Ordered Probabilistic Choice

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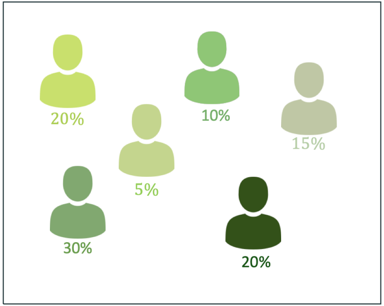
Think of a stochastic choice coming from a heterogeneous population (or repeated choices of a single person).

S $\dashrightarrow$

Decision Problem



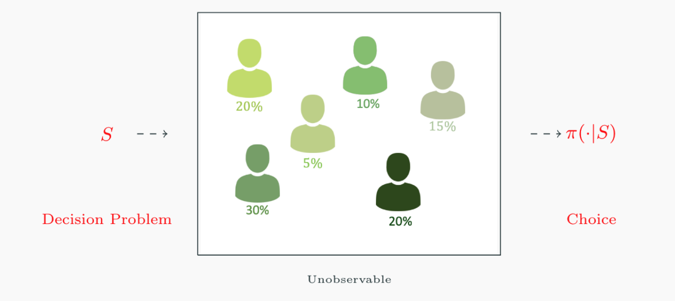
Unobservable

$\dashrightarrow \pi(\cdot|S)$

Choice

$\pi(x|S) =$ frequency of types choosing x from S

Random Utility Model (RUM)

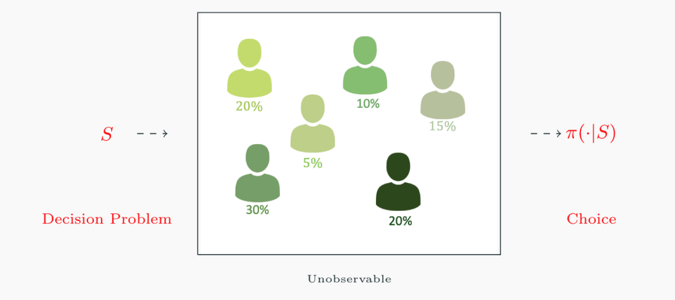


- RUM
  - each type is a utility maximizer
  - μ: probability distribution over all preference relations

$$\pi(x|S) = \sum_{x \text{ is } \succ-\text{best in } S} \mu(\succ)$$

  - a single individual vs a group of individuals
  - an important tool across fields

Limitations of RUM



- Each type must be “rational”
- Distribution of types is not unique

Motivation

- Random Utility Model
  - RUM: rational types $\Rightarrow$ non-unique
- Apesteguia, Ballester, Lu (ECMA, 2017)
  - SCRUM: rational types + "one dimensional heterogeneity" $\Rightarrow$ unique
- Filiz-Ozbay and Masatlioglu (JPE, 2023)
  - PRC: all types + "one dimensional heterogeneity" $\Rightarrow$ unique
  - L-PRC: less-is-more + "one dimensional heterogeneity" $\Rightarrow$ unique

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Today's Talk

- ??? $\Rightarrow$ unique
- what makes “one dimensional heterogeneity” special?
- some random observations

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Ordered Types aka “one dimensional heterogeneity”

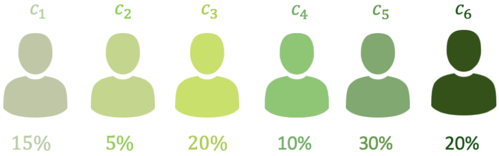
$\rhd$: the reference order

- e.g., policies ordered by being environmental friendly

$\{c_t\}$: Ordered types

- choice types ordered based on being environmentally conscious

less environmental ... more environmental



Progressive

Definition

A collection of choice functions is progressive with respect to $\triangleright$ if it can be sorted $\{c_1, c_2, \ldots, c_T\}$ such that $c_t(S) \trianglerighteq c_s(S)$ for all S and for any $t \geq s$.

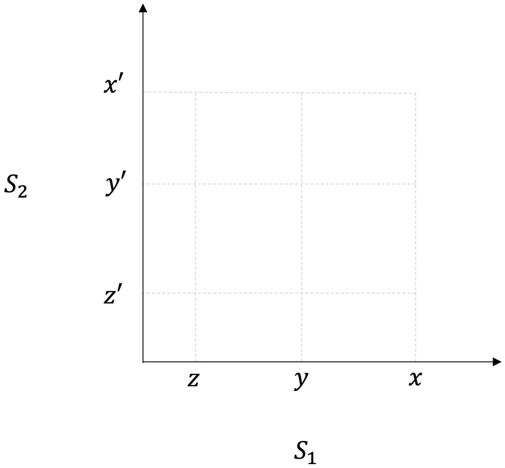
	c_1	c_2	c_3	c_4	c_5
$\{g, h, e\}$	g	g	h	h	e
$\{g, h\}$	g	h	h	h	h
$\{g, e\}$	g	g	g	e	e
$\{h, e\}$	h	h	h	h	e

$g \triangleleft h \triangleleft e$

How to Construct a Joint Distribution

- How to construct joint distribution from marginals?

How to Construct a Joint Distribution



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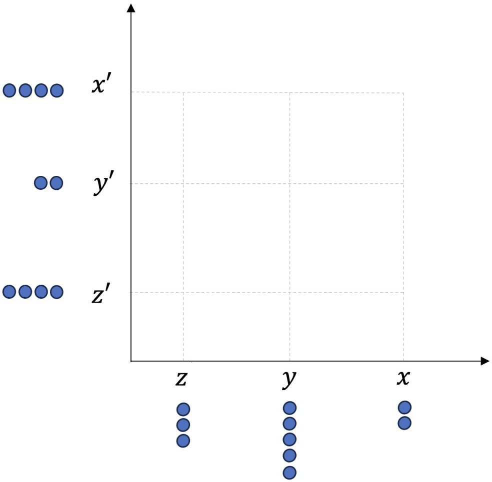

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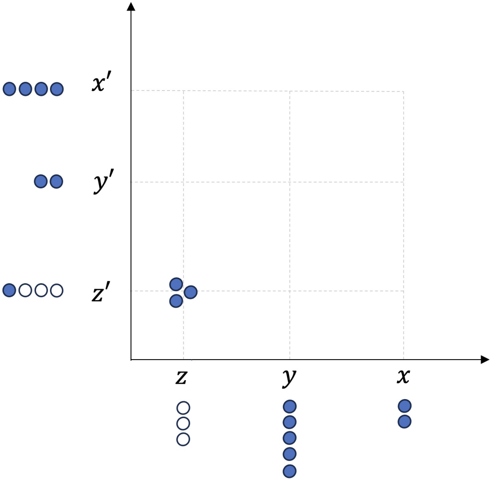
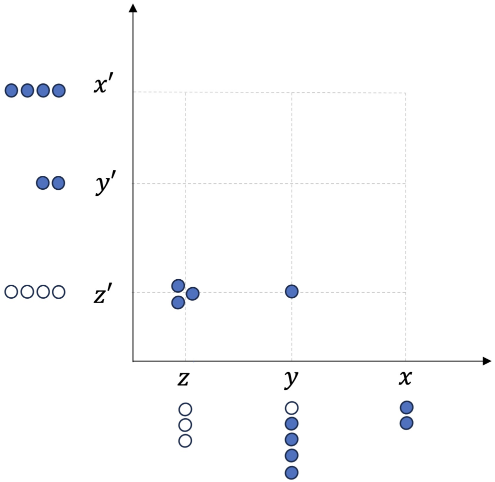
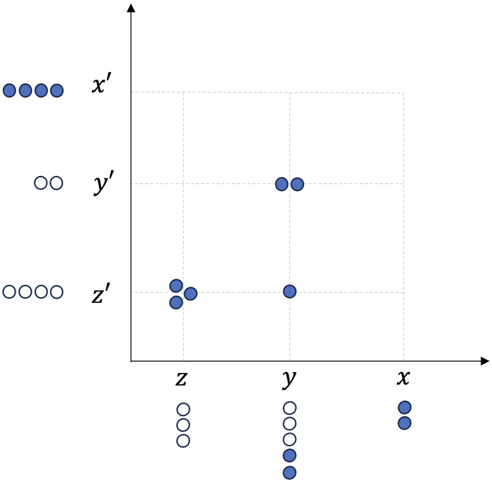



x'
y'
z'
z
y
x

How to Construct a Joint Distribution



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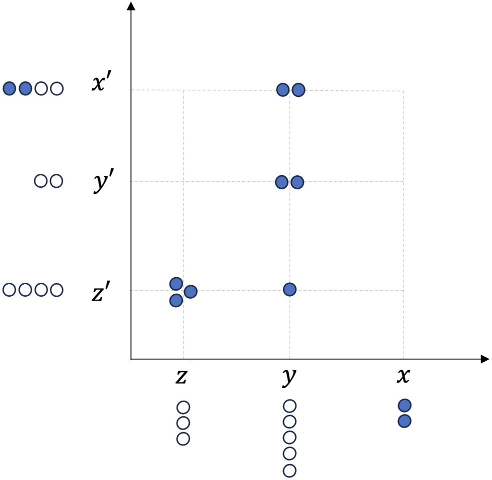

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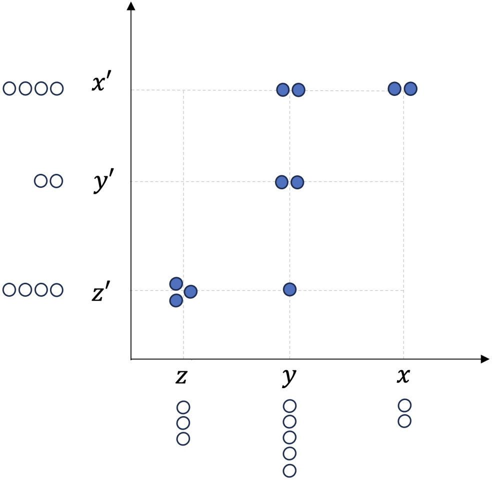

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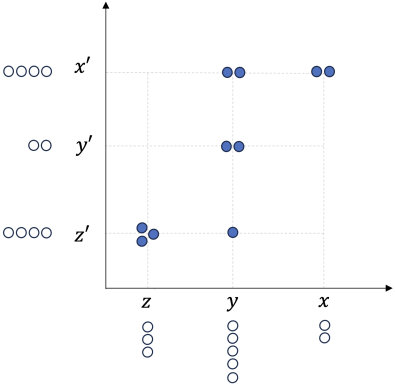
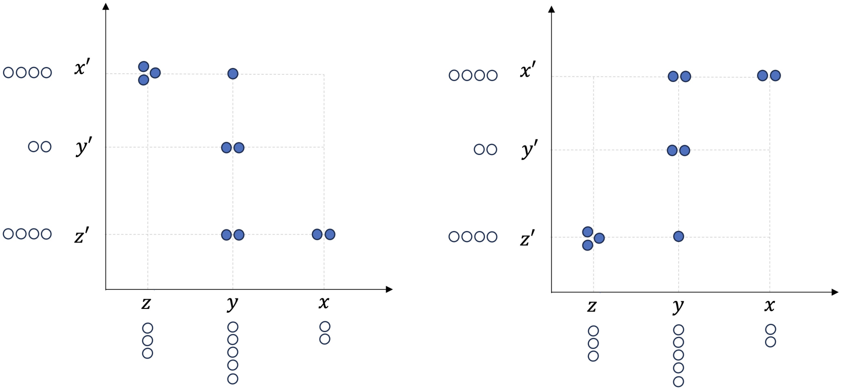


	c_1	c_2	c_3	c_4	c_5
S_1	z	y	y	y	x
S_2	z'	z'	y'	x'	x'
	0.3	0.1	0.2	0.2	0.2

How to Construct a Joint Distribution



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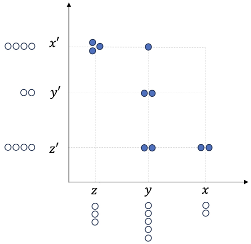
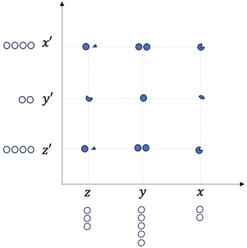
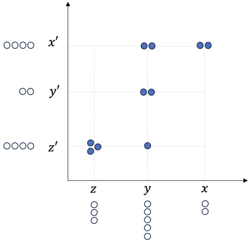


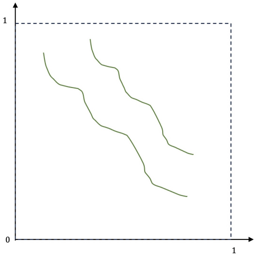




Copula

Copula

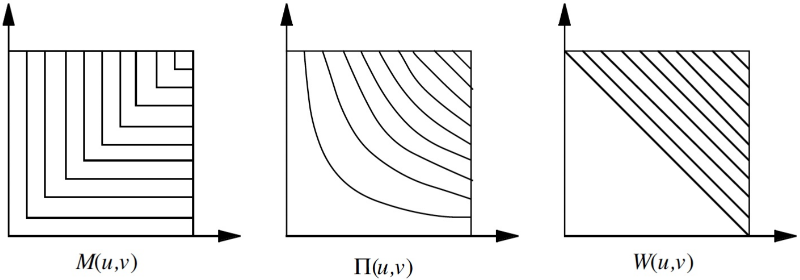
$C: [0,1]^2 \rightarrow \mathbb{R}$



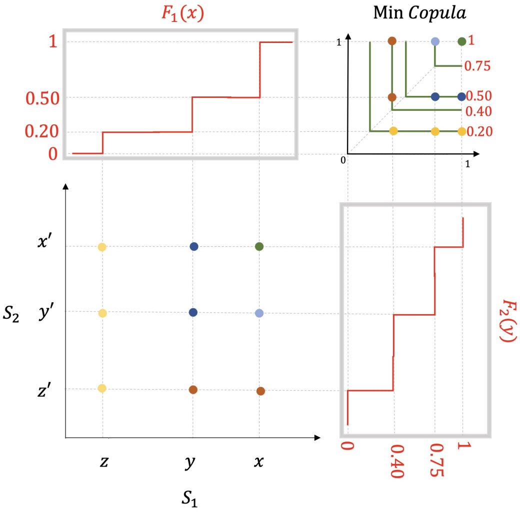
- $C(u, 0) = 0 = C(0, v)$
- $C(u, 1) = u$ and $C(1, v) = v$
- For any $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C\left(u_2, v_2\right) - C\left(u_2, v_1\right) - C\left(u_1, v_2\right) + C\left(u_1, v_1\right) \geq 0$$

- $M(u, v) = \min\{u, v\}$
- $\Pi(u, v) = uv$
- $W(u, v) = \max\{u + v - 1, 0\}$

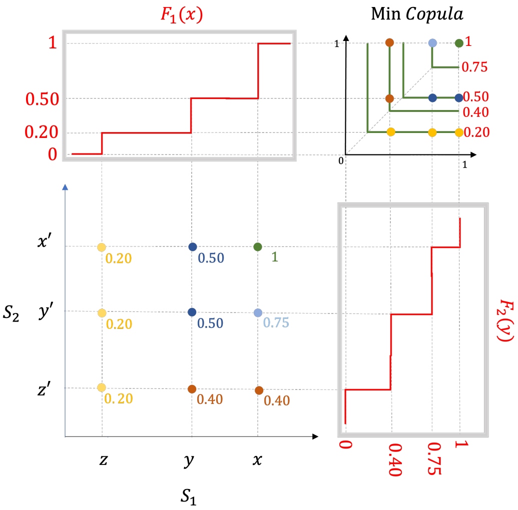


Min Copula



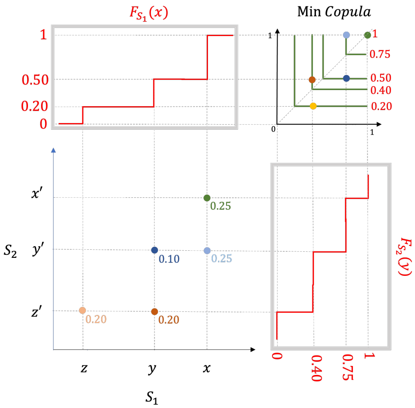
$F_1(x)$
1
0.50
0.20
0
Min *Copula*
1
0.75
0.50
0.40
0.20
0
1
x'
y'
z'
S_2
z
y
x
S_1
$F_2(y)$
0
0.40
0.75
1

Min Copula



$F_1(x)$
Min Copula
1
0.50
0.20
0
1
0.75
0.50
0.40
0.20
0
1
x'
y'
z'
S_2
0.20
0.50
1
0.20
0.50
0.75
0.20
0.40
0.40
z
y
x
S_1
$F_2(y)$
0
0.40
0.75
1

Min Copula



	c_1	c_2	c_3	c_4	c_5
S_1	z	y	y	x	x
S_2	z'	z'	y'	y'	x'
	0.2	0.2	0.1	0.25	0.25

Copula

- Define a function $C : [0, 1]^n \to \mathbb{R}$
- Three properties
  - $C(u_1, u_2, \cdots, u_n) = 0$ whenever any $u_i = 0$
  - $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ any i
  - any $n-$ dimensional cube $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, C must be non-negative
- Special ones
  - $M(u_1, u_2, \cdots, u_n) = \min\{u_1, u_2, \cdots, u_n\}$
  - $\Pi(u_1, u_2, \cdots, u_n) = \prod u_i$
  - $W(u_1, u_2, \cdots, u_n) = \max\{\sum u_i + 1 - n, 0\}$

Observation #1

- $M(u_1, u_2, \cdots, u_n) = \min\{u_1, u_2, \cdots, u_n\}$ is the identification method for
  - SCRUM
  - L-PRC
  - PRC
  - any kind of restriction + "one dimensional heterogeneity"
- Functional form!!!

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  - PRC
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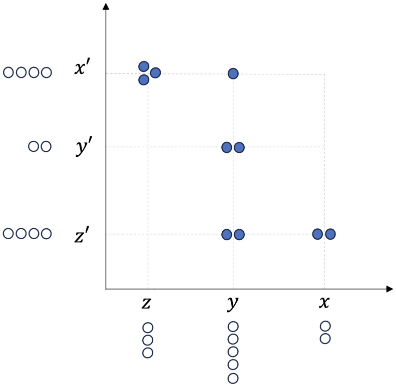
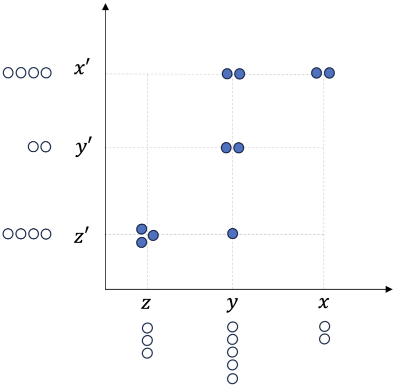
- Functional form!!!

Observation #2

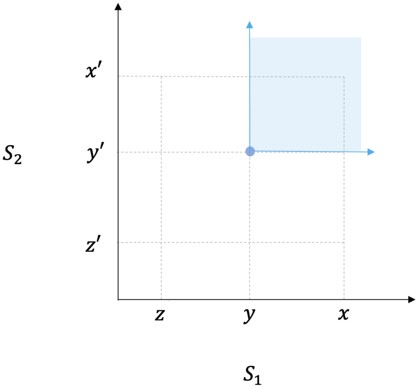
- Take any stochastic choice rule, π, within SCRUM
- Let μ^* be the SCRUM representation.
- Since SCRUM $\subset$ RUM, π might have other RUM representations $\{\mu_i\}$
- How $\{\mu_i\}$ and μ^* are related?

How to Compare Representations





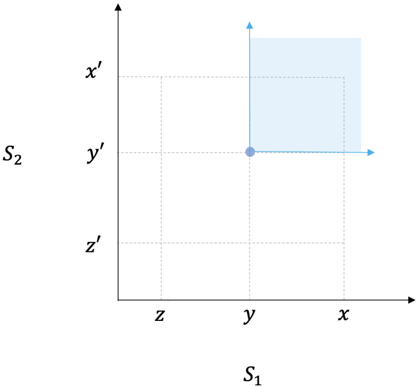
Dominance



	c
S_1	y
S_2	y'

Observation #2

- SCRUM assigns the highest weight to the choice type consistent with the underlying order.
- Given a particular "rational" choice type c, SCRUM assigns the highest weight to all "rational" choice types dominating c



	c
S_1	y
S_2	y'

Observation #3

- Take any stochastic choice rule, π, within RUM but outside of SCRUM
- PRC representation assigns the highest weight to the choice type consistent with the underlying order.

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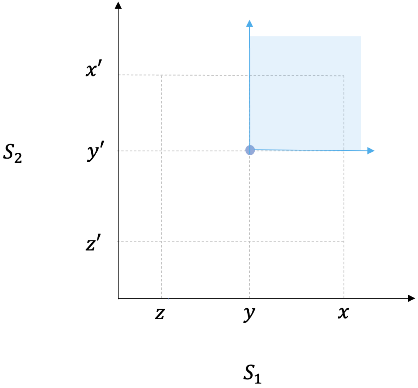

	c
S_1	y
S_2	y'

Proposition

Let π be a stochastic choice and μ be a probability distribution over choice types that generates π. Then, for each choice type c, the PRC representation of π assigns a higher probability to the set of types dominating c compared to μ.

Frechet-Hoeffding Bounds

Consider a copula C. Then

$$\max\{\sum_{i=1}^{n} u_i + 1 - n, 0\} \leq C(u_1, u_2, \cdots, u_n) \leq \min\{u_1, u_2, \cdots, u_n\}$$

What is next

- So what?
- the FH-upper bound gives us “one-dimensional heterogeneity”
- what about the FH-lower bound?

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HF-Lower Bound

- What about $W(u_1, u_2, \cdots, u_n) = \max\{\sum_{i=1}^n u_i + 1 - n, 0\}$?
- $W(u_1, u_2, \cdots, u_n)$ is not a copula in general
- The corresponding model must have empirical content!!!

1-mistake model

- Consider a set of individuals tries to maximize the reference order, $\triangleright$,
- Each deterministic type is
  - either completely rational (free of mistakes) or
  - makes one mistake in only one choice problem.

An example

- let $S_1 = \{x, y, z\}$, $S_2 = \{x, y\}$, and $S_3 = \{x, z\}$
- the underlying order is $x \rhd y \rhd z$
- nearly $\rhd$-optimal choice functions

	c_1	c_2	c_3	c_4	c_5
$\{x, y, z\}$	x	x	x	y	z
$\{x, y\}$	x	x	y	x	x
$\{x, z\}$	x	z	x	x	x

1-mistake model

Definition

A probabilistic choice function π has a 1-mistake representation with respect to $\triangleright$ if there exits a probability distribution μ over nearly $\triangleright$-optimal choice functions such that

$$\pi(x, S) = \sum_{c:c(S)=x} \mu(c)$$

1-mistake model

Axiom Let $\bar{s}$ be the $\triangleright$-best element in S. Then we have

$$\sum_S (1 - \pi(\bar{s}, S)) \leq 1$$

Characterization

Let π be a stochastic choice and $\bar{s}$ be the $\triangleright$-best element in S for all S. Then, π has 1-mistake representation if and only if $\sum_S (1 - \pi(\bar{s}, S)) \leq 1$.

- If ρ is a 1-mistake model, then ρ is identified by the FH-lower bound.

Observation #3

- $W(u_1, u_2, \cdots, u_n)$ is the identification method for 1-mistake model
- Representation is unique
- Given a particular choice type c, μ^* assigns the lowest weight to all choice types dominated by c

What next?

Other Identifying Methods

$$C_\alpha(u_1, u_2, \cdots, u_n) := \alpha M(u_1, u_2, \cdots, u_n) + (1-\alpha)W(u_1, u_2, \cdots, u_n)$$

$$C_t(u_1, u_2, \cdots, u_n) := \begin{cases} \max\{\sum_{i=1}^n u_i + 1 - n, t\} & \text{if } u_i \geq t \ \forall i \\ M(u_1, u_2, \cdots, u_n) & \text{otherwise} \end{cases}$$

$$C_k(u_1, u_2, \cdots, u_n) := W(M(u_1, u_2, \cdots, u_k), M(u_{k+1}, \cdots, u_k))$$

$$C(u_1, u_2, \cdots, u_n) := W(\cdots W(W(u_1, u_2), u_3), \cdots), u_n)$$

$$C_k(u_1, u_2, \cdots, u_n) := M(W(u_1, u_2, \cdots, u_k), W(u_{k+1}, \cdots, u_k))$$

Some Answers

- ??? $\Rightarrow$ unique
  Copula is the key
- what makes single-crossing/progressive special?
  Frechet-Hoeffding upper bound
- some random observations
  we had some

the end

References