BEHAVIORAL INFLUENCE*

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Abstract. In the context of stochastic choice, we introduce a model which admits a cardinal notion of interactive influence among individuals. The model presumes that individual choice is not only determined by idiosyncratic evaluations of the alternatives but also by the influence from the observed behavior of others. We establish that the equilibrium defined by the model is unique, stable and falsifiable. Moreover the underlying preference and influence parameters are uniquely identified from, arguably, limited data. The baseline model includes two individuals with conformity motives. Generalizations to multi-individual settings and negative interactions are also introduced and analyzed. Our analysis can be interpreted as providing an empirically meaningful foundation for network learning models.

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1. Introduction

Individual choices are directly influenced by the choices of others. Behavioral evidence on whether social interactions alter individual behavior is conclusive and indisputable.[1] We know that *it happens*. What is less clear is *how it happens*. How exactly does influence from others alter one's behavior? More importantly, viewing behavior as resulting from an unobservable cognitive process, how can we identify the extent to which one's behavior is attributed to social influence as opposed to one's own preferences?

This paper aims to provide a novel perspective to identification of social influence. We introduce a simple model of decision making for interacting individuals that enables inference of underlying unobserved parameters out of observable behavior. Our model is designed with two features in mind: First, the effect of social interactions on individual decisions is isolated from other individual factors. These other factors we classify under preferences. Second, without any prior information or assumption on the links between individuals, the underlying influence network is fully revealed by observable behavior.

Identification of peer influence from observable behavior has been one of the main goals of empirical social interactions research. In an early work, (Manski, 1993), Manski observed that in classical peer-influence models, there is an inherent non-identification problem. Manski refers to this non-identification as the *reflection problem* and it is now known as the "Manski critique." Ever since, a fundamental challenge has been to disentangle endogenous effects (direct interdependence between choices) from contextual effects (effects of predetermined social factors such as gender, age, race, etc. on individual behavior). This identification problem is a matter of simultaneous equations: Behavior of group members is itself determined by the behavior of other group

[1]Peer behavior has a significant influence not only on a student's school achievement (Calvo-Armengol et al., 2009), but also on social behavior such as consumption of recreational activities, drinking, smoking, etc. (Sacerdote, 2011). High productivity co-workers are found to increase one's own productivity (Mas and Moretti, 2009). Involvement in crime (Glaeser et al., 1996), job search (Topa, 2001), adolescent pregnancy (Case and Katz, 1991), college major choice (De Giorgi et al., 2010) are other prominent examples in which social interactions are shown to be crucial constituents of individual behavior.

members. Hence, data on outcomes on its own do not reveal whether group behavior actually affects individual behavior.

Most analyses of the problem have focused on econometric methods in parametric models. There is an extensive economic literature in this direction.[2] We contribute to this literature by taking a novel approach to the very same problem. Rather than ex-post estimation techniques, we focus on the micro-foundations of interactions. We adopt the tools of choice theory in order to identify social influence from observable behavior.

The novelty in our approach lies in the introduction of a new source of variation for network interaction models. Specifically, we imagine varying the set of available options from which individuals choose. Without any variation in the choice set, the reflection problem cannot be solved. However with minimal variation, *e.g.*, observations from two choice sets rather than one, it becomes possible to identify not only endogenous effects and preferences but also the structure of the underlying influence network. Our main contribution is to provide an intuitive and tractable model which affords a meaningful, and measurable, definition of "influence" as derived from choice behavior alone. Thus, our work provides a meaningful behavioral language for discussing social influence in an abstract framework.

Our model describes a simple procedure that individuals use to determine their choices. There are two essential parameters: An individual preference parameter and an individual influence parameter. The latter captures interdependence of behavior across individuals. The individual preference parameter is more standard. It can be interpreted as the intrinsic utility of the underlying alternatives; the subjective value of the alternative absent any social effects.[3] Social influence transpires through the *observed behavior* of the other individual(s). Individual behavior arises as the outcome of a weighted linear aggregation process. The subjective value of each alternative

[2]For a comprehensive review of early works on the identification of social interactions see Blume et al. (2011), for recent works see De Paula et al. (2019).

[3]In social interactions literature, the non-influence parameters that affect individual behavior are defined via types of variables such as predetermined social factors like gender, age, race, etc. Our model abstracts away from these effects, classifying them under the individual preference parameter.

is adjusted by a weighted version of the observed behavior of others regarding that alternative.

Our model can accommodate the idea that individuals are influenced differently by different agents.[4] Individual choice behavior reflects the relative attraction of each alternative in a given menu adjusted by the aggregate social influence. More precisely, the choice frequency of each alternative from a budget set is equal to the relative attraction of this alternative under social influence, with respect to all other available alternatives.

Linear aggregation is relatively standard in social interaction models. See, for example, (Blume et al., 2011, 2015), as well as naive learning models of social learning literature (DeGroot, 1974; Golub and Jackson, 2010). In our context, linearity is crucial. Because of the linearity of the adjustment process, the probabilistic behavior envisioned in the model can be also be understood as an aggregate model of deterministic behavior. Specifically, behavior of one agent is defined as a function of the choice frequencies by the other agent. This raises the question of what it means to observe and respond to a probabilistic behavior. As we show in section 2, the linearity of the adjustment procedure enables an interpretation of the world where each individual observes and responds to a deterministic choice by the other. Then our model simply summarizes the aggregate behavior at the equilibrium.

Following is a simple example demonstrating our identification strategy. It illustrates how we avoid the "reflection problem."

An example: Consider two colleagues, Dan and Bob, who potentially influence each other's choice of news sources. There are two different online news sources: BBC (B) and Daily Mail (D). Their browser histories suggest that Dan uses B approximately 71% of the time whereas this frequency is 78% for Bob, as summarized on the left panel of the table below. Assume that these browsing frequencies constitute the only information available to an outside observer, who aims to learn about the underlying preferences as well as peer influence over online news sources. Absent any further

[4]Aral and Walker (2012) investigate this heterogeneity over social media networks, Frey and Meier (2004) for prosocial behavior of university students and Glaeser et al. (1996) for criminal behavior.

information, the reflection problem remains. It is not possible to infer whether Dan and Bob have similar preferences and this is why they ended up with similar behavior or one of them is a strong influencer. However, when a new online news source, the Conversation (C) is launched, these frequencies change, as presented on the right panel:

	Dan	Bob	Dan	Bob
BBC	0.71	0.78	0.60	0.70
Conversation	-	-	0.14	0.11
Daily Mail	0.29	0.22	0.26	0.19

This pair of behaviors are consistent with our model, hence we can reveal the underlying preferences and the interaction parameters uniquely. Interestingly, our identification implies that although Dan and Bob's choice frequencies are aligned over the news sources, their idiosyncratic preferences are not aligned. For Bob, indeed the weight of B is the highest and D is the lowest, whereas for Dan, the exact opposite holds. However Bob's behavior has great influence on Dan. To be precise, the weights of B, C, D for Dan and Bob are $0.1, 0.3, 0.6$ and $0.8, 0.08, 0.12$, with interaction parameters 5 and 1, respectively. This means for Dan, Bob's behavior is five times more important then his own subjective evaluation, whereas for Bob they are equivalent. Thus strong conformity motives have resulted in the observed behavior. □

We establish in subsection 2.4 that our model is falsifiable, by providing its empirical content in terms of choice. Three behavioral properties are sufficient to characterize the model. All of these properties are built around a cross-elasticity type parameter that evaluates the relative rate of change in the individual choice frequency of an alternative as a response to a comparative change in the behavior of the other individual(s). In contrast to standard models of individual choicee, this influence parameter is derived from the choice behaviors of *all* of the individuals jointly as opposed to the behavior of only one individual. Hence, these characterizing properties are entirely novel.[5]

[5] A natural question is whether *any* individual choice behavior could be rationalized by the presence of some unobserved individual's influence. While this question is certainly interesting, it is tangental to what we are doing. Our model postulates a given, observable set of individuals, and tests *whether* these individuals' behavior is in line with our predictions. This is much in the same spirit as the theory of consumer choice. Afriat (1967) characterizes the empirical content of such choice, but Varian (1988) shows that, in principle, if some commodities are unobservable, then any behavior can be rationalized.

The parameters of our model define an "equilibrium," where the choice behavior of each individual is a function of idiosyncratic utility and influence parameters; as well as the behavior of the other individual(s). Unlike many other discrete choice models (Brock and Durlauf, 2001; Blume et al., 2011), the equilibrium defined by our model is unique. Moreover, it is also stable, in the sense that a dynamic adjustment procedure always tends to this unique equilibrium. Before two individuals start interacting, we have no particular reason to suppose that their behavior conforms to our model. We show under reasonably general conditions that, through time, as each individual responds to the other's choices via the linear adjustment procedure, the predictions of our model will be borne out. In other words, if we believe that each individual aggregates behaviorally according to our procedure, we should expect their behavior to conform to our model in the long-run. There are two critical implications of this result. The first implication is more practical: if one individual mistakenly chooses, or one of them misobserves the other's choices at some period in time, their behavior will still revert to the predictions of our model in the long run. Second and more importantly, identification of the underlying parameters from dynamic data is also possible. Then in the absence of equilibrium choice behavior, we can use a similar identification strategy over consecutive choice data. Subsection 2.5 elaborates on this.

Our baseline model involves two individuals with conformity motives, as in the example above. An action's choice probability increases as the action is chosen more frequently by one's peer. However our model easily adapts to more individuals and accommodates other types of interaction. We present two simple extensions. The first incorporates multi-individual interaction, where individuals have different degrees of influence on the behaviors of their peers, and second "negative" influence. Despite its simplicity, our model is versatile enough to capture a wide range of social phenomena involving interactions. Let us provide a few applications to exemplify this.

Homophily: Homophily refers to the tendency to create social ties with people that are similar to oneself (McPherson et al., 2001; Blackwell and Lichter, 2004; Currarini et al., 2009). Since both homophily and peer-influence result in behavioral resemblance among peers, an identification problem arises. For instance, consider a group of high school students with a tendency towards delinquent behavior. The interpretation is twofold: It might be the case that all of these kids have high aspirations towards criminal behavior, and that is why they hang out together. On the other hand, it might instead be the case that conformity motives with one (or some) influential members have resulted in this group behavior. Diagnosing the correct interaction dynamic is imperative for effective treatment of the issue. Since our model allows for unique and full identification of the underlying parameters, it enables the differentiation of homophily (similar underlying preferences) from peer-influence (high conformity parameters), as long as the observed behavior satisfies the characterizing properties of our model.

Deidentification among siblings: A fascinating topic of research amongst adolescence researchers and psychologists is how different siblings are from each other despite their common genetic make-up and environment (Eckstein et al., 2010). Many common inter-sibling differences in behavioral patterns and personalities have been documented. For example, higher academic achievements of first born children (Black et al., 2005) as opposed to better athletic achievements of second-born children (Hopwood et al., 2015). One of the channels through which siblings end up with different behavioral patterns is the process of "deidentification". This refers to a choice of different paths by the siblings for the sake of differentiating themselves, especially performed by the second-born in order to avoid sibling competition (Schachter et al., 1976; Sulloway, 2010). Our model accommodates this process simply by the use of negative interaction coefficients.

Social norms: One of the most fundamental concepts in the study of social influence is that of social norms. Thanks to the versatility of our model, we can accommodate different attitudes towards social norms in our framework. We can treat the behavior induced by social norms as the behavior of an exogenous "hypothetical" individual. In this case, different levels of compliance with social norms can easily be captured by different individual interaction parameters. Alternatively, the formation of social norms can also be modeled in our framework, in the dynamic setting of subsection 2.5. In that case, we can treat social norms as a hypothetical individual, but this time with a high level of dependence on the behavior of the others. The equilibrium then pictures a society with established and stable social norms and yet different levels of compliance with them.

We provide three distinct and well-known social influence settings where the behavior produced by our model can be reproduced under certain assumptions. We refer to these as 'foundational justifications' for our model since each of them can be seen as an economic mechanism underlying our model of influence. The first of those is utility maximization in a discrete choice setting with peer effects, whereas the second one incorporates strategic interactions, introducing a simultaneous game setting whose Quantal Response Equilibrium happens to coincide with our model. The last mechanism is a basic naive learning set up as in DeGroot (1974). All of these models are distinguished from our model as we use menu variability in our setting. Our model is a stochastic choice model that assumes consistent behavior across all budget sets. Critically, this menu variability grants us unique identification. In other words behavior consistent with our model can be traced back to underlying preference and influence parameters uniquely. Moreover, our identification strategy does not suffer from a common handicap of identification in revealed preference or decision-theoretic models: arguably unrealistic data requirements. Many choice theoretic models require a rich dataset, typically individual choices from all menus, for identification purposes.[6] As we show in Subsection 2.3, observations from only two menus are sufficient for unique identification for our baseline model, involving two individuals. For identification of influence networks involving more than two individuals, observations from two menus can still be sufficient as long as there are sufficiently many alternatives in the menus. We elaborate more on this in Section 3.

The organization of the paper is as follows. The next subsection is devoted to literature review. Section 2 presents a detailed analysis of the baseline model with two individuals with conformity motives, including identification, falsifiability and stability results. Section 3 introduces the generalization to multi-individual settings, whereas

[6]For a recent exception to this common trend as well as a discussion on the topic, see Dardanoni et al. (2020).

Section 4 incorporates negative influence to these settings. Finally, we conclude. All proofs are left to an appendix.

1.1. **Related Literature.** Economics research on identification of social interactions has mainly utilized econometrics tools and techniques in order to separate contextual effects from endogeneous effects. Most of these studies employ *linear social interaction models* (Manski, 1993; Blume et al., 2011; Jackson, 2011; Blume et al., 2015), where individual utility of an action is defined as a linear additive function with two components: an individual private utility and a social utility. Blume et al. (2015) provide micro-foundations to these linear interaction models by showing that under certain parametric assumptions they can be reproduced as the Bayesian-Nash equilibrium of an incomplete information game where individuals choose an action to maximize their expected utility given their type and the public types of others.

Linear social interaction models are defined for continuous choice variables. An alternative to this is developed by incorporating the linear additive utility function with interaction effects into a discrete choice setting (Blume, 1993; Brock and Durlauf, 2001, 2006). Binary or multinomial discrete choice models with social interactions make use of random fields models to study the equilibrium. Three critical assumptions ensure tractability of the model. First, the assumption of constant strategic complementarity: the cross-partial of social utility is a positive constant that is the same for all individuals. Second, rational expectations: the expected average behavior is simply the objective average behavior. Finally, the error terms follow a relevant extreme value distribution. These assumptions are sufficient to produce individual choice outcomes that are consistent with logistic choice with multiple equilibria. The majority of these papers assume large populations in order to justify the assumption that each individual ignores the effect of their own choice on the average choice of the society. An exception to this is Soetevent and Kooreman (2007), where they consider interaction in small groups in which choices of other individuals is fully observable. Thus, the choice of an individual directly depends on the observed behavior of the others. Our model also uses this intuition. Indeed, under certain assumptions the behavior produced by a multinomial discrete choice model with social interactions coincides with the behavior produced by our baseline model. This requires a different error distribution then the one commonly assumed for those works. We clarify this connection in subsection 2.2.

In this strand of literature social interactions has typically been taken to be generated by group specific averages. Incorporating network theory in the study of identification of social interactions has enabled a much richer analysis of the microstructure of interactions. Early works on this assumed a *known* network structure, based on common observables or self-reported, elicited data (Bramoullé et al., 2009; Lee et al., 2010; De Giorgi et al., 2010). However both of these methods bear shortcomings for econometric methods or practical reasons related to collecting data (De Paula, 2017). A first improvement on this was suggested by Blume et al. (2015) by assuming only partial information on the structure of the underlying network. De Paula et al. (2019) advances on this by assuming no a priori information on the network structure and provides sufficient conditions for full identification of social interactions with panel data. Our paper is complementary to this literature since our general model also encompasses an influence network, where the structure of the relations do not need to be known a priori. Instead it is fully revealed by the behaviors thanks to our identification strategy.

It is important to note that almost all theoretical models of peer influence are restricted by strategic complementarity: individual utility over an action increases with the number of peers taking the action. However empirical evidence points out to negative interactions as well. For instance, Glaeser et al. (1996) suggests the existence of negative interactions among criminals due to competition for "resources"; Conley and Topa (2002) note the presence of negative correlations in unemployment rates of individuals when ethnic differences are large. Cont and Lowe (2010) suggest fashions and fads as an example of negative interactions. As we show in an extension, our model is flexible enough to capture negative interactions.

The use of choice theoretic tools to study social interactions is quite recent. As far as we know the first choice-theoretic work investigating influence across individuals is Cuhadaroglu (2017). This work introduces a deterministic model of two-stage optimization where the first stage involves maximization of own preferences (transitive but not necessarily complete), and the second stage accommodates social influence to further refine first stage outcomes. Recently, Borah and Kops (2019) propose a choice procedure in a group setting that makes use of 'a consideration set' approach. According to their model, individuals only consider those alternatives that are chosen sufficiently enough by the members of their reference group. Then, in a second stage, they choose their personal best out of those considered. The main difference of our work from this model is about the channel through which others' behavior influence the individual. Our model presumes that social influence alters one's preferences, whereas Borah and Kops' model assume a limitation of the choice set due to social influence.[7]

Fershtman and Segal (2018) also consider a social interaction set up where individual behavior not only depends on one's own preferences but also on the behavior of other agents. Each individual possesses a private vNM utility and a perfectly observable vNM utility. A social influence function converts the private utility of the agent and the observable utilities of everyone else to an observable utility for the agent. They study certain properties of social influence functions and their implications for the equilibrium without proposing an explicit behavioral model.

Finally, our work is related to the literature discussing the revealed preference implications of solution concepts in games; for example, Sprumont (2000); Lee (2012). One interpretation of the mathematics of our model is as formalizing, for each choice set, a game and a solution concept. Thus, our model provides observable predictions of our concept as strategy sets vary. The aforementioned papers also study the predictions of game theory as strategy sets vary.

[7]The traditional assumption of exogenous and fixed individual preferences has also been frequently challenged by economic theorists over the last couple of decades; see, for instance, Bowles (1998); Bisin and Verdier (2001); Bar-Gill and Fershtman (2005); Fehr and Hoff (2011); Doepke and Zilibotti (2017). Many findings from social psychology or experimental economics literatures support the notion that social influence alters one's preferences. For instance, Kremer and Levy (2008) show that alcohol consumption by one roommate is more likely to influence the alcohol consumption of another roommate via a preference change rather than a modification of the choice set. Kenrick and Gutierres (1980) show that individual evaluations of physical attractiveness of random people are directly altered by evaluations from peers. According to the notion of (mis)identification in social psychology, when some alternatives become identified with certain identities, they become more likely to be preferred by aspiring individuals, whereas despising individuals avoid them in order not to be misindentified (Berger, 2016).

2. Behavioral Influence

2.1. **The Model.** Let X be a finite set of *alternatives* with $|X| > 2$. There are two individuals, 1 and 2. A *stochastic choice rule* is a map $p : 2^X \setminus \{\emptyset\} \to \bigcup_{E \in X} \Delta_{++}(E)$ such that for all $E \subseteq X$, $p(E) \in \Delta_{++}(E)$.[8]

We propose a simple model of influence. Each individual is influenced by the choices of the other individual. The observable behaviour is a pair of stochastic choice rules (p_1, p_2) where p_i stands for individual i's choices. We use the notation $i, j \in \{1, 2\}$ with $i \neq j$ for the individuals in general.

The primitives of our setting are idiosyncratic weights and influence parameters. Let $w_i : X \to (0, 1)$ with $\sum_{x \in X} w_i(x) = 1$ measure the idiosyncratic weight of the available alternatives for individual i. These can be interpreted as intrinsic utilities of the alternatives absent any social influence effects. We postulate that the choice behavior of individual j regarding an alternative $x \in S$ directly influences individual i's evaluation of that alternative for the same choice set. Specifically individual i now adjusts her evaluation of alternative x to $w_i(x) + \alpha_i p_j(x, S)$, where α_i measures the degree of influence of j on i. For the baseline model, we assume that $\alpha_i \geq 0$, hence α_i acts as a conformity parameter. The higher the probability that j chooses x from S, the higher is i's evaluation of x in S. The choice frequency of x is equal to the relative evaluation of x in S with respect to all other available alternatives. Formally:

Definition. (p_1, p_2) has a **dual interaction** representation if there exist two functions $w_1, w_2 : X \to (0, 1)$, with $\sum_{x \in X} w_1(x) = \sum_{x \in X} w_2(x) = 1$ and $\alpha_1, \alpha_2 \in \Re^+$ such that

$$p_i(x, S) = \frac{w_i(x) + \alpha_i p_j(x, S)}{\sum_{y \in S} [w_i(y) + \alpha_i p_j(y, S)]} \tag{1}$$

for all $x \in S, S \in 2^X \setminus \emptyset$ and $i, j \in \{1, 2\}$ with $j \neq i$.

When (p_1, p_2) has a dual interaction representation with parameters $(w_1, w_2, \alpha_1, \alpha_2)$, we say that $(w_1, w_2, \alpha_1, \alpha_2)$ *represent* (p_1, p_2).

[8]The notation Δ_{++} refers to the set of probability distributions with full support.

At the essence of the dual interaction model lies the premise that an action by an individual is the outcome of underlying preferences as well as the influence from the observed actions of the other individual. In abstract terms,

$$p_i = f_i(w_i, p_j),$$

where f_i refers to the cognitive mechanism that aggregates preferences and influence, in our case the map given in equation (1). Observe that *influence* is then the idea that as p_j changes, say to p'_j, we would expect a corresponding change in p_i to p'_i, where p'_i is closer to p'_j and p_i is closer to p_j. This is precisely the intuition that equation (1) captures.

Our model refers to a specific f_i that uses a weighted linear aggregation mechanism over probabilistic actions. The mathematics of the model might be interpreted in several ways. In the most basic, each individual directly observes and responds to the probabilistic choice of the other. However, the dual interaction model also affords an interpretation of frequency of deterministic choices. More specifically, i responds to a deterministic choice by j upon each observation, yet the aggregate behavior at the equilibrium is stochastic, as it corresponds to the choice frequencies.

Let us elaborate on this. For notational simplicity, consider $S = \{x, y\}$. Then a deterministic choice of x by j can be expressed as $(1, 0)$.[9] Whenever i observes j choosing x, she takes the action given by $f_i(w_i, (1, 0))$ and so on. Since j chooses x with a frequency of $p_j(x, S)$, the overall choice probabilities are indeed given by

$$p_i(S) = (p_i(x, S), p_i(y, S)) = p_j(x, S)f_i(w_i, (1, 0)) + p_j(y, S)f_i(w_i, (0, 1))$$

Critically, owing to the linearity of our model, this is indeed equivalent to:

$$p_i(S) = f_i(w_i, p_j(S))$$

[9]Let us note that although we restrict our attention to strictly positive stochastic choice rules (hence considered $w_i(\cdot) \in (0, 1)$), it is possible to extend the model to allow $w_i(\cdot) \in [0, 1]$. In this case two additional properties dealing with 0 probabilities are required for characterization of the model. Although this is a rather straightforward extension, the proof becomes tedious, hence we choose the restricted setting. The proof is available upon request.

To see this explicitly, notice that for our model f_i corresponds to:

$$f_i(w_i,(p_j(x,S),p_j(y,S)) = \left(\frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(S)+\alpha_i}, \quad \frac{w_i(y)+\alpha_i p_j(y,S)}{w_i(S)+\alpha_i}\right)$$

$$f_i(w_i,(1,0)) = \left(\frac{w_i(x)+\alpha_i}{w_i(S)+\alpha_i}, \quad \frac{w_i(y)}{w_i(S)+\alpha_i}\right)$$

$$f_i(w_i,(0,1)) = \left(\frac{w_i(x)}{w_i(S)+\alpha_i}, \quad \frac{w_i(y)+\alpha_i}{w_i(S)+\alpha_i}\right)$$

Our model defines an equilibrium where given $(w_1, w_2, \alpha_1, \alpha_2)$, each p_i is defined implicitly by the procedure in equation (1). Note that p_1 is not explicitly defined: p_2 needs to be known in order to determine p_1 and vice versa. However we can obtain an explicit representation by solving the system of simultaneous equations, arriving at:

(2) $$p_i(x,S) \equiv \lambda_i(S)\frac{w_i(x)}{\sum\limits_{x\in S} w_i(x)} + (1-\lambda_i(S))\frac{w_j(x)}{\sum\limits_{x\in S} w_j(x)}$$

for $\lambda_i(S) \in (0,1)$, defined explicitly below. Hence each p_i can be expressed as a linear combination of their Luce ratios, where the weights in the combination depend on S.[10]

Equation 2 helps to explain why we think of α_i as a measure of influence. The stochastic choice of i from choice set S is, geometrically, a convex combination of i's Luce choices and j's Luce choices. As α_i increases, this combination tends to be closer to j's Luce choices.

Observe this is "as if" each individual knows exactly not only her own intrinsic utilities but also those of the other individual, which are not necessarily observable. Notice that in our original formulation, each individual utilizes each others' observable choice behavior rather than their unobservable Luce weights. We believe influence based on observed behavior rather than an unobserved parameter is behaviorally and procedurally more plausible. Nevertheless, this explicit formulation provides more

[10]A stochastic choice rule p has a Luce representation if there exists a weight function $w: X \to (0,1)$ with $\sum\limits_{x\in X} w(x) = 1$ such that $p(x,S) = \dfrac{w(x)}{\sum\limits_{y\in S} w(y)}$ for all $x \in S, S \in 2^X \setminus \emptyset$. This ratio of relative weights is known as the "Luce ratio."

insight about the model. Here, the weight attached to each individual's Luce ratio depends on the budget set. That is,

$$\lambda_i(S) = \frac{w_i(S)[w_j(S) + \alpha_j]}{w_i(S)w_j(S) + \alpha_i w_j(S) + \alpha_j w_i(S)}$$

where $w_i(S) = \sum_{x \in S} w_i(x)$. So, the parameter $\lambda_i(S)$ is decreasing in α_i and increasing in α_j. In other words, the more influenced by the other person the more weight attached to other individual's Luce ratio. In the extreme case, when $\alpha_i = 0$, $\lambda_i(S)$ is equal to 1, independent of the budget set, and the model boils down to standard Luce model.[11]

Another important implication of this formulation is about uniqueness of the behavior produced, which is not obvious from the equilibrium description of the model. Since (p_1, p_2) can explicitly be expressed as functions of the preference parameters, for a given $(w_1, w_2, \alpha_1, \alpha_2)$, there is a unique pair (p_1, p_2) consistent with the dual interaction model.

2.2. **Foundations.** Why does the dual interaction model make sense as a decision procedure that incorporates social influence? We provide three different foundational justifications below. In other words, we provide three different mechanisms that produce behavior consistent with the dual interaction model. Each environment differs from the classical stochastic choice setting. To this end, we strip the menu-richness of the choice argument away and focus on a single budget set, say X. We suppress the menu dependence in the notation of this subsection. All of the following can be reproduced for any menu S.

The first mechanism we introduce reproduces dual interaction behavior in a discrete choice setting as the outcome of utility maximization, whereas the second one incorporates strategic interactions. The main link between these two and our model is built around the use of a logistic distribution. However as we show in the third mechanism,

[11]It is worth noting that p_i consistent with dual interaction model does not satisfy IIA, the characterizing property of Luce model, ($\frac{p_i(x,S)}{p_i(y,S)} = \frac{p_i(x,T)}{p_i(y,T)}$ for all S, T and $x, y \in S \cap T$) in general; it only does so when $\alpha_i = 0$ or $\alpha_i \to \infty$. In the former, there is no ifluence, hence i behaves according to w_i, whereas in the latter, i fully mimics j.

the logistic set-up is dispensable. This last part introduces a simple naive learning mechanism that also reproduces dual interaction behavior in the limit.

2.2.1. *Random Utility with linear social interactions:* The standard econometric tools to study social interactions include discrete choice models with peer effects (Blume, 1993; Brock and Durlauf, 2001, 2006). These models regard individual utility as a linear additive function of observed and unobserved individual characteristics as well as social influence. Under the assumption of i.i.d extreme value unobserved characteristics, utility maximization yields choice frequencies as a function of individual characteristics and social influence. The dual interaction model can also be reproduced in a multinomial discrete choice setting. Two specific assumptions are sufficient to achieve this: a logarithmic transformation of the utility and a relevant extreme value distribution. To see how this works, assume a multiplicative form for individual utility as follows:

$$U_i(x) = V_i(x)\varepsilon_i(x) \quad \text{where } V_i(x) = w_i(x) + \alpha_i p_j(x)$$

Under the assumption that the disturbances are i.i.d. with a Log-logistic distribution (i.e., $\log \varepsilon_i$ follows a Type 1 extreme value distribution) with $f(\log \varepsilon_i) = e^{-\log \varepsilon_i} e^{-e^{-\log \varepsilon_i}}$, maximization of log-utility yields:

$$\begin{aligned}
\log U_i(x) &= \log V_i(x) + \log \varepsilon_i(x) \\
p_i(x) &= Prob\left(\log V_i(x) + \log(\varepsilon_i(x)) > \log V_i(y) + \log(\varepsilon_i(y)), \;\; \forall y \neq x\right) \\
&= Prob\left(\log \varepsilon_i(y) < \log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right), \;\; \forall y \neq x\right)
\end{aligned}$$

Then for a given $\varepsilon_i(x)$, using $F(\log \varepsilon_i)$:

$$Prob\left(x|\varepsilon_i(x)\right) = \prod_{y \neq x} \exp\left\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)}\right\}$$

which leads to:

$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_{y \neq x} \exp\left\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)}\right\}\right) e^{-\log \varepsilon_i} \exp\{-e^{-\log \varepsilon_i}\} d\log(\varepsilon_i)$$

$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_y \exp\Big\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} \right) e^{-\log \varepsilon_i} d\log(\varepsilon_i)$$

The second line above is observed by collecting terms in the exponent of e given that $\dfrac{V_i(x)}{V_i(x)} = 1$.

$$\begin{aligned} p_i(x) &= \int_{-\infty}^{+\infty} \exp\Big\{ - \sum_y e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} e^{-\log \varepsilon_i} d\log(\varepsilon_i) \\ &= \int_{-\infty}^{+\infty} \exp\Big\{ - e^{-\log \varepsilon_i} \sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} e^{-\log \varepsilon_i} d\log(\varepsilon_i) \end{aligned}$$

Apply a transformation of variables as $t = e^{-\log(\varepsilon_i(x))}$ such that $dt = -e^{-\log(\varepsilon_i(x))} d\log(\varepsilon_i)$. Note that as $\log(\varepsilon_i)$ approaches infinity, t approaches zero, and as $\log(\varepsilon_i)$ approaches negative infinity, t becomes infinitely large.

$$\begin{aligned} p_i(x) &= \int_{\infty}^{0} - \exp\Big\{ - t \sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} dt \\ &= \int_{\infty}^{0} - \exp\Big\{ - t \sum_y \frac{V_i(y)}{V_i(x)} \Big\} dt \\ &= \left. \frac{e^{-t\frac{\sum V_i(y)}{V_i(x)}}}{\frac{\sum V_i(y)}{V_i(x)}} \right|_{\infty}^{0} = \frac{V_i(x)}{\sum_y V_i(y)} = \frac{w_i(x) + \alpha_i p_j(x)}{\sum_y (w_i(y) + \alpha_i p_j(y))} \end{aligned}$$

Thus, a logarithmic transformation of the individual utility and a relevant extreme value distribution for the error terms in a discrete choice setting with social interactions lead to the behavior described by the dual interaction model.

2.2.2. *Logit Quantal Response Equilibrium:* In a similar fashion to the preceding part, our model appears conceptually related to Quantal Response Equilibrium (QRE), which is a solution concept for normal form games (McKelvey and Palfrey, 1995). And indeed, it is possible to reproduce the behavior granted by our model as a logit

QRE. To see this, consider a normal form game with two players 1 and 2, with $S = S_1 \times S_2 = X \times X$ as the set of strategy profiles and s_i represents a pure strategy for player i. Let Σ_i denote the set of probability distributions over S_i and an element $\sigma_i \in \Sigma_i$ is a mixed strategy, and $\sigma_i(s_i)$ is the probability that player i chooses pure strategy s_i with Σ as the set of mixed strategy profiles. The pay-off functions $u_i : S \to \Re$ are such that $u_i(x, y)$ represents the utility of player i when player 1 consumes x and player 2 consumes y. In particular, assume that $u_1(s) = u_1(x, y) = w_1(x) + \alpha_1 \mathbf{1}\{x = y\}$ and $u_2(s) = u_2(x, y) = w_2(y) + \alpha_2 \mathbf{1}\{y = x\}$. In other words each player receives a consumption utility $w_i(x)$ and additional utility α_i when their consumptions match.

Hence, for each mixed-strategy profile $\sigma \in \Sigma$, player i's expected payoff is $u_i(\sigma) = \sum_{s \in S} \sigma_i(s)\sigma_j(s)u_i(s)$ and the expected payoff for adopting the pure strategy s when the other player uses σ is $u_i(s, \sigma_j) = \sum_{s' \in S_j} \sigma_j(s')u_i(s, s') = \sigma_j(s)(w_i(s) + \alpha_i) + (1 - \sigma_j(s))w_i(s) = w_i(s) + \alpha_i\sigma_j(s)$.

Now, similar to the mechanism introduced in 2.2.1, under the assumption that $U_i(s, \sigma) = u_i(s, \sigma)\varepsilon_{is}$ with $u_i(s, \sigma_j) = w_i(s) + \alpha_i\sigma_j(s)$ and i.i.d. log-logistic errors ε_{is}, the QRE outcome coincides with (p_1, p_2) of the dual interaction model. The derivation is exactly the same, hence omitted.

2.2.3. *Naive learning with anchors:* Naive learning over social networks (also known as the DeGroot model) envisions a non-Bayesian updating of individual beliefs by repeatedly taking weighted averages of one's neighbors' beliefs (Golub and Jackson, 2010). It is well-know that for any group of strongly connected (each pair of individuals is directly or indirectly connected) and closed group of individuals, aperiodicity of the transition matrix is necessary and sufficient for the convergence of beliefs to a unique consensus (DeGroot, 1974). The behavior produced by the dual interaction model can also be reproduced in a particular naive learning setting. In our case, convergence obviously will not entail consensus. This is because w_1 and w_2 act as anchors for the individuals although they keep getting influenced by each other's observed behavior.

In a DeGroot setting, each agent $n \in N$ has a belief $p_i(t) \in [0,1]$ at time $t \in \{0,1,2,...\}$. Given the stochastic interaction matrix $T_{n\times n}$, where T_{ij} captures the influence of agent j on i, the updating rule is simply $p(t) = Tp(t-1) = T^t p(0)$, where $p(\cdot)$ stands for the vector of beliefs of all agents. Now, to see the relationship to our model let $N = 2$ -although the extension to the n individual case is immediate. Let,

$$p(0) = \begin{pmatrix} w_1(x) & w_2(x) & p_1^0(x) & p_2^0(x) \end{pmatrix}'$$

for some alternative $x \in X$, where $p_i^0 \in [0,1]$ is any initial behavior.[12] Note that we only focus on one alternative to keep things simple. The same can be done for all alternatives in the menu. The transition matrix T

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{1}{1+\alpha_1} & 0 & 0 & \dfrac{\alpha_1}{1+\alpha_1} \\ 0 & \dfrac{1}{1+\alpha_2} & \dfrac{\alpha_2}{1+\alpha_2} & 0 \end{pmatrix}$$

will result in

$$p(1) = Tp(0) = p(1) = \begin{pmatrix} w_1(x) & w_2(x) & \dfrac{w_1(x) + \alpha_1 p_2^0(x)}{1+\alpha_1} & \dfrac{w_2(x) + \alpha_2 p_1^0(x)}{1+\alpha_2} \end{pmatrix}.$$

As we also prove in subsection 2.5, in the limit $p(t) = T^t p(0)$ indeed converges to a $\begin{pmatrix} w_1(x) & w_2(x) & p_1^*(x) & p_2^*(x) \end{pmatrix}$ with $(p_1^*(x), p_2^*(x))$ as defined by the dual interaction model.

Overall, these three settings indicate that behavior postulated by our model can be justified by an underlying utility maximization as well as a simple learning mechanism. The main difference of our model lies in the menu variability of our setting. Our model

[12] Friedkin and Johnsen (1990) suggest a generalization of the DeGroot model where updating at each period also involves agents' initial beliefs. They also show convergence to a non-consensus state. The slight difference with the dynamic version of our model, as we examine in subsection 2.5 is that, for their model the initial behavior is equal to the initial belief. Instead we show convergence to the behavior dictated by our model for any p_0.

is a stochastic choice model that assumes consistent behavior across menus. Critically this menu variability grants us unique identification of the underlying unobserved parameters as we show next.

2.3. **Identification.** Assume we observe (p_1, p_2) consistent with the dual interaction model. How can we identify the underlying preference and interaction parameters?

A powerful feature of our model is that our identification strategy requires observation of behavior from only two menus: The universal set X and any menu S that has at least two distinct alternatives, say x and y. To see how, first define for each $i = 1, 2$, for any pair (x, S) with $x \in S$, $d_i : (x, S) \mapsto \Re$, by

$$d_i(x, S) := p_i(x, S) - p_i(x, X).$$

The quantity $d_i(x, S)$ is simply the change in the probability of i's choosing x as the set X shrinks to S. When $\alpha_i = 0$, this change is always nonnegative.[13] In a larger set, there are more alternatives from which to choose. In the dual interaction model, this change instead is governed by two separate effects. First, there is the individual effect. A larger set includes more alternatives, rendering any given alternative relatively less attractive. In addition, there is also a social influence effect imposed by the change of the other individual's choice probability, $d_j(x, S)$. Since $\alpha_i > 0$, as the set enlarges, this indirect effect contributes to the loss in choice probability of any given alternative. Let us decompose $d_i(x, S)$ into these two effects explicitly for the model:

$$\begin{aligned} d_i(x, S) =& p_i(x, S) - p_i(x, X) \\ =& \frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S) + \frac{w_i(S) + \alpha_i}{1 + \alpha_i} p_i(x, S) - \frac{1 + \alpha_i}{1 + \alpha_i} p_i(x, X) \\ =& \frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S) + \frac{w_i(x) + \alpha_i p_j(x, S)}{1 + \alpha_i} - \frac{w_i(x) + \alpha_i p_j(x, X)}{1 + \alpha_i} \\ =& \underbrace{\frac{1 - w_i(S)}{1 + \alpha_i} p_i(x, S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{1 + \alpha_i} d_j(x, S)}_{\text{social influence}} \end{aligned}$$

[13] Indeed this refers to the well-known Regularity property.

The third line follows from the description of the model. Notice what is captured by the individual counterpart. In Luce's model, this loss is equal to $d(x,S) = \hat{p}(x,S) - \hat{p}(x,X) = \dfrac{w(x)}{w(S)} - w(x) = (1 - w(S))\hat{p}(x,S)$, where $\hat{p}(x,S)$ is the corresponding Luce probability. In our decomposition the individual counterpart captures a similar effect, but weighted by $1/(1+\alpha_i)$.

We will make use of this decomposition to infer α_i. One way of achieving this is to make use of a normalization and the decomposition of $d_i(y,S)$ to cancel out the individual counterparts. To this end, take an alternative $y \in S \setminus \{x\}$ and normalize both of the decompositions by the respective observed probabilities as follows:

$$\frac{d_i(x,S)}{p_i(x,S)} = \frac{\frac{1-w_i(S)}{1+\alpha_i} p_i(x,S)}{p_i(x,S)} + \frac{\frac{\alpha_i}{1+\alpha_i} d_j(x,S)}{p_i(x,S)}$$

$$\frac{d_i(y,S)}{p_i(y,S)} = \frac{\frac{1-w_i(S)}{1+\alpha_i} p_i(y,S)}{p_i(y,S)} + \frac{\frac{\alpha_i}{1+\alpha_i} d_j(y,S)}{p_i(y,S)}$$

The difference between these two expressions yields:

$$\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} = \frac{\alpha_i}{1+\alpha_i}\left[\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}\right] \tag{3}$$

revealing α_i uniquely whenever there exists (x,y,S) such that $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \neq 0$.

For the inference of $w_i(x)$, we simply make use of the description of the model for choices from X, yielding:

$$w_i(x) = p_i(x,X) + \alpha_i(p_i(x,X) - p_j(x,X)).$$

Obviously each $w_i(x)$ is identified uniquely with $\sum_X w_i(x) = 1$. Let us state these results in a proposition for completeness purposes.

Proposition 1. *Let (p_1, p_2) have a dual interaction representation. Then $(w_1, w_2, \alpha_1, \alpha_2)$ that represent (p_1, p_2) are identified uniquely.*

Identification above relies on the availability of data from two sets, the universal set X and any other menu S with at least two alternatives. This begs the question whether it is possible to do any inference when choices from X are not available? Indeed it is possible to identify the underlying preference and interaction parameters from any two sets S, T as long as they have at least two common elements, say x, y, although the identification strategy gets slightly more complicated. To see how, let any two distinct sets S, T with $x, y \in S \cap T$ and $S \cup T = X$ and reproduce equation (3) for any two such S, T:

$$\begin{aligned}
d_i(x,S,T) =& p_i(x,S) - p_i(x,T) \\
=& \frac{w_i(T)-w_i(S)}{w_i(T)+\alpha_i} p_i(x,S) + \frac{w_i(S)+\alpha_i}{w_i(T)+\alpha_i} p_i(x,S) - p_i(x,T) \\
=& \frac{w_i(T)-w_i(S)}{w_i(T)+\alpha_i} p_i(x,S) + \frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(T)+\alpha_i} - \frac{w_i(x)+\alpha_i p_j(x,T)}{w_i(T)+\alpha_i} \\
=& \underbrace{\frac{w_i(T)-w_i(S)}{w_i(T)+\alpha_i} p_i(x,S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{w_i(T)+\alpha_i} d_j(x,S,T)}_{\text{social influence}}
\end{aligned}$$

Normalizing the decompositions for distinct $x, y \in S$ and taking the difference will result in:

$$\frac{d_i(x,S,T)}{p_i(x,S)} - \frac{d_i(y,S,T)}{p_i(y,S)} = \underbrace{\frac{\alpha_i}{w_i(T)+\alpha_i}}_{\gamma_i(x,y,S,T)} \left[\frac{d_j(x,S,T)}{p_i(x,S)} - \frac{d_j(y,S,T)}{p_i(y,S)} \right]$$

Thus two identifying equations are:

$$\gamma_i(x,y,S,T) = \frac{\alpha_i}{w_i(T)+\alpha_i} \text{ and } \gamma_i(x,y,T,S) = \frac{\alpha_i}{w_i(S)+\alpha_i}.$$

Unlike the case with data from X, we now have one too many parameters for unique identification only from γ_is. The third identity we need comes from the normalization assumption $w_i(X) = 1$. Yet as the behavior from X is not observed, we need to decompose it consistently over S and T. Since $w_i(S) + w_i(T \setminus S) = 1$, by definition of the model $w_i(x) = [\alpha_i + w_i(S)]p_i(x,S) - \alpha_i p_j(x,S)$ yields:

$$w_i(T \setminus S) = [\alpha_i + w_i(T)] \sum_{x \in T\setminus S} p_i(x,T) - \alpha_i \sum_{x \in T\setminus S} p_j(x,T) = 1 - w_i(S),$$

resulting in the last equation sufficient for unique identification combined with the two above.

2.4. **Falsifiability.** For identification we assumed a pair of choice behaviors (p_1, p_2) consistent with the dual interaction model. We now need to express explicitly how one can detect the consistency of the data with the model. In other words for given (p_1, p_2), which properties of these behaviors ensures that these two individuals are behaving as if they are choosing according to dual interaction model?

We have three falsifiable characterizing properties built around the decomposition of $d_i(x,S)$ into individual and social counterparts as we have used in subsection 2.3. Specifically, for any $S \neq X$ and $x \in S$, $d_i(x,S)$ is composed of two counterparts: the individual effect (as there are more options in X than S for i's attraction) and the social influence effect (same goes for j's attraction).

Our characterizing properties build on the premise that one can eliminate the unobserved individual effects for $x \in S$ by cancelling them out with those of $d_i(y,S)$ for some distinct $y \in S$. The remainder will then be a function of the social influence effect. Specifically, it will be a linear function. Formally, take any S and $x, y \in S$ with $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \neq 0$ and define $\beta_i(x,y,S)$ as follows:

$$\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} = \beta_i(x,y,S)\left[\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}\right] \tag{4}$$

Three properties that impose conditions on these two variables $\beta_1(x,y,S)$ and $\beta_2(x,y,S)$ are sufficient for the characterization of the dual interaction model.

Independence [*I*]. *$\beta_i(x,y,S)(:= \beta_i)$ is independent of S, x, y. Moreover β_i satisfies (4) for all $S \neq X$ and distinct $x, y \in S$.*

Uniform Boundedness [*UB*]. *$\beta_i(x, y, S) < \min_{z \in X} \left\{ \frac{p_i(z, X)}{p_j(z, X)} \right\}$ for all S, and distinct $x, y \in S$.*

Non-negativeness [*Nn*]. *$\beta_i(x, y, S) \geq 0$ for all S, and distinct $x, y \in S$.*

Independence is the property that restores the additive linear influence structure among individuals. $\beta_i(x, y, S)$ is defined for all those observations with a non-zero $\frac{d_j(x, S)}{p_i(x, S)} - \frac{d_j(y, S)}{p_i(y, S)}$. The first part of Independence ensures that $\beta_i(x, y, S)$ is indeed constant across observations, hence defining β_i. The second part of Independence guarantees that this β_i satifies equation (4) even for those observations with $\frac{d_j(x, S)}{p_i(x, S)} - \frac{d_j(y, S)}{p_i(y, S)} = 0$. Uniform Boundedness guarantees that idiosyncratic evaluations of alternatives, w_i are positive. And finally, Non-negativeness restricts the interaction among individuals to conformity behavior rather than diversification.

The characterization result is stated for pairs of stochastic choice rules with some variation in the overall behavior, i.e., $p_1 \neq p_2$. This is because one cannot learn much from the data when $p_1 = p_2$. Having exactly the same behavior in any choice set might be due to identical preferences of 1 and 2, i.e, $w_1 = w_2$; or it might be because one of the individuals only cares about imitating the other individual. It is not possible to distinguish between these cases without any additional information, such as their choice behavior in isolation.

Theorem 1. *Let $p_1 \neq p_2$. Then (p_1, p_2) has a **dual interaction** representation with $\alpha_i \geq 0$ and $w_i \gg 0$ for each i if and only if it satisfies Independence, Uniform Boundedness, and Non-negativeness.*

The proof constructs the model thanks to the structure granted by Independence and by the help of restrictions imposed by the remaining two axioms. We take $\alpha_i(x, y, S) := \alpha_i = \frac{\beta_i}{1 - \beta_i}$ (well-defined by the first two properties and non-negative by the latter two) and $w_i(x) := p_i(x, X) + \alpha_i(p_i(x, X) - p_j(x, X))$ (positive by Uniform Boundedness). We then show that for any S and $x, y \in S$, Independence builds up to

$$\frac{p_i(x, S)}{p_i(y, S)} = \frac{w_i(x) + \alpha_i p_j(x, S)}{w_i(y) + \alpha_i p_j(y, S)}.$$

The fact that this holds for each pair of alternatives immediately gives us the dual interaction model.

2.5. **Stability.** The dual interaction model involves an adjustment procedure where an individual's evaluation of an alternative is adjusted by the other's behavior as well as the level of susceptibility to influence. We now embed this adjustment procedure in a dynamic setting, where individuals start interaction from possibly unrelated behaviors. Specifically let (p_1^t, p_2^t) denote the behaviors of 1 and 2 at period $t > 0$ and assume that their initial behaviors (p_1^1, p_2^1) are given. One can think of new roommates or teenagers just enrolled in a new school as examples. Below we show that although these individuals start interacting from possibly unrelated behaviors, as long as they adjust consistently, eventually they converge to (p_1^*, p_2^*), the unique pair of behaviors that the model yields for the given set of parameters. In other words, the behavior produced by the dual interaction model constitutes a stable equilibrium when embedded in a dynamic environment.

Theorem 2. *Take $w_i \gg 0$, $\alpha_i \geq 0$, $p_i^*(S) \in \Delta_{++}(S)$ for all $S \in 2^X \setminus \{\varnothing\}$ and for each $i \in \{1, 2\}$ and let $(w_1, w_2, \alpha_1, \alpha_2)$ represent (p_1^*, p_2^*). Further, let $(p_1^1, p_2^1) \in \Delta(S) \times \Delta(S)$. Define for each $i \in \{1, 2\}$ and $t \geq 2$, $p_i^t(\cdot, S) \in \Delta(S)$ via*

$$p_i^t(x, S) \equiv \frac{w_i(x) + \alpha_i p_j^{t-1}(x, S)}{\sum_{y \in S} w_i(y) + \alpha_i p_j^{t-1}(y, S)}.$$

Then for each $i \in \{1, 2\}$, $\lim_{t \to \infty} p_i^t = p_i^*$.

An interesting implication of this dynamic environment involves identification. Although the observed behavior changes over time, because it changes in a consistent way, our identification strategy still holds for the underlying preference and interaction parameters $(w_1, w_2, \alpha_1, \alpha_2)$. Similar to the static setting, the data requirement is minimal: only choice behavior from two different sets need be observed. However, since now observations are from different time periods, inference of α_i demands data from two successive periods. To see how, let us reproduce equation (3) for this dynamic environement. Take any $S \neq X$ with $x, y \in S$ such that $\dfrac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \dfrac{d_j^{t-1}(y,S)}{p_i^t(y,S)} \neq 0$ and let:

$$\beta_i(x,y,S) = \frac{\dfrac{d_i^t(x,S)}{p_i^t(x,S)} - \dfrac{d_i^t(y,S)}{p_i^t(y,S)}}{\dfrac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \dfrac{d_j^{t-1}(y,S)}{p_i^t(y,S)}}.$$

Then,
$$\begin{aligned} d_i^t(x,S) =& p_i^t(x,S) - p_i^t(x,X) \\ =& \frac{1-w_i(S)}{1+\alpha_i} p_i^t(x,S) + \frac{w_i(S)+\alpha_i}{1+\alpha_i} p_i^t(x,S) - \frac{1+\alpha_i}{1+\alpha_i} p_i^t(x,X) \\ =& \frac{1-w_i(S)}{1+\alpha_i} p_i^t(x,S) + \frac{w_i(x)+\alpha_i p_j^{t-1}(x,S)}{1+\alpha_i} - \frac{w_i(x)+\alpha_i p_j^{t-1}(x,X)}{1+\alpha_i} \\ =& \underbrace{\frac{1-w_i(S)}{1+\alpha_i} p_i^t(x,S)}_{\text{individual}} + \underbrace{\frac{\alpha_i}{1+\alpha_i} d_j^{t-1}(x,S)}_{\text{social influence}} \end{aligned}$$

Then, the difference between normalized decompositions for distinct $x, y \in S$ yields:

$$\frac{d_i^t(x,S)}{p_i^t(x,S)} - \frac{d_i^t(y,S)}{p_i^t(y,S)} = \frac{\alpha_i}{1+\alpha_i}\left[\frac{d_j^{t-1}(x,S)}{p_i^t(x,S)} - \frac{d_j^{t-1}(y,S)}{p_i^t(y,S)}\right]$$

Hence, we have

$$\beta_i(x,y,S) = \frac{\alpha_i}{1+\alpha_i}$$

as before. Identification of $w_i(x)$ is achieved via:

$$w_i(x) = (1+\alpha_i)p_i^t(x,X) - \alpha_i p_j^{t-1}(x,X).$$

We conclude the analysis of our baseline model by stating this identification result.

Proposition 2. *Let $(p_1^{t-1}, p_2^{t-1}, p_1^t, p_2^t)$ such that for each $i \in \{1,2\}$ and $p_i^t(\cdot,S) \in \Delta(S)$*

$$p_i^t(x,S) \equiv \frac{w_i(x) + \alpha_i p_j^{t-1}(x,S)}{\sum_{y\in S} w_i(y) + \alpha_i p_j^{t-1}(y,S)}.$$

Then $(w_1, w_2, \alpha_1, \alpha_2)$ that represent (p_1, p_2) are identified uniquely.

3. Multi-agent Interaction

One of the strengths of our model is that it is easily generalizable to multi individual settings with more intricate forms of social interactions. We can easily capture the heterogeneities that drive different behavioral outcomes in a social context. Not only individuals have different preferences but they also have different levels of susceptibility to influence. Or similarly, different people might influence an individual in different ways. The generalization of our model to multi individual settings allow for these variations, by providing a complementary approach to the identification of social interactions over social networks. In particular, it allows the identification of a weighted social network from choice behavior.

Early works on social networks have assumed known network structure, based on common observables or self-reported, elicited data (Bramoullé et al., 2009; Lee et al., 2010; De Giorgi et al., 2010), that is rather costly to collect (De Paula, 2017). A first improvement on this was suggested by Blume et al. (2015) by assuming only partial information on the structure of the underlying network. De Paula et al. (2019) advances on this by assuming no a priori information on the network structure and provides sufficient conditions for full identification of social interactions with panel data. Similarly, our generalized model do not require any exogenous network structure. On the contrary, our representation theorem reveals the unknown network of social influence in addition to individual preferences and influence patterns. Specifically, given the behavior of a group of individuals that is consistent with our characterizing properties, we can *uniquely* identify the underlying preferences, represented by w_i, and the interaction patterns, represented by α_{ij}, capturing how individual i is influenced by the behavior of individual j for all pairs of individuals i and j. Note that the interaction between i and j might be asymmetric, i.e., α_{ij} need not be equal to α_{ji}.

Let us now formally introduce the multi individual model. Let N denote a set of $n < +\infty$ individuals interacting. As before, for each choice problem, $S \in 2^X \setminus \emptyset$, we observe agent i's stochastic choice, $p_i(x, S)$. Let $\boldsymbol{p}_{-i}(x, S) \in \Re^{n-1}$ denote the vector of $p_j(x, S)$ and $\boldsymbol{d}_{-i}(x, S) \in \Re^{n-1}$ the vector of $d_j(x, S)$ for all $j \neq i$.

Definition. $(p_1, p_2, ..., p_n)$ has a **social interaction** representation if for each $i \in N$ there exist $w_i : X \to (0,1)$ with $\sum\limits_{x \in X} w_i(x) = 1$ and $\boldsymbol{\alpha}_i \in \Re_+^{n-1}$ such that

$$p_i(x, S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x, S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y, S)]}$$

for all $x \in S$ and for all S.

The parameter $\boldsymbol{\alpha}_i$ captures different levels of susceptibility to influence from different individuals, i.e., agent i can be influenced differently by different j's. Let α_{ij} denote the entry of $\boldsymbol{\alpha}_i$ relating to the influence of individual j on i. If $\alpha_{ij} = 0$ for all $j \neq i$, once again i's choice behavior reduces down to Luce.

The identification strategy and the characterizing properties are similar to those of the baseline model. Notice that for any $S \neq X$, and any two distinct $x, y \in S$, now there might be multiple vectors $\boldsymbol{\gamma}_i \in \Re^{n-1}$ satisfying the following equation:

$$\boldsymbol{\gamma}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x, S)}{p_i(x, S)} - \frac{\boldsymbol{d}_{-i}(y, S)}{p_i(y, S)} \right) = \frac{d_i(x, S)}{p_i(x, S)} - \frac{d_i(y, S)}{p_i(y, S)}. \tag{5}$$

We will be interested in the ones that satifsy it for all observations.

$$\mathcal{B}_i = \{\boldsymbol{\gamma}_i \in \Re^{n-1} \, | \boldsymbol{\gamma}_i \text{ solves (5) for any } S \text{ and distinct } x, y \in S\}$$

The first characterizing property ensures that $\mathcal{B}_i$ is nonempty, hence there is at least one solution to the system of equations given by (5) for all S and $x, y \in S$. The last one puts bounds on it.

N-Independence [*N-I*]. *$\mathcal{B}_i$ is nonempty.*

N-Positive Uniform Boundedness. [***N-PUB***] *For all $z \in X$, $p_i(z, X) > \boldsymbol{\gamma}_i \cdot \boldsymbol{p}_{-i}(z, X)$ for some $\boldsymbol{\gamma}_i \in \mathcal{B}_i$ with $\boldsymbol{\gamma}_i \in \Re_+^{n-1}$.*

N-Independence implies that there exists a vector, say $\boldsymbol{\beta}_i$, that satisfies (5) independent of S, x, y. As before, $\boldsymbol{\alpha}_i$ is to be identified from $\boldsymbol{\beta}_i$. Specifically, $\alpha_{ij} = \dfrac{\beta_{ij}}{1 - \sum\limits_{j \neq i} \beta_{ij}}$.

However, unique identification requires more than two observations this time, simply because there are more unknowns now. Indeed, equation (5) has $(n-1)$ unknowns, α_{ij} for each $j \neq i$. Hence, the number of linearly independent equations required to solve the system is $(n-1)$. Notice that this does not mean we necessarily need data from $(n-1)$ different menus. All that is required is $(n-1)$ observations; data from two different menus is sufficient as long as there are at least $(n-1)$ common pairs of alternatives in these two menus.

Unique identification of the underlying preferences is then achieved via

$$w_i(x) = p_i(x, X) + \sum_{j \neq i} \alpha_{ij}[p_i(x, X) - p_j(x, X)]. \tag{6}$$

Theorem 3. *Let $\{p_i\}_{i \in N}$. Then, $\{p_i\}_{i \in N}$ has a **social interaction** representation if and only if N-Independence and N-Positive Uniform Boundedness hold.*

As before, the equilibrium defined by the model always exists and is unique. Moreover, when embedded in a dynamic adjustment process, as in subsection 2.5, the limit behavior happens to be the equilibrium defined by our model. The following theorem formalizes these.

Theorem 4. *Take $w_i \gg 0, \alpha_{ij} \geq 0$ for all $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. Then, there is a unique $(p_1^*, \ldots, p_N^*) \in \Delta_{++}(S)^N$ for which*

$$p_i^*(x, S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^*(x, S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^*(y, S)]}$$

and for any $(p_1^1(\cdot|S), \ldots, p_N^1(\cdot|S)) \in \Delta_{++}(S)^N$, the iterative map

$$p_i^t(x, S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^{t-1}(x, S)}{\sum_{y \in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}^{t-1}(y, S)]}$$

converges to $(p_1^*, \ldots, p_N^*)$.

4. Negative Interactions

Most of the theoretical tools developed to study social interactions are restricted by strategic complementarity or conformity type assumptions. This is because they only focus on positive interactions, where the individual payoff of an action increases the more it is chosen by one's peers. However in certain contexts, where individuals especially do not want to behave similarly, negative interactions are in play. An intuitive example to this is fashions and fads. A trend setter happens to be the one that initially behaves differently than everyone else. The choice of a fashion product not only signals which social group you would like to identify with but also signals who you would like to differentiate from (Pesendorfer, 1995). Among criminals competition for resources governs the need for negative interactions (Glaeser et al., 1996). Conley and Topa (2002) note the presence of negative correlations in unemployment rates of individuals when ethnic differences are large.

The versatility of the dual interaction model allows us to extend it to capture negative interactions in a rather straightforward way. We refer to a *negative interaction* between i and j as the following phenomenon. Consider our benchmark model, with two individuals i and j, and a pair (x, S) with $x \in S$. We can imagine two hypothetical behaviors from individual j, say $p_j(x, S)$ and $q_j(x, S)$, where $p_j(x, S) > q_j(x, S)$. *Negative interaction* refers to the property that if $p_i(x, S) = \dfrac{w_i(x) + \alpha_i p_j(x, S)}{w_i(S) + \alpha_i}$ and $q_i(x, S) = \dfrac{w_i(x) + \alpha_i q_j(x, S)}{w_i(S) + \alpha_i}$, then $p_i(x, S) < q_i(x, S)$. In other words, as j increases their propensity to choose x from S, i decreases her propensity in response. Notice that a negative α_i does not necessarily imply negative interaction.[14] Thus, we employ a simple reparametrization of the model in order to avoid confusion. For the two-agent model, let $\delta_i \equiv \dfrac{1}{1 + \alpha_i}$, and observe that

$$p_i(x, S) = \frac{\delta_i w_i(x) + (1 - \delta_i) p_j(x, S)}{\delta_i w_i(S) + 1 - \delta_i}.$$

We maintain the premise that w_i remains a "weight of choice" absent any influence, so we hypothesize that $\delta_i > 0$. Observe that the case of $\alpha_i \geq 0$ corresponds to $\delta_i \leq 1$.

Now with negative interactions in play, existence and stability are not straightforward implications of the model. In particular certain parametric restrictions are required

[14]Crucially, for values of $\alpha_i < 0$, whenever $w_i(S) < |\alpha_i|, p_i(x, S)$ puts a negative weight on $w_i(x)$ and a positive weight on $p_j(x, S)$, quite contrary to the essence of negative interactions.

to ensure that the linear aggregation procedure defines a probability, and the model remains meaningful for the dynamic adjustment procedure. As a first observation, suppose that $p_j(x,S) = 0$. It follows that

$$p_i(x,S) = \frac{\delta_i w_i(x)}{\delta_i w_i(S) + 1 - \delta_i}.$$

This term will be non-negative and well-defined exactly when $\delta_i w_i(S) + 1 - \delta_i > 0$. So, this is the first necessary condition for the dynamic adjustment procedure to make sense. We can suppose that it holds for all S, including S which are singleton. As a second observation, suppose that $p_j(x,S) = 1$. It follows that $p_i(x,S) = \frac{\delta_i w_i(x) + 1 - \delta_i}{\delta_i w_i(S) + 1 - \delta_i}$. Given that $\delta_i w_i(S) + 1 - \delta_i > 0$, this term will be nonnegative exactly when $\delta_i w_i(x) + 1 - \delta_i \geq 0$. Observe next that if $\delta_i w_i(x) + 1 - \delta_i \geq 0$ for each x, then for any S for which $|S| \geq 2$ and any $x \in S$, we have $\delta_i w_i(S) + 1 - \delta_i > \delta_i w_i(x) + 1 - \delta_i \geq 0$, so that the first inequality is redundant.

Our goal is to generalize this observation to the multi-agent situation, and to establish that these conditions are sufficient for stability and convergence of the iterative procedure. To this end, recall our general model, whereby for each $i, j \in N$ with $i \neq j$, α_i^j is the influence that j exerts on i. We can now define

$$\delta_i \equiv \frac{1}{1 + \sum_{j \neq i} \alpha_{ij}},$$

so long as $1 + \sum_{j \neq i} \alpha_{ij} \neq 0$, and by defining

$$\delta_{ij} \equiv \frac{\alpha_{ij}}{1 + \sum_{j \neq i} \alpha_{ij}},$$

we obtain a representation via:

$$p_i(x,S) = \frac{\delta_i w_i(x) + \sum_{j \neq i} \delta_{ij} p_j(x,S)}{\delta_i w_i(S) + \sum_{j \neq i} \delta_{ij}}.$$

Observe that $\delta_i + \sum_{j \neq i} \delta_{ij} = 1$. In the case of non-negative interaction, $(\delta_i, \boldsymbol{\delta}_{-i})$ is a probability measure over N; a member of $\Delta(N)$, for which $\delta_i > 0$. In this more general environment, it is simply a vector which sums to one.

Now, we may perform the same exercise as in the binary case. We want to ensure that, no matter what is p_{-i} or (x,S), that $p_i(x,S)$ is a well-defined probability. To this end, we can first imagine that $p_{-i}(x,S)=\mathbf{0}$. Then $p_i(x,S)=\dfrac{\delta_i w_i(x)}{\delta_i w_i(S)+\sum_{j\neq i}\delta_{ij}}$. Consequently, we establish that as $\delta_i>0$, it follows that $\delta_i w_i(S)+\sum_{j\neq i}\delta_{ij}>0$. Secondly, we can suppose that $p_{-i}(x,S)=\mathbf{1}_{\{j:\delta_{ij}<0\}}$. This then implies that $p_i(x,S)=\dfrac{\delta_i w_i(x)+\sum_{j\neq i}\min\{0,\delta_{ij}\}}{\delta_i w_i(S)+\sum_{j\neq i}\delta_{ij}}$. We must ensure that

$$\delta_i w_i(x)+\sum_{j\neq i}\min\{0,\delta_{ij}\}\geq 0$$

for all x, and in particular, for every S, that there exists some x for which the inequality is strict. In particular, it is only a slight loss of generality to assume that the inequality is strict *for every* x. Observe then that we have the sufficient condition that for all x:

$$\delta_i w_i(x)+\sum_{j\neq i}\min\{0,\delta_{ij}\}>0. \tag{7}$$

Equation (7) is a sufficient and almost necessary condition for the iterative procedure to always result in a probability measure. It is necessary and sufficient for the iterative procedure to always map any probability measure into a full-support probability measure.

Equation (7) has a very simple interpretation. Recall that the numerator of the expression defining choice reflects the relative propensity to choose. Equation (7) requires that this propensity to choose be positive, independently of the choices of others.

Definition. $(p_1,p_2,...,p_n)$ has a **general social interaction** representation if for each $i\in N$ there exist $w_i:X\to(0,1)$ with $\sum_{x\in X}w_i(x)=1$, $\delta_i>0$, and $\boldsymbol{\delta}_i\in\Re^{n-1}$ such that

(1) $\delta_i+\sum_{j\neq i}\delta_{ij}=1$

(2) For every x, $\delta_i w_i(x)+\sum_{j\neq i}\min\{0,\delta_{ij}\}>0$

and

$$p_i(x,S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x,S)}{\delta_i w_i(S) + \boldsymbol{\delta}_i \cdot \mathbf{1}}$$

for all S and all $x \in S$.

The identification strategy and the characterization of this general model is very similar to that of the social interaction model. The identification equation, equation (5) remains the same, hence *N-Independence* functions as the main characterizing property. Since the main difference between these two models is the set of admissible values for the interaction coefficients, a general boundedness property, that takes care of the bounds on the revealed $\boldsymbol{\gamma}_i$ is required.

GN-Uniform Boundedness. [***GN-UB***] *For all $z \in X$, $p_i(z,X) > \boldsymbol{\gamma}_i \cdot \boldsymbol{p}_{-i}(z,X) - \sum_{j \neq i} min\{0, \gamma_{ij}\}$ for some $\boldsymbol{\gamma}_i \in \mathcal{B}_i$ with $\sum \gamma_{ij} < 1$.*

Theorem 5. *Let $\{p_i\}_{i \in N}$. Then, $\{p_i\}_{i \in N}$ has a **general social interaction** representation if and only if N-Independence and GN-Uniform Boundedness hold.*

Following is our most general stability result. All other stability results in the paper are corollaries of this.

Theorem 6. *Take $w_i \gg 0$, $\delta_i > 0$, and $\boldsymbol{\delta}_i \in \Re^{N \setminus \{i\}}$, where $w_i(X) = 1$, and*

(1) $\delta_i + \sum_{j \neq i} \delta_{ij} = 1$

(2) For every x, $\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} > 0$.

Then, there is a unique $(p_1^*, \ldots, p_N^*) \in \Delta_{++}(S)^N$ for which

$$p_i^*(x,S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}^*(x,S)}{\sum_{y \in S}[\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}^*(y,S)]}$$

and for any $(p_1^1(\cdot|S), \ldots, p_N^1(\cdot|S)) \in \Delta_{++}(S)^N$, the iterative map

$$p_i^t(x,S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}^{t-1}(x,S)}{\sum_{y \in S}[\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}^{t-1}(y,S)]}$$

converges to $(p_1^*, \ldots, p_N^*)$.

Given the w and δ parameters, an explicit representation for this unique representation is possible in terms of inverses of matrices. This expression is standard, and appears in the proof of Theorem 6. This expression demonstrates, for example, that $(p_1^*, \ldots, p_N^*) \in \Delta_{++}(X)^N$ is an affine function of $(w_1, \ldots, w_N)$.[15]

5. Concluding Remarks

The identification of social interactions from observable behavior is an important and highly topical agenda for economists. We believe that the use of choice theoretic tools to study social interactions introduces a new perspective. This model, and others, should prove useful for identification of unobservable underlying interaction structures and parameters. We suggest the dual interaction and social interaction models as simple tools for this purpose.

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[15]The same does not hold true for $S \subset X$, $S \neq X$ in general.

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6. Appendix

Proof of Theorem 1. ($\Rightarrow$) Let (p_1, p_2) with $p_1 \neq p_2$ have a dual interaction representation with $(w_1, w_2, \alpha_1, \alpha_2)$.

First we assume that β_i is well-defined and show that Equation 4 holds for all x, y and S. Define $\beta_i \equiv \dfrac{\alpha_i}{1+\alpha_i}$. Then $\beta_i \left(\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \right)$ is equal to

$$
\begin{aligned}
&= \frac{\alpha_i}{1+\alpha_i} \left(\frac{p_j(x,S) - p_j(x,X)}{p_i(x,S)} - \frac{p_j(y,S) - p_j(y,X)}{p_i(y,S)} \right) \\
&= \frac{w_i(x) + \alpha_i p_j(x,S) - w_i(x) - \alpha_i p_j(x,X)}{(1+\alpha_i) p_i(x,S)} - \frac{w_i(y) + \alpha_i p_j(y,S) - w_i(x) - \alpha_i p_j(y,X)}{(1+\alpha_i) p_i(y,S)} \\
&= \frac{(w_i(S) + \alpha_i) p_i(x,S) - (1+\alpha_i) p_i(x,X)}{(1+\alpha_i) p_i(x,S)} - \frac{(w_i(S) + \alpha_i) p_i(y,S) - (1+\alpha_i) p_i(y,X)}{(1+\alpha_i) p_i(y,S)} \\
&= \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)} \\
&= \frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}.
\end{aligned}
$$

Since this holds for all $S \neq X$ and distinct $x, y \in S$, Equation 4 holds for all x, y and S.

Now we show that β_i is indeed well-defined. We have three exhaustive cases. Fix $i, j \in \{1, 2\}$ with $i \neq j$ and first let $\alpha_i \neq 0$. We will show that for some $S \neq X$ and distinct x, y, we have $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \neq 0$, hence, $\beta_i(x, y, S)$ exists. Assume for a contradiction that $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Then,

$$
\begin{aligned}
\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} = 0 &\Rightarrow \frac{p_j(x,S) - p_j(x,X)}{p_i(x,S)} = \frac{p_j(y,S) - p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{\alpha_i p_j(x,S) - \alpha_i p_j(x,X)}{p_i(x,S)} = \frac{\alpha_i p_j(y,S) - \alpha_i p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{w_i(x) + \alpha_i p_j(x,S) - w_i(x) - \alpha_i p_j(x,X)}{p_i(x,S)} = \frac{w_i(y) + \alpha_i p_j(y,S) - w_i(y) - \alpha_i p_j(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{[w_i(S) + \alpha_i] p_i(x,S) - [1 + \alpha_i] p_i(x,X)}{p_i(x,S)} = \frac{[w_i(S) + \alpha_i] p_i(y,S) - [1 + \alpha_i] p_i(y,X)}{p_i(y,S)} \\
&\Rightarrow \frac{p_i(x,X)}{p_i(x,S)} = \frac{p_i(y,X)}{p_i(y,S)}
\end{aligned}
$$

But since this holds for all S, x, y, then IIA would be satisfied, establishing a contradiction with $\alpha_i \neq 0$. Now consider $\alpha_i = 0$ and $\alpha_j \neq 0$. Then p_i has a Luce representation and $\dfrac{d_i(x,S)}{p_i(x,S)} - \dfrac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and $x, y \in S$. We now show that for some S and distinct $x, y \in S$, $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} \neq 0$ so that I is satisfied for $\beta_i = \dfrac{\alpha_i}{1+\alpha_i} = 0$. Assume for a contradiction that for all S and distinct $x, y \in S$, $\dfrac{d_j(x,S)}{p_i(x,S)} - \dfrac{d_j(y,S)}{p_i(y,S)} = 0$. Since $\alpha_j \neq 0$, we have

$$
\frac{\alpha_j}{1+\alpha_j}\left(\frac{d_i(x,S)}{p_j(x,S)} - \frac{d_i(y,S)}{p_j(y,S)}\right) = \frac{d_j(x,S)}{p_j(x,S)} - \frac{d_j(y,S)}{p_j(y,S)} \tag{8}
$$

for all S and distinct $x, y \in S$, as we have shown above. Take S and $x, y \in S$ with $\frac{d_i(x,S)}{p_j(x,S)} \neq \frac{d_i(y,S)}{p_j(y,S)}$ and substitute $d_j(x,S)$ by $d_j(y,S)p_i(x,S)/p_i(y,S)$ in (8):

$$\begin{aligned}
\frac{\alpha_j}{1+\alpha_j}\left(\frac{d_i(x,S)}{p_j(x,S)} - \frac{d_i(y,S)}{p_j(y,S)}\right) =& \frac{d_j(y,S)p_i(x,S)}{p_j(x,S)p_i(y,S)} - \frac{d_j(y,S)}{p_j(y,S)} \\
\frac{\alpha_j}{1+\alpha_j}\left(\frac{d_i(x,S)p_j(y,S) - d_i(y,S)p_j(x,S)}{p_j(x,S)p_j(y,S)}\right) =& \frac{d_j(y,S)p_i(x,S)p_j(y,S) - d_j(y,S)p_j(x,S)p_i(y,S)}{p_j(x,S)p_i(y,S)p_j(y,S)} \\
\frac{\alpha_j}{1+\alpha_j} =& \frac{d_j(y,S)[p_i(x,S)p_j(y,S) - p_i(y,S)p_j(x,S)]}{p_i(y,S)[d_i(x,S)p_j(y,S) - d_i(y,S)p_j(x,S)]}
\end{aligned}$$

As p_i has a Luce representation, $d_i(x,S) = p_i(x,S)(1-w_i(S))$. We can then simplify the expression as follows:

$$\frac{\alpha_j}{1+\alpha_j} = \frac{d_j(y,S)}{p_i(y,S)(1-w_i(S))} = \frac{d_j(y,S)}{d_i(y,S)}.$$

But then,

$$\begin{aligned}
\frac{\alpha_j}{1+\alpha_j} = \frac{d_j(y,S)}{d_i(y,S)} &\Rightarrow \frac{\alpha_j p_i(y,S) - \alpha_j p_i(y,X)}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{w_j(y) + \alpha_j p_i(y,S) - w_j(y) - \alpha_j p_i(y,X)}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{p_j(y,S)[w_j(S)+\alpha_j] - p_j(y,X)[1+\alpha_j]}{1+\alpha_j} = p_j(y,S) - p_j(y,X) \\
&\Rightarrow \frac{p_j(y,S)[w_j(S)+\alpha_j]}{1+\alpha_j} - p_j(y,X) = p_j(y,S) - p_j(y,X)
\end{aligned}$$

Contradiction since $w_j(S) \neq 1$.

Finally, let $\alpha_i = \alpha_j = 0$. We claim that there exists S and distinct $x, y \in S$ such that $\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)} \neq 0$ so that $\beta_i = \frac{\alpha_i}{1+\alpha_i} = 0$ solves (4) for all S and distinct $x, y \in S$. Assume for a contradiction not. Since p_j allows a Luce representation, $d_j(x,S) = (1-w_j(S))p_j(x,S)$. But then, $\frac{d_j(x,S)}{p_i(x,S)} = \frac{d_j(y,S)}{p_i(y,S)}$ implies $\frac{p_j(x,S)}{p_i(x,S)} = \frac{p_j(y,S)}{p_i(y,S)}$. Since this would be the case for all S and $x, y \in S$, we would have $p_i = p_j$, contradiction. Thus, we have established I for all cases with $\beta_i \equiv \beta_i(x,y,S) = \frac{\alpha_i}{1+\alpha_i}$.

Nn follows directly. UB follows from $w_i(x) > 0$ for all x since $w_i(x) = (1 + \alpha_i)p_i(x, X) - \alpha_i p_j(x, X)$. Then we have $\frac{p_i(x, X)}{p_j(x, X)} > \beta_i$, establishing necessity.

($\Leftarrow$) Let $p_1 \neq p_2$ satisfy the axioms. Now define $\beta_i \equiv \beta_i(x, y, S)$ by I. UB implies $\beta_i \neq 1$ since otherwise $1 < \frac{p_i(x, X)}{p_j(x, X)}$ for all $x \in X$ yields $p_i(x, X) > p_j(x, X)$, from which it follows that $1 = \sum_{x \in X} p_i(x, X) > \sum_{x \in X} p_j(x, X) = 1$, a contradiction. Since $\beta_i \neq 1$, define $\alpha_i := \frac{\beta_i}{1 - \beta_i}$.

We claim that $\alpha_i \geq 0$. Observe that by UB, $\beta_i < 1$. Joint with Nn, this means $\beta_i \in [0, 1)$. Hence it follows that $\alpha_i = \frac{\beta_i}{1 - \beta_i} \geq 0$.

Next, we define weights for each alternative:

$$w_i(x) \equiv p_i(x, X) + \alpha_i(p_i(x, X) - p_j(x, X)).$$

Observe that $\sum_{x \in X} w_i(x) = 1$.

Now take any $S \neq X$ with distinct $x, y \in S$. Then:

$$\begin{aligned}
\frac{d_i(x, S)}{p_i(x, S)} - \frac{d_i(y, S)}{p_i(y, S)} &= \frac{\alpha_i}{1 + \alpha_i}\left[\frac{d_j(x, S)}{p_i(x, S)} - \frac{d_j(y, S)}{p_i(y, S)}\right] \\
\frac{p_i(y, X)}{p_i(y, S)} - \frac{p_i(x, X)}{p_i(x, S)} &= \alpha_i\left[\frac{d_j(x, S) - d_i(x, S)}{p_i(x, S)} - \frac{d_j(y, S) - d_i(y, S)}{p_i(y, S)}\right] \\
\frac{p_i(x, X) + \alpha_i d_j(x, S) - \alpha_i d_i(x, S)}{p_i(x, S)} &= \frac{p_i(y, X) + \alpha_i d_j(y, S) - \alpha_i d_i(y, S)}{p_i(y, S)} \\
\frac{p_i(x, X) + \alpha_i d_j(x, S) - \alpha_i d_i(x, S) + \alpha_i p_i(x, S)}{p_i(x, S)} &= \frac{p_i(y, X) + \alpha_i d_j(y, S) - \alpha_i d_i(y, S) + \alpha_i p_i(y, S)}{p_i(y, S)}.
\end{aligned}$$

The last equality is obtained by adding α_i to both sides of the previous equality. Notice that as $-\alpha_i d_i(x, S) + \alpha_i p_i(x, S) = \alpha_i p_i(x, X)$, the numerators of both of the sides are nonzero. Hence:

$$
\begin{aligned}
\frac{p_i(x,S)}{p_i(y,S)} =& \frac{p_i(x,X)+\alpha_i d_j(x,S)-\alpha_i d_i(x,S)+\alpha_i p_i(x,S)}{p_j(y,X)+\alpha_i d_j(y,S)-\alpha_i d_i(y,S)+\alpha_i p_i(y,S)} \\
=& \frac{p_i(x,X)+\alpha_i(p_i(x,X)-p_j(x,X))+\alpha_i p_j(x,S)}{p_i(x,X)+\alpha_i(p_i(x,X)-p_j(x,X))+\alpha_i p_j(x,S)} \\
=& \frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(y)+\alpha_i p_j(y,S)}.
\end{aligned}
$$

Observe in particular that this equality holds even in the case $x = y$. Now, for any $x, y \in S$, we have

$$p_i(y,S) = p_i(x,S)\frac{w_i(y)+\alpha_i p_j(y,S)}{w_i(x)+\alpha_i p_j(x,S)}$$

so that

$$\sum_{y\in S} p_i(y,S) = \sum_{y\in S} p_i(x,S)\frac{w_i(y)+\alpha_i p_j(y,S)}{w_i(x)+\alpha_i p_j(x,S)}.$$

Conclude

$$1 = p_i(x,S)\frac{\sum_{y\in S}(w_i(y)+\alpha_i p_j(y,S))}{w_i(x)+\alpha_i p_j(x,S)}.$$

Consequently,

$$p_i(x,S) = \frac{w_i(x)+\alpha_i p_j(x,S)}{\sum_{y\in S}(w_i(y)+\alpha_i p_j(y,S))}.$$

We finally show that $w_i(x) > 0$ for all $x \in X$. For all $x \in X$, $\dfrac{p_i(x,X)}{p_j(x,X)} > \beta_i = \dfrac{\alpha_i}{1+\alpha_i}$. Here, we obtain $(\alpha_i+1)p_i(x,X) > \alpha_i p_j(x,X)$ for all x. Consequently, $w_i(x) = p_i(x,X) + \alpha_i[p_i(x,X) - p_j(x,X)] > 0$ for all x.

■

Proof of Theorem 3. ($\Rightarrow$) Let $(p_1, p_2, ..., p_n)$ be a social interaction model. For any i, define $\boldsymbol{\beta}_i \in R^{n-1}$ such that $\beta_{ij} = \dfrac{\alpha_{ij}}{1+\sum\limits_{j\neq i}\alpha_{ij}}$ for all $j \neq i$. We will first show $\boldsymbol{\beta}_i \in \boldsymbol{\mathcal{B}}_i$.

First let $\alpha_{ij} = 0$ for all i and j with $i \neq j$. Then, for all i, p_i has a Luce representation and hence $d_i(x,S) = (1 - w_i(S))p_i(x,S)$. Moreover $\dfrac{d_i(x,S)}{p_i(x,S)} - \dfrac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Hence $\boldsymbol{\beta}_i = \mathbf{0}$ is an element in $\boldsymbol{\mathcal{B}}_i$.

Now let $\boldsymbol{\alpha}_i \neq \mathbf{0}$ for some i. Take any S and any distinct $x, y \in S$. Then $\boldsymbol{\beta}_i \cdot \left(\dfrac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \dfrac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right)$ is equal to

$$
\begin{aligned}
&=\sum_j \frac{\beta_{ij}(p_j(x,S)-p_j(x,X))}{p_i(x,S)} - \sum_j \frac{\beta_{ij}(p_j(y,S)-p_j(y,X))}{p_i(y,S)}\\
&=\sum_j \frac{\alpha_{ij}(p_j(x,S)-p_j(x,X))}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \sum_j \frac{\alpha_{ij}(p_j(y,S)-p_j(y,X))}{(1+\sum_j \alpha_{ij})p_i(y,S)}\\
&=\frac{w_i(x)+\sum_j \alpha_{ij}p_j(x,S) - w_i(x) - \sum_j \alpha_{ij}p_j(x,X)}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \frac{w_i(y)+\sum_j \alpha_{ij}p_j(y,S) - w_i(y) - \sum_j \alpha_{ij}p_j(y,X)}{(1+\sum_j \alpha_{ij})p_i(y,S)}\\
&=\frac{p_i(x,S)[w_i(S)+\sum_j \alpha_{ij}] - p_i(x,X)[1+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})p_i(x,S)} - \frac{p_i(y,S)[w_i(S)+\sum_j \alpha_{ij}] - p_i(y,X)[1+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})p_i(y,S)}\\
&=\frac{[w_i(S)+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})} - \frac{p_i(x,X)}{p_i(x,S)} - \frac{[w_i(S)+\sum_j \alpha_{ij}]}{(1+\sum_j \alpha_{ij})} + \frac{p_i(y,X)}{p_i(y,S)}\\
&=\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)},
\end{aligned}
$$

establishing $\boldsymbol{\beta}_i \in \mathcal{B}_i$.

Certainly, $\boldsymbol{\beta}_i \in R_+^{n-1}$ as $\alpha_{ij} \geq 0$ for all i, j with $i \neq j$. *N-PUB* then follows from $w_i(x) > 0$ for all x, since $w_i(x) = p_i(x,X) + \sum_{j \neq i} \alpha_{ij}(p_i(x,X) - p_j(x,X)) > 0 \Rightarrow (1 + \sum_{j \neq i} \alpha_{ij})p_i(x,X) > \sum_{j \neq i} \alpha_{ij}p_j(x,X) \Rightarrow p_i(x,X) > \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X)$.

($\Leftarrow$) Take $(p_1, p_2, ..., p_n)$ satisfying our axioms. Take any $i \in N$, x, y and S and by *N-I*, take $\boldsymbol{\beta}_i \in \mathcal{B}_i$, which also satisfies *N-PUB*. Further, define $\boldsymbol{\alpha}_i \in R^{n-1}$ such that $\alpha_{ij} = \dfrac{\beta_{ij}}{1 - \sum_{j \neq i} \beta_{ij}}$. We first show that $\boldsymbol{\alpha}_i$ is well-defined and nonnegative since $\sum_{j \neq i} \beta_{ij} < 1$. This is because by *N-PUB*, $p_i(x,X) > \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X)$ for all x, we have $1 = \sum_{x \in X} p_i(x,X) > \sum_{x \in X} \boldsymbol{\beta}_i \cdot \boldsymbol{p}_{-i}(x,X) = \sum_{j \neq i} \beta_{ij}$. Hence, $\boldsymbol{\alpha}_i \in R_+^{n-1}$ is well-defined for all $\boldsymbol{\beta}_i$ as claimed.

Notice we then have $\frac{1}{1+\sum_{j\neq i}\alpha_{ij}}\boldsymbol{\alpha}_i = \boldsymbol{\beta}_i$. Now define

$$w_i(x) := p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)]$$

where $\mathbf{1} \in R^{n-1}$ is a vector of ones and observe that

$$\begin{aligned}
\sum_{x\in X} w_i(x) &= \sum_{x\in X} \left(p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)]\right) \\
&= 1 + \boldsymbol{\alpha}_i \cdot \left[\sum_{x\in X} p_i(x,X)\mathbf{1} - \sum_{x\in X} \boldsymbol{p}_{-i}(x,X)\right] \\
&= 1 + \boldsymbol{\alpha}_i(\mathbf{1} - \mathbf{1}) \\
&= 1.
\end{aligned}$$

By *N-I*,

$$\frac{1}{1+\sum_{j\neq i}\alpha_{ij}}\boldsymbol{\alpha}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \frac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right) = \frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}$$

$$\frac{(1+\sum_{j\neq i}\alpha_{ij})p_i(x,X) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S) - \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,X)}{p_i(x,S)} = \frac{(1+\sum_{j\neq i}\alpha_{ij})p_i(y,X) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{p_i(y,S)} - \frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,X)}{p_i(y,S)}.$$

Notice that numerators in both of the sides are positive since $p_j(x,S) > p_j(x,X)$ for all j, x and S. Hence

$$\begin{aligned}
\frac{p_i(x,S)}{p_i(y,S)} &= \frac{p_i(x,X) + \boldsymbol{\alpha}_i \cdot [p_i(x,X)\mathbf{1} - \boldsymbol{p}_{-i}(x,X)] + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{p_i(y,X) + \boldsymbol{\alpha}_i \cdot [p_i(y,X)\mathbf{1} - \boldsymbol{p}_{-i}(y,X)] + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)} \\
&= \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}.
\end{aligned}$$

But then, since this claim holds for all $y \in S$:

$$p_i(y,S) = p_i(x,S)\frac{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}$$

$$\sum_{y\in S} p_i(y,S) = \sum_{y\in S} p_i(x,S)\frac{w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}$$

$$1 = p_i(x,S)\frac{\sum_{y\in S}[(w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)]}{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}$$

$$p_i(x,S) = \frac{w_i(x) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(x,S)}{\sum_{y\in S}[w_i(y) + \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_{-i}(y,S)]}.$$

We finally show that $w_i(x) > 0$ for all $x \in X$. This is established by *N-PUB*. Since $p_i(x,X) > \boldsymbol{\beta}_i \boldsymbol{p}_{-i}(x,X)$ and $1+\sum_{j\neq i} \alpha_{ij} > 0$, then, $(1+\sum_{j\neq i} \alpha_{ij})p_i(x,X) > \boldsymbol{\alpha}_i \boldsymbol{p}_{-i}(x,X) \Rightarrow w_i(x) > 0$. ■

Proof of Theorem 5. ($\Rightarrow$) Let $(p_1, p_2, ..., p_n)$ be a general social interaction model. We will first show $\boldsymbol{\delta}_i \in \mathcal{B}_i$.

First let $\delta_{ij} = 0$ for all i and j with $i \neq j$. Then, for all i, p_i has a Luce representation and hence $d_i(x,S) = (1 - w_i(S))p_i(x,S)$. Moreover $\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)} = 0$ for all S and distinct x, y. Hence $\boldsymbol{\delta}_i = \boldsymbol{0}$ is an element in $\mathcal{B}_i$.

Now let $\boldsymbol{\delta}_i \neq \boldsymbol{0}$ for some i. Take any S and any distinct $x, y \in S$. Then $\boldsymbol{\delta}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x,S)}{p_i(x,S)} - \frac{\boldsymbol{d}_{-i}(y,S)}{p_i(y,S)}\right)$ is equal to

$$=\sum_j \frac{\delta_{ij}(p_j(x,S) - p_j(x,X))}{p_i(x,S)} - \sum_j \frac{\delta_{ij}(p_j(y,S) - p_j(y,X))}{p_i(y,S)}$$

$$=\frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \boldsymbol{p_{-i}}(x,S) - \delta_i w_i(x) - \boldsymbol{\delta}_i \boldsymbol{p_{-i}}(x,X)}{p_i(x,S)} - \frac{\delta_i w_i(y) + \boldsymbol{\delta}_i \boldsymbol{p_{-i}}(y,S) - \delta_i w_i(y) - \boldsymbol{\delta}_i \boldsymbol{p_{-i}}(y,X)}{p_i(y,S)}$$

$$=\frac{p_i(x,S)[\delta_i w_i(S) + \sum_j \delta_{ij}] - p_i(x,X)}{p_i(x,S)} - \frac{p_i(y,S)[\delta_i w_i(S) + \sum_j \delta_{ij}] - p_i(y,X)}{p_i(y,S)}$$

$$=\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)},$$

establishing $\boldsymbol{\delta}_i \in \mathcal{B}_i$.

Since $\delta_i > 0$, we have $\sum_j \delta_{ij} < 1$. *GN-UB* then follows from $\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} > 0$ for all x, since $\delta_i w_i(x) = p_i(x, X) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, S)$.

($\Leftarrow$) Take $(p_1, p_2, ..., p_n)$ satisfying our axioms. Take any $i \in N$, x, y and S and by *N-I*, take $\boldsymbol{\delta}_i \in \mathcal{B}_i$, which also satisfies *GN-UB*. Let $\delta_i = 1 - \sum \delta_{ij}$. Notice $\delta_i > 0$ by *GN-UB*.

Now define

$$w_i(x) := \frac{p_i(x, X) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, X)}{\delta_i}$$

and observe that

$$\sum_{x \in X} w_i(x) = \frac{1 - \sum \delta_{ij}}{\delta_i} = 1.$$

GN-UB then ensures that $\delta_i w_i(x) + \sum_{j \neq i} \min\{0, \delta_{ij}\} > 0$ and hence $w_i(x) > 0$ for all $x \in X$.

By *N-I*,

$$\boldsymbol{\delta}_i \cdot \left(\frac{\boldsymbol{d}_{-i}(x, S)}{p_i(x, S)} - \frac{\boldsymbol{d}_{-i}(y, S)}{p_i(y, S)} \right) = \frac{p_i(y, X)}{p_i(y, S)} - \frac{p_i(x, X)}{p_i(x, S)}$$

$$\frac{p_i(x, X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, X)}{p_i(x, S)} = \frac{p_i(y, X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, X)}{p_i(y, S)}.$$

Notice that numerators in both of the sides are positive by *GN-UB*. Hence

$$\begin{aligned} \frac{p_i(x, S)}{p_i(y, S)} &= \frac{p_i(x, X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, X)}{p_i(y, X) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, S) - \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, X)} \\ &= \frac{\delta_i w_i(x) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(x, S)}{\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, S)}. \end{aligned}$$

But then, since this claim holds for all $y \in S$, as before, we arrive at

$$p_i(x, S) = \frac{\delta_i w_i(x) + \boldsymbol{\delta}_{\boldsymbol{i}} \cdot \boldsymbol{p}_{-i}(x, S)}{\sum_{y \in S} [\delta_i w_i(y) + \boldsymbol{\delta}_i \cdot \boldsymbol{p}_{-i}(y, S)]}$$

establishing the proof. ∎

Proof of Theorems 2, 4, and 6. Let us suppose without loss that $|S| \geq 2$. Consider first an affine function $f : \Re^m \to \Re^m$, which is given by $f(x) = Ax + b$, where $A \in \Re^{m \times m}$ and $b \in \Re^m$. We claim that if A has a spectrum (maximal degree of eigenvalue) less than one, then there is a unique solution $f(x^*) = Ax^* + b$, and that the iterative process $x^t = f(x^{t-1})$ converges to x^*, independently of choice of x^1. This is because if there were two distinct solutions, x^* and y^*; then it would follow that $(x^* - y^*) = A(x^* - y^*)$, in which case $x^* - y^*$ is an eigenvector of A with value one, contradicting the fact that all eigenvalues have degree less than one.

We use the standard parametrization with $\alpha_i^j \equiv \dfrac{\delta_{ij}}{\delta_i}$, in the case of Theorem 6.

Let us consider the linear process $f(x) = Ax + b$, where $x \in \mathbb{R}^{|N||S|}$, $A \in \mathbb{R}^{|N||S| \times |N||S|}$, and $b \in \mathbb{R}^{|N||S|}$. It is well-known that there is a unique $x^* \in \mathbb{R}^{|N||S|}$ for which for any x^1, the process $x^t = f(x^{t-1})$ converges to x^* if the maximal absolute value of an eigenvalue of A has value less than 1. See for example, Varga (1962), Theorem 1.4. This unique fixed point will be a member of $\Delta(S)^N$, because $f(\Delta(S)^N) \subseteq \Delta(S)^N$ in our case.

To this end, let us describe the matrix A and vector b in which we take interest. To ease the exposition, let $\hat{\alpha}_{ij} = \dfrac{\alpha_{ij}}{w_i(S) + \sum_{j \neq i} \alpha_{ij}}$ for $j \neq i$, and $\hat{w}_i(x) = \dfrac{w_i(x)}{w_i(S) + \sum_{j \neq i} \alpha_{ij}}$. Here, each 0 is the $S \times S$ matrix of zeroes, and I denotes the identity matrix in $S \times S$.

Now, we let the matrix $A = \begin{bmatrix} 0 & \hat{\alpha}_{12} I & \dots & \hat{\alpha}_{1n} I \\ \hat{\alpha}_{21} I & 0 & \dots & \hat{\alpha}_{2n} I \\ \vdots & \vdots & & \vdots \\ \hat{\alpha}_{n1} I & \hat{\alpha}_{n2} I & \dots & 0 \end{bmatrix}$ and let $b = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_n \end{bmatrix}$, so that the iterated vector is of the form $p^t = \begin{bmatrix} p_1^t \\ p_2^t \\ \vdots \\ p_n^t \end{bmatrix}$.

Finally, by Corollary 1 on p. 17 of Varga (1962), we conclude that the maximal absolute value of an eigenvalue is bounded above by $\max_i \sum_{j \neq i} |\hat{\alpha}_{ij}|$. But for each i, we know that $\sum_{j\neq i} |\hat{\alpha}_{ij}| = \sum_{j\neq i} \frac{|\alpha_{ij}|}{w_i(S) + \sum_{j\neq i} \alpha_{ij}}$. Now, by assumption, and since $|S| \geq 2$, $0 < w_i(S) + 2\sum_{j\neq i} \min\{0, \alpha_{ij}\}$.[16] Observe then that, by adding to each side of this strict inequality $\sum_{j\neq i} |\alpha_{ij}|$ we obtain $\sum_{j\neq i} |\alpha_{ij}| < w_i(S) + \sum_{j\neq i} \alpha_{ij}$. Therefore, by definition of $\hat{\alpha}$, we conclude $\sum_{j\neq i} |\hat{\alpha}_{ij}| < 1$, which is what we wanted to show.

As a last point, we observe that the solution p^* is the unique vector satisfying $p^* = Ap^* + b$, or in other words, $p^* = (I - A)^{-1}b$. ■

[16]Let $x, y \in S$ for which $x \neq y$. Then $w_i(x) + \sum_{j\neq i} \min\{0, \alpha_{ij}\} > 0$ and $w_i(y) + \sum_{j\neq i} \min\{0, \alpha_{ij}\} > 0$, so that $0 < w_i(x) + w_i(y) + 2\sum_{j\neq i} \min\{0, \alpha_{ij}\} \leq w_i(S) + 2\sum_{j\neq i} \min\{0, \alpha_{ij}\}$.