The Solow Model in Discrete Time

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1 Technology

There is a single good Y_t produced by means of two factors of production – capital K_t in place at the beginning of the period and labor L_t according to a production function

$$Y_t = F(K_t, L_t) \tag{1}$$

We make the following assumptions on $F(K, L)$: 1) twice continuously differentiable, with positive first derivatives and negative second partial derivatives; 2) homogeneous of degree one (that is, $F(\lambda K, \lambda L) = \lambda F(K, L)$ for any constant λ); 3) each input is strictly necessary (that is, $F(0, L) = F(K, 0) = 0$); 4) "Inada" conditions on $F(K, 1)$, that is, $\lim\limits_{K \to 0} F_K(K, 1) = \infty$ and $\lim\limits_{K \to \infty} F_K(K, 1) = 0$, where the notation $F_K(K, 1)$ refers to the partial derivative of $F(K, 1)$ with respect to K.

To begin, assume there is no technical progress, that is, the production function $F\left(K, L\right)$ does not shift up over time.

2 Inputs

Labor supply grows exponentially at exogenous rate n, the rate of population growth. That is, labor is supplied inelastically:

$$L_{t+1} = \left(1 + n\right) L_t \tag{2}$$

Gross investment is the change in the capital stock plus capital depreciation, that is,

$$I_t \equiv K_{t+1} - K_t + \delta K_t \tag{3}$$

where the rate of depreciation is δ, where $0 < \delta < 1$. Net investment is gross investment minus depreciation, that is, the net change in the capital stock $K_{t+1} - K_t$.

Gross investment equals aggregate saving, meaning capital is also supplied inelastically. The key assumption is that saving is a fixed fraction s of gross output, where $0 < s < 1$, that is

$$I_t = sY_t \tag{4}$$

Y_t. Hence, Solow based this assumption on stylized fact about economies, rather than on optimizing. It is equivalent to assuming a constant marginal propensity to consume (MPC). Combining the above two equations, capital evolves according to:

$$K_{t+1} = sY_t + \left(1 - \delta\right) K_t \tag{5}$$

3 Evolution of the Economy

The key question is – Given initial values of capital and labor (denoted K_0 and L_0), how does this economy evolve? That is, how do Y_t, K_t, and L_t evolve? We know the answer for labor from (2). Dividing (5) through by L_t we may write

$$\frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = s\frac{Y_t}{L_t} + (1-\delta)\frac{K_t}{L_t}$$

Define a new variable, the capital-labor ratio $k_t = K_t/L_t$. If we know how it evolves, we can multiply by L_t at any date to find how K_t evolves. Using (2), we have:

$$k_{t+1} = \frac{s}{1+n}\frac{Y_t}{L_t} + \frac{1-\delta}{1+n}k_t \tag{6}$$

Using the first-degree homogeneity of the production function, we may write:

$$\frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = F(k_t, 1) \equiv f(k_t) \tag{7}$$

where the last equality indicates the definition of a new function, which "takes on" the properties of as assumed in assumptions 1), 3), and 4) in section 1 above. Hence, $f(0) = 0$, $f'(k) = F_K(K, 1) > 0$, $f''(k) < 0$, and so on. (You should demonstrate this.)

One may then write a dynamic equation in a single *state variable* k_t that completely describes the evolution of the economy:

$$k_{t+1} = \frac{s}{1+n}f(k_t) + \frac{1-\delta}{1+n}k_t \equiv g(k_t) \tag{8}$$

State variables are a set of variables that fully describe at each point in time the state of the system, that is the value of *all* the system's variables, given the exogenous variables. (As in this case, there may be only one variable). Generally, one has the goal of reducing a system to the minimal level of state variables. (8) is a basic *first-order difference equation*. To solve it one needs a single *boundary condition*, which here is simply the starting point, namely $k_0 = K_0/L_0$. The evolution of all variables in this economy can be found once we know the evolution of the state variable k_t.

One can analyze this in a discrete time phase diagram, plotting $k_{t+1} = g(k_t)$ in $k_t - k_{t+1}$ space, and using the 45° line in this space (that is, the line $k_t = k_{t+1}$) to derive the dynamics and the stationary or rest point, that is, the *steady state*. From (8), we have

$$\begin{aligned} g'(k_t) &= \frac{s}{1+n}f'(k_t) + \frac{1-\delta}{1+n} > 0 && \text{(9a)} \\ g''(k_t) &= \frac{s}{1+n}f''(k_t) < 0 && \text{(9b)} \end{aligned}$$

so that $g(\cdot)$ is an increasing, strictly concave function. Figure 1 presents the phase diagram.

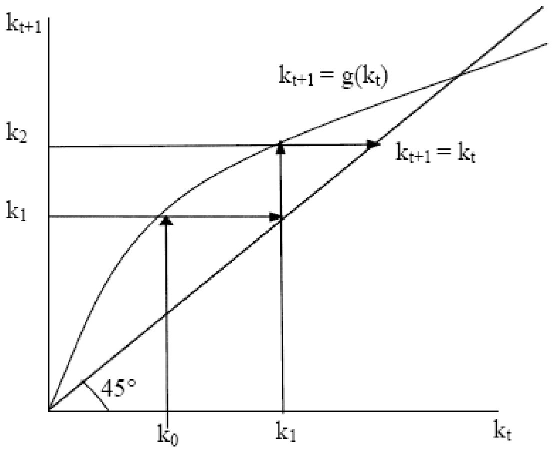

Figure 1: Dynamics in the Solow Model

4 Characteristics of the Steady State

A steady state is a value $k = k_t = k_{t+1}$, that is a solution to $k = g(k)$ from (8). This is termed a balanced growth path, with all extensive variables – K_t, L_t, and Y_t – growing at the same rate n.

4.1 Existence of a steady state

Using the characteristic $f(0) = 0$, one steady state, that is, solution to $k = g(k)$, is obviously $k = 0$. Another (unique) solution is a value of k such that:

$$\frac{f(k)}{k} = \frac{n+\delta}{s} \tag{10}$$

This is the *Harrod-Domar condition*, where pre-Solovian growth theory was seen as arguing that: 1) that it was unlikely to be satisfied (that is, a steady state was unlikely to exist, implying that steady state or *balanced* growth was unlikely);[1] and 2) if it were satisfied, it would not be stable.

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[1] On existence, Domar was seen as arguing that with a fixed-proportions or Leontief prodction function the capital-output ratio (the inverse of the left-hand side of (10)) was technologically fixed and thus unlikely to be equal to the exogenous ratio $\frac{f(k)}{k}$ on the right-hand side of (10). (Domar argued, it seems correctly, that his article was misinterpreted and never made this claim, and he quickly "disowned" this interpretation of the Harrod-Domar condition. Harrod roughly argued that even though the output-capital ratio is not technologically fixed, the monetary side of the economy fixes the interest rate, which then determines $\frac{f(k)}{k}$.

One of Solow's key contribution in light of growth theory at the time was to show that given the above assumptions on the production function (that is, on the nature of production), balanced growth was possible and that the balanced growth solution was stable. Specifically, one may show that the Inada conditions imply that $\frac{f(k)}{k}$ approaches 0 as k approaches infinity and that $\frac{f(k)}{k}$ approaches infinity as k approaches 0, that is, that $\frac{f(k)}{k}$ can take any value between 0 and ∞. (See the Appendix for a proof.) Since $\frac{n+s}{\delta}$ also lies between 0 and ∞, for any value of the right-hand side of (10), there is a at least one value of k such that (10) is satisfied. Since $f(k)$ is continuous and concave in k, this interior value, call it k^{SS}, is unique.

Note that the ratio of net output, $y_t - \delta k_t$, to k_t is also constant along a balanced growth path (see Krusell and Smith, 2015). We return to this later in the term when we evaluate the controversy over Piketty's work on the evolution of the capital output ratio over time.

4.2 Convergence (Stability of steady state)

In the phase diagram, we see that k_t is increasing to the left of k^{SS}, where $g(k_t) > k_t$, and decreasing to the left of k^{SS}, where $g(k_t) < k_t$. We cannot "prove" convergence from a diagram, but we can use the diagrams to gain intuition about what we need to prove. From a phase diagram, one can see that a necessary and sufficient condition for convergence/stability is that at the rest point k^{SS}, $g(k_t)$ cut the 45° line from above, that is, $g'\left(k^{SS}\right) < 1$, or, using (9a),

$$f'\left(k^{SS}\right) < \frac{n+\delta}{s} \tag{11}$$

at k^{SS}. Since $\frac{n+s}{\delta} = \frac{f\left(k^{SS}\right)}{k^{SS}}$from (10), (11) becomes

$$f'\left(k^{SS}\right) < \frac{f\left(k^{SS}\right)}{k^{SS}} \tag{12}$$

which is guaranteed by the strict concavity of $f(\cdot)$.

4.3 Extensive Growth

In steady state the capital stock K_t and total output Y_t are growing at rate n, while k_t and y_t are constant. So there is "extensive" growth, but not "intensive" (per-capita) growth. The marginal products of labor and capital are also constant. If these are equal to the wage and the net interest rate (net of depreciation, that is, $r_t = f'(k_t) - \delta$), as would be true with factor payments determined competitively, these will be constant as well along a balanced growth path.

5 Technical Progress

Suppose that there is technical progress, so that the production function is shifting up over time, that is, $Y_t = F(K_t, L_t, t)$, where $\partial F(\cdot)/\partial t > 0$. Balanced growth requires a constant rate of technical progress, but it requires more. In addition, for there to be a balanced growth path either technical progress must be purely "labor augmenting" or the elasticity of substitution between capital and labor (as in the Cobb-Douglas case) must be unity. (To think about the case where this elasticity is below one, as much evidence may suggest, see Grossman, Helpman, Oberfield, and Sampson [2017].[2])

"Labor-augmenting technical progress" means that the *effect* of technical progress can be mathematically represented as if each physical worker's productivity is being increased or 'augmented' by a factro each period. That is, we can write the production function inclusive of technical progress as

$$F\left(K_t, L_t, t\right) = F\left(K_t, A_t L_t\right)$$

where $A_t L_t$ are "effective" labor units at time t. Note that labor-augmenting technical progress does *not* refer to the way technical progress makes factors more productive, i.e., better-trained workers, etc.) but only the mathematical effect of technical progress.

For there to be a balanced growth path, the rate of labor-augmenting technical progress must be constant, say $A_{t+1} = (1+a)\, A_t$, where a is a constant. Then, along a balanced growth path, output Y_t will be growing at rate γ, where $\gamma = (1+a)\,(1+n) - 1 \approx a + n$. Dividing equation (5) by Y_t and using $\frac{K_{t+1}}{Y_{t+1}} = \frac{K_t}{Y_t}$ along a balanced growth path, the Harrod-Domar condition (10) can be written

$$\frac{K_t}{Y_t} = \frac{s}{\gamma + \delta} \tag{13}$$

that is, with γ replacing n. Since labor grows at rate n but output grows at rate $\gamma > n$. there is intensive growth, with $\frac{Y_t}{L_t}$ growing over time. The interest rate r_t will be constant, while the wage per unit of labor will grow at rate γ.

APPENDIX: The Behavior of $\frac{f(k)}{k}$ in the Limit

We want to show that $\frac{f(k)}{k} \in (0, \infty)$.

a) $K \lim\limits_{k \to \infty} \frac{f(k)}{k} = K \lim\limits_{L \to 0} \frac{L}{K} F\left(\frac{K}{L}, 1\right) = \lim\limits_{L \to 0} F\left(K, L\right) = F\left(K, 0\right) = 0.$

b) $\lim\limits_{k \to 0} \frac{f(k)}{k} = \lim\limits_{k \to 0} \frac{f(k) - f(0)}{k} = f'\left(0\right) = \lim\limits_{K \to 0} F_K\left(K, 1\right) = \infty.$

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[2] Grossman, G., E. Helpman. E. Oberfield, and T. Sampson (2017), "Balanced Growth Despite Uzawa," American Economic Review 107(4), 1293–1312.