A Random Reference Model

Özgür Kıbrıs
Sabanci U

Yusufcan Masatlıoglu
U Maryland

Elchin Suleymanov
Purdue U

Why is choice random?

- Agent's choices are deterministic but the outside observer only receives aggregate/stochastic choices
  - Randomly fluctuating tastes
  - Noisy signals
  - Attention is random
  - Experimentation (experience goods)
  - TODAY: Random reference point

Reference Dependence

- Since Markowitz [1952], Kahneman and Tversky [1979], and Tversky and Kahneman [1991]
- Reference-dependence widely accepted as a fundamental feature of decision making

Reference-Dependence

- Can explain observed behavior, such as
  - Default bias (in pension and insurance choice, selection of internet privacy, organ donation), golf players, poker players, cab drivers, physicians, fishermen, deer hunters, drivers
- Can explain "economic puzzles" such as
  - Attitudes towards risk, equity premium puzzle, annuitization puzzle, disposition effect in financial markets and in housing markets

Multiple Potential Reference Points

- Existing theories on reference-dependent choice assume a single alternative serves as the reference point
- Many real-life situations present a multitude of candidates for reference points
  - Kahneman [1992], March and Shapira [1992], Baucells et al. [2011], Koop and Johnson [2012], Baillon et al. [2020]
- Kahneman [1992]:

> *There are many situations in which people are fully aware of the multiplicity of relevant reference points, and the question of how they experience such outcomes and think about them must be raised. There appears to have been little discussion of this issue in behavioral decision research.*

Multiple Potential Reference Points

- When reference points are not observable

  multiple reference points $\Rightarrow$ additional challenge in predicting behavior

- An outside observer must identify

  1. which reference points are used
  2. how frequently they are employed
  3. how reference points affect preferences

Research Question

- provide random reference models
  - parametric
  - non-parametric

- illustrate when and how one can identify parameters of the model from observed choice data

- Identifying model's parameters important for a policy maker

  1. to make out-of-sample predictions

  2. evaluate the impact of policy interventions

A Model to Encompass Multiple Reference Points

Random Reference Model (RAR)

- Each feasible alternative serves as the reference point with a certain probability
- Each reference point induces a specific reference-dependent preference
- Agent maximizes the reference-dependent preference to choose an alternative

Model

Basic Definitions

- X: finite set of alternatives
- $\emptyset \neq S \subset X$: choice problem
- $p(x|S) \in [0,1]$
- $p(x|S) = 0$ if $x \notin S$
- $\sum_{x \in S} p(x|S) = 1$

Reference-dependent Preferences

- First component of our model: $\{\succ_x\}_{x \in X}$
  - $\succ_x$: reference-dependent preference under $\mathbf{x}$

- alternative is more desirable when it is the reference point (SQB)
  - If $x \succ_z y$, then $x \succ_x y$
  - If $x \succ_y y$, then $x \succ_z y$ for all z

Logit Random Reference Model

Logit Random Reference Model: L-RAR

- This is a parametric model of reference point formation
- p is consistent with the **Logit random reference model (L-RAR)** if
  - there is $\{\succ_x\}_{x \in X}$ and $\{w_x\}_{x \in X}$ such that
  - for each $S \in \mathcal{X}$ and $x \in S$,

$$p(x|S) = \sum_{y \in S} \underbrace{\left(\frac{w_y}{\sum_{z \in S} w_z} \right)}_{\substack{\text{probability of } y \text{ being} \\ \text{the reference point}}} \underbrace{\mathbb{1}(x \text{ is } \succ_y\text{-best in } S)}_{x \text{ is the maximizer of } \succ_y} .$$

Logit Random Reference Model: Example

	Ref-Dep Pref.			Choices				
	$\succ_x$	$\succ_y$	$\succ_z$	$p(.\|S)$	$\{x,y,z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$
	x	y	x	x	$1/2$	$1/3$	1	–
	y	x	z	y	$1/2$	$2/3$	–	$2/3$
	z	z	y	z	0	–	0	$1/3$
w	1	2	1					

L-RAR: Special Cases

- If all $\succ_x = \succ_y$, then
  - L-RAR = Rational Choice Model
- If x is $\succ_x$-best for all x, then
  - L-RAR = Luce (Logit) Model
  - an alternative is chosen only when it is the reference point
  - resembles personal equilibrium of Kőszegi and Rabin [2006]
  - personal equilibrium is equivalent to extreme status quo bias in our framework

Revealed Information

Revealed Information

- Assume p is consistent with L-RAR
- Use p to reveal: preferences and reference weights
- (Find all $\{\succ_x, w_x\}_{x \in X}$ representations of p: what do they look like?)
  - There is random attention on multiple reference points
    $\Rightarrow$
    Difficult to reveal reference dependent preferences.
  - Relative choice probabilities of two alternatives depend on the availability of other alternatives (context-dependence).
    $\Rightarrow$
    Difficult to reveal relative salience parameters.
- Yet, "unique" identification even with limited data

Revealed Information: Extreme Cases

Example 1: p consistent with Luce model

- Reference weights are uniquely determined (up to a normalization)
- Only restriction on preferences: each x is $\succ_x$-best (multiplicity)

Example 2: p consistent with rational choice model

- No restriction on reference weights (multiplicity)
- All preferences identical and uniquely identified

Multiple Representations

If two representations disagree on the relative ranking of two alternatives (under some reference point)

- we cannot make any conclusions regarding the ranking of these two alternatives

Similarly, if two representations disagree on the relative reference weights of two alternatives

- we cannot make any conclusions

Before making any conclusions, require that all representations agree on these revelations.

But how to check all possible representations of p to reveal information? **Need a practical method.**

Revealed Preference

For $y \neq z$ in S, we learn that $y \succ_x z$ if

1. $p(y|S) = 1$,
   - $\implies y$ is $\succ_w$-best in S for all $w \in S$

$$\implies y \succ_z z \overset{\text{(by SQB)}}{\implies} y \succ_x z \text{ for any } x \in X$$

2. $p(y|S) > 0$ and $x = y$
   - $\implies y$ is $\succ_w$-best in S for some $w \in S$

$$\implies y \succ_w z \overset{\text{(by SQB)}}{\implies} y \succ_y z$$

3. $p(y|S) > p(y|S \setminus x)$
   - $\implies y$ is $\succ_x$-best in S
   - $\implies y \succ_x z$

Revealed Preference

Let yP_xz if one of the three patterns above is observed.

- P_x must be transitive
- However, it might not be complete (e.g. Luce)

- P_x must be part of the revealed reference-dependent preferences
- Furthermore, there is no other revelation:
  - any reference-dependent preference that respects $\{P_x\}_{x \in X}$ represents p.

Proposition 1

(Revealed Preference) *Suppose p admits an L-RAR representation. Then y is revealed to be preferred to z under reference point x if and only if yP_xz.*

“Unique” Revealed Preference

- If $\{P_x\}_{x \in S}$ contain few revelations, undesirable
  - Hard for policy makers to make welfare evaluations
- This is not the case!
  - Relative ranking of any two alternatives under any reference point can be revealed as long as it matters for choice
- Relative ranking of y and z under x must be revealed unless both are inferior to x
  - An even in that case, we sometimes reveal preference (SQB)

Proposition 2

(Uniqueness of Revealed Preference) *Suppose p admits an L-RAR representation where at least one of $x \succ_x y$ and $x \succ_x z$ does not hold. Then, assuming $y \neq z$, either yP_xz or zP_xy must hold.*

Revealed Reference Weights

Revealing reference weight of x relative to y

1. if $p(x|\{x,y\}) \in (0,1)$,

$$\alpha_{xy} = \frac{p(x|\{x,y\})}{p(y|\{x,y\})},$$

   - $x \succ_x y$ and $y \succ_y x \implies p(x|\{x,y\})/p(y|\{x,y\}) = w_x/w_y$

2. if $p(x|\{x,y,z\}) = 0$ and $p(z|\{x,y,z\}) > p(z|\{y,z\})$,

$$\alpha_{xy} = \frac{p(z|\{x,y,z\})}{p(y|\{x,y,z\})} - \frac{p(z|\{y,z\})}{p(y|\{y,z\})}$$

   - $z \succ_x$-best (hence $z \succ_z$-best) and $y \succ_y$-best in $\{x,y,z\}$
   - $p(z|\{x,y,z\})/p(y|\{x,y,z\}) = w_{xz}/w_y$, $p(z|\{y,z\})/p(y|\{y,z\}) = w_z/w_y$

3. if $(n-1)$ of $\{\alpha_{x_i x_{i+1}}\}_{i=1}^{n}$ are known, $(n+1=1)$

the last one can be identified through the equality

$$\prod_{i=1}^{n} \alpha_{x_i x_{i+1}} = 1.$$

Revealed Reference Weights

Let xWy if α_{xy} is defined by one of the three patterns above.

- W must be symmetric and transitive
- However, it might not be complete (e.g. rational choice model)

- The α's in W must be part of the revealed reference weights
- Furthermore, there is no other revelation:
  - any weight vector $\{w_x\}_{x \in X}$ that respects $\{\alpha_{xy} \mid (x, y) \in W\}$ represents p.

Proposition 3

(Revealed Reference Weights) *Suppose p admits a RAR representation. Then the reference weight of x relative to y is revealed to be α_{xy} if and only if xWy.*

"Unique" Revealed Reference

- Similar to revealed preference
  - relative reference weights are "unique".
- A relative reference weight can be revealed as long as it matters for choice
  - Relative weight of x to y must be revealed as long as there is an S containing them and where no alternative is chosen with probability 1.

    Otherwise, relative weight of x to y can never influence choice behavior

Proposition 4

(Uniqueness of Revealed Reference) *Suppose p admits an L-RAR representation. Then, for any x and y, if there exists $S \supseteq \{x, y\}$ such that $p(z|S) = 1$ for no $z \in S$, we have xWy.*

Random Reference Model

Random Reference Model (RAR)

L-RAR is very tractable

Is this due to its parametric (Logit) structure?

Generalize L-RAR: only assume that the reference probabilities $\rho(\cdot|S)$

1. are strictly positive

$$\rho(x|S) > 0$$

2. satisfy strict regularity

$$x \in T \subsetneq S \implies \rho(x|T) > \rho(x|S)$$

L-RAR, random utility representations with full support, random consideration model [Manzini and Mariotti, 2014], weighted linear stochastic choice model [Chambers et al., 2021] satisfy both conditions.

Random Reference Model (RAR)

- p is consistent with the **random reference model (RAR)** if
  - there is $\{\succ_x\}_{x \in X}$ and ρ such that
  - for each $S \in \mathcal{X}$ and $x \in S$,

$$p(x|S) = \sum_{y \in S} \rho(y|S)\, \mathbb{1}(x \text{ is } \succ_y\text{-best in } S).$$

How are the revealed preferences in RAR related to L-RAR?

Random Reference Model (RAR)

Revealed preferences are the same in RAR and L-RAR!

RAR is also very tractable

Proposition 5

(Revealed Preference) *Suppose p admits a RAR representation. Then y is revealed to be preferred to z under reference point x if and only if yP_xz.*

This result is important for two reasons

1. preference revelations are not driven by the parametric structure of L-RAR
2. any generalization of L-RAR that satisfies the two basic assumptions of RAR shares the same revelations

RAR vs RUM (Random Utility Model)

- Important differences:
  - RAR $\rightarrow$ set of preferences used context-dependent (at most $|S|$)

    RUM $\rightarrow$ all preferences always used
  - RAR $\rightarrow$ violates regularity

    RUM $\rightarrow$ **regularity axiom** (Block-Marschak Inequalities)

    $$x \in T \subset S \implies p(x|T) \geq p(x|S)$$
  - RAR $\rightarrow$ preferences related via SQB condition

    RUM $\rightarrow$ preferences arbitrary
  - RAR $\rightarrow$ set of rationalizing preferences "almost unique"

    RUM $\rightarrow$ set of rationalizing preferences not unique
- RAR $\cap$ RUM = rational choice and Luce

Regularity and the Attraction Effect

- Luce model and all other random utility models
  - regularity axiom (Suppes and Luce [1965])
- An established behavioral pattern: attraction effect
  - addition of an asymmetrically dominated alternative improving the choice share of the dominating option
  - involves a regularity violation (Huber et al. [1982])

Attraction Effect in Repeated Choice Environments

- There are deterministic models that explain the attraction effect
  - e.g. Kıbrıs et al. [2021]
- Attraction effect recently demonstrated in repeated choice environments
  - Berkowitsch et al. [2014], Mohr et al. [2017]
- Deterministic models can not explain
  - don't allow fluctuating choice
- Luce, RUM and variants can not explain
  - since they satisfy regularity
- RAR can explain attraction effect in repeated choice environments

Behavioral Postulates of RAR

Behavioral Postulates of RAR

Behavioral postulates that characterize RAR

- ordinal axioms (1-4)
- cardinal axiom (5)

Weak Regularity

Axiom 1

If $p(x|\{x,y\}) = 0$, then $p(x|S) = 0$ for every S that contains y.

- if x is not chosen against y then x is not chosen whenever y is available

Axiom 1 is a significant weakening of **regularity**

Eliminating Unchosen Alternative

Axiom 2

If $p(x|S) = 0$, then there is $y \in S$ such that $p(y|\{x, y\}) = 1$ and either $p(y|S) = 1$ or $p(y|S) > p(y|S \setminus x)$.

- if x is not chosen in S then there must be an alternative y which beats x in a binary comparison and in that case either y beats every other alternative in S or eliminating x induces a regularity violation for y

Eliminating Chosen Alternative

Axiom 3

If $p(x|S) > 0$, then $p(y|S) \leq p(y|S \setminus x)$ for any $y \in S \setminus x$.

- elimination of a chosen alternative can not induce a regularity violation

Asymmetry in Regularity Violations

Axiom 4

If $p(x|S) > p(x|S \setminus z)$ and $x, y, z \in T \cap S$, then $p(y|T) \leq p(y|T \setminus z)$.

- if elimination of z induces a regularity violation on x when y is available

$$p(x|S) > p(x|S \setminus z)$$

- elimination of z cannot induce a regularity violation for y when x is available

$$p(y|T) \leq p(y|T \setminus z)$$

Behavioral Postulates of RAR

The ordinal Axioms 1-4 not sufficient.

$p(\cdot, S)$	$\{x, y, z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$
x	0.7	1	0.5	–
y	0	0	–	0.1
z	0.3	–	0.5	0.9

Satisfies Axioms 1-4, but cannot be represented by RAR. Suppose not

- $x \succ_y y \succ_y z$, $x \succ_x y, z$ and $z \succ_z x, y$.
- $\rho(y|\{y, z\}) = p(y|\{y, z\}) = 0.1$.
- Similarly, $\rho(x|\{x, z\}) = p(x|\{x, z\}) = 0.5$.
- By strict regularity, $\rho(x|\{x, y, z\}) < 0.5$ and $\rho(y|\{x, y, z\}) < 0.1$.
- This is a contradiction since $p(x|\{x, y, z\}) > 0.1 + 0.5$.

Behavioral Postulates of RAR

New axiom: limits the size of regularity violations p exhibits

- Guarantees that ρ satisfies positivity and strict regularity

Similar in spirit to Block-Marschak inequalities which characterize RUM

- Closely related to Motzkin's transposition theorem [Motzkin, 1936]
  - The system $Ax \leq b$ and $Bx < c$ has a solution x iff
  - for all vectors $y \geq 0$ and $z \geq 0$, $A^T y + B^T z = 0 \implies b^T y + c^T z \geq 0$ (with strict inequality if $z \neq 0$)

- A "theorem of alternatives"
  - e.g. Farkas' Lemma (used in linear programming)

Motzkin Axiom

Notation: for $x, y \in S \subseteq X$: $\lambda_{xy}(S) \geq 0$ a constant.

λ: vector of all $\lambda_{xy}(S)$ ($\lambda_x(S)$ means $\lambda_{xx}(S)$)

$$V(\lambda, p) = \sum_{S \in \mathcal{X}} \sum_{x \in S \subseteq X} \lambda_x(S) p(x|S) + \sum_{S \in \mathcal{X}} \sum_{x,y \in S \subseteq X \text{ s.t. } x \neq y} \lambda_{xy}(S)(p(x|S \setminus y) - p(x|S)).$$

$$\Gamma_\lambda(x, S) = \lambda_x(S) + \sum_{w \notin S} \lambda_{xw}(S \cup w) - \sum_{z \in S \setminus x} \lambda_{xz}(S)$$

Axiom 5

For any $\lambda \geq 0$ that satisfies $\Gamma_\lambda(x, S) = \Gamma_\lambda(y, S)$ for all $x, y \in S$ such that either $p(x|S) = 1$ or $p(x, S) > p(x, S \setminus y)$, we have $V(\lambda, p) \geq 0$, with strict inequality if $\lambda \neq 0$.

Example violates Motzkin Axiom

- Let $\lambda \geq 0$ be such that
  - $\lambda_{xy}(\{x,y,z\}) = \lambda_{yx}(\{x,y,z\}) = 1$
  - every other term in λ is zero

- If part of the axiom is satisfied
  - $\Gamma_\lambda(x,\{x,y,z\}) = -1 = \Gamma_\lambda(y,\{x,y,z\})$
  - $\Gamma_\lambda(x,\{x,y\}) = 0 = \Gamma_\lambda(y,\{x,y\})$

- However,

$$V(\lambda,p) = p(x|\{x,z\}) - p(x|\{x,y,z\}) + p(y|\{y,z\}) - p(y|\{x,y,z\}) = -0.1 < 0$$

$p(\cdot,S)$	$\{x,y,z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$
x	0.7	1	0.5	–
y	0	0	–	0.1
z	0.3	–	0.5	0.9

Characterization Theorem

Theorem 1

A random choice rule p satisfies Axioms 1–5

if and only if

p has a RAR representation.

- RAR satisfies the Ordinal Axioms and Motzkin Axiom
- and
- any p that satisfies the Ordinal Axioms and Motzkin Axiom

is consistent with RAR

Behavioral Postulates of L-RAR

Behavioral postulates that characterize L-RAR

- ordinal axioms (1-4) SAME
- cardinal axioms (6-9) Replace Motzkin's Axiom

Conclusion

- Two simple models of random reference-dependent choice
  1. L-RAR (a parametric model: attention rule Logit)
  2. RAR (nonparametric generalization of L-RAR)

- **Key innovation:** observations of stochastic choice behavior might be a product of stochastic attention (reference formation) even though, from the point of view of the decision maker, choices are deterministic.

- Demonstrate how to identify parameters.
  - Parameters uniquely identified whenever they affect choice
  - Identification possible with incomplete data

  - Not due to Logit structure on reference point formation
  - RAR and L-RAR have same revealed preference

- Characterize behavioral postulates of RAR (L-RAR in the Appendix)

Conclusion: Extension 1

L-RAR can accommodate regularity violations (e.g. the attraction effect)

Only happen when an alternative is chosen with zero probability

- Straightforward to create a modification where
  1. all alternatives are chosen with positive probability
  2. regularity violations continue to occur
- Consider the following model ($\varepsilon > 0$):

$$p(x|S) = \sum_{y \in S} \left(\frac{w(y)}{\sum_{z \in S} w(z)} \right) \left(\varepsilon \mathbb{1}(x = y) + (1-\varepsilon)\mathbb{1}(x \text{ is } \succ_y\text{-best in } S)\right)$$

  - As $\varepsilon \to 0$, this model converges to L-RAR.
  - Has a bounded rationality interpretation:

    maximize $\succ$ with probability $1 - \varepsilon$
    pick reference point with probability ε

Conclusion: Extension 2

Another possible extension

Allows an exogenous reference point r to probabilistically affect the reference formation process:

$$p_r(x|S) = \sum_{y \in S} \left(\frac{w_r(y)}{\sum_{z \in S} w_r(z)} \right) \mathbb{1}(x \text{ is } \succ_y\text{-best in } S)$$

where $w_r(r) = w(r) + b(r)$
and $w_r(z) = w(z)$ for all $z \neq r$.

- The exogenous r receives a boost $b(r)$
- $\implies$ r becomes the reference point with a higher probability.
  - As $b(r) \to \infty$, this model converges to a deterministic choice model where the DM maximizes $\succ_r$ when her reference point is r
  - As $b(r) \to 0$, the model converges to RAR

THANK YOU

Appendix

Behavioral Postulates of L-RAR

Behavioral postulates that characterize L-RAR

- ordinal axioms (1-4)
- cardinal axioms (6-9)

Cardinal Axiom: Luce IIA for Reference Probabilities

In singleton, binary and trinary choice sets construct $q(x|S)$

Reflects the probability that x is the reference point in S

- For any $\{x, y\}$ where $p(a|\{x, y\}) > 0$ for all $a \in \{x, y\}$, let

$$q(a|\{x, y\}) = p(a|\{x, y\}).$$

- For any $\{x, y, z\}$ where $p(a|\{x, y, z\}) > 0$ for all $a \in \{x, y, z\}$, let

$$q(a|\{x, y, z\}) = p(a|\{x, y, z\}).$$

- For $p(x|\{x, y, z\}) > 0$, $p(y|\{x, y, z\}) > 0$, and $p(z|\{x, y, z\}) = 0$.
  - If $p(x|\{x, y, z\}) > p(x|\{x, y\})$ (Axiom 2)
    $\implies q(y|\{x, y, z\}) = p(y|\{x, y, z\})$.
    $\implies q(x|\{x, y, z\}) = \frac{p(x|\{x,y\})p(y|\{x,y,z\})}{p(y|\{x,y\})}$.
    $\implies q(z|\{x, y, z\}) = p(x|\{x, y, z\}) - \frac{p(x|\{x,y\})p(y|\{x,y,z\})}{p(y|\{x,y\})}$.

Cardinal Axiom: Luce IIA for Reference Probabilities

Let $\mathcal{T}$ denote all singleton, binary, and trinary choice sets for which q is defined as above.

Axiom 6

For any $S_1, S_2, \dots, S_N \in \mathcal{T}$ and any $x_1, \dots, x_N \in X$ such that $\{x_i, x_{i+1}\} \subseteq S_i$ for $i < N$ and $\{x_1, x_N\} \subseteq S_N$,

$$\frac{q(x_1|S_N)}{q(x_N|S_N)} = \prod_{i=1}^{N-1} \frac{q(x_i|S_i)}{q(x_{i+1}|S_i)}.$$

- Reguires IIA for all sets in $\mathcal{T}$
- $\mathcal{T}$ does not contain all binary and trinary sets

$$\text{e.g.} \qquad p(x\{x,y\}) = 1 \text{ or } p(x\{x,y,z\}) = 1$$

Cardinal Axiom: Luce IIA for Choice Probabilities

Axiom 7

If $p(x|S)p(y|S)p(z|S) > 0$ and $p(z|S) \leq p(z|S \setminus t)$ for all $t \in S \setminus z$, then

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)}.$$

- Consider a chosen alternative $z \in S$

$$p(x|S)p(y|S)p(z|S) > 0$$

- such that elimination of no alternative in S induces a regularity violation for z.

$$p(z|S) \leq p(z|S \setminus t) \text{ for all } t \in S \setminus z$$

- When such a "well-behaved" alternative z is eliminated from S, choices must satisfy IIA

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)}$$

Cardinal Axiom: IIA Violation 1 (Constant Ratio)

Compares the effect of elimination of z on x/y in two different sets: S and $S \setminus t$

Axiom 8

If $p(x|S \setminus t) \geq p(x|S) > p(x|S \setminus z)$, $p(y|S) > 0$, and $p(t|S) = 0$, then

$$\frac{p(x|S)/p(y|S)}{p(x|S \setminus z)/p(y|S \setminus z)} = \frac{p(x|S \setminus t)/p(y|S \setminus t)}{p(x|S \setminus \{t,z\})/p(y|S \setminus \{t,z\})}.$$

- x and y are chosen, and z and t are not chosen from S
- elimination of z from S induces a regularity violation on x

$p(x|S) > p(x|S \setminus z)$
IIA will not be satisfied for x/y
on both S and $S \setminus t$

- elimination of t from S does not induce a regularity violation for x

$p(x|S \setminus t) \geq p(x|S)$

Cardinal Axiom: IIA Violation 2 (Constant Difference)

Compares the effect of elimination of z on x/y in two different sets: S and $S \setminus t$

Axiom 9

If $p(x|S) > \max\{p(x|S \setminus z), p(x|S \setminus t)\}$ and $p(y|S) > 0$, then

$$\frac{p(x|S)}{p(y|S)} - \frac{p(x|S \setminus z)}{p(y|S \setminus z)} = \frac{p(x|S \setminus t)}{p(y|S \setminus t)} - \frac{p(x|S \setminus \{z,t\})}{p(y|S \setminus \{z,t\})}.$$

- x and y are chosen, and z and t are not chosen from S
- elimination of z from S induces a regularity violation on x

$p(x|S) > p(x|S \setminus z)$
IIA will not be satisfied for x/y
on both S and $S \setminus t$

- elimination of t from S induces a regularity violation for x

$p(x|S) > p(x|S \setminus t)$

Axioms:

- Ordinal Axioms (concerning regularity)

  1. Weak Regularity (zero probabilities)
  2. Eliminating Unchosen Alternative (regularity violation)
  3. Eliminating Chosen Alternative (no regularity violation)
  4. Asymmetry in Regularity Violations

- Cardinal Axioms (concerning Luce IIA)

  6. IIA for Reference Probabilities
  7. IIA for Choice Probabilities
  8. IIA Violation 1 (constant ratio)
  9. IIA Violation 2 (constant difference)

Characterization Theorem

Theorem 2

A random choice rule p satisfies Axioms 1-4 and 6-9

if and only if

p has a L-RAR representation.

- L-RAR satisfies the Ordinal Axioms and the Cardinal Axioms
- and
- any p that satisfies the Ordinal Axioms and the Cardinal Axioms

is consistent with L-RAR

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