American Economic Review 2016, 106(9): 2760–2782
http://dx.doi.org/10.1257/aer.20140973

A Behavioral Analysis of Stochastic Reference Dependence†

By Yusufcan Masatlioglu and Collin Raymond*

We examine the reference-dependent risk preferences of Kőszegi and Rabin (2007), *focusing on their choice-acclimating personal equilibria. Although their model has only a trivial intersection* (*expected utility*) *with other reference-dependent models*, *it has very strong connections with models that rely on different psychological intuitions. We prove that the intersection of rank-dependent utility and quadratic utility*, *two well-known generalizations of expected utility*, *is exactly monotone linear gain-loss choice-acclimating personal equilibria. We use these relationships to identify parameters of the model*, *discuss loss and risk aversion*, *and demonstrate new applications.* (*JEL* D11, D81)

The notion of reference dependence was first introduced in economics by Markowitz (1952) and was formalized by Kahneman and Tversky (1979). Their reference-dependent model has become popular because it accommodates common behavior that is anomalous within the expected utility framework. However, Kahneman and Tversky, in both their original formulation and follow-up work, did not specify how the reference point is formed; this makes it difficult to derive general predictions and tests.

Recently, Kőszegi and Rabin (2006, 2007) proposed a model of reference dependence that specifies how individuals form reference points. In their model, consumers care about consumption utility as well as gain-loss utility (i.e., utility over deviations from the reference point). The reference point is determined by the probabilistic beliefs of the decision maker about the choice sets she will face, and the decision she will make for each choice set (i.e., expectations). Since beliefs

* Masatlioglu: Department of Economics, University of Maryland, 3114 Tydings Hall, College Park, MD 20742 (e-mail: masatlioglu@gmail.com); Raymond: Department of Economics, Amherst College, 100 Boltwood Avenue, Amherst, MA 01002 (e-mail: collinbraymond@gmail.com). Previous drafts circulated under the titles "Stochastic Reference Points, Loss Aversion and Choice under Risk" and "Drs. Kőszegi and Rabin or: How I Learned to Stop Worrying and Love Reference Dependence." For their helpful comments, we would like to thank three anonymous referees; seminar participants at Amherst College; Boston University; BRIC; FUR; Harvard University; London Business School; London School of Economics; Ludwig-Maximilians-Universitat Munich; Monash University; Penn State University; Queen Mary University of London; University of California, Berkeley; University of Copenhagen; University of Michigan; University of Oxford; University of Pennsylvania; and Johannes Abeler; Daniel Benjamin; Ian Crawford; Vince Crawford; David Dillenberger; Andrew Ellis; David Freeman; David Gill; David Huffman; Matthew Rabin; Neslihan Uler; and Jill Westfall. We would also like to thank Yang Lu for excellent assistance. Any remaining errors are ours. The authors declare that they have no relevant or material financial interests that relate to the research described in this paper.

† Go to http://dx.doi.org/10.1257/aer.20140973 to visit the article page for additional materials and author disclosure statement(s).

determine the reference point, Kőszegi and Rabin provide a solution concept that determines expectations endogenously. Their framework has inspired numerous applications.[1]

Despite its popularity, it can be difficult to understand the implications of Kőszegi and Rabin's model for behavior, even in simple domains, due to its complicated functional form. For example, examining choice over risk, little is known about how to distinguish their theory from other models of reference dependence; these include earlier models of Gul (1991), Bell (1985), and Loomes and Sugden (1986). All of these models have similar formulations as Kőszegi and Rabin's (2007) but specify a different process of reference point formation. More generally, it also is not clear how Kőszegi and Rabin's model relates to other models of nonexpected utility theory that rely on completely different psychological intuitions (e.g., rank-dependent utility).

We focus on preferences induced by Kőszegi and Rabin's (2007) choice-acclimating personal equilibrium with linear gain-loss utility and refer to the functional form that they use as *CPE* (see Section IV for nonlinear gain-loss utility). Kőszegi and Rabin (2007) discuss how *CPE* captures the idea of a decision maker committing to a choice long before uncertainty is resolved (e.g., insurance decisions). Therefore, in line with the motivation provided by Kőszegi and Rabin (2007), the results in this paper should be interpreted in the context of choice where uncertainty will not be resolved immediately but rather in the future, so that the chosen lottery has time to become the reference point.

We describe the functional form used in *CPE* as well as other generalizations of expected utility in Section I. Our first result is to characterize when preferences with a *CPE* representation respect first-order stochastic dominance—we refer to this functional form as monotone *CPE*.

Section II provides a characterization of monotone *CPE*. We show that the intersection of rank-dependent utility (*RDU*) and quadratic utility (*Q*), two well-known generalizations of expected utility, is exactly monotone *CPE*. To be precise, a preference has both *RDU* and *Q* representations if and only if it has a monotone *CPE* representation. We also show that the value of the coefficient of loss aversion is tightly linked to a decision maker's attitudes toward the convexification of indifferent lotteries. Using this result, we describe the equivalent quadratic and rank-dependent representations of *CPE*.

Our characterization is interesting because it implies that there is an equivalence between correct beliefs but nonstandard utility (à la *CPE*), and a type of distorted beliefs but standard utility (à la *RDU*). This may be surprising because, as Kőszegi and Rabin (2007, p. 1048) note; "We assume that a person correctly predicts her probabilistic environment and her own behavior in that environment, so that her beliefs fully reflect the true probability distribution of outcomes." Moreover, looking purely at the functional form, it would seem to be the case that *CPE* should generate similar behavior to cumulative prospect theory (formalized by Tversky

[1] For example, Heidhues and Kőszegi (2008, 2014); Sydnor (2010); Herweg, Müller, and Weinschenk (2010); Abeler et al. (2011); Card and Dahl (2011); Crawford and Meng (2011); Pope and Schweitzer (2011); Carbajal and Ely (2012); Karle and Peitz (2014); and Eliaz and Spiegler (2014). In related work, Freeman (2012) characterizes Kőszegi and Rabin's (2007) notion of preferred personal equilibrium.

and Kahneman 1992) but without the effects of probability weighting. As we make clear, in the case of linear gain-loss utility, the correct comparison is actually the opposite—*CPE* is a subset of cumulative prospect theory, but with only probability weighting and no gain-loss utility.

In addition to *CPE*, there are other models that attempt to capture similar psychological intuitions regarding reference dependence and appear to be quite close in nature. Kőszegi and Rabin (2007, p. 1049) themselves say, "Except that we specify the reference point as a lottery's full distribution rather than its certainty equivalent, [our] concept is similar to the disappointment-aversion models of Bell (1985), Loomes and Sugden (1986), and Gul (1991)." However, a corollary of our characterization is that the intersection of *CPE* and other classical models of endogenous reference points is only expected utility. In other words, despite trying to capture the same intuition about the effect of expectations on preferences, these models do so in distinct ways.[2] In fact, when a decision maker exhibits preferences represented by *CPE* and either Gul's or Bell-Loomes-Sugden's models, then they must be expected utility maximizers; in other words, they must not exhibit any reference dependence at all.

In Section III, we use our results to discuss the relationship between economic behavior, such as risk aversion and first-order risk aversion, and the parameters in *CPE*. We first identify what specifications of *CPE* are consistent with classical notions of risk aversion (i.e., aversion to mean-preserving spreads). Our results point to a tight linkage, for loss-averse decision makers, between preferences respecting two different orderings: the one induced by first-order stochastic dominance and the one induced by mean-preserving spreads. We then go on to relate the coefficient of loss aversion in monotone *CPE* to aversion to small-stakes lotteries (i.e., first-order risk aversion).

In Section IV, we consider generalizations of *CPE*, where the gain-loss utility function may not be linear. We discuss whether and how our results from previous sections extend when more general functional forms are allowed.

In Section V, we provide an example of why our results are useful in terms of applications. We show that *CPE* suffers from a very similar calibration critique to the one Rabin (2000) leveled against expected utility; plausible choices over small-stakes lotteries imply implausible choices over large-stakes lotteries. Thus, in order to address the Rabin critique, we must look beyond linear gain-loss functions. Section VI concludes, while the Appendix contains additional results and proofs.

I. Preliminaries and Functional Forms

Consider an interval $[w, b] = X \subset \mathbb{R}$ of money. Let Δ_X be the set of all simple lotteries (i.e., probability measures with finite support) on X. A lottery $f \in \Delta_X$ is a function from X to $[0,1]$ such that $\sum_{x \in X} f(x) = 1$ and the number of prizes with nonzero probability is finite. $f(x)$ represents the probability assigned to the

[2] The results derived in this paper consider an arbitrary number of outcomes. If we examine choice over restricted domains, the relationships can differ. If, as in many experiments, certainty equivalents are elicited from lotteries defined on only two outcomes, *CPE*, Gul's and Bell-Loomes-Sugden's models are all subsets of *RDU*. Thus, many studies provide limited data to distinguish many nonexpected utility models from one another.

outcome x in lottery f (we denote the cumulative distribution function of f as F). For any lotteries f, g, we let $\alpha f + (1-\alpha) g$ be the lottery that yields x with probability $\alpha f(x) + (1-\alpha) g(x)$. Denote by δ_x the degenerate lottery that yields x with probability 1 (i.e., $\delta_x(x) = 1$). We will also refer to δ_x simply as x. $\succsim$ is a weak order over Δ_X, which represents the decision maker's preferences over lotteries. For three outcomes $x, y, z \in X$, we denote the unit simplex of possible lotteries over those three outcomes as $\Delta_{x,y,z}$, or for an arbitrary set of three outcomes, Δ_3. In this case, we will refer to the best outcome as $\overline{\delta}$, the worst outcome as $\underline{\delta}$, and the middle outcome as $\hat{\delta}$.

Choice-Acclimating Personal Equilibria.—Kőszegi and Rabin (2006), building on Bowman, Minehart, and Rabin (1999), extend the notion of reference dependence introduced in Kahneman and Tversky (1979) by having an individual's utility depend both on gain-loss utility (i.e., the comparison of outcomes to a reference point) and consumption utility (which depends only on the absolute value of the outcomes, rather than a comparison to a referent). This formulation is applied to lotteries in Kőszegi and Rabin (2007), which introduced *CPE*. As mentioned, *CPE* is meant to capture situations where, at the time of the resolution of uncertainty, the choice is the reference point. Hence, the value of a lottery f is the sum of two separate components. The first is consumption utility (or just the expected Bernoulli utility of f). The second is the gain-loss utility, where, ex post (after a realization) an individual compares what she actually received (for example x) to what she expected to receive, which is the distribution implied by f. The individual compares x to each y that could have been expected and weighs those comparisons by the probability that y could have been realized. From an ex ante perspective, the individual takes the weighted average of these ex post comparisons, weighting by the probability that each x occurs. Thus, the utility value of a lottery f is[3]

$$V_{CPE}(f) = \underbrace{\sum_x u(x) f(x)}_{\text{consumption utility}} + \underbrace{\sum_x \sum_y \mu(u(x) - u(y))\, f(x) f(y)}_{\text{gain-loss utility}},$$

where u is a continuous increasing consumption (Bernoulli) utility function over final wealth and μ is the gain-loss function

$$\mu(z) = \begin{matrix} z & \text{if } z \geq 0 \\ \lambda z & \text{if } z < 0 \end{matrix}$$

where λ is the coefficient of loss aversion.[4] Loss aversion occurs when $\lambda \geq 1$, while loss-loving occurs when $\lambda \leq 1$. If $\lambda = 1$, the preferences simplify to expected utility. We say *a preference has a CPE representation if it can be represented using* $V_{CPE}(f)$.

[3] This form was independently developed by Delquié and Cillo (2006) as a model of disappointment without prior expectation.

[4] Although here the gain-loss functional is linear in the difference between the consumption utility levels, we discuss more general gain-loss functionals later in the paper. We focus on linear gain-loss functionals because it is both the focus of Kőszegi and Rabin (2007) as well as most of the applications of their model.

For much of this paper, we will focus on preferences that respect first-order stochastic dominance and refer to these as *monotone* preferences. Proposition 7 of Kőszegi and Rabin (2007) points out that if loss aversion is a strong enough factor in preferences, then a decision maker will avoid any risk even though every outcome in the lottery is better than the outside option, thus violating first-order stochastic dominance. Our first result is to extend this intuition and characterize the class of monotone preferences; when λ is strictly greater than 2 or strictly less than 0, then there exists a nondegenerate lottery *strictly* worse than the worst degenerate lottery.[5]

PROPOSITION 1: *Let a preference have a CPE representation. Then it respects first-order stochastic dominance if and only if* $0 \leq \lambda \leq 2$.

There are a variety of other preferences that are meant to capture behavior over risky outcomes. The standard model used in the literature is expected utility, which we refer to as *EU*. Different types of models generalize *EU* in a variety of ways. Here, we discuss three of them.

Quadratic.—One generalization of expected utility is quadratic preferences. A utility functional is said to be quadratic in probabilities if it can be expressed in the form

$$V_Q(f) = \sum_x \sum_y \phi(x,y)\, f(x) f(y),$$

where $\phi : X \times X \rightarrow \mathbb{R}$ is a continuous function.[6] The quadratic functional form was introduced in Machina (1982) and further developed in Chew, Epstein, and Segal (1991, 1994). One can think of ϕ as a function that compares any given outcome to any other given outcome (e.g., it gives the value of x when y is the reference point). The value of a lottery is then the average value of all of those comparisons over the outcomes with positive support. Viewed this way, the intuition for *Q* is very similar to that of *CPE*.

Rank-Dependent.—A utility functional V_{RDU} is a rank dependence expected utility functional if there exists a continuous function u and a strictly increasing, continuous function $w : [0,1] \rightarrow [0,1]$, with $w(0) = 0$ and $w(1) = 1$, such that

$$V_{RDU}(f) = \sum_x u(x) \left[w\Big(\sum_{y \geq x} f(y) \Big) - w\Big(\sum_{y > x} f(y) \Big) \right].$$

This form was introduced in Quiggin (1982) and has been examined by myriad authors.[7] We will use *RDU* to denote the class of rank-dependent utility functionals. Observe that when w is the identity function, V_{RDU} reduces to the expected utility.

[5] Many applications of reference dependence set $\lambda \geq 2$, which, in combination with *CPE*, implies that preferences are not monotone.

[6] Chew, Epstein, and Segal (1991) assume ϕ to be symmetric: $\phi(x,y) = \phi(y,x)$ for all x, y. There is no loss of generality in restricting ϕ to be symmetric, since an arbitrary $\phi(x,y)$ can always be replaced by $\frac{\phi(x,y) + \phi(y,x)}{2}$.

[7] See Abdellaoui (2002) for a recent characterization and references to the larger literature.

Otherwise, w acts to distort the decumulative distribution function associated with lottery f. The term $w\left(\sum_{y \geq x} f(y)\right) - w\left(\sum_{y > x} f(y)\right)$ measures the marginal probability contribution of x to the distorted decumulative distribution function.[8]

Reference-Dependent Models.—We now describe other reference-dependent models that feature endogenous reference points. They attempt to capture similar psychological intuitions, use similar functional forms, and appear to be quite close in nature. Recall that in Kőszegi and Rabin (2007) the reference point is a lottery's full distribution. In contrast, in the disappointment theory developed by Bell (1985) and Loomes and Sugden (1986), the reference point is expected (Bernoulli) utility without disappointment, while in Gul's (1991) model, the reference point includes the expected losses due to disappointment—the reference point is determined by the full certainty equivalent of the lottery.

Gul's (1991) theory of disappointment aversion is a special case of a more general class of preferences. Since they share the same intuition, we describe this more general class, betweenness preferences, which was introduced by Chew (1983), Fishburn (1983), and Dekel (1986). Betweenness, B, functionals have the form

$$V_B(f) = \sum_x \nu(x, V_B(f)) f(x),$$

where ν is continuous and an increasing function of its first argument. These preferences feature a type of endogenous reference dependence, where $V_B(f)$ is the reference point used to evaluate outcomes.

The model of disappointment theory introduced by Bell (1985) and Loomes and Sugden (1986) has proven quite popular in applications, as the reference point is neither stochastic nor recursively defined, but is instead simply the expected consumption (Bernoulli) utility of the lottery. Given a function u, denote the expected value of u given lottery f as $E_u(f)$. The value of a lottery is then

$$V_{BLS}(f) = \sum_x u(x) f(x) + \sum_x \mu(u(x) - E_u(f)) f(x),$$

where μ is a piecewise linear function with $\mu(0) = 0$ (as in V_{CPE}).[9]

II. Relationships

Our main result highlights the connection between well known but seemingly unrelated models. We show that the intersection of quadratic and rank-dependent models is exactly equal to Kőszegi and Rabin's model. In other words, the intersection of two models that people have found useful in capturing behavior has its own independent attraction.[10]

[8] *RDU* accommodates probability weighting while ensuring preferences respect first-order stochastic dominance.

[9] The original papers introducing this model do not require μ to be piecewise linear; however we make this restriction in order to make the model as comparable to *CPE* as possible.

[10] Chew, Epstein, and Segal (1991) provide an example showing that the intersection of *RDU* and *Q* is nonempty. Their example is a special case of monotone *CPE*, although their functional form makes the representation appear different at first glance.

THEOREM 1: *A preference has both Q and RDU representations if and only if it has a monotone CPE representation.*

Theorem 1 highlights the fact that the monotone *CPE* model has a very strong predictive power. In addition, Theorem 1 completely characterizes preferences with monotone *CPE* representations. Since both quadratic and rank-dependent models have already axiomatic foundations, Theorem 1 indirectly provides axiomatic foundations for Kőszegi and Rabin's model. *Q* has been characterized using preferences in Chew, Epstein and Segal (1991, 1994). There exist numerous characterizations of *RDU* using preferences; a recent one is Abdellaoui (2002).

In order to relate the parameters of the model, u and λ, to behavior and conduct comparative statics, as we do in the following section and the online Appendix, we first need to know to what extent u and λ are uniquely identified from observed behavior.

PROPOSITION 2: *For any preference with a monotone CPE representation, u is unique up to affine transformation and λ is unique.*[11]

Analogous to the expected utility, u is unique up to affine transformation. Moreover, λ (the parameter governing loss aversion) is uniquely identified from observed behavior. However, our characterization does not identify which agents have $\lambda \geq 1$ or $\lambda \leq 1$. In order to do so, we discuss two well-known relaxations of independence axiom. The first is mixture aversion. Preferences satisfy mixture aversion if, given two lotteries that are indifferent, any mixture of them is worse than the original lotteries (mixture aversion is often called "quasiconvexity" in the literature). Mixture-loving can be defined analogously.

Mixture Aversion (Loving): If $f \sim g$, then $\alpha f + (1-\alpha)g \precsim (\succsim) f$ for all $\alpha \in [0,1]$.

Mixture aversion is a necessary condition of neither *Q* nor *RDU*. However, *CPE* preferences are always mixture-averse if λ is greater than 1 and mixture-loving if λ is smaller than 1. For monotone preferences, this result is a corollary of Wakker (1994), who showed that for preferences in *RDU*, pessimism (optimism) is equivalent to mixture aversion (loving).

PROPOSITION 3: *For any preference with a CPE representation (u, λ), it satisfies mixture aversion (loving) if and only if $\lambda \geq 1$ ($\lambda \leq 1$).*

Theorem 1 immediately implies that any preference with a monotone *CPE* representation has both *Q* and *RDU* representations. We now formally relate V_{CPE} to both V_Q and V_{RDU}. To see the first relationship, simply define quadratic functional ϕ as follows:

$$\phi(x,y) = \frac{1}{2}(u(x) + u(y)) + \frac{1}{2}(1 - \lambda)|u(x) - u(y)|.$$

[11] This uniqueness result can be extended to any preference with a nonmonotone *CPE* representation.

The first component of ϕ, $\frac{1}{2}(u(x) + u(y))$, simply reflects the consumption utility term of V_{CPE}, while the second component, $\frac{1}{2}(1 - \lambda)|u(x) - u(y)|$, reflects the gain-loss utility. This makes it transparent that V_{CPE} is more restrictive than V_Q.

Seeing the relationship between V_{CPE} and V_{RDU} is not as straightforward. Again, V_{CPE} is more restrictive than V_{RDU}.[12] The following proposition formally relates monotone V_{CPE} and V_{RDU}.[13]

PROPOSITION 4: *Any preference with a monotone CPE representation* (u, λ) *has also an RDU representation* (u, w_λ) *where* $w_\lambda(z) = (2 - \lambda)z + (\lambda - 1)z^2$.

Note that both representations share the same consumption utility u. The difference comes from distortions introduced by either gain-loss or a probability weighting function. For example, when $\lambda \in [1, 2]$, w_λ is a convex function, which means that preferences are "pessimistic," or, equivalently, worse outcomes are overweighted. Hence, Proposition 4 implies that loss aversion $(\lambda \geq 1)$ can be considered a type of pessimism, in that loss-averse individuals overweight bad outcomes.

We finish this section by relating *CPE* with other reference-dependent models that share similar psychological intuitions. The next result proves that these models of endogenous reference dependence are capturing reference dependence in distinct ways.

PROPOSITION 5: *If a decision maker's preference is represented by both* V_{CPE} *and* V_{BLS} *(or* V_B*), she must be an expected utility maximizer.*

Any preferences that are fully consistent with *CPE*'s notion of reference dependence, as well as Bell-Loomes-Sugden's, must not exhibit any reference dependence at all—they must be *EU*. The same applies for any preference with a betweenness representation. These distinctions point out that models of reference dependence capture intuitions not only about loss aversion (or first-order risk aversion), but also about other important behavior, such as attitudes toward randomization. Consider two lotteries *f* and *g*, that a decision maker is indifferent between. If her preferences can be represented by V_B, then she must also be indifferent between *f* and any mixture of *f* and *g*. In contrast, if her preferences are represented by V_{CPE}, she must weakly prefer *f* to any mixture of *f* and *g*. These distinctions enable us to distinguish between models of reference dependence.

Figure 1 summarizes the results of this section, showing how *CPE* relates to other models of nonexpected utility.[14]

[12] More generally, we can show that preferences in *CPE* have a representation that is exactly the same as V_{RDU} but dropping the restriction that w is a strictly increasing function.

[13] Delquié and Cillo (2006) derive an equivalent result in the context of their model, although the way they prove their result is formally distinct from our proof.

[14] For a demonstration that *BLS* and *B* intersect only at the expected utility please see the proof of Proposition 5.

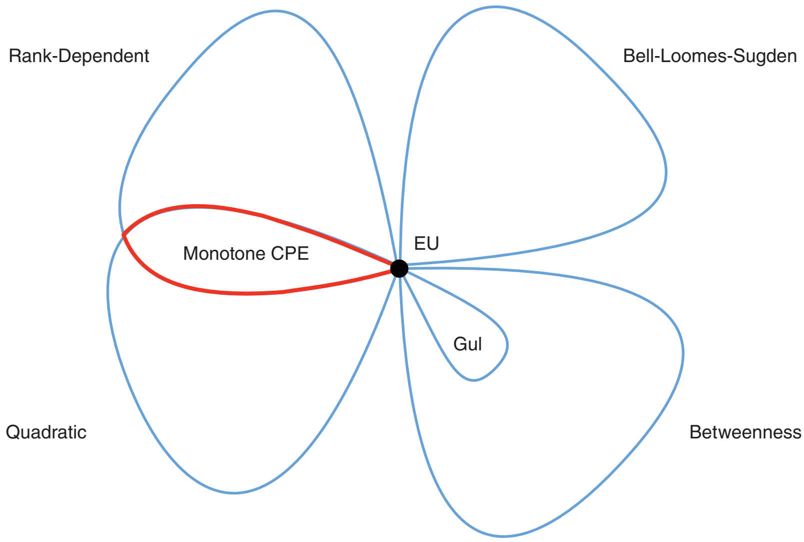

Figure 1. Summary of Relationship between Non-EU Models

III. The Economic Meaning of Parameters: u and λ

Given the uniqueness of the *CPE* representation, we can analyze what the economic interpretations of the parameters of the model are. We show that they are closely tied to the well-studied phenomena of (second-order) risk aversion and first-order risk aversion.[15] First, we examine when individuals' observed preferences are in accordance with the classical notion of risk aversion—aversion to mean-preserving spreads.[16]

DEFINITION 1: *A decision maker is risk-averse if whenever g differs from f by a mean-preserving spread, she prefers f over g.*

We show that, so long as $1 \leq \lambda \leq 2$, u has the standard interpretation: concavity is equivalent to risk aversion. Intuitively, it would seem that loss aversion should enhance any aversion to mean-preserving spreads that u alone induces. This intuition is true if u is linear. However, more generally it is not the case that there is a trade-off between risk and loss aversion in terms of observed behavior. Instead, both a concave consumption utility u and loss aversion $\lambda \geq 1$ are necessary conditions for an individual to be risk-averse. These conditions are not sufficient though. An individual who has a nonlinear u, and is also so loss-averse so that their preferences are no longer monotone, will not always be averse to mean-preserving spreads.

[15] Similar analyses can be done for other models of reference dependence. For example, Gul (1991) provides similar linkages between parameters and behavior in his model of disappointment aversion.

[16] An alternative way of defining risk aversion is that the certainty equivalent of a lottery is less than the expected value of that lottery. However, certainty equivalent is not always well-defined for preferences with a nonmonotone *CPE* representation.

PROPOSITION 6: *Suppose* (u, λ) *represents a decision maker's preference. Then:*

(*i*) *if* $\lambda < 1$, *then the decision maker is not risk-averse*;

(*ii*) *if* $1 \leq \lambda \leq 2$, *then the decision maker is risk-averse if and only if* u *is concave*;

(*iii*) *if* $2 < \lambda$, *then the decision maker is risk-averse if and only if* u *is linear.*

Kőszegi and Rabin (2007) develop intuitions relating riskiness of a lottery to preferences when u is linear (e.g., their Proposition 13): they demonstrate that adding mean-preserving risk to a degenerate lottery reduces the value of the lottery. Proposition 6 implies that these intuitions relating riskiness to preferences generalize only when u is concave and $\lambda \in [1, 2]$. In fact, when u is nonlinear, the only time a decision maker will be risk-averse is when the loss aversion parameter is between 1 and 2. Thus, intuitions developed around increases in risk for monotone *CPE* will generally not extend to nonmonotone ones. Moreover, Kőszegi and Rabin (2007, p. 1060) mention that violations of first-order stochastic dominance could be interpreted as a form of risk aversion: "We also feel that the preference for a stochastically dominated lottery captures in extreme form the strong risk aversion consumers display." As Proposition 6 points out, if individuals are loss-averse, then unless u is linear, violations of first-order stochastic dominance are inconsistent with standard notions of risk aversion—violations of first-order stochastic dominance by preferences imply violations of the ordering imposed by mean-preserving spreads.

We now turn to examining the distinct effects of the loss aversion parameter. Given the results just derived, we will focus on the case of monotone preferences.[17] In the behavioral literature, λ has been associated with attitudes to small-stakes lotteries, and $\lambda \geq 1$ has been thought to capture something described as loss aversion: see Bowman, Minehart, and Rabin (1999). In line with this, Kőszegi and Rabin (2007) specifically describe $\lambda > 1$ as capturing aversion to small-stakes risk and provide sufficient conditions so that an individual will always choose a degenerate lottery over a degenerate lottery plus some noise.

In order to provide a characterization that relates λ to small-stakes risk preferences we will rely on Segal and Spivak's (1990) analysis of first-order risk aversion. Individuals who are first-order risk-averse display an aversion to small-stakes lotteries. As in Segal and Spivak (1990), we will measure the extent to which individuals dislike (or enjoy) small-stakes lotteries using the notion of risk premium (i.e., the difference between the expected value and certainty equivalent of a lottery). Denote $\pi(f)$ as the risk premium of the lottery f. An individual has first-order risk attitudes if the derivative of the risk premium of a fair lottery does not go to zero as the stakes in the lottery become arbitrarily small. We will focus on situations in which individuals have a wealth level w and are facing a lottery ϵf, where ϵ is a scalar that multiplies the sizes of all the outcomes in lottery f. We denote this situation as $w + \epsilon f$.

[17] The results can easily be extended to include nonmonotone preferences.

DEFINITION 2: *A decision maker exhibits first-order risk aversion* (*loving*) *at wealth level w if for all* $f \neq \delta_0$, *where* $E(f) = 0$, $\frac{\partial \pi(w + \epsilon f)}{\partial \epsilon}|_{\epsilon = 0^+}$ *is* $< (>) 0$.

If $\frac{\partial \pi(w + \epsilon f)}{\partial \epsilon}|_{\epsilon = 0^+} \neq 0$ then the individual is not risk-neutral over arbitrarily small lotteries. Moreover, as Segal and Spivak (1990) observe, if $\frac{\partial \pi(w + \epsilon f)}{\partial \epsilon}|_{\epsilon = 0^+} < 0$, then the individual will refuse all better than fair lotteries that are sufficiently small. *CPE* preferences can exhibit first-order risk preferences; in addition, their attitudes are governed entirely by λ. We show that $\lambda > 1$, as is commonly assumed, is equivalent to first-order risk aversion. In order to simplify our statements, we will make the assumption that u is differentiable everywhere on its domain (which we refer to as "everywhere differentiable").[18]

PROPOSITION 7: *Suppose* (u, λ) *represents a decision maker's preference with u everywhere differentiable. Then the decision maker is first-order risk-averse* (*loving*) *at all wealth levels if and only if* $\lambda > 1$ ($\lambda < 1$).

Our result thus tightly links the parameter λ, described as the parameter that captures loss aversion by Kőszegi and Rabin (2007), to the phenomenon of first-order risk aversion. We view this as supporting the opinion of Köbberling and Wakker (2005, p. 125), who state that "…first-order risk aversion, discussed mostly for rank-dependent utility, may be driven by loss aversion to a great extent."

In conjunction with previous results, Proposition 7 indicates an interesting behavioral equivalence that occurs in *CPE*. Individuals are first-order risk-averse (i.e., loss-averse) if and only if they are mixture-averse.

IV. Nonlinear Gain-Loss Functionals

Up until this point, we have focused on the case of linear gain-loss utility; however, there is no reason to assume that this is always the case. In this section, we consider a more general structure, where the gain-loss function does not have to be linear. This can be thought of as a way of enabling the degree of exhibited loss aversion to be stake-dependent. We define the general *CPE* (*GCPE*) functional as

$$V_{GCPE}(f) = \sum_x u(x)f(x) + \sum_x \sum_y \mu(u(x) - u(y))f(x)f(y),$$

where

$$\mu(z) = \begin{cases} \nu(z) & \text{if } z \geq 0 \\ -\lambda\nu(-z) & \text{if } z < 0 \end{cases}$$

and λ is the loss aversion parameter.

[18] If u is not differentiable, then because it is monotonically increasing, it must be that it is differentiable almost everywhere and the definitions and propositions can be appropriately modified.

Moreover, ν is a continuous, strictly increasing function that maps from the positive reals to the positive reals, $\nu(0) = 0$, and ν is differentiable everywhere but 0. In line with the literature, we will also focus on the case where ν exhibits diminishing sensitivity: ν is concave.[19]

Considering this more general functional form allows us to understand more clearly the role that the linearity of the gain-loss functional plays in V_{CPE}. In particular, that the linearity of gain-loss utility is equivalent having a rank-dependent representation. In other words, linearity guarantees a consistent weighting function given a rank-dependent representation.

Mirroring our previous analysis, we first characterize when, given a particular ν, preferences will respect first-order stochastic dominance. Because *GCPE* models have different gain-loss utility functions ν, we must interpret the weighting of gains relative to losses (i.e., λ) differently in terms of behavior.[20]

PROPOSITION 8: *Let a preference be represented by* V_{GCPE}. *Then it respects first-order stochastic dominance if and only if* $-1 \leq (1 - \lambda)\nu'(0) \leq 1$.

This result indicates that as $\nu'(0)$ increases, or as individuals become more sensitive to receiving a very small gain or loss, the range of λs that generate behavior consistent with first-order stochastic dominance shrinks. For example, if $\nu(z) = \log(z + 1)$, then $\nu'(0) = 1$; so $0 \leq \lambda \leq 2$ generates preferences that respect first-order stochastic dominance (as in *CPE*). In contrast, if $\nu(z) = \sqrt{z}$, then $\lim_{z \to 0^+} \nu'(z) = \infty$, and so the only λ that allows preferences to be monotone is $\lambda = 1$ (i.e, the expected utility).

Many of the relationships between *CPE* and other non-*EU* models of choice discussed in Section II extend to *GCPE*.

PROPOSITION 9: *Any preference represented by* V_{GCPE} *also admits a quadratic representation.*

Proposition 9 immediately implies that if a decision maker's preference can be represented by both V_{GCPE} and V_B (or V_{BLS}), she must be an expected utility maximizer.[21] However, any preference represented by a nonlinear V_{GCPE} does not admit an *RDU* representation. This is true even if we restrict ourselves to monotone preferences.

Despite the fact the *RDU* toolkit is no longer applicable, we can still use methods developed for the quadratic utility functionals to understand *GCPE*. These include understanding when an individual with a *GCPE* representation is risk-averse. An immediate implication of Chew, Epstein, and Segal (1991) is that a decision

[19] This is more restrictive than the assumptions A0–A4 Kőszegi and Rabin (2007) make on μ. Similar results can be derived using only A0–A4 but are harder to interpret and verify, as they depend on global properties of μ.

[20] This raises an important point for calibration exercises—that one must be careful about taking estimates of λ derived from one model and applying them to a second with a different ν, as the same λ can generate quite different behavior depending on ν.

[21] In fact, the result of Proposition 9 extends more generally, as it will hold for any arbitrary gain-loss function μ.

maker with a *GCPE* representation is risk-averse if and only if $u(x) + u(y) + (1 - \lambda)\nu(|u(x) - u(y)|)$ is concave in x for all y.[22]

Recall that we were able to demonstrate a previously unknown relationship between loss aversion/loving behavior and attitudes toward mixing lotteries within the *CPE* framework. This provides a powerful test of the predictions of *CPE*. We extend this result and show that mixture aversion is also equivalent to $\lambda \geq 1$ in *GCPE*.

PROPOSITION 10: *If a preference has a GCPE representation, then, for any* Δ_3, (i) *indifference curves are ellipses and* (ii) *preferences are mixture-averse* (*loving*) *if and only if* $\lambda \geq 1$ ($\lambda \leq 1$).

Although we have focused our attention on situations where ν exhibits decreasing marginal sensitivity, the case of increasing marginal sensitivity is also interesting. Mean-variance preferences, where individuals have preferences over the first two moments of lotteries, is a special case of *GCPE* in this case: if $\nu(z) = z^2$, it is easy to verify that $V_{GCPE}(f) = \sum_x u(x)f(x) + (1 - \lambda)(\mathrm{Var}_f(u(x)))$, and so mean-variance preferences occur when u is linear.

V. Discussion

We believe our theoretical results are useful in and of themselves, because they help illuminate relationships between models that are attempting to explain the same set of stylized facts. In addition, we also think our results provide new ways to link *CPE* to common choice patterns, a point we explore in this section with a stylized example.

An important argument against the plausibility of the expected utility is the Rabin (2000) critique: under very mild conditions it is impossible to find expected utility preferences that generate plausible behavior over both small- and large-stakes lotteries. Rabin's calibration result and Safra and Segal's (2005) extension show how local behavior relates to global behavior.

As we will show, monotone *CPE* representations also suffer from a modified version of Rabin's critique. Theorem 5 in Safra and Segal (2005) shows that if (i) preferences are in *RDU*; (ii) u has either decreasing absolute risk aversion everywhere or increasing absolute risk aversion everywhere; and (iii) the decision maker plausibly rejects small-stakes lotteries when added to any gamble defined over relevant wealth levels, then the decision maker should also (implausibly) reject very attractive large-stakes lotteries. By Proposition 4, a decision maker whose preferences have a monotone *CPE* representation satisfying condition (ii) also suffers from this calibration critique.[23] For example, assume u exhibits decreasing absolute

[22] Although it might seem that this condition could be easily satisfied, this is not the case. In fact, in many situations, even "reasonable-looking" parameterizations of *GCPE* will not be risk-averse. For example, if u is linear, and ν is strictly concave, then preferences with a *GCPE* representations are risk-averse if and only if they are expected utility. To see this, observe that necessary conditions for concavity are that both $u''(x) + u''(x)(1 - \lambda)\nu'(|u(x) - u(y)|) + (u')^2(1 - \lambda)\nu''(|u(x) - u(y)|)$ and $u''(x) - (1 - \lambda)u''(x)\nu'(|u(x) - u(y)|) + (1 - \lambda)(u')^2\nu''(|u(x) - u(y)|)$ are less than 0 and $\lambda \geq 1$. The first condition reduces to $(1 - \lambda)\nu''(z) \leq 0$, which contradicts the third condition unless $\lambda = 1$.

[23] Neilson (2001) makes a related calibrational critique of *RDU*.

risk aversion. Then, as Safra and Segal (2005) demonstrate, if a decision maker whose preferences have a monotone *CPE* representation rejects a lottery that gives -100 with probability 0.5 and 110 with probability 0.5 when added to all gambles defined over a large enough wealth level with a lower bound of w, she will reject a lottery that gives $-20{,}000$ with probability 0.0054 and $100{,}000 - \zeta$ with probability 0.9946 for a sequence of ζs converging to 0 at wealth level w.

Linearity of *CPE* is crucial for the calibration result of Safra and Segal (2005). If the assumption that μ (the gain-loss function) is linear is relaxed, it is possible to generate plausible small- and large-stakes risk aversion. Kőszegi and Rabin's (2007) Table 1 does exactly this. However, the most tractable form of Kőszegi and Rabin's (2007) model, that with linear gain-loss utility, cannot avoid an extension of the Rabin critique. Thus, in order to model individuals who exhibit plausible behavior over both small- and large-stakes lotteries, we must turn to nonlinear gain-loss functionals.

VI. Conclusion

This paper contributes to understanding behavior under loss aversion and endogenous reference-point formation. In particular, understanding where *CPE* fits within the taxonomy of non-EU theory can be extremely helpful for both theoretical and empirical researchers. It allows researchers to make use of a larger toolkit of methods and to better understand how to distinguish models of reference dependence from one another. In particular, the relationships developed in Section II provide new opportunities to relate theory to data.

Theorem 1 implies that we can test *CPE* using existing data originally designed to test other models of choice under risk (e.g., *RDU*). Thus, subject to the experimental design correctly capturing the psychology underlying *CPE*, our results allow us to bring over 20 years of existing experimental evidence to bear on *CPE*. For example, the weighting function of *CPE* in its rank-dependent representation must be strictly convex, a prediction at odds with much of the existing literature (for one example, among many, see Gonzalez and Wu 1999).

More generally, we discuss multiple classes of models which incorporate endogenous reference points. All these classes accommodate many of the same stylized facts, including small-stakes risk aversion. Thus, many experimental tests of these behaviors cannot serve to distinguish between competing explanations. However, as the results of our paper make clear, these models differ in their predictions regarding attitudes toward randomization. As Proposition 3 demonstrates, preferences with a loss-averse *CPE* representation are always mixture-averse. In contrast, other models of reference-dependent preferences exhibit distinct attitudes toward mixing: Gul's (1991) preferences are both mixture-averse and mixture-loving; while Bell-Loomes-Sugden's preferences are mixture-averse on part of their domain and mixture-loving on part of the domain. Thus, examining these attitudes represents a potentially useful area of research in order to better understand reference-point formation.

Appendix

We prove only a subset of all results here. The rest are contained in the online Appendix. We first prove Propositions 9 and 8 since they are useful in the proof of Proposition 1 and Theorem 1.

PROPOSITION 9: *Any preference represented by V_{GCPE} also admits a quadratic representation.*

PROOF:
Define

$$\phi(x,y) = \frac{1}{2}(u(x) + u(y)) + \frac{1}{2}\mu(u(x) - u(y)) + \frac{1}{2}\mu(u(y) - u(x)).$$

The quadratic functional defined using these ϕ is easily shown to generate V_{GCPE}.[24] ■

PROPOSITION 8: *Let a preference be represented by V_{GCPE}. Then it respects first-order stochastic dominance if and only if $-1 \leq (1 - \lambda)\nu'(0) \leq 1$.*

PROOF:
Since any preference represented by V_{GCPE} also admits a quadratic representation we can use the fact from Chew, Epstein, and Segal (1991) that the necessary and sufficient conditions for preferences to respect first-order stochastic dominance are that $\phi(x,x) \geq \phi(y,y)$ whenever $x > y$ and $\phi(x,y)$ is nondecreasing in x for all y. So long as u is monotone, our first condition is satisfied. We need to verify the second. Because ν is increasing, the condition is satisfied at $x = y$ or where the argument of ν is 0. Thus, we need to check everywhere else on the domain, which implies the argument of ν is not 0 and so ν is differentiable.

Fix $\hat{y}$. If $x > \hat{y}$ then ϕ is $0.5(u(x) + u(\hat{y}) + (1 - \lambda)\nu(u(x) - u(\hat{y})))$. The first derivative has the same sign as $u'(x) + u'(x)(1 - \lambda)\nu'(u(x) - u(\hat{y}))$. Then $u'(x) + u'(x)(1 - \lambda)\nu'(u(x) - u(\hat{y})) \geq 0$ if and only if $1 + (1 - \lambda)\nu'(u(x) - u(\hat{y})) \geq 0$ or $1 \geq (\lambda - 1)\nu'(u(x) - u(\hat{y}))$. If $\lambda \leq 1$ then this is always true, so we focus on the case where $\lambda \geq 1$. Then we get $\frac{1}{\lambda - 1} \geq \nu'(z)$ for $z \geq 0$. Since ν exhibits decreasing marginal sensitivity $\frac{1}{\lambda - 1} \geq \nu'(0)$ if and only if $\frac{1}{\lambda - 1} \geq \nu'(z)$ for all $z \geq 0$.

If $x < \hat{y}$ then ϕ is $0.5(u(x) + u(\hat{y}) + (1 - \lambda)\nu(u(\hat{y}) - u(x)))$ and so the derivative has the same sign as $u'(x) - (1 - \lambda)u'(x)\nu'(u(\hat{y}) - u(x))$. Then $u'(x) - (1 - \lambda)u'(x)\nu'(u(\hat{y}) - u(x)) \geq 0$ if and only if $1 \geq (1 - \lambda)\nu'(u(\hat{y}) - u(x))$. If $\lambda \geq 1$ then this is always true, so we focus on the case where $\lambda \leq 1$. Then this is equivalent to $\frac{1}{1 - \lambda} \geq \nu'(z)$. Decreasing marginal sensitivity again implies that this is true if the inequality holds for $\nu'(0)$.

[24] There is no overlap between *GCPE* and *B* other than the expected utility because of the relationships between *B* and *Q*.

So we have $\frac{1}{\lambda - 1} \geq \nu'(0)$ if $\lambda \geq 1$, and $\frac{1}{1 - \lambda} \geq \nu'(0)$ if $\lambda \leq 1$. Thus, $-1 \leq (1 - \lambda)\nu'(0) \leq 1$. ∎

PROPOSITION 1: *Let a preference have a CPE representation. Then it respects first-order stochastic dominance if and only if* $0 \leq \lambda \leq 2$.

PROOF:

This is an immediate corollary of Proposition 8. ∎

THEOREM 1: *A preference has both Q and RDU representations if and only if it has a monotone CPE representation.*

PROOF:

First we need to show that a preference with a monotone *CPE* representation also has a *Q* representation. This is an immediate corollary of Proposition 9. Second, we illustrate that if a preference has a monotone *CPE* representation then it also has a *RDU* representation.

CLAIM 1: *Every preference with a monotone CPE representation must have a RDU representation.*

PROOF:

This proof is constructive. Let (u, λ) be a monotone *CPE* representation of $\succsim$. Then we illustrate that w_λ is a particular probability weighting structure such that (i) $w_\lambda(z) = (\lambda - 1)z^2 + (2 - \lambda)z$, (ii) (u, w_λ) is a rank-dependent representation of $\succsim$. Initially, we prove this claim for lotteries where all outcomes are equally likely. Since this set of lotteries is dense in Δ_X this proves the claim for all lotteries (see Chew, Epstein, and Segal 1991).

Take a lottery f such that all outcomes are equally likely and consider the set of outcomes x such that $f(x) \neq 0$. Label them $x_1, \ldots, x_n$ in increasing order. We now utilize the quadratic representation, that is, $V_{CPE}(f) = \sum_{x_i} \sum_{x_j} \phi(x_j, x_i) f(x_j) f(x_i)$. We fix the second argument x_i in the quadratic representation of *CPE* and explicitly expand $\sum_{x_j} \phi(x_j, x_i) f(x_j)$.

$$\frac{1}{2n}\big(u(x_1) + u(x_i) - (\lambda - 1)(u(x_i) - u(x_1)) + \cdots + u(x_n)$$

$$+ u(x_i) - (\lambda - 1)(u(x_n) - u(x_i))\big).$$

That is equal to

$$\frac{1}{2n}\Big\{[(n + 1) - (\lambda - 1)(i - 1) + (\lambda - 1)(n - i)]u(x_i)$$

$$+ \sum_{j<i} \lambda u(x_j) + \sum_{j>i} (2 - \lambda)u(x_j)\Big\}.$$

Now we take the additional sum over the second argument x_i. That is,

$$V_{CPE}(f) = \sum_{x_i}\sum_{x_j}\phi(x_j, x_i)f(x_i)f(x_j)$$
$$= \frac{1}{2n^2}\sum_i\Big([(n+1)+(\lambda-1)(n-2i+1)]u(x_i)+\sum_{j<i}\lambda u(x_j)+\sum_{j>i}(2-\lambda)u(x_j)\Big).$$

Notice that if $j < i$ then $u(x_j)$ has a weight of λ. if $j > i$ then $u(x_j)$ has a weight of $2-\lambda$. We now identify the coefficient in front of $u(x_i)$ in the previous formula. The ith term will appear $n-i$ times as the lower outcome in pairs of outcomes and $i-1$ times as the greater outcome in pairs of outcomes. Thus, the coefficient in front of $u(x_i)$ is

$$\frac{1}{2n^2}((n+1)+(\lambda-1)(n-2i+1)+(i-1)(2-\lambda)+(n-i)\lambda),$$

which is equal to

$$\frac{2\lambda n+4(1-\lambda)i+2(\lambda-1)}{2n^2}.$$

Hence, we can rewrite the entire equation as follows:

$$\sum u(x_i)\left(\lambda\frac{1}{n}+(1-\lambda)\frac{2i-1}{n^2}\right).$$

We will write the equation above in terms of *RDU* representation. Consider

$$w_\lambda(z) = (\lambda-1)z^2+(2-\lambda)z.$$

Notice that w is convex for $1 \leq \lambda \leq 2$, and is concave for $0 \leq \lambda \leq 1$. It is routine to check that

$$w_\lambda\left(\frac{n-i+1}{n}\right) - w_\lambda\left(\frac{n-i}{n}\right) = \lambda\frac{1}{n}+(1-\lambda)\frac{2i-1}{n^2}$$

is equivalent to the quadratic representation.

Hence, this proves the claim for lotteries where all outcomes are equally likely. Since this set of lotteries is dense in Δ_X this proves the claim for all lotteries by continuity. Notice, when $\lambda > 2$ (or $\lambda < 0$), w_λ is not increasing. Hence, it will violate first-order stochastic dominance. Moreover, the relationship is strict because w_λ is a particular probability weighting structure. ∎

We now prove the opposite direction: any preference with both *Q* and *RDU* representations has also a monotone *CPE* representation. We first prove a simple claim required for the rest of the proof.

CLAIM 2: *If preferences have a rank-dependent representation and quadratic representation then the probability weighting function in the rank-dependent representation must be differentiable.*

PROOF:

Consider the rank-dependent representation of the preference and the indifference curves they induce:

$$u(\overline{\delta})[w(q) - w(0)] + u(\hat{\delta})[w(1-p) - w(q)] + u(\underline{\delta})[w(1) - w(1-p)] = k.$$

We can rewrite this as $w(q)[u(\overline{\delta}) - u(\hat{\delta})] + w(1-p)[u(\hat{\delta}) - u(\underline{\delta})] + u(\underline{\delta}) = k$. The slopes of the indifference curves are $\frac{w'(1-p)[u(\hat{\delta}) - u(\underline{\delta})]}{w'(q)[u(\overline{\delta}) - u(\hat{\delta})]}$.

Note that w is a continuous monotone function so it must be nondifferentiable on a measure 0 set. Suppose that the weighting function w is not differentiable at $1-p$. Then there exists a q such that $\frac{w'(1-p)[u(\hat{\delta}) - u(\underline{\delta})]}{w'(q)[u(\overline{\delta}) - u(\hat{\delta})]}$ is not defined. Thus, at (p,q) the indifference curves must have a kink: i.e., they cannot be smooth in the space Δ_3. Observe that if preferences have a quadratic representation then slopes of the indifference curves can be written as $-\frac{Bq + 2Cp + E}{Bp + 2Aq + D}$ for some parameters A, B, C, D, and E. This, by construction, has no kinks in the space Δ_3. Thus, w must be differentiable. ∎

CLAIM 3: *Any preference with both Q and RDU representations has also a monotone CPE representation.*[25]

PROOF:

Now, let (u,w) be an RDU representation of $\succsim$. Given any two outcomes we normalize the utility of the better outcome to 1 and utility of the lesser outcome to 0. Choose $x(p)$ such that

$$f_1 = (0, 1-p; x(p), p) \sim (1,1) = f_2.$$

We have $V(f_1) = u(x(p))w(p) + u(0)(1 - w(p)) = u(1)w(1) = V(f_2)$. Then we must have

$$u(x(p)) = \frac{1}{w(p)}.$$

Notice that $x(p) > 1$. Since we have a quadratic functional, we must have the following: for any f_1, f_2,

$$\text{(1)} \qquad \frac{d}{d\alpha} V(\alpha f_1 + (1-\alpha) f_2) = 0 \text{ if and only if } \alpha = \frac{1}{2}.$$

[25] We would like to thank one of the referees for suggesting the following the proof strategy. It immensely simplified the proof.

We first write the lottery $\alpha f_1 + (1 - \alpha) f_2$, that is,

$$\alpha f_1 + (1 - \alpha) f_2 = (x(p), \alpha p; 1, (1 - \alpha); 0, \alpha(1 - p)).$$

Hence,

$$V(\alpha f_1 + (1 - \alpha) f_2) = u(x(p)) w(\alpha p) + u(1)[w(1 - \alpha + \alpha p) - w(\alpha p)].$$

Given that w is differentiable, we know that

$$\frac{dV}{d\alpha} = \frac{d}{d\alpha}\left\{\left(\frac{1}{w(p)} - 1\right) w(\alpha p) + w(1 - \alpha + \alpha p)\right\}$$

$$= p\left(\frac{1}{w(p)} - 1\right) w'(\alpha p) - (1 - p) w'(1 - \alpha + \alpha p).$$

By equation (1), if we evaluate $\frac{dV}{d\alpha}$ at $\alpha = \frac{1}{2}$, we must get 0. That is,

$$\frac{dV}{d\alpha}\Big|_{\alpha=\frac{1}{2}} = p\left(\frac{1}{w(p)} - 1\right) w'\left(\frac{p}{2}\right) - (1 - p) w'\left(\frac{1 + p}{2}\right) = 0.$$

Then we end up with the following differential equation:

$$p(1 - w(p)) w'\left(\frac{p}{2}\right) = (1 - p) w(p) w'\left(\frac{1 + p}{2}\right). \tag{2}$$

We now use the method of power series; that is, we look for a solution of the form:

$$w(p) = \sum_{n=0}^{\infty} c_n p^n.$$

Since w is a weighting function, we have $w(0) = 0$ and $w(1) = 1$. While the former implies $c_0 = 0$, the latter implies $\sum_{n=0}^{\infty} c_n = 1$. Then we take the derivative:

$$w'(p) = \sum_{n=1}^{\infty} n c_n p^{n-1} = \sum_{n=0}^{\infty} (n + 1) c_{n+1} p^n.$$

If we plug these into equation (2), we get

$$p\left\{1 - \sum_{n=0}^{\infty} c_n p^n\right\}\left\{\sum_{n=0}^{\infty} (n + 1)\frac{c_{n+1}}{2^n} p^n\right\}$$

$$= (1 - p)\left\{\sum_{n=0}^{\infty} c_n p^n\right\}\left\{\sum_{n=0}^{\infty} (n + 1)\frac{c_{n+1}}{2^n}(1 + p)^n\right\}.$$

The left-hand side of equation (2) is

$$c_1 p + (c_2 - c_1 c_1) p^2 + \left(\frac{3c_3}{2^2} - c_1 \frac{2c_2}{2} - c_2 c_1\right) p^3$$

$$+ \left(\frac{4c_4}{2^3} - c_1 \frac{3c_3}{2^2} - c_2 \frac{2c_2}{2} - c_3 c_1\right) p^4 + \cdots.$$

The right-hand side of equation (2) is

$$c_1 \left\{ \sum_{n=1}^{\infty} \frac{nc_n}{2^{n-1}} \right\} p + \left\{ (c_2 - c_1) \sum_{n=1}^{\infty} \frac{nc_n}{2^{n-1}} + c_1 \sum_{n=2}^{\infty} \frac{n(n-1)c_n}{2^{n-1}} \right\} p^2 + \cdots.$$

Since the coefficients of p in both sides must be equal, we must have

$$\sum_{n=1}^{\infty} \frac{nc_n}{2^{n-1}} = c_1 + c_2 + \frac{3c_3}{4} + \frac{4c_4}{8} + \cdots = 1.$$

Given this and the coefficients of p^2 in both sides must be equal, we should have:

$$\sum_{n=2}^{\infty} \frac{n(n-1)c_n}{2^{n-1}} = c_2 + \frac{3c_3}{2} + \frac{3c_4}{2} + \frac{5c_5}{4} + \cdots = 1 - c_1.$$

Hence, we have two equations:

$$c_1 + c_2 + \frac{3c_3}{4} + \frac{4c_4}{8} + \frac{5c_5}{16} \cdots = 1 \quad \text{and} \quad c_1 + c_2 + \frac{3c_3}{2} + \frac{3c_4}{2} + \frac{5c_5}{4} + \cdots = 1.$$

They imply

$$c_i = 0 \text{ for all } i \geq 3 \quad \text{and} \quad c_1 + c_2 = 1.$$

Since w is increasing, we must have $c_1 + 2c_2 p \geq 0$ for all $p \in [0,1]$. When $p = 0$, we must have $c_1 \geq 0$. When $p = 1$, we have $c_1 + 2(1 - c_1) \geq 0$, which implies $c_1 \leq 2$. Therefore,

$$w_\lambda(p) = (2 - \lambda)p + (\lambda - 1)p^2$$

is the only solution for the differential equation where $2 \geq \lambda \geq 0$. ∎

By Claim 1, for every *RDU* representation in the form of (u, w_λ), there exists a corresponding *CPE* representation in the form of (u, λ), which proves the theorem. ∎

PROPOSITION 2: *For any preference with a monotone CPE representation, u is unique up to affine transformation and* λ *is unique.*

PROOF:

Consider a quadratic function ϕ that represents $\succsim$ which is not equivalent to expected utility. Then ϕ is unique up to affine transformations by Theorem 2 of Chew, Epstein, and Segal (1991). This implies u is unique up to an affine transformation and λ is unique. To see one direction, consider an affine transformation of $u' = \alpha u + \beta$ where $\alpha > 0$ and define a ϕ':

$$\phi'(x,y) = \frac{1}{2}(\alpha u(x) + \beta + \alpha u(y) + \beta) + \frac{1}{2}(1 - \lambda)|\alpha u(x) + \beta - \alpha u(y) - \beta|$$

$$= \alpha \frac{1}{2}(u(x) + u(y)) + \alpha \frac{1}{2}(1 - \lambda)|u(x) + u(y)| + \beta$$

$$= \alpha\phi(x,y) + \beta.$$

This is an affine transformation of ϕ, hence it represents $\succsim$ by Chew, Epstein, and Segal (1991). For the other direction, assume (u,λ) and (v,λ') both represent $\succsim$. Let ϕ and ϕ' be the quadratic representation of (u,λ) and (v,λ'), respectively. By Chew, Epstein, and Segal (1991), there exist $\alpha > 0$ and β such that $\phi'(x,y) = \alpha\phi(x,y) + \beta$. Thus, $\phi'(x,x) = \alpha\phi(x,x) + \beta$. Since by construction $\phi(x,x) = u(x)$, we must have $v(x) = \alpha u(x) + \beta$. Then it is routine to check that $\lambda = \lambda'$. ∎

PROPOSITION 3: *For any preference with a CPE representation* (u,λ), *it satisfies mixture aversion* (*loving*) *if and only if* $\lambda \geq 1$ ($\lambda \leq 1$).

PROOF:

The proof for monotone *CPE* is discussed in the text. For nonmonotone *CPE*, observe that Observation 2 of Wakker (1994), which proves that a convex weighting function implies a convex V_{RDU} (in terms of probabilities), and so mixture aversion (and similar a concave weighting function implies mixture-loving) does not depend on the monotonicity assumption of the weighting function. Moreover, looking at the proof of Theorem 1, it is clear that we can represent any preferences in *CPE* and represented by (u,λ) with a representation that has the same functional form as *RDU* with representation (u,w) but is not in the actual class *RDU*, because w is not monotone. However, w is still convex (concave) if and only if $\lambda \geq (\leq) 1$.

To go the other way, observe that λ is a scalar, so the weighting function must be either (globally) convex or concave. ∎

REFERENCES

Abdellaoui, Mohammed. 2002. "A Genuine Rank-Dependent Generalization of the Von Neumann-Morgenstern Expected Utility Theorem." *Econometrica* 70 (2): 717–36.

Abeler, Johannes, Armin Falk, Lorenz Goette, and David Huffman. 2011. "Reference Points and Effort Provision." *American Economic Review* 101 (2): 470–92.

Bell, David E. 1985. "Disappointment in Decision Making under Uncertainty." *Operations Research* 33 (1): 1–2.

Bowman, David, Deborah Minehart, and Matthew Rabin. 1999. "Loss Aversion in a Consumption-Savings Model." *Journal of Economic Behavior & Organization* 38 (2): 155–78.

Carbajal, Juan Carlos, and Jeffrey C. Ely. 2012. "Optimal Contracts for Loss Averse Consumers." University of Queensland Economics Discussion Paper 460.

Card, David, and Gordon B. Dahl. 2011. "Family Violence and Football: The Effect of Unexpected Emotional Cues on Violent Behavior." *Quarterly Journal of Economics* 126 (1):103–43.

Chew, Soo Hong. 1983. "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox." *Econometrica* 51 (4): 1065–92.

Chew, Soo Hong, Larry G. Epstein, and Uzi Segal. 1991. "Mixture Symmetry and Quadratic Utility." *Econometrica* 59 (1): 139–63.

Chew, Soo Hong, Larry G. Epstein, and Uzi Segal. 1994. "The Projective Independence Axiom." *Economic Theory* 4 (2): 189–215.

Crawford, Vincent P., and Juanjuan Meng. 2011. "New York City Cab Drivers' Labor Supply Revisited: Reference-Dependent Preferences with Rational-Expectations Targets for Hours and Income." *American Economic Review* 101 (5): 1912–32.

Dekel, Eddie. 1986. "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom." *Journal of Economic Theory* 40 (2): 304–18.

Delquié, Philippe, and Alessandra Cillo. 2006. "Disappointment without Prior Expectation: A Unifying Perspective on Decision under Risk." *Journal of Risk and Uncertainty* 33 (3): 197–215.

Eliaz, Kfir, and Ran Spiegler. 2014. "Reference Dependence and Labor Market Fluctuations." In *NBER Macroeconomics Annual 2013, Volume 28*, edited by Jonathan A. Parker and Michael Woodford, 159–200. Chicago: University of Chicago Press.

Fishburn, Peter C. 1983. "Transitive Measurable Utility." *Journal of Economic Theory* 31 (2): 293–317.

Freeman, David. 2012. "Revealed Preference Foundations of Expectations-Based Reference Dependence." http://www.sfu.ca/~dfa19/EBRDa.pdf (accessed November 20, 2015).

Gonzalez, Richard, and George Wu. 1999. "On the Shape of the Probability Weighting Function." *Cognitive Psychology* 38 (1): 129–66.

Gul, Faruk. 1991. "A Theory of Disappointment Aversion." *Econometrica* 59 (3): 667–86.

Heidhues, Paul, and Botond Kőszegi. 2008. "Competition and Price Variation when Consumers Are Loss Averse." *American Economic Review* 98 (4): 1245–68.

Heidhues, Paul, and Botond Kőszegi. 2014. "Regular Prices and Sales." *Theoretical Economics* 9 (1): 217–51.

Herweg, Fabian, Daniel Müller, and Philipp Weinschenk. 2010. "Binary Payment Schemes: Moral Hazard and Loss Aversion." *American Economic Review* 100 (5): 2451–77.

Kahneman, Daniel, and Amos Tversky. 1979. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica* 47 (2): 263–92.

Karle, Heiko, and Martin Peitz. 2014. "Competition under Consumer Loss Aversion." *RAND Journal of Economics* 45 (1): 1–31.

Köbberling, Veronika, and Peter P. Wakker. 2005. "An Index of Loss Aversion." *Journal of Economic Theory* 122 (1): 119–31.

Kőszegi, Botond, and Matthew Rabin. 2006. "A Model of Reference-Dependent Preferences." *Quarterly Journal of Economics* 121 (4): 1133–65.

Kőszegi, Botond, and Matthew Rabin. 2007. "Reference-Dependent Risk Attitudes." *American Economic Review* 97 (4): 1047–73.

Loomes, Graham, and Robert Sugden. 1986. "Disappointment and Dynamic Consistency in Choice under Uncertainty." *Review of Economic Studies* 53 (2): 271–82.

Machina, Mark J. 1982. "'Expected Utility' Analysis without the Independence Axiom." *Econometrica* 50 (2): 277–323.

Markowitz, Harry. 1952. "The Utility of Wealth." *Journal of Political Economy* 60 (2): 151–58.

Neilson, William S. 2001. "Calibration Results for Rank-Dependent Expected Utility." *Economics Bulletin* 4 (10): 1–5.

Pope, Devin G., and Maurice E. Schweitzer. 2011. "Is Tiger Woods Loss Averse? Persistent Bias in the Face of Experience, Competition, and High Stakes." *American Economic Review* 101 (1): 129–57.

Quiggin, John. 1982. "A Theory of Anticipated Utility." *Journal of Economic Behavior & Organization* 3 (4): 323–43.

Rabin, Matthew. 2000. "Risk Aversion and Expected-Utility Theory: A Calibration Theorem." *Econometrica* 68 (5): 1281–92.

Safra, Zvi, and Uzi Segal. 2005. "Are Universal Preferences Possible? Calibration Results for Non-Expected Utility Theories." http://fmwww.bc.edu/EC-P/wp633.pdf (accessed November 20, 2015).

Segal, Uzi, and Avia Spivak. 1990. "First Order versus Second Order Risk Aversion." *Journal of Economic Theory* 51 (1): 111–25.

Sydnor, Justin. 2010. "(Over)insuring Modest Risks." *American Economic Journal: Applied Economics* 2 (4): 177–99.

Tversky, Amos, and Daniel Kahneman. 1992. "Advances in Prospect Theory: Cumulative Representation of Uncertainty." *Journal of Risk and Uncertainty* 5 (4): 297–323.

Wakker, Peter. 1994. "Separating Marginal Utility and Probabilistic Risk Aversion." *Theory and Decision* 36 (1): 1–44.