n} M_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned} M_1 &\equiv -\Lambda^{-1} L_1 \Lambda^{-1} \\ &= -\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i' - \Lambda\right)\right)\Lambda^{-1} = O_p\left(1\right) \end{aligned}$$

$$\begin{aligned} M_2 &\equiv \Lambda^{-1} L_1 \Lambda^{-1} L_1 \Lambda^{-1} \\ &= \Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i' - \Lambda\right)\right)\Lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(z_i z_i' - \Lambda\right)\right)\Lambda^{-1} = O_p\left(1\right) \end{aligned}$$

Lemma 6

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\varepsilon_i = C_0$$

where

$$C_0 \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\varepsilon_i = O_p(1)$$

Lemma 7

$$\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\varepsilon_i\right) = \mathbb{N}_0 + \frac{1}{\sqrt{n}}\mathbb{N}_1 + \frac{1}{n}\mathbb{N}_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
\mathbb{N}_0 &\equiv \lambda'\Lambda^{-1}C_0 = O_p(1)\\
\mathbb{N}_1 &\equiv \lambda' M_1 C_0 + A_1'\Lambda^{-1}C_0 = O_p(1)\\
\mathbb{N}_2 &\equiv A_1' M_1 C_0 + \lambda' M_2 C_0 = O_p(1)
\end{aligned}$$

Proof. From Lemmas 3, 5, and 6, we obtain

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} z_i\varepsilon_i\right)\\
=\;&\left(\lambda + \frac{1}{\sqrt{n}}A_1\right)'\left(\Lambda^{-1} + \frac{1}{\sqrt{n}}M_1 + \frac{1}{n}M_2 + o_p\left(\frac{1}{n}\right)\right)C_0\\
=\;&\lambda'\Lambda^{-1}C_0 + \frac{1}{\sqrt{n}}\left(\lambda' M_1 C_0 + A_1'\Lambda^{-1}C_0\right) + \frac{1}{n}\left(A_1' M_1 C_0 + \lambda' M_2 C_0\right)
\end{aligned}$$

■

Lemma 8

$$\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right) = \mathbb{D}_0 + \frac{1}{\sqrt{n}}\mathbb{D}_1 + \frac{1}{n}\mathbb{D}_2 + o_p\left(\frac{1}{n}\right)$$

where

$$\begin{aligned}
\mathbb{D}_0 &\equiv \lambda'\Lambda^{-1}\lambda = O_p(1)\\
\mathbb{D}_1 &\equiv 2\lambda'\Lambda^{-1}A_1 + \lambda' M_1\lambda = O_p(1)\\
\mathbb{D}_2 &\equiv \lambda' M_2\lambda + 2\lambda' M_1 A_1 + A_1'\Lambda^{-1}A_1 = O_p(1)
\end{aligned}$$

Proof. From Lemmas 3, and 5, we obtain

$$\begin{aligned}
&\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)'\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i\right)\\
=\;&\left(\lambda + \frac{1}{\sqrt{n}}A_1\right)'\left(\Lambda^{-1} + \frac{1}{\sqrt{n}}M_1 + \frac{1}{n}M_2 + o_p\left(\frac{1}{n}\right)\right)\left(\lambda + \frac{1}{\sqrt{n}}A_1\right)\\
=\;&\lambda'\Lambda^{-1}\lambda\\
&+\frac{1}{\sqrt{n}}\left(2\lambda'\Lambda^{-1}A_1 + \lambda' M_1\lambda\right)\\
&+\frac{1}{n}\left(\lambda' M_2\lambda + 2\lambda' M_1 A_1 + A_1'\Lambda^{-1}A_1\right)
\end{aligned}$$

■

Alternative Parametrization of the Stationarity Property in Dynamic Panel Models with Weak Identification: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

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We consider the dynamic panel model

$$\begin{aligned} y_{it} &= (1-\beta_n)\,\alpha_i + \beta_n y_{it-1} + \varepsilon_{it} \\ \varepsilon_{it} &\sim N(0,\sigma_\varepsilon^2) \\ \alpha_i &\sim N(0,\sigma_\alpha^2) \\ y_{i0} &\sim N(\alpha_i, \frac{\sigma_y^2}{1-\beta_n^2}) \end{aligned} \tag{1}$$

where we assume that $t = 1, ..., T$ and $i = 1, ..., n$ with T being fixed and $n \to \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/n)$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{n} + o(n^{-1})$ for any constant k. We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = y_{i0} - \alpha_i$ such that

$$\begin{aligned} \Delta y_{it} &= -(\beta_n - 1)\,\beta_n^{t-1}\alpha_i + (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + (\beta_n - 1)\,\beta_n^{t-1} y_{i0} + \varepsilon_{it} \\ &= (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + (\beta_n - 1)\,\beta_n^{t-1}\eta_{i0} + \varepsilon_{it}. \end{aligned}$$

We define $g_{i1}(\beta) = [u_{it}\Delta u_{is}(\beta), u_{iT}\Delta u_{ij}(\beta), \overline{u}_i\Delta u_{ik}(\beta)]'$ and $g_{i2}(\beta) = [\Delta u_i(\beta)' y_{i0}]$ such that

$$g_n(\beta) = n^{-3/2}\sum_{i=1}^{n}[g_{i1}(\beta)', g_{i2}(\beta)']'$$

with the optimal weight matrix $\Omega = \lim_n E g_n(\beta_n) g_n(\beta_n)'$. The infeasible 2SLS of a possibly transformed set of moment conditions then solves

$$\beta_{2SLS} = \arg\min_{\beta} g_n(\beta)' C (C'\Omega C)^{+} C' g_n(\beta)$$

where C is $n \times r$ matrix for $1 \le r \le G$ such that C is of full column rank and $rank(C(C'\Omega C)^{+}C') \ge 1$. We use Ω^{+} to denote the Moore-Penrose inverse (see Magnus and Neudecker p. 32, 1988). We thus allow the use of a singular weight matrix. Choosing r less than G allows to exclude certain moment conditions. We define the vectors

$$\begin{aligned} f_{i,1} &= \left[u_{it}\Delta y_{is-1}, u_{iT}\Delta y_{ij-1}, \overline{u}_i \Delta y_{ik-1}\right]' \\ f_{i,2} &= \left[\Delta y_{i1} y_{i0}, ..., \Delta y_{iT-1} y_{i0}\right]' \end{aligned}$$

for $s = 2, ..., T; t = 1, ...s-2; j = 2, ..., T-1; k = 2, ..., T$ and let $f_n = n^{-3/2} \sum_{i=1}^{n} \left[f_{i1}', f_{i,2}'\right]'$ such that the infeasible 2SLS estimator can be written as

$$\beta_{2SLS} - \beta_{n0} = \left(f_n' C (C'\Omega C)^{+} C' f_n\right)^{-1} f_n' C \left(C'\Omega C\right)^{+} C' g_n(\beta_{n0}). \tag{2}$$

Lemma 1 *Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \to \infty$*

$$n^{-1} \sum_{i=1}^{n} f_{i,1} \xrightarrow{p} \mu_1, n^{-1} \sum_{i=1}^{n} g_{i,1}(\beta_0) \xrightarrow{p} 0$$

with $\mu_1' = \left[0, ..., 0, \sigma_\varepsilon^2 T^{-1} \mathbf{1}_{T-1}'\right]$ and

$$n^{-1} \sum_{i=1}^{n} \left[f_{i,2}', g_{i,2}'(\beta_0)\right]' \xrightarrow{d} \left[\xi_x', \xi_y'\right]'$$

where $\left[\xi_x', \xi_y'\right] \sim N(\mu_2, \Sigma)$ with $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\mu_2' = \left[\mathbf{1}_{T-1}' \sigma_y^2/2, 0, ..., 0\right]$ and $\Sigma_{11} = \delta I, \Sigma_{12} = \delta M_1, \Sigma_{22} = \delta M_2$ where $\delta = \frac{\sigma_y^2 \sigma_\varepsilon^2}{c}$,

$$M_1 = \begin{bmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

and $\Sigma_{12} = \Sigma_{21}'$.

An immediate consequence of this result is that an estimator based solely on the condition

$$E \overline{u}_i \Delta u_{ik}(\beta_0) = 0 \tag{3}$$

is consistent. If we consider estimators that do use all the moment conditions except (3) then we are in a situation where again the moment conditions in $g_{i1}(\beta)$ become asymptotically redundant. Now, however the limiting distribution of the non-redundant conditions has a noncentrality parameter.

More explicitly, if β_{2SLS} is the GMM estimator based on $g_{i2}(\beta)$ then it has the following nonstandard limiting distribution under the weak identification asymptotics adopted here

$$\beta_{2SLS} - \beta_0 \xrightarrow{d} \frac{(\mu + \xi_x)' C (C'\Omega C)^{-1} C' \xi_y}{(\mu + \xi_x)' C (C'\Omega C)^{-1} C' (\mu + \xi_x)} = X$$

where $\mu = -\mathbf{1}_{T-1} \sigma_y^2/2$. It turns out that this limiting distribution is the same as the one obtained when $\beta = \exp(-c/\sqrt{n})$ as is shown in a separate Appendix. There we carry out some numerical calculations which indicate that this limiting distribution leads to small biases and does not capture the type of issues we are most concerned with.

Proof of Lemma (1). Consider $Ef_{i,1}$ with

$$\begin{aligned} Eu_{it}\Delta y_{is-1} &= Eu_{it}\left[(\beta_n-1)\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right] \\ &= (\beta_n-1)\beta_n^{t-s-1}\sigma_\varepsilon^2 = O(n^{-1}) \end{aligned}$$

and $\text{Var}(u_{it}\Delta y_{is-1}) = Eu_{it}^2(\Delta y_{is-1})^2-(\beta_n-1)^2\beta_n^{2(s-t)}\sigma_\varepsilon^4$ where

$$\begin{aligned} Eu_{it}^2(\Delta y_{is-1})^2 &= \beta_n^{2(s-2)}(\sigma_\varepsilon^2+(1-\beta_n)^2\sigma_\alpha^2)\frac{\sigma_y^2(\beta_n-1)^2}{1-\beta_n^2}+(\sigma_\varepsilon^2+(1-\beta_n)^2\sigma_\alpha^2)\sigma_\varepsilon^2 \\ &\quad +(\beta_n-1)^2(\sigma_\varepsilon^2+(1-\beta_n)^2\sigma_\alpha^2)\sigma_\varepsilon^2\sum_{r=1}^{s-2}\beta_n^{2(r-1)}+2(\beta_n-1)^2\sigma_\varepsilon^4\beta^{2(s+1-t)} \end{aligned}$$

such that $\text{Var}(u_{it}\Delta y_{is-1})=\sigma_\varepsilon^4+O(n^{-1})$. By independence of $u_{it}\Delta y_{is-1}$ across i it follows by the Markov inequality that $P\left(\left|n^{-1}\left(\sum_{i=1}^n u_{it}\Delta y_{is-1}-Eu_{it}\Delta y_{is-1}\right)\right|>\epsilon\right)\le\frac{n^{-1}(\sigma_\varepsilon^2+\sigma_\alpha^2)\sigma_\varepsilon^2+o(n^{-1})}{\epsilon}$ for any $\epsilon>0$ which shows that $n^{-1}\sum_{i=1}^n(u_{it}\Delta y_{is-1}-Eu_{it}\Delta y_{is-1})=o_p(1)$. Since $n^{-1}\sum_{i=1}^n Eu_{it}\Delta y_{is-1}=O(n^{-1/2})$ it follows that $n^{-1}\sum_{i=1}^n u_{it}\Delta y_{is-1}=o_p(1)$. For $\overline{u}_i\Delta y_{is-1}$ note that

$$\frac{1}{T}E\left[\sum\nolimits_{t=1}^T u_{it}\left((\beta_n-1)\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)\right]=\frac{\sigma_\varepsilon^2}{T}+\frac{\sigma_\varepsilon^2(\beta_n-1)}{T}\sum_{t=1}^{s-1}\beta_n^{r-1}$$

and

$$\begin{aligned} &E\left[T^{-1}\sum\nolimits_{t=1}^T u_{it}\left((\beta_n-1)\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)\right]^2 \\ &\quad=(\beta_n-1)^2\beta_n^{2(s-2)}T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\eta_{i0}^2+T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\varepsilon_{is-1}^2 \\ &\qquad+2(\beta_n-1)T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\varepsilon_{is-1}\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r} \\ &\qquad+\frac{(\beta_n-1)^2}{T^2}E\sum_{t,s=1}^T u_{it}u_{is}\sum_{q,r=1}^{k-2}\beta_n^{r-1}\beta^{q-1}\varepsilon_{ik-1-q}\varepsilon_{ik-1-r} \\ &\quad=\beta_n^{2(s-2)}T^{-2}\left(T^2(1-\beta_n)^2\sigma_\alpha^2+T\sigma_\varepsilon^2\right)\frac{\sigma_y^2(\beta_n-1)^2}{1-\beta_n^2}+\frac{(1-\beta_n)^2\sigma_\alpha^2\sigma_\varepsilon^2+\sigma_\varepsilon^4}{T}+\frac{2\sigma_\varepsilon^4}{T^2} \\ &\quad+(\beta_n-1)\frac{\sigma_\varepsilon^4}{T^2}\sum_{r=1}^{s-2}\beta_n^{r-2}+\frac{(\beta_n-1)^2}{T^2}\left(T^2(1-\beta_n)^2\sigma_\alpha^2\sigma_\varepsilon^2+(T-1)\sigma_\varepsilon^4\right)\sum\nolimits_{r=1}^{s-2}\beta_n^{2r-2} \\ &\qquad+(\beta_n-1)^2\frac{\sigma_\varepsilon^4}{T^2}\left(\sum_{t=1}^{k-1}\beta^{k-1-r}\right)^2+(\beta_n-1)^2\frac{2\sigma_\varepsilon^4}{T^2}\sum\nolimits_{r=1}^{s-2}\beta_n^{2r-2}=\frac{(T+2)\sigma_\varepsilon^4}{T^2}+O(n^{-1}). \end{aligned}$$

such that by a law of large numbers $n^{-1}\sum_{i=1}^n\overline{u}_i\Delta y_{ik-1}\xrightarrow{p}\sigma_\varepsilon^2/T$ using the same arguments as before.

Next consider

$$\begin{aligned} E\Delta y_{it}y_{i0} &= Ey_{i0}\left((\beta_n-1)\beta_n^{t-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right) \\ &= \beta_n^{t-2}\sigma_y^2(\beta_n-1)/(1-\beta_n^2)=-\frac{\sigma_y^2}{2}+O(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned}
E\left(\Delta y_{it} y_{i0}\right)^2 &= E y_{i0}^2 \left(\left(\beta_n - 1\right) \beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + \left(\beta_n - 1\right) \sum\nolimits_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right)^2 \\
&= \beta_n^{2(t-2)} \frac{(\beta_n - 1)^2 \sigma_y^2}{\left(1-\beta_n^2\right)} \sigma_\alpha^2 + 3\beta_n^{2(t-2)} \frac{\sigma_y^4 (\beta_n - 1)^2}{\left(1-\beta_n^2\right)^2} \\
&\quad + \left[\sigma_\varepsilon^2 + \left(\beta_n - 1\right)^2 \sigma_\varepsilon^2 \sum\nolimits_{r=1}^{s-2} \beta_n^{2(r-1)} \right] \left(\sigma_\alpha^2 + \sigma_y^2 \left(1-\beta_n^2\right)^{-1}\right) \\
&= \frac{\sigma_\varepsilon^2 \sigma_y^2}{2c} n + o(n).
\end{aligned}$$

such that $\text{Var}(n^{-1} \sum_{i=1}^n \Delta y_{it} y_{i0}) = O(1)$.

For $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0)$ consider $E g_{i,1}(\beta_0) = 0$ which is implied by the orthogonality conditions and $E\left(u_{it} \Delta u_{is}\right)^2 = 2((1-\beta_n)^2 \sigma_\alpha^2 + \sigma_\varepsilon^2) \sigma_\varepsilon^2 = 2\sigma_\varepsilon^2 + o(n^{-1})$ by independence. For $E(\bar{u}_i \Delta u_{ik}) = T^{-1} \sum_{t=1}^T E u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) = 0$ while

$$E(\bar{u}_i \Delta u_{ik})^2 = T^{-2} E \left(\sum_{t=1}^T u_{it} (\varepsilon_{ik} - \varepsilon_{ik-1}) \right)^2 = 2(1-\beta_n)^2 \sigma_\alpha^2 \sigma_\varepsilon^2 + T^{-1} \sigma_\varepsilon^4 + 2T^{-2} \sigma_\varepsilon^2$$

such that by the independence of $g_{i,1}(\beta_0)$ across i and a standard WLLN it follows that $n^{-1} \sum_{i=1}^n g_{i,1}(\beta_0) \xrightarrow{p} 0$. For $n^{-1} \sum_{i=1}^n g_{i,2}(\beta_0)$ we have from the moment conditions that $E g_{i,2}(\beta_0) = 0$ and $\text{Var}(\Delta u_i(\beta_0) y_{i0}) = 2\sigma_\varepsilon^2 (\sigma_\alpha^2 + \sigma_y^2 / (1-\beta_n^2)) = \frac{\sigma_\varepsilon^2 \sigma_y^2}{c} n + O(1)$.

The joint limiting distribution of $n^{-1} \sum_{i=1}^n \left[f_{i,2}' - E f_{i,2}', g_{i,2}(\beta_0)' \right]'$ can now be obtained from a triangular array CLT. By previous arguments

$$E\left[f_{i,2}', g_{i,2}(\beta_0)' \right] = \left[\begin{array}{cccc} \mu' & 0 & \cdots & 0 \end{array} \right]$$

with $\mu = \mathbf{1}_{T-1} \sigma_y^2 / 2 + O(n^{-1})$ where $\mathbf{1}_{T-1}$ is the $T-1$ dimensional vector with elements 1. Then $E\left[f_{i,2}' - E f_{i,2}', g_{i,2}(\beta_0)' \right]' \left[f_{i,2}' - E f_{i,2}', g_{i,2}(\beta_0)' \right] = \Sigma_n$ where

$$\Sigma_n = \left[\begin{array}{cc} \Sigma_{11,n} & \Sigma_{12,n} \\ \Sigma_{21,n} & \Sigma_{22,n} \end{array} \right]$$

By previous calculations we have found the diagonal elements of $\Sigma_{11,n}$ and $\Sigma_{22,n}$ to be $\frac{\sigma_\varepsilon^2 \sigma_y^2}{2c} n$ and $\frac{\sigma_\varepsilon^2 \sigma_y^2}{c} n$ The off-diagonal elements are found to be

$$\begin{aligned}
E \Delta y_{it} \Delta y_{is} y_{i0}^2 &= E \left[y_{i0}^2 \left(\left(\beta_n - 1\right) \beta_n^{t-2} \eta_{i0} + \varepsilon_{is-1} + \left(\beta_n - 1\right) \sum\nolimits_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{is-1-r} \right) \right. \\
&\quad \left. \times \left(\left(\beta_n - 1\right) \beta_n^{t-2} \eta_{i0} + \varepsilon_{it-1} + \left(\beta_n - 1\right) \sum\nolimits_{r=1}^{s-2} \beta_n^{r-1} \varepsilon_{it-1-r} \right) \right] \\
&= \beta_n^{t-2} \beta_n^{s-2} \frac{(\beta_n - 1)^2}{\left(1-\beta_n^2\right)} (\sigma_\alpha^2 \sigma_y^2 + 3\sigma_y^4 \left(1-\beta_n^2\right)^{-1}) + O(n^{-1}) = -\frac{3\sigma_y^4}{2} + O(1)
\end{aligned}$$

which is of lower order of magnitude while $n^{-1} \left(E \Delta y_{it} y_{i0}\right)^2 = O(1)$. Thus $n^{-1} \Sigma_{11,n} \to \text{diag}(\frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{2c}, ..., \frac{\sigma_\varepsilon^2 \sigma_\alpha^2}{2c})$. The off-diagonal elements of $\Sigma_{22,n}$ are obtained from

$$E \Delta u_{it} \Delta u_{is} y_{i0}^2 = \begin{cases} \frac{-\sigma_\varepsilon^2 \sigma_y^2}{2c} n + O(1) & t = s+1 \text{ or } t = s-1 \\ 0 & \text{otherwise} \end{cases}$$

zero and for $\Sigma_{12,n}$ we consider

$$E\Delta y_{it}\Delta u_{is}y_{i0}^2 = \begin{cases} \frac{\sigma_\varepsilon^2\sigma_y^2}{c}n + O(1) & \text{if } t = s \\ -\frac{\sigma_\varepsilon^2\sigma_y^2}{c}n + O(1) & \text{if } t = s-1 \\ 0 & \text{otherwise} \end{cases}$$

It then follows that for $\ell \in \mathbb{R}^{T(T+1)/+2T-6}$ such that $\ell'\ell = 1$ $n^{-1}\sum_{i=1}^n \ell'\Sigma_n^{-1/2}\left[f_{i,2}' - Ef_{i,2}, g_{i,2}(\beta_0)'\right]' \xrightarrow{d} N(0,1)$ by the Lindeberg-Feller CLT for triangular arrays. It then follows from a straightforward application of the Cramer-Wold theorem and the continuous mapping theorem that $n^{-1}\sum_{i=1}^n \left[f_{i,2}', g_{i,2}(\beta_0)'\right]' \xrightarrow{d} \left[\xi_x', \xi_y'\right]'$ where $\left[\xi_x', \xi_y'\right]' \sim N((\mu', 0)', \Sigma)$. ■

Note on Near Unit Root Asymptotics in Dynamic Panel Models when Unity is Approached at $n^{-1/2}$: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

Jinyong Hahn   Jerry Hausman   Guido Kuersteiner

June, 2002

We consider the dynamic panel model

$$\begin{aligned} y_{it} &= \alpha_i + \beta_n y_{it-1} + \varepsilon_{it} \\ \varepsilon_{it} &\sim N(0, \sigma_\varepsilon^2) \\ \alpha_i &\sim N(0, \sigma_\alpha^2) \\ y_{i0} &\sim N\left(\frac{\alpha_i}{1-\beta_n}, \frac{\sigma_\varepsilon^2}{1-\beta_n^2}\right) \end{aligned} \tag{1}$$

where we assume that $t = 1, ..., T$ and $i = 1, ..., n$ with T being fixed and $n \to \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/\sqrt{n})$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{\sqrt{n}} + o(n^{-1/2})$ for any constant k. We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = \alpha_i - (1-\beta_n) y_{i0}$ such that

$$\begin{aligned} \Delta y_{it} &= \beta_n^{t-1}\alpha_i + (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + (\beta_n - 1)\beta_n^{t-1}y_{i0} + \varepsilon_{it} \\ &= (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + \beta_n^{t-1}\eta_{i0} + \varepsilon_{it}. \end{aligned}$$

We define $g_{i1}(\beta) = [u_{it}\Delta u_{is}(\beta), u_{iT}\Delta u_{ij}(\beta), \overline{u}_i \Delta u_{ik}(\beta)]'$ and $g_{i2}(\beta) = [\Delta u_i(\beta)' y_{i0}]$ such that

$$g_n(\beta) = n^{-1}\sum_{i=1}^{n}[g_{i1}(\beta)', g_{i2}(\beta)']'$$

with the optimal weight matrix $\Omega = \lim_n E g_n(\beta_n) g_n(\beta_n)'$. The infeasible 2SLS of a possibly transformed set of moment conditions then solves

$$\beta_{2SLS} = \arg\min_\beta g_n(\beta)' C(C'\Omega C)^+ C' g_n(\beta)$$

where C is $n \times r$ matrix for $1 \leq r \leq G$ such that $C'C = I$ and $rank(C(C'\Omega C)^+ C') \geq 1$. We use Ω^+ to denote the Moore-Penrose inverse (see Magnus and Neudecker p. 32, 1988). We thus allow the use of a singular weight matrix. Choosing r less than G allows to exclude certain moment conditions. We define the vectors

$$\begin{aligned} f_{i,1} &= \left[u_{it}\Delta y_{is-1}, u_{iT}\Delta y_{ij-1}, \overline{u}_i \Delta y_{ik-1}\right]' \\ f_{i,2} &= \left[\Delta y_{i1} y_{i0}, ..., \Delta y_{iT-1} y_{i0}\right]' \end{aligned}$$

for $s = 2, ..., T; t = 1, ...s - 2; j = 2, ..., T - 1; k = 2, ..., T$ and let $f_n = n^{-1} \sum_{i=1}^{n} \left[f'_{i1}, f'_{i,2}\right]'$ such that the infeasible 2SLS estimator can be written as

$$\beta_{2SLS} - 1 = \left(f'_n C (C'\Omega C)^+ C' f_n\right)^{-1} f'_n C \left(C'\Omega C\right)^+ C' g_n(\beta_{n0}). \tag{2}$$

Lemma 1 *Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \to \infty$*

$$n^{-1} \sum_{i=1}^{n} f_{i,1} \xrightarrow{p} \mu_1, n^{-1} \sum_{i=1}^{n} g_{i,1}(\beta_0) \xrightarrow{p} 0$$

with $\mu'_1 = \left[0, ..., 0, \sigma^2_\varepsilon T^{-1} \mathbf{1}'_{T-1}\right]$ and

$$n^{-1} \sum_{i=1}^{n} \left[f'_{i,2}, g'_{i,2}(\beta_0)\right]' \xrightarrow{d} \left[\xi'_x, \xi'_y\right]'$$

where$\left[\xi'_x, \xi'_y\right] \sim N(\mu_2, \Sigma)$ with $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ where $\mu'_2 = \left[-\mathbf{1}'_{T-1}\sigma^2_\varepsilon/2, 0, ..., 0\right]$ and $\Sigma_{11} = \delta I, \Sigma_{12} = \delta M_1, \Sigma_{22} = \delta M_2$ where $\delta = \frac{\sigma^2_\alpha \sigma^2_\varepsilon}{c}$,

$$M_1 = \begin{bmatrix} -1 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

and $\Sigma_{12} = \Sigma'_{21}$.

An immediate consequence of this result is that an estimator based solely on the condition

$$E\overline{u}_i \Delta u_{ik}(\beta_0) = 0 \tag{3}$$

is consistent. If we consider estimators that do use all the moment conditions except (3) then we are in a situation where again the moment conditions in $g_{i1}(\beta)$ become asymptotically redundant. Now, however the limiting distribution of the non-redundant conditions has a noncentrality parameter.

The limiting normal distribution of Lemma (1) is singular. An alternative representation of the limiting distribution can be given by eliminating redundant dimensions from $\left[\xi'_x, \xi'_y\right]'$. Let $\xi_y = \left[\xi'_{y0}, \xi_{y1}\right]'$ where ξ_{y1} is the last element of ξ_y and $\xi_u = \left[\xi'_x, \xi_{y1}\right]'$. Then $\xi_y = H\xi_z$ where H is a $T - 1 \times T$ matrix defined as

$$H = \begin{bmatrix} -1 & 1 & & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & 0 & -1 & 1 & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix}.$$

It can be checked easily that the covariance matrix of vector $\left[\xi_x', \xi_u' H'\right]'$ is Σ as required. Using Lemma (1) the limiting distribution of $\beta_{2SLS} - \beta_n$ is stated in the next corollary. For this purpose we define the augmented vectors $\xi_x^{\#} = \left[0, ..., 0, \xi_x'\right]'$ and $\xi_y^{\#} = \left[0, ..., 0, \xi_y\right]$ and partition $C = [C_0', C_1']'$ such that $C'\xi_x^{\#} = C_1'\xi_x$. Let r_1 denote the rank of C_1.

More explicitly, if β_{2SLS} is the GMM estimator based on $g_{i2}(\beta)$ then it has the following nonstandard limiting distribution under the weak identification asymptotics adopted here

$$\beta_{2SLS} - \beta_0 \xrightarrow{d} \frac{(\mu + \xi_x)' \, C_1 (C_1' \Omega C_1)^{-1} C_1' H \xi_u}{(\mu + \xi_x)' \, C_1 (C_1' \Omega C_1)^{-1} C_1' \, (\mu + \xi_x)} = X$$

where $\mu = -\mathbf{1}_{T-1}\sigma_\varepsilon^2/2$.

We report bias and MSE for the long difference (LD) and Arellano and Bond/Ahn and Schmidt estimators in Tables 1 and 2. As is immediately obvious from the tables bias and MSE are much lower than what we observe in actual Monte Carlo experiments.

Proof of Lemma (1). Consider $Ef_{i,1}$ with

$$\begin{aligned} Eu_{it}\Delta y_{is-1} &= Eu_{it}\left[\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right] \\ &= (\beta_n-1)\,\beta_n^{t-s-1}\sigma_\varepsilon^2 = O(n^{-1/2}) \end{aligned}$$

and $\operatorname{Var}(u_{it}\Delta y_{is-1}) = Eu_{it}^2\,(\Delta y_{is-1})^2 - (\beta_n-1)^2\,\beta_n^{2(s-t)}\sigma_\varepsilon^4$ where

$$\begin{aligned} Eu_{it}^2\,(\Delta y_{is-1})^2 &= \beta_n^{2(s-t)}(\sigma_\varepsilon^2+\sigma_\alpha^2)\frac{\sigma_\varepsilon^2(\beta_n-1)^2}{1-\beta_n^2}+(\sigma_\varepsilon^2+\sigma_\alpha^2)\sigma_\varepsilon^2 \\ &\quad + (\beta_n-1)^2\,(\sigma_\varepsilon^2+\sigma_\alpha^2)\sigma_\varepsilon^2\sum_{r=1}^{s-2}\beta_n^{2(r-1)} \end{aligned}$$

such that $\operatorname{Var}(u_{it}\Delta y_{is-1}) = \sigma_\varepsilon^4 + O(n^{-1/2})$. By independence of $u_{it}\Delta y_{is-1}$ across i it follows by the Markov inequality that $P\left(\left|n^{-1}\left(\sum_{i=1}^n u_{it}\Delta y_{is-1} - Eu_{it}\Delta y_{is-1}\right)\right| > \epsilon\right) \le \frac{n^{-1}(\sigma_\varepsilon^2+\sigma_\alpha^2)\sigma_\varepsilon^2+o(n^{-1/2})}{\epsilon}$ for any $\epsilon > 0$ which shows that $n^{-1}\sum_{i=1}^n (u_{it}\Delta y_{is-1} - Eu_{it}\Delta y_{is-1}) = o_p(1)$. Since $n^{-1}\sum_{i=1}^n Eu_{it}\Delta y_{is-1} = O(n^{-1/2})$ it follows that $n^{-1}\sum_{i=1}^n u_{it}\Delta y_{is-1} = o_p(1)$. For $\overline{u}_i\Delta y_{is-1}$ note that

$$\frac{1}{T}E\left[\sum\nolimits_{t=1}^T u_{it}\left(\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)\right] = \frac{\sigma_\varepsilon^2}{T}+\frac{\sigma_\varepsilon^2\,(\beta_n-1)}{T}\sum_{t=1}^{s-1}\beta_n^{r-1}$$

and

$$\begin{aligned} &E\left[T^{-1}\sum\nolimits_{t=1}^T u_{it}\left(\beta_n^{s-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)\right]^2 \\ &= \beta_n^{2(s-2)}T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\eta_{i0}^2 + T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\varepsilon_{is-1}^2 \\ &\quad + 2\,(\beta_n-1)\,T^{-2}E\left(\sum\nolimits_{t=1}^T u_{it}\right)^2\varepsilon_{is-1}\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r} \\ &\quad + \frac{(\beta_n-1)^2}{T^2}E\sum_{t,s=1}^T u_{it}u_{is}\sum_{q,r=1}^{k-2}\beta_n^{r-1}\beta^{q-1}\varepsilon_{ik-1-q}\varepsilon_{ik-1-r} \\ &= \beta_n^{2(s-2)}T^{-2}\left(T^2\sigma_\alpha^2+T\sigma_\varepsilon^2\right)\frac{\sigma_\varepsilon^2(\beta_n-1)^2}{1-\beta_n^2}+\sigma_\alpha^2\sigma_\varepsilon^2+\frac{3\sigma_\varepsilon^4}{T} \\ &\quad + 4(\beta_n-1)T^{-2}\sigma_\varepsilon^4\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1} \\ &\quad + (\beta_n-1)^2\left(\sigma_\alpha^2\sigma_\varepsilon^2+\frac{3\sigma_\varepsilon^4}{T}\right)\sum_{r=1}^{s-2}\beta_n^{2r-4}+(\beta_n-1)^2\frac{\sigma_\varepsilon^4}{T^2}\left(\sum_{t=1}^{k-1}\beta^{k-1-r}\right)^2 \\ &= \sigma_\alpha^2\sigma_\varepsilon^2+\frac{3\sigma_\varepsilon^4}{T}+O(n^{-1/2}). \end{aligned}$$

such that by a law of large numbers $n^{-1}\sum_{i=1}^n \overline{u}_i\Delta y_{ik-1} \xrightarrow{p} \sigma_\varepsilon^2/T$ using the same arguments as before.

Next consider

$$\begin{aligned} E\Delta y_{it}y_{i0} &= Ey_{i0}\left(\beta_n^{t-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right) \\ &= \beta_n^{t-2}\sigma_\varepsilon^2\,(\beta_n-1)\,/(1-\beta_n^2) = -\frac{\sigma_\varepsilon^2}{2}+O(n^{-1/2}) \end{aligned}$$

and

$$
\begin{aligned}
E\left(\Delta y_{it}y_{i0}\right)^2 &= Ey_{i0}^2\left(\beta_n^{t-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)^2 \\
&= \beta_n^{2(t-2)}\frac{(\beta_n-1)^2\sigma_\varepsilon^2}{\left(1-\beta_n^2\right)\left(1-\beta_n\right)^2}\sigma_\alpha^2+3\beta_n^{2(t-2)}\frac{\sigma_y^4(\beta_n-1)^2}{\left(1-\beta_n^2\right)^2} \\
&\quad+\left[\sigma_\varepsilon^2+\left(\beta_n-1\right)^2\sigma_\varepsilon^2\sum\nolimits_{r=1}^{s-2}\beta_n^{2(r-1)}\right]\left(\sigma_\alpha^2\left(1-\beta_n\right)^{-2}+\sigma_\varepsilon^2\left(1-\beta_n^2\right)^{-1}\right) \\
&= \frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c}n+o(n).
\end{aligned}
$$

such that $\mathrm{Var}(n^{-1}\sum_{i=1}^n \Delta y_{it}y_{i0})=O(1)$.

For $n^{-1}\sum_{i=1}^n g_{i,1}(\beta_0)$ consider $Eg_{i,1}(\beta_0)=0$ which is implied by the orthogonality conditions and $E\left(u_{it}\Delta u_{is}\right)^2=2(\sigma_\alpha^2+\sigma_\varepsilon^2)\sigma_\varepsilon^2$ by independence. For $E(\bar{u}_i\Delta u_{ik})=T^{-1}\sum_{t=1}^T Eu_{it}(\varepsilon_{ik}-\varepsilon_{ik-1})=0$ while

$$
E(\bar{u}_i\Delta u_{ik})^2=T^{-2}E\left(\sum_{t=1}^T u_{it}(\varepsilon_{ik}-\varepsilon_{ik-1})\right)^2=2\sigma_\alpha^2\sigma_\varepsilon^2+T^{-1}\sigma_\varepsilon^4+2T^{-2}\sigma_\varepsilon^2
$$

such that by the independence of $g_{i,1}(\beta_0)$ across i and a standard WLLN it follows that $n^{-1}\sum_{i=1}^n g_{i,1}(\beta_0)\xrightarrow{p}0$. For $n^{-1}\sum_{i=1}^n g_{i,2}(\beta_0)$ we have from the moment conditions that $Eg_{i,2}(\beta_0)=0$ and $\mathrm{Var}(\Delta u_i(\beta_0)y_{i0})=2\sigma_\varepsilon^2(\sigma_\alpha^2/(1-\beta_n)^2+\sigma_\varepsilon^2/(1-\beta_n^2))=\frac{2\sigma_\varepsilon^2\sigma_\alpha^2}{c}n+O(1)$.

The joint limiting distribution of $n^{-1}\sum_{i=1}^n\left[f_{i,2}'-Ef_{i,2}',g_{i,2}(\beta_0)'\right]'$ can now be obtained from a triangular array CLT. By previous arguments

$$
E\left[f_{i,2}',g_{i,2}(\beta_0)'\right]=\left[\begin{array}{cccc}\mu' & 0 & \cdots & 0\end{array}\right]
$$

with $\mu=-\mathbf{1}_{T-1}\sigma_\varepsilon^2/2+O(n^{-1})$ where $\mathbf{1}_{T-1}$ is the $T-1$ dimensional vector with elements 1. Then $E\left[f_{i,2}'-Ef_{i,2}',g_{i,2}(\beta_0)'\right]'\left[f_{i,2}'-Ef_{i,2}',g_{i,2}(\beta_0)'\right]=\Sigma_n$ where

$$
\Sigma_n=\left[\begin{array}{cc}\Sigma_{11,n} & \Sigma_{12,n}\\ \Sigma_{21,n} & \Sigma_{22,n}\end{array}\right]
$$

By previous calculations we have found the diagonal elements of $\Sigma_{11,n}$ and $\Sigma_{22,n}$ to be $\frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c}n$. The off-diagonal elements are found to be

$$
\begin{aligned}
E\Delta y_{it}\Delta y_{is}y_{i0}^2 &= E\left[y_{i0}^2\left(\beta_n^{t-2}\eta_{i0}+\varepsilon_{is-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{is-1-r}\right)\right. \\
&\quad\left.\times\left(\beta_n^{t-2}\eta_{i0}+\varepsilon_{it-1}+(\beta_n-1)\sum\nolimits_{r=1}^{s-2}\beta_n^{r-1}\varepsilon_{it-1-r}\right)\right] \\
&= \beta_n^{t-2}\beta_n^{s-2}\frac{(\beta_n-1)^2\sigma_\varepsilon^2}{\left(1-\beta_n^2\right)}\left(\frac{\sigma_\alpha^2}{\left(1-\beta_n\right)^2}+3\sigma_\varepsilon^2\left(1-\beta_n^2\right)^{-1}\right)+O(n^{-1})=-\frac{3\sigma_\varepsilon^2\sigma_\alpha^2}{2}n^{1/2}+O(1)
\end{aligned}
$$

which is of lower order of magnitude while $n^{-1}\left(E\Delta y_{it}y_{i0}\right)^2=O(1)$. Thus $n^{-1}\Sigma_{11,n}\to\mathrm{diag}(\frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c},...,\frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c})$. The off-diagonal elements of $\Sigma_{22,n}$ are obtained from

$$
E\Delta u_{it}\Delta u_{is}y_{i0}^2=\begin{cases}\frac{-\sigma_\varepsilon^2\sigma_\alpha^2}{c}n+O(1) & t=s+1\text{ or }t=s-1\\ 0 & \text{otherwise}\end{cases}
$$

zero and for $\Sigma_{12,n}$ we consider

$$E\Delta y_{it}\Delta u_{is}y_{i0}^2 = \begin{cases} \frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c}n + O(1) & \text{if } t = s \\ -\frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c}n + O(1) & \text{if } t = s-1 \\ 0 & \text{otherwise} \end{cases}$$

It then follows that for $\ell \in \mathbb{R}^{T(T+1)/+2T-6}$ such that $\ell'\ell = 1$ $n^{-1}\sum_{i=1}^n \ell'\Sigma_n^{-1/2}\left[f_{i,2}' - Ef_{i,2}, g_{i,2}(\beta_0)'\right]' \xrightarrow{d} N(0,1)$ by the Lindeberg-Feller CLT for triangular arrays. It then follows from a straightforward application of the Cramer-Wold theorem and the continuous mapping theorem that $n^{-1}\sum_{i=1}^n \left[f_{i,2}', g_{i,2}(\beta_0)'\right]' \xrightarrow{d} \left[\xi_x', \xi_y'\right]'$ where $\left[\xi_x', \xi_y'\right]' \sim N((\mu', 0)', \Sigma)$. ■

Table 1: Bias of root-n Near Unit Root Approximations

T	AS/AB	LD
5	-0.016404	0.010327
6	-0.01391	0.0061784
7	-0.011778	0.0043041
8	-0.0094988	0.0041043
9	-0.0081541	0.0020319
10	-0.006843	0.0014522

Table 2: MSE of root-n Near Unit Root Approximations

T	AS/AB	LD
5	0.017079	0.022672
6	0.0095719	0.013495
7	0.0058787	0.0095064
8	0.003924	0.0068071
9	0.0027829	0.0052526
10	0.0019628	0.0040317

Note on Near Unit Root Asymptotics for the Blundell and Bond Estimator: Supplementary Appendix for "Long Difference Instrumental Variables Estimation for Dynamic Panel Models with Fixed Effects"

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We consider the dynamic panel model

$$\begin{aligned} y_{it} &= \alpha_i + \beta_n y_{it-1} + \varepsilon_{it} \\ \varepsilon_{it} &\sim N(0, \sigma_\varepsilon^2) \\ \alpha_i &\sim N(0, \sigma_\alpha^2) \\ y_{i0} &\sim N\left(\frac{\alpha_i}{1-\beta_n}, \frac{\sigma_\varepsilon^2}{1-\beta_n^2}\right) \end{aligned} \tag{1}$$

where we assume that $t = 1, ..., T$ and $i = 1, ..., n$ with T being fixed and $n \to \infty$. We consider a sequence of generating measures indexed by $\beta_n = \exp(-c/n)$ for some constant $c > 0$. Note that $\beta_n^k = 1 - \frac{kc}{\sqrt{n}} + o(n^{-1/2})$ for any constant k. We also define the innovation $u_{it} = \alpha_i + \varepsilon_{it}$. Let $\eta_{i0} = \alpha_i - (1-\beta_n) y_{i0}$ such that

$$\begin{aligned} \Delta y_{it} &= \beta_n^{t-1}\alpha_i + (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + (\beta_n - 1)\beta_n^{t-1} y_{i0} + \varepsilon_{it} \\ &= (\beta_n - 1)\sum_{s=1}^{t-1}\beta_n^{s-1}\varepsilon_{it-s} + \beta_n^{t-1}\eta_{i0} + \varepsilon_{it}. \end{aligned}$$

In particular it holds that

$$\Delta y_{i1} = \eta_{i0} + \varepsilon_{i1}.$$

In this note we are interested in the behavior of the additional moment condition (4.4) proposed by Blundell and Bond (1998). As Blundell and Bond point out this moment condition can be combined with the other moment conditions propsed by Ahn and Schmidt (1995) to construct a first order asymptotically efficient estimator. Here, we are however interested in the behavior of the Blundell and Bond estimator under weak instrument conditions and thus do not need to focus on the remaining conditions which do not produce consistent estimators.

We define

$$g_{i,AB} = [y_{i,0}\Delta u_{i2}, y_{i,0}\Delta u_{i3}, y_{i1}\Delta u_{i3}, y_{i,0}\Delta u_{i4} ..., y_{i2}\Delta u_{i4}, ..., y_{i0}\Delta u_{iT}, ..., y_{iT-2}\Delta u_{iT}]'$$

and

$$g_{i,BB}(\beta) = [u_{i2}(\beta)\Delta y_{i1}, ..., u_{iT}(\beta)\Delta y_{iT-1}]$$

We define

$$f_{i,BB}(\beta) = \left[y_{i1}(\beta)\,\Delta y_{i1}, ..., y_{iT-1}(\beta)\,\Delta y_{iT-1}\right]'$$

Lemma 1 *Assume $\beta_n = \exp(-c/n)$ for some $c > 0$. For T fixed and as $n \to \infty$*

$$n^{-2}\sum_{i=1}^{n} f_{i,BB} \xrightarrow{p} \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{c}\mathbf{1}_{T-1}, n^{-1}\sum_{i=1}^{n} g_{i,BB}(\beta_n) \xrightarrow{p} 0.$$

and

$$E\left[g_{i,AB}g'_{i,BB}\right] = O(1).$$

Define the Blundell and Bond (1998) estimator by letting $f_n = n^{-2}\sum_{i=1}^{n}\left[f'_{i,AB}, f'_{i,BB}\right]'$, $g_n = n^{-2}\sum_{i=1}^{n}\left[g'_{i,AB}, g'_{i,BB}\right]'$ and the weight matrix $\Omega_n = E\left[g_n g'_n\right]$ such that the infeasible 2SLS estimator can be written as

$$b_{BB} - 1 = \left(f'_n\Omega_n^+ f_n\right)^{-1} f'_n\Omega_n^+ g_n\left(\beta_n\right). \tag{2}$$

It follows from earlier results and the Lemma that for $q_3 = T\,(T+1)/2 - 2$

$$n\Omega_n \to \begin{bmatrix} 0_{q_3,q_3} & \cdots & 0_{q_3,T-1} \\ \vdots & \Sigma_{22} & \vdots \\ 0_{T-1,q_3} & \cdots & 0_{T-1,T-1} \end{bmatrix}$$

This suggests that the optimally weighted Blundell and Bond estimator becomes ill-defined under the weak instrument asymptotic approximation because $n^{-1}f'_n\Omega_n^+ f_n \xrightarrow{p} 0$. An inefficient version of the estimator is better behaved. Define

$$b_{IBB} - 1 = \left(f'_n f_n\right)^{-1} f'_n g_n\left(\beta_n\right).$$

Then it follows that

$$b_{IBB} - 1 \xrightarrow{p} 0$$

from Lemma (1) and results in the main part of the paper. In other words, exploiting stationarity leads to consistent estimators under weak instrument asymptotics. This result confirms the analysis in Blundell and Bond (1998).

Proof of Lemma (1). Note that

$$\begin{aligned} y_{it} &= \frac{\left(1-\beta_n^t\right)\alpha_i}{(1-\beta_n)} + \beta_n^t y_{i0} + \sum_{r=1}^{t-1} \beta_n^{r-1}\varepsilon_{it-r} + \varepsilon_{it} \\ &= \frac{\alpha_i}{(1-\beta_n)} + \frac{\beta_n^t \eta_{i0}}{(1-\beta_n)} + \varepsilon_{it} + \sum_{r=1}^{t-1} \beta_n^{r-1}\varepsilon_{it-r}. \end{aligned}$$

Consider $Ef_{i,BB}$ with

$$\begin{aligned} Ey_{it}\Delta y_{it} &= Ey_{it}\left[\beta_n^t\eta_{i0} + \varepsilon_{it} + (\beta_n - 1)\sum_{r=1}^{t-1}\beta_n^{r-1}\varepsilon_{it-r}\right] \\ &= \frac{\sigma_\alpha^2}{(1-\beta_n)} + \frac{\beta_n^{2t}\sigma_\varepsilon^2}{(1-\beta_n)} + (\beta_n - 1)\sum_{r=1}^{t-1}\beta_n^{2(r-1)}\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{c}n + O(n^{-1}) \end{aligned}$$

and $\text{Var}(y_{it}\Delta y_{it}) = E\left(y_{it}^2\right)E\left(\Delta y_{is-1}\right)^2 + \left(E\left(y_{it}\Delta y_{is-1}\right)\right)^2$ where

$$\begin{aligned} E\left(y_{it}^2\right)E\left(\Delta y_{is-1}\right)^2 &= \left(\frac{\sigma_\alpha^2}{(1-\beta_n)} + \frac{\beta_n^{2t}\sigma_\varepsilon^2}{(1-\beta_n)} + \sum_{r=0}^{t-1}\beta_n^{r-1}\sigma_\varepsilon^2\right)\left(\frac{\beta_n^{2t}\sigma_\varepsilon^2}{(1-\beta_n)} + \sigma_\varepsilon^2 + (\beta_n-1)^2\sum_{r=0}^{t-1}\beta_n^{2(r-1)}\sigma_\varepsilon^2\right) \\ &= \frac{\sigma_\varepsilon^2\sigma_\alpha^2}{c^2}n^2 + O(n). \end{aligned}$$

such that $\text{Var}(y_{it}\Delta y_{is-1}) = \frac{\sigma_\varepsilon^2\sigma_\alpha^2 + \left(\sigma_\varepsilon^2+\sigma_\alpha^2\right)^2}{c^2}n^2 + O(n)$. It then follows that $n^{-2}\sum_{i=1}^n Ey_{it}\Delta y_{is-1} = O(1)$ and $n^{-2}\sum_{i=1}^n y_{it}\Delta y_{is-1} = \frac{\sigma_\varepsilon^2+\sigma_\alpha^2}{c} + o_p(1)$.

For $g_{i,BB}(\beta)$ consider

$$E\left(u_{it}\Delta y_{it-1}\right) = E\left(\alpha_i + \varepsilon_{it}\right)\left[\beta_n^{t-1}\eta_{i0} + \varepsilon_{it-1} + (\beta_n-1)\sum_{r=1}^{t-2}\beta_n^{r-1}\varepsilon_{it-1-r}\right] = 0$$

and

$$\begin{aligned} \text{Var}\left(u_{it}\Delta y_{it-1}\right) &= E\left(u_{it}\Delta y_{it-1}\right)^2 = Eu_{it}^2 E\left(\Delta y_{it-1}\right)^2 \\ &= \left(\sigma_\alpha^2 + \sigma_\varepsilon^2\right)\left(\beta_n^{t-1}\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + (\beta_n-1)^2\sum_{r=1}^{t-2}\beta_n^{2(r-1)}\sigma_\varepsilon^2\right) \\ &= 2\left(\sigma_\alpha^2+\sigma_\varepsilon^2\right)\sigma_\varepsilon^2 + o(1). \end{aligned}$$

such that

$$n^{-1}\sum_{i=1}^n g_{i,BB}(\beta) \xrightarrow{p} 0 \ .$$

For the correlation between $g_{i,BB}$ and $g_{i,AB}$ consider

$$\begin{aligned} E\left[u_{it}\Delta y_{it-1}\Delta u_{is}y_{i0}\right] &= E\left[u_{it}\Delta u_{is}\right]E\left[\Delta y_{it-1}y_{i0}\right] + E\left[\Delta u_{is}\Delta y_{it-1}\right]E\left[y_{i0}u_{it}\right] \\ &= \begin{cases} (\beta_n-1)\left(\beta_n^{t-1-s} - \beta_n^{t-s}\right)\sigma_\varepsilon^2\frac{\sigma_\alpha^2}{1-\beta_n} & t \geq s+1 \\ \sigma_\varepsilon^2\beta_n^{t-1}\frac{\sigma_\varepsilon^2(\beta_n-1)}{1-\beta_n^2} - \frac{(1-\beta_n)\sigma_\varepsilon^2\sigma_\alpha^2}{1-\beta_n} & s = t \\ -\sigma_\varepsilon^2\beta_n^{t-1}\frac{\sigma_\varepsilon^2(\beta_n-1)}{1-\beta_n^2} & t = s-1 \end{cases} = O(1) \end{aligned}$$

and

$$E\left[u_{it}\Delta y_{it-1}\Delta u_{is}u_{ik}\right] = E\left[u_{it}\Delta u_{is}\right]E\left[\Delta y_{it-1}u_{ik}\right] + E\left[\Delta u_{is}\Delta y_{it-1}\right]E\left[u_{ik}u_{it}\right] = O(1)$$

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