

A Regional Approach to Saliency

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November 26, 2017

Abstract

We propose a novel *regional preference model (RPM)* where the framing of the decision problem affects the saliency of a product through the region in which it lies, and the product's saliency affects the agent's evaluation of it. RPM retains the key ingredients of the salient thinking model of Bordalo et al. [2013b] and allows us to understand the crux of their approach. Our general framework also encompasses the loss aversion model of Tversky and Kahneman [1991] and the status quo bias model of Masatlioglu and Ok [2005]. We specialize RPM to provide a behavioral foundation for the salient thinking model to highlight the saliency model's strong predictions and distinguish it from other existing models.

*We thank David Dillenberger, Erik Eyster, Andrei Schleiffer, Rani Spiegler, Tomasz Strazlecki, Collin Raymond, and conference/seminar participants at BRIC 2017, CETC 2017, SAET 2017, Lisbon Meetings 2017, Brown, and Harvard for helpful comments and discussions. This project began at ESSET Gerzensee, whose hospitality is gratefully acknowledged.

1 Introduction

Psychologists have long known that the *framing* of a decision problem – the context in which a decision takes place – plays a significant role in decision making.¹ Each frame selects some features of the alternatives to highlight and makes them more *salient* in the agent’s subsequent evaluation. As described by Taylor and Thompson [1982], this causes “the phenomenon that when one’s attention is differentially directed to one portion on the environment rather than to others, the information contained in that portion will receive disproportionate weighing in subsequent judgments.”

In this paper, we propose and axiomatize a model of salience, which we call the *regional preference model (RPM)*. RPM retains the key ingredients of the approach taken by the salient thinking model of Bordalo et al. [2013b] (henceforth, BGS). Although relatively new, BGS is becoming one of the leading models of individual decision making due to its appealing psychological motivation and ability to accommodate a variety of disparate evidence.² Freed from the functional form assumptions of BGS, RPM allows us to understand the crux of their approach and its implications for choice. Our results reveal a sharp distinction between BGS and that of other models sharing similar psychological intuitions, including Gabaix and Laibson [2006], Kőszegi and Szeidl [2013], Bhatia and Golman [2013], Gabaix [2014], Bushong et al. [2015]. Moreover, our framework provides an unexpected connection between BGS and other behavioral models that rely on completely different psychological intuitions, including Tversky and Kahneman [1991] and Masatlioglu and Ok [2005].

In RPM, the decision maker (henceforth, DM) chooses among alternatives that have distinct and easily observable attributes in the presence of an exogenously specified frame.³ Each frame divides the product space into different *regions* according to these attributes. The model’s key feature is that the DM evaluates trade-offs between attributes differently depending on the region in which the product lies. Formally, the DM maximizes a weighted sum of the object’s attributes, and in different regions, the weight on each attribute varies. The same product under distinct frames has attributes that are differentially salient and

¹See e.g. Tversky and Kahneman [1981], Slovic et al. [1982], Fischer et al. [1986], Rowe and Puto [1987], Frisch [1993], Levin et al. [1998].

²For some recent applications, see Hastings and Shapiro [2013], Busse et al. [2015], Bordalo et al. [2016], Karlan et al. [2016], Gurun et al. [2016], Dertwinkel-Kalt [2016].

³In this paper, the frames are exogenously given. Of course, it is possible that the choice problem itself has an influence on determining the frame. We believe that studying endogenous frame formation is an important and difficult subject, which we leave for future research.

may therefore be evaluated differently. In BGS, for example, the attributes are price and quality and the frame is the average price and quality. The regions are determined by which attribute is further from the frame (average), and this salient dimension receives higher weight than the others.

Despite its generality, RPM makes testable predictions and excludes certain type of modeling choices. We provide a complete characterization of the choice behavior equivalent to representation by RPM. First, ranking within a region is independent of the frame. Second, preference within a region is well-behaved, in the sense that versions of well-known axioms hold within it. Specifically, the preference is monotonic, continuous and linear inside the region. Overall, RPM enjoys an intuitive and simple axiomatic foundation.

This result highlights an important distinction between model of salience. Of the models cited above that attempt to capture salience, only BGS is an RPM. In other words, even the most general version of BGS exclude these models. Hence BGS offers a completely different method of modeling choice under salience than other existing models thereof. In addition to the salient thinking model of BGS, we show that RPM nests many seemingly unrelated models that have received a great deal of attention in economics. In particular, it includes the constant loss aversion model of Tversky and Kahneman [1991] (TK) and the linear status quo bias model of Masatlioglu and Ok [2005] (MO).⁴ The loss aversion model is one of the most influential models of behavioral economics. MO is one of the first papers to the study status quo bias phenomenon by utilizing a choice theoretic approach. As such, RPM can account for many phenomena that the traditional rational model cannot, including asymmetric price elasticities, insensitivity to bad news, endowment effects, and buying-selling price gaps, decoy and compromise effects, and context dependent willingness to pay, among others (Camerer [2004], Bordalo et al. [2012, 2013a,b], Crawford and Meng [2011], Ericson and Fuster [2011]).

BGS, MO and TK have been developed to account for similar decision-making behavior, but they differ substantially in terms of their functional forms. RPM fulfills the challenge of integrating these theories into one cohesive and general model of salience. It allows us to compare these seemingly unrelated models, and the RPM axioms highlight behavior common to all three.

⁴In this paper, for comparison purposes, we consider a straight-forward extension of their model in which the reference point might be unavailable.

We now turn to the model’s distinct predictions for choice. To this end, we provide a complete behavioral characterization of BGS. While TK and MO both provide axiomatizations of their models, the characterization of BGS was an open question. BGS strengthens RPM in two important ways. First, it imposes a particular structure on the regions. We characterize this structure and show that the regions generated by any parametrization are identical, reference point by reference point. Second, it imposes a particular structure on the utility functions. The most general property is a consistency one: the reference point only affects preference through the regions it generates. The properties more specific to BGS are that preference reflects across the 45-degree line, and that attribute i gets more weight when the product is in the i -salient region.

Our results highlight trade-offs between the different modeling approaches. For instance, BGS maintains a stronger consistency condition across frames than does TK, but TK, unlike BGS, satisfies monotonicity across regions (in addition to within regions). The strong consistency property is normatively appealing – as long as neither alternatives’ salience changes, their relative ranking does not change – and ideally we would like a model that satisfies both. However, the trade-off between the two is a general property of reference-dependent regional preferences. Any reference-dependent regional preference that exhibits salient thinking and satisfies the strong consistency condition either violates monotonicity or has a very particular and hard to interpret structure. This finding does not favor one model over the other, but make their differences clear. If the strong consistency is desired in an application, then BGS could be a better fit to the job than TK. If monotonicity is a necessary property, TK would be ahead of BGS.

Interpreting regional preferences as arising from differential attention to attributes, RPM has a close relationship with the literature studying how limited attention affects decision making. Masatlioglu et al. [2012] and Manzini and Mariotti [2014] study a DM who has limited attention to the alternatives available. The DM maximizes a fixed preference relation over the consideration set, a subset of the alternatives actually available. In contrast, in RPM the DM the considers all available alternatives but maximizes a preference relation distorted by her attention. Caplin and Dean [2015], de Olivera et al. [2016] and Ellis [2018] study a DM who has limited attention to information. In contrast to RPM, attention is chosen rationally to maximize ex ante utility, rather determined by the framing of the decision, and choice varies across states of the world. The most related interpretation considers attributes as payoffs in a fixed state. In addition to choices varying across states, each alternative has

the same weights on each attribute, similar to Kőszegi and Szeidl [2013].

2 Model

In this paper, each product has several attributes. For simplicity, we focus on two-attributes case. We let X denote the set of alternatives, and assume that $X = I_1 \times I_2$ for open intervals I_1 and I_2 .⁵ For $x, y \in X$ and $\alpha \in [0, 1]$, denote by $x\alpha y$ the coordinate-by-coordinate mixture of x and y , i.e. $[x\alpha y]_i = \alpha x_i + (1 - \alpha)y_i$ for all i .

We denote the set of frames by \mathcal{F} . Each $f \in \mathcal{F}$ does not affect the material outcome of alternatives but may affect their salience. Examples of frames include the reference or default option of the consumer, the intensity of advertising, and a lottery over reference bundles (as in Kőszegi and Rabin [2006]). In the first example, we set $\mathcal{F} = X$, in the second, we set $\mathcal{F} = [0, 1]$, and in the final one, we set $\mathcal{F} = \Delta X$. In general, \mathcal{F} can be any non-empty, connected subset of a metric space.

Finally, for each frame $f \in \mathcal{F}$, the DM maximizes a complete and transitive preference relation, denoted by \succsim_f . As usual, \succ_f denotes strict preference and \sim_f indifference. Our observable data is thus a family of such preferences indexed by the set of frames, $\{\succsim_f\}_{f \in \mathcal{F}}$.

The first idea of our model is that, given a frame f , the product space is divided into different regions according to their salience. Each region corresponds to a different saliency and changes as the frame changes. We allow the regions to have a very general structure.

Definition 1. A vector-valued function $\mathcal{R} = (R_1, R_2, \dots, R_n)$ is a *regional function* if each $R_i : \mathcal{F} \rightarrow 2^X$ satisfies the following properties:

1. $R_i(f)$ is a non-empty, connected, open set,
2. $\bigcup_{i=1}^n R_i(f)$ is dense,
3. $R_i(f) \cap R_j(f) = \emptyset$ for all $i \neq j$, and
4. $R_i(\cdot)$ is continuous.⁶

⁵For the purposes of our representation theorem, X can be any open, bounded, convex subset of \mathbb{R}^n without changing the arguments.

⁶That is, each R_i is both upper and lower semicontinuous when viewed as a correspondence.

We interpret the properties as follows. Every region contains some bundle for every frame. If x belongs to the region, then so do all y that are close enough to x . There is a path that stays within the region between any two points, so regions cannot be the union of “islands.” Almost every alternative is in at least one region, and none are in two regions. Finally, if the frame does not change too much, then neither do the regions.

In our model, a consumer values each good with an affine utility function. Utility depends not only on consumption of a product, as in the standard neoclassical model, but also on the region to which the product belongs. Suppose the consumption x lies in the region i when the frame f , that is, $x \in R_i(f)$. Then, the value of consumption x is represented by $u_i(x|f)$. We require that the frame does not affect the utility trade-off within a region. In other words, changing frame does not alter the relative ranking of two alternatives as long as both of the alternatives lie in the same region. To capture this feature, we assume that $u_i(\cdot|f)$ is a positive and affine transformation of $u_i(\cdot|f')$. We formally define the regional preference model for a regional function \mathcal{R} as follows.

Definition 2. The family $\{\succsim_f\}_{f \in \mathcal{F}}$ conforms to the *regional preference model (RPM)* under a given \mathcal{R} if there exist n families $\{u_i(\cdot|f)\}_{f \in \mathcal{F}}$ of strictly increasing, affine functions, where $u_i(\cdot|f)$ is a positive affine transformation of $u_i(\cdot|f')$ for every $f, f' \in \mathcal{F}$, such that for all $x \in R_i(f)$ and $y \in R_j(f)$,

$$x \succsim_f y \text{ if and only if } u_i(x|f) \geq u_j(y|f).$$

Figure 1 provides an illustration for RPM (à la Köszegi and Rabin [2006]) for a fixed frame f . The frame is a lottery of two deterministic reference points r and r' with equal probability. In this example, there are nine different regions.

3 A Behavioral Foundation

In this section, we provide a set of behavioral postulates characterizing RPM. This behavior represents the key features of the model. Theorem 1 shows they hold if and only if the DM is representable by RPM, rendering the model behaviorally testable.

The first postulates states that changing the frame does not alter the relative ranking

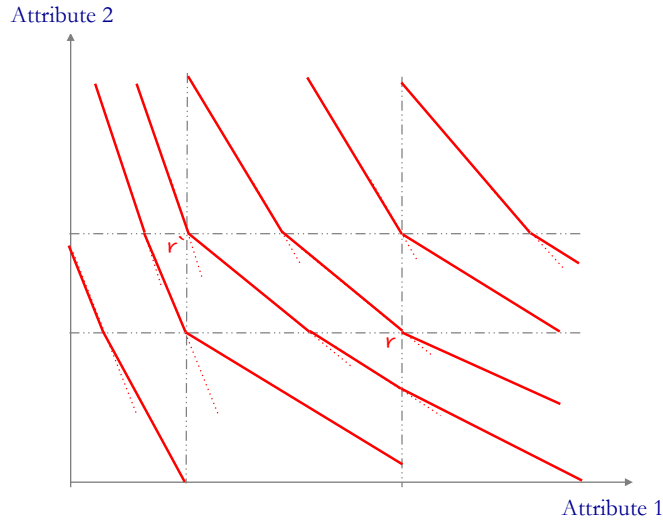


Figure 1: A Regional Preference Model (with nine regions)

of two alternatives as long as both of the alternatives lie in the same region.

Axiom 1 (Frame Independence). If $x, y \in R_i(f) \cap R_i(f')$, then $x \succsim_f y \iff x \succsim_{f'} y$.

The next postulate states that the indifference curves are straight and parallel lines for a given region.

Axiom 2 (Regional Linearity). For any $x, y, a, b \in X$ and $\alpha \in (0, 1]$ such that $x, x\alpha y, y \in R_i$ and $a, a\alpha b, b \in R_j$: if $x \succsim_f a$ and $y \succsim_f b$, then $x\alpha y \succsim_f a\alpha b$, strictly whenever $x \succ_f a$.

For one region, Regional Linearity is equivalent to the usual independence axiom when preference is complete and transitive.

We assume that each product consists of desirable attributes. Monotonicity means that if a product x contains more of some or all attributes, but no less of any, than another product y , then x is preferred to y . The next postulate assumes that monotonicity is maintained within a region for a given frame.

Axiom 3 (Regional Monotonicity). For any $x, y \in R_i(f)$, if $y \geq x$, then $y \succsim_f x$, strictly whenever $y \neq x$.

The final property requires continuity of preferences within a region.

Axiom 4 (Regional Continuity). For any x and f and region R_i , the sets $\{y \in R_i(f) : y \succ_f x\}$ and $\{y \in R_i(f) : x \succ_f y\}$ are open.

Our main result states that these four postulates characterize RPM.

Theorem 1. *If \mathcal{R} is a regional function, then $(\{\succ_f\}_{f \in \mathcal{F}}, \mathcal{R})$ satisfies Frame Independence, Regional Linearity, Regional Monotonicity and Regional Continuity if and only if $\{\succ_f\}_{f \in \mathcal{F}}$ conforms to RPM under \mathcal{R} .*

First, Theorem 1 highlights the fact that the regional preference model has strong predictive power. Second, it provides a complete characterization of regional preferences. That is, Axioms 1-4 are both necessary and sufficient for the model. In other words, Theorem 1 lays out all possible implications of regional preferences. This is crucial since some of these implications are hard to see by looking at the description of the model. This emphasizes the important role of Theorem 1 in understanding of regional preferences. Third, the result provides a set of unifying behavior that underlies and unifies special cases of RPM, like TK and BGS. Finally, Theorem 1 makes it possible to show that there are other models that attempt to capture salience but that are not outside of our framework. These include the models of Gabaix and Laibson [2006], Kőszegi and Szeidl [2013], Bhatia and Golman [2013], Gabaix [2014], Bushong et al. [2015].⁷

The closest results of which we are aware are found in Segal [1992] and Chapter 2.4 of Schmidt [1998].⁸ Both are concerned with the existence of an additively separable or affine utility functions on subsets of product space or lotteries, such as our regions. The former provides conditions for existence of an additively separable utility function on subsets of product spaces, including that the indifference sets are connected. The latter studies piecewise affine utility representations over lotteries, where each piece is σ -convex. None of their results do not apply in this setting because their assumptions rule out regions of the form assumed by BGS. Our stronger independence axiom allows us to overcome this limitation. Additionally, neither considers how preferences change with frames.

⁷See appendix for the functional forms and discussion.

⁸We thank David Dillenberger for bringing the latter to our attention.

3.1 Revealing Regions

For our representation result, we took the regions as given. While this assumption simplifies exposition and statement of the axioms, it places significant demands on the data observed. Nevertheless, if we make stronger assumptions about the slope of the indifference curves, we can identify the regions directly from the DM's choices. This subsection discusses how we do so.

We first identify a potential region around x by an open set where the DM's choices reveal straight and parallel indifference curves.

Definition 3. An open set A is a *revealed region around x for frame f* if $x \in A$, when $a, b, aac, bac \in A$ we have $a \succsim_f b \iff aac \succsim_f bac$.

That is, a revealed region is an open set for which the preference satisfies our linearity axiom, or equivalently has linear indifference curves. When the underlying preference conforms to RPM where each region's indifference curves have distinct slopes, we can identify the entire region to which each point belongs. Specifically, the largest revealed region around x , in the sense of set inclusion, corresponds to the region specified by the region function.

Proposition 1 (Revealing Regions). *Suppose $\{\succsim_f\}_{f \in \mathcal{F}}$ conforms to RPM for some regional function \mathcal{R} where $u_i(\cdot|f)$ is not an affine transformation of $u_j(\cdot|f)$ when $i \neq j$. Then for any $x \in X$ and $f \in \mathcal{F}$, there exists a unique maximal region around x for f that equals $R_i(f)$ when $x \in R_i(f)$ and equals \emptyset when x belongs to no region.*

The proposition allows us to identify the regions directly from the preference relation. In particular, it provides a test for whether a given set belongs to the same region as x . Either the set is a subset of the same region as x , or we can find a violation of linearity within the set. This test allows us to decide whether the properties asserted by Proposition 1 hold from looking only at the reference point and the agent's behavior.

3.2 Reference point as frame

In the rest of the paper, we focus primarily on a special case of the model that includes BGS, MO and TK: reference-dependent frames. In this subsection, we formally define regions so

that the salience of a good is determined solely by its position relative to a “reference good.” This reference good is the only feature that alters the salience of goods.⁹ This special case is of interest in its own right since it covers several well-known models.

We first adjust our definition of \mathcal{R} to focus specifically on cases where the reference point is the frame. Here, the reference good is the focal point that splits the alternatives around it. While it does not belong to any particular region, the comparison between the bundles and the reference create the regions. Hence, all regions surround the reference point. To that end, formally, we consider the following modification of a regional function.

Definition 4. The function $\mathcal{R} = (R_1, R_2, \dots, R_n)$ is a *reference-dependent regional function* if $\mathcal{F} = X$ and each $R_i : \mathcal{F} \rightarrow 2^X$ satisfies properties 2, 3 and 4 of Definition 1 and:¹⁰

1. $R_i(r)$ is a non-empty open set such that $R_i(r) \cup \{r\}$ is connected, and
2. $r \in \bigcap_{i=1}^n bd(R_i(r))$.

It is easy to see that RPM with a reference dependent regional function (henceforth, RD-RPM) is a special case of RPM with a standard regional function. We do so by treating as equivalent any regions with the same utility, provided they are close to the reference point.¹¹ This requires that regions look like slices of a pie centered around the reference point, and each region may be more than one slice. This rules out, for instance, regions a la Köszegi and Rabin [2006] that do not meet at a single point (see Figure 1). Moreover, it requires that the regions change with the reference point.

4 Foundations for BGS

BGS propose an intuitive and descriptive behavioral model based on salience, but they focus on understanding interesting applications rather than connecting the components of

⁹We do not specify the reference good, which could be (i) the default option (Tversky and Kahneman [1991], Johnson and Goldstein [2003], Samuelson and Zeckhauser [1988]), (ii) status quo (Tversky and Kahneman [1991], Johnson and Goldstein [2003], Samuelson and Zeckhauser [1988]), (iii) the average level of each attribute in the choice set (Kivetz et al. [2004], Bordalo et al. [2013b]), (iv) aspirations/expectations/goals (Payne et al. [1980]) or (v) actual choice (Köszegi and Rabin [2006]).

¹⁰To make the distinction we use r instead of f for the reference-dependent regional function.

¹¹That is we replace $R_1(r), \dots, R_n(r)$ with $R'_1(r), \dots, R'_m(r)$ where, e.g., $R'_1(r) = \bigcup_i R_i(r)$ where i ranges over $\{i : u_i(\cdot|f) = u_1(\cdot|f) \text{ for all } f \in \mathcal{F}\}$.

their model to observed choice behavior. It is still an open question what are the full set of behavioral postulates characterizing BGS' original formulation. This section focuses on closing that question.

4.1 The BGS Model

In the BGS model, the salience of an attribute depends on the value of the product's attribute and the reference level of that attribute.¹² The amount of salience is determined by a *salience function* σ , and the attribute with the largest salience is called the salient attribute for that good. In general, the different attributes are salient for different goods. The salience function does not depend on either the identity of the products or the particular attribute considered. Each salience function creates two regions: one where attribute 1 is salient and the other attribute where 2 is salient.

More formally, let (r_1, r_2) be the reference level. Then attribute 1 is salient for good x if $\sigma(x_1, r_1) > \sigma(x_2, r_2)$, and attribute 2 is salient for good x if $\sigma(x_1, r_1) < \sigma(x_2, r_2)$. BGS assume that σ is continuous and symmetric, i.e. $\sigma(a, b) = \sigma(b, a)$. They require two additional properties: Ordering and Homogeneity of Degree Zero. Homogeneity of Degree Zero imposes that for all $\alpha > 0$, $\sigma(\alpha a, \alpha b) = \sigma(a, b)$. Ordering states that salience increases in contrast. Formally, when $\epsilon, \epsilon' \geq 0$ with $\epsilon + \epsilon' > 0$, if $a > b$, then $\sigma(a + \epsilon, b - \epsilon') > \sigma(a, b)$, and if $a < b$, then $\sigma(a - \epsilon, b + \epsilon') > \sigma(a, b)$. We say that σ is a *salience function* if it is continuous, symmetric and satisfies Ordering and Homogeneity of Degree Zero. In Proposition 2, we characterize the regions generated by any salience function. Surprisingly, they coincide: any two salience functions generate the same regions.

BGS theorize how salience distorts the valuation of a good. For each product, the DM ranks puts higher weight on the salient attribute. In other words, if attribute i is salient for product x , then attribute i attracts more attention than attribute j and receives greater decision weight for the valuation of x . In particular, they are represented by the function

$$V_{BGS}(x|r) = \begin{cases} wx_1 + (1-w)x_2 & \text{if } \sigma(x_1, r_1) > \sigma(x_2, r_2) \\ (1-w)x_1 + wx_2 & \text{if } \sigma(x_2, r_2) > \sigma(x_1, r_1) \end{cases} \quad (1)$$

¹²In the original paper, BGS illustrate their model in an environment where one attribute desirable (quality) and the other is undesirable (price). We provide an graphical illustration for such cases in the Appendix.

where $w \in (0.5, 1)$ increases in the severity of salient thinking.¹³ The left panel in Figure 2 shows BGS regions and the resulting indifference curves for a fixed reference point. In Section 4.3, we provide axioms equivalent to representation by V_{BGS} .

4.2 A Characterization of the BGS Regions

We first ask when a reference-dependent regional function can be derived from a salience function. To study BGS, we assume $X = (0, \bar{x}) \times (0, \bar{x})$, where $\bar{x} > 0$ throughout this section.¹⁴ Since there are two attributes, there are two regions. Fix a reference-dependent regional function $\mathcal{R} = (R_1, R_2)$. We interpret $R_i(r)$ as the goods that are i -salient for the reference point r . We now state properties which are defined for a reference-dependent regional function. Then we show these properties characterize the BGS's salience function.

S1 (Moderation) For any $\lambda \in (0, 1]$ and $r \in X$:

if $x \in R_{-i}(r)$ and $y_i = \lambda x_i + (1 - \lambda)r_i$ and $y_{-i} = x_{-i}$, then $y \in R_{-i}(r)$.

S2 (Equal Salience) For any $x, r \in X$: if $\frac{x_1}{r_1} = \frac{x_2}{r_2}$ or $\frac{x_1}{r_1} = \frac{r_2}{x_2}$, then $x \notin R_i(r)$ for $i = 1, 2$.

S3 (Regular regions) For all $r \in X$ and $i = 1, 2$: $R_i(r)$ is a regular open set.¹⁵

The properties have natural interpretations. First, making the attribute of a bundle closer to the reference point's attribute decreases the salience of that attribute. That is, when x and y differ only in attribute j , and y is closer to the reference in that attribute, if x is i -salient, then so is y . Second, if every attribute of x differs from the reference point by the same percentage, then none of the attributes stands out. More formally, if the percentage difference between x_i and r_i is the same across attributes, then x is not i -salient for any i . Finally, regions are regular open sets. That is, there is no bundle completely surrounded by i -salient bundles that is not an i -salient bundle itself. Any bundle that can be approached by a sequence of i -salient bundles is either i -salient or on the boundary of the i -salient region.

¹³BGS parametrize by $\delta \in (0, 1]$, so their utility function is $2V_{BGS}(x|r)$ for $w = \frac{1}{1+\delta}$.

¹⁴BGS focus on the case where one attribute, price, is a "bad." We can accommodate this by instead taking $X = (-\bar{x}, 0) \times (0, \bar{x})$ and letting attribute 1 represent the bad. This version of model is illustrated in the Appendix. To do so, we must replace x_1 with $-x_1$ in S1-S3 and alter Axiom 7 as discussed in Footnote 16.

¹⁵Recall that a set A is regular open if $A = \text{int}(\text{cl}(A))$.

Proposition 2. *The following are equivalent:*

- (i) *The reference-dependent regional function (R_1, R_2) satisfies S1-S3,*
- (ii) *There exists a salience function σ s.t. $x \in R_i(r) \iff \sigma(x_i, r_i) > \sigma(x_{-i}, r_{-i}),$*
- (iii) *For any salience function σ , $x \in R_i(r) \iff \sigma(x_i, r_i) > \sigma(x_{-i}, r_{-i}).$*

This proposition provides a characterization for BGS's salience function. In other words, Proposition 2 translates the functional form assumptions on the salience function in terms properties on the regions. An immediate implication is that *any* specification of the salience function leads to the same regions.

4.3 A Characterization for BGS

We now identify the additional assumptions to pin down the exact formulation of BGS. These assumptions identify the additional structure imposed by BGS in addition to Axioms 1-4. Each corresponds to a particular feature imposed by BGS.

The first property guarantees that the salient attribute gets a higher weight in the utility.

Axiom 5 (Salient Dimension Overweighted; SDO).

If $y \in R_i(r) \cap R_i(r')$, $x \in R_i(r) \cap R_{-i}(r')$, $x \succsim_r y$ and $x_i > x_j$, then $x \succ_{r'} y$

SDO requires that regions correspond to the dimension that gets the most weight. That is, the DM overvalues bundles with a best attribute of i in region i . To see why, suppose dimension 1 is the best for x . SDO requires that when x is at least as good as y when dimension 2 stands out for both, the DM chooses x over y for sure when 1 stands out for it.

Second, changing reference point does not reverse the ranking of two products unless it also changes their salience.

Axiom 6 (Strong Frame Independence; SFI).

If $x \in R_i(r) \cap R_i(r')$ and $y \in R_j(r) \cap R_j(r')$, then $x \succsim_r y$ if and only if $x \succsim_{r'} y$.

For the general RPM, the reference point influences choice through two channels: salience and valuation. The axiom eliminates the latter. When comparing two alternatives across

different reference points, the DM’s relative ranking does not change when neither’s salience changes. This property greatly limits the effect of the reference point. In fact, a sufficiently small change in the reference never leads to a preference reversal. Notice that Frame Independence is the special case of SFI where $i = j$.

Finally, because both salience and preference are symmetric across attributes, permuting the attributes of all objects in the same way does not change rankings. Thus, our last additional axiom requires that the preference “reflects” about the 45 degree line.

Axiom 7 (Reflection).

$$(x_1, x_2) \succsim_{(r_1, r_2)} (y_1, y_2) \text{ if and only if } (x_2, x_1) \succsim_{(r_2, r_1)} (y_2, y_1).$$

Note that both the reference point and the values of each attribute reverse.¹⁶ Relaxing the symmetry across attributes in terms of either preference or salience would break reflection. Hence, the behavioral implications of the symmetry assumption is exactly the reflection property.

Theorem 2. *Let $\{\succsim_r\}_{r \in X}$ conform to RD-RPM under $\mathcal{R} = (R_1, R_2)$. Then, \mathcal{R} satisfies S1-S3 and $(\{\succsim_r\}_{r \in X}, \mathcal{R})$ satisfies SDO, Strong Frame Independence and Reflection if and only if $\{\succsim_r\}_{r \in X}$ has a BGS representation.*

Theorem 2 highlights the fact that the salience model of BGS has strong predictions and can be distinguished from other existing models. In addition, Theorem 2 provides a behavioral foundation for the salience model. Hence it is possible to test the BGS model non-parametrically by using a revealed-preference technique. The contrast between the hypotheses of Theorem 1 and Theorem 2 reveals the additional structure imposed by the BGS model. The key properties of BGS are Strong Frame Independence and SDO. One can relax its other properties, i.e. reflection and S1-S3, to increase explanatory power without sacrificing the main idea of the model.

¹⁶If attribute 1 takes only negative values, as discussed above, then the axiom becomes: “For any $x, r, y \in X$, $(x_1, x_2) \succsim_{(r_1, r_2)} (y_1, y_2)$ if and only if $(-y_2, -y_1) \succsim_{(-r_2, -r_1)} (-x_2, -x_1)$.” The intuition is similar, but we must take into account that we are replacing a bad with a good and keep in mind our domain restriction.

5 Comparing Models

We have established that RPM nests the salient thinking model of Bordalo et al. [2013b]. In this section, we also show that RPM also covers the constant loss aversion model of Tversky and Kahneman [1991] and the linear status quo bias model of Masatlioglu and Ok [2005]. Our general framework makes it possible to compare these seemingly unrelated models. We highlight some of the behaviors in common between these models and allows us to distinguish their distinct predictions for choice.

5.1 Models

We first provide a summary for TK and MO in the language of regional preferences. BGS, TK and MO all feature a reference point that divides alternatives into regions as above. Therefore, taking the frame equal to the reference point, they are all RD-RPM. Figure 2 shows their regions and the resulting indifference curves for a fixed reference point. Our aim is to compare and contrast the behavior they allow.

Constant Loss Aversion Model

TK provide a behavioral model that extends the Prospect Theory to the case of riskless consumption bundles. Each bundle is evaluated relative to reference point r , and losses loom larger than gains. In the absence of losses, the DM values each product bundle with a linear utility function, $(x_1 - r_1) + (x_2 - r_2)$, which attaches equal weights to each attribute. If she experiences a loss in attribute i , then she inflates the weight attached to that attribute by $\lambda_i > 1$. This captures the phenomenon of loss aversion.¹⁷ The linear utility implies that the sensitivity to a given gain (or loss) on dimension i does not depend on whether the reference bundle is distant or near in that dimension. For any $x \in X$ such that $x_1 \neq r_1$ and $x_2 \neq r_2$, we have

¹⁷We exclude $\lambda_i = 1$ because the model becomes the classical model.

$$V_{TK}(x|r) = \begin{cases} (x_1 - r_1) + (x_2 - r_2) & \text{if } x_1 > r_1 \text{ and } x_2 > r_2 \\ \lambda_1(x_1 - r_1) + (x_2 - r_2) & \text{if } x_1 < r_1 \text{ and } x_2 > r_2 \\ (x_1 - r_1) + \lambda_2(x_2 - r_2) & \text{if } x_1 > r_1 \text{ and } x_2 < r_2 \\ \lambda_1(x_1 - r_1) + \lambda_2(x_2 - r_2) & \text{if } x_1 < r_1 \text{ and } x_2 < r_2 \end{cases}$$

Notice that there are four different regions in the TK formulation: (i) gain in both dimensions, (ii) gain in the first dimension and loss in the second dimension, (iii) loss in the first dimension and gain in the second dimension, and (iv) loss in both dimensions (see Figure 2). We model this as $\mathcal{R} = (R_{GG}, R_{GL}, R_{LG}, R_{LL})$ where $R_{GG}(r) = \{x : x \gg r\}$, $R_{LL}(r) = \{x : x \ll r\}$, $R_{GL}(r) = \{x : x_1 > r_1 \text{ and } x_2 < r_2\}$ and $R_{LG}(r) = \{x : x_1 < r_1 \text{ and } x_2 > r_2\}$.

Linear Status Quo Bias Model

In the MO model, individuals may experience some form of psychological discomfort when they have to abandon their status quo option. This discomfort imposes an additional utility cost. Of course, if an alternative is unambiguously superior to the status quo, the DM does not feel any psychological discomfort to forgo the status quo; in such cases there will be no cost. Formally, $Q(r)$ is a closed set denoting the alternatives that are unambiguously superior to the default option r (see Figure 2). If an alternative does not belong to this set, then the DM pays a cost $c(r) > 0$, which may depend on the reference point, to move away from the status quo. For any $x \neq r$, we have

$$V_{MO}(x|r) = \begin{cases} x_1 + x_2 & \text{if } x \in Q(r) \\ x_1 + x_2 - c(r) & \text{if } x \notin Q(r) \end{cases} \quad (2)$$

5.2 Comparison

Our goal is to understand what choices are compatible with TK, BGS, MO and the classic model. As they all belong to RPM, Theorem 1 describes the behavior that they have in common. We search for predictions that do not depend on a particular parametrization, but rather hold true for any specification of the model.

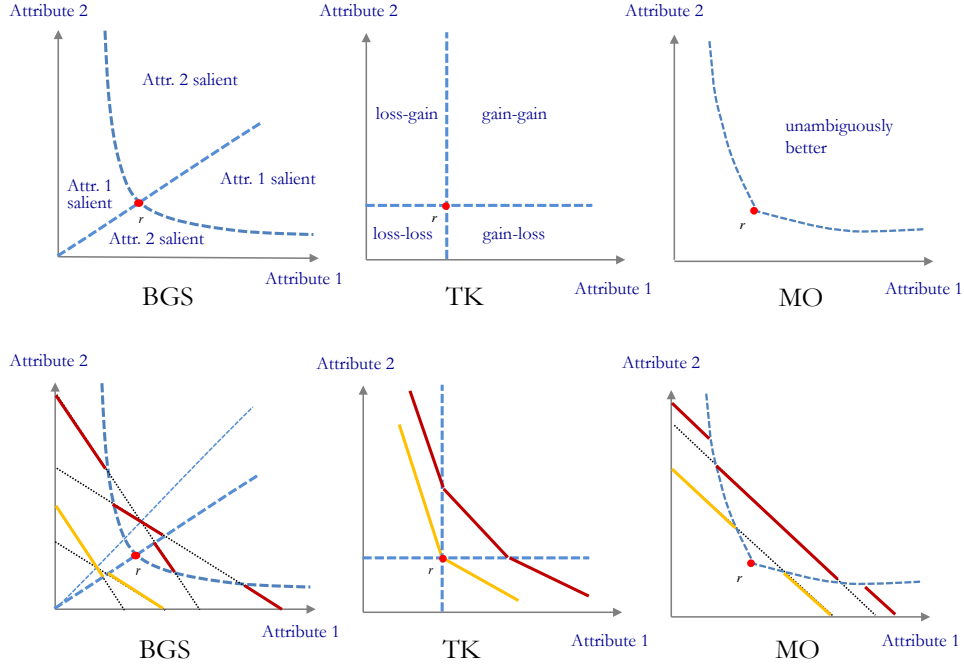


Figure 2: Reference-Dependent Regional Functions

If the sole aim is to differentiate models, one might look for examples that are consistent with one but not the other. An obvious starting point is the motivating examples of these models. Then one might check whether these examples are consistent with both models. In case of comparing BGS and TK, the constant loss aversion model can accommodate the motivating examples from BGS, as illustrated in the Appendix. Given that this strategy did not work, we might look for other examples. However, this exercise would be a shot in the dark.

Instead, we make use of Theorem 2 and Tversky and Kahneman [1991]’s characterization theorem for TK, which shows that the following axiom holds.

Axiom 8 (Cancellation). For all $x_1, y_1, z_1 \in I_1$, and $x_2, y_2, z_2 \in I_2$, and $r \in X$, if $(x_1, z_2) \succsim_r (z_1, y_2)$ and $(z_1, x_2) \succsim_r (y_1, z_2)$, then $(x_1, x_2) \succsim_r (y_1, y_2)$.

We first provide a plausible example violating the cancellation axiom, and hence behavior inconsistent with TK. Then, we illustrate BGS can accommodate this example without requiring a shift in the reference point. While the example is one simple test to distinguish BGS from TK, it is also powerful as it works for a fixed reference point.

Example 1. Consider a consumer who visits the same wine bar regularly. The bartender

occasionally offers promotions. The customer prefers to pay \$8 for a glass of French Syrah rather than having a free bottle of water. At the same time, she prefers a free Australian Shiraz to paying \$5 for the French Syrah.¹⁸ However, without any promotion in the store, she prefers paying \$5 for water to paying \$8 for Australian Shiraz.¹⁹

The intuition behind this example is consistent with BGS. When prices are low, the consumer focuses on prices and her choices imply that the marginal benefit of soda over water is strictly less than \$1. However, when there is no promotion, prices are no longer salient, hence she is willing to pay an additional \$1 to receive soda. Notice that this explanation does not require that the reference points must be different. Since the consumer visits this store regularly, intuitively, her reference point should be fixed and stable.

TK's characterization theorem helped us identify choices consistent with BGS but not TK. We turn now to the other direction: what choices are consistent with TK but not BGS? We first investigate the differences more systematically in terms of choice behavior between the models more thoroughly, focusing on two key desirable properties of a DM's choices and argue that regional preference models of salience face a tradeoff between the two. The first property is Strong Frame Independence (Axiom 6), a key property for BGS, that states changing reference point does not reverse the ranking of two products unless it also changes their salience.

The second property of interest is Monotonicity.

Axiom 9 (Monotonicity). For all $x, y, r \in X$: if $y \geq x$, then $y \succsim_r x$.

Monotonicity requires that if x exceeds y in every dimension, then the DM chooses x over y for any reference point. Theorem 1 shows that every regional preference satisfies Regional Monotonicity. The above, more standard, version, also known as asymmetric dominance, is more demanding. Theorem 1 leaves open the possibility that RPM violates Monotonicity.

Table 1 compares the four models in terms of Strong Frame Independence, Monotonicity and Cancellation. Only the classical theory satisfies all conditions; none of the other three do. On the one hand, BGS satisfies SFI but violates Monotonicity. On the other, TK maintains Monotonicity but violates SFI. Finally, MO satisfies both of them.

¹⁸Implicitly, the example reveals that the customer prefers French Syrah to Australian Shiraz to water.

¹⁹Verify that $(-8, q_f) \succ_r (0, q_w)$, $(0, q_a) \succ_r (-5, q_f)$ and $(-5, q_w) \succ_r (-8, q_a)$ when $q_f = 12.875$, $q_a = 7.85$, and $q_w = 5$ for the reference point $r = (\frac{1}{2}(-5 + -8), \frac{1}{2}(5 + 12.875))$ using BGS formula and $w = 0.51$.

	Classical	BGS	TK	MO
Monotonicity	✓	✗	✓	✓
Strong Frame Independence	✓	✓	✗	✓
Cancellation	✓	✗	✓	✗

Table 1: Comparisons of Models

While Theorem 1 highlights the fact that Monotonicity is not necessarily satisfied in the regional preference model, it does not inform us when it is violated, nor which features of regional preferences is responsible for violation of Monotonicity. Moreover, Theorem 1 does not communicate how serious Monotonicity violations are in this class. For example, there may only a small fraction of regional preference violates Monotonicity.

The remainder of this section seeks to understand this trade-off better. Within the class of Regional Preferences, are there models that accommodate both Saliency, Monotonicity, and Strong Frame Independence? The first step towards providing an answer is to investigate the implications of SFI for RPM. We then explore the set of RPM satisfying SFI and Monotonicity, and show that they rule out reference dependent saliency.

The next proposition translates SFI into the language of utilities.

Proposition 3. *Suppose that for any R_i and R_j and any $r \in X$, there exist $x \in R_i(r)$ and $y \in R_j(r)$ with $x \sim_r y$. Then, the family $(\{\succsim_r\}_{r \in X}, \mathcal{R})$ satisfies Axioms 1-4 and Strong Frame Independence if and only if it has a RPM representation under \mathcal{R} with $u_i(\cdot|r) = u_i(\cdot|r')$ for all $r, r' \in X$. Moreover, any RPM of $(\{\succsim_r\}_{r \in X}, \mathcal{R})$ has $u_i(R_i(r)|r) \cap u_j(R_j(r)|r) \neq \emptyset$ for all r .*

While BGS focus on a particular class of utility function and a particular regional function, their main idea is not limited to these restrictions. While we drop these restrictions, we keep the core idea of BGS: regions and Strong Consistency. Hence, Proposition 3 provides a behavioral foundation for the idea of saliency free from these particular restrictions.

5.3 Monotonicity and Strong Frame Independence

The remainder of our analysis highlights a general tension between Strong Frame Independence and Monotonicity. Salient thinkings occurs when the region to which the alternative

belongs alters the trade-off between the dimensions. In particular, salience should not allow a region to make every alternative in a region worse, as in MO. Given salient thinking, we encounter a tradeoff: either the reference point affects the agent's evaluation or the DM does not respect asymmetric dominance. Formally, SFI implies a violation of Monotonicity and Monotonicity implies a violation of SFI.

In BGS, the weights on the utility function always add up to 1. We assume that for each i , there is a distinct $w^i \in (0, 1)$ such that $u_i(x|r) = w^i x_1 + (1 - w^i)x_2$; if this holds, we call $\{u_i(\cdot|r)\}_i$ *salience utilities*. Intuitively, the tradeoff between attributes differs across regions, and objects with the same level of each attribute have the same evaluation, regardless of the region to which they belong. BGS is an example of salience utilities where $w_1 = 1 - w_2$. Any such specification satisfies SFI by Proposition 3.

Proposition 4. *Let there exist $\bar{x} \in \mathbb{R}$ with $(\bar{x}, \bar{x}) \in X$. If $(\{\succsim_r\}_{r \in X}, \mathcal{R})$ is an RD-RPM with at least two regions and salience utilities, then \succsim_r violates Monotonicity for some r .*

Given reference-dependent regions and salient thinking, the proposition states that Monotonicity fails, regardless of how we specify the regions in a salience model. This holds no matter how one specifies the regions or the weights on utilities. Hence, independent of the regional function \mathcal{R} or the number of regions, there is a clash between salient thinking and Monotonicity.

Finally, we consider a behavioral definition of salience in a model with two regions. Our idea is based on the idea that increasing the value of an alternative's salient dimension improves it relative to an alternative whose non-salient dimension is improved.

Definition 5. The family $(\{\succsim_r\}_{r \in X}, (R_1, R_2))$ exhibits *salient thinking* if for all $r \in X$, there exist $x, x', x'' \in R_1(r)$ and $y, y', y'' \in R_2(r)$ such that $x \sim_r y$, $x' \succ_r y'$, $y'' \succ_r x''$, $x' - x = y' - y = (\epsilon_1, 0)$ and $y'' - y = x'' - x = (0, \epsilon_2)$ for some $\epsilon_1, \epsilon_2 > 0$.

To interpret, consider bundles x, y, x', y', x'', y'' as in the definition. Suppose that $R_i(r)$ is the bundles for which dimension i is salient and that the DM overweights the salient dimension. Since x' and y' (x'' and y'') improve the same amount in dimension 1 (dimension 2), x' (y'') should improve more than y' (x''). Hence, if $x \sim_r y$, then $x' \succ_r y'$ and $y'' \succ_r x''$. Observe that BGS exhibits salient thinking.

Salient thinking, SFI and Monotonicity can coexist in a RD-RPM model, but only with a very particular structure on regions and utilities.

Proposition 5. *Let $(\{\succsim_r\}_{r \in X}, (R_1, R_2))$ be an RD-RPM under \mathcal{R} . If $(\{\succsim_r\}_{r \in X}, (R_1, R_2))$ exhibits salient thinking and satisfies SFI, then either:*

- (i) \succsim_r violates Monotonicity for some r , or
- (ii) there exists $i \in \{1, 2\}$ such that $u_i(x|r) > u_j(x|r)$ for all $x, r \in X$, $cl(R_i(r)) = \bigcup_{x \in cl(R_i(r))} \{y : y \geq x\}$ and $cl(R_j(r)) = \bigcup_{x \in cl(R_j(r))} \{y : y \leq x\}$.

Fixing utilities, Strong Frame Independence and Monotonicity imply restrictions on X . Even with such a restricted X , there is not much scope for salience. Up to a meager set, R_i is closed under \geq and R_{-i} is closed under \leq . MO interpret the former regions as the bundles that are “unambiguously better” than the reference point. It is hard to interpret such regions as resulting from Salience, since it means that dimension i stands out for all “unambiguously better” bundles. The result thus demonstrates a general tradeoff between SFI and Monotonicity with salient thinking.

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A Appendix

BGS in Price-Quality Domain

In the original paper, BGS motivates and illustrates their model by using price-quality pairs. Here, we would like to provide an illustration where one of the attributes is not desirable. This illustration will help readers who are familiar with the original domain. First notice that the regions do not affected by the fact that the first attribute is undesirable. On the other hand, indifference curves drastically change compare to cases where both attributes are desirable. Since price is a “bad” (undesirable), less is preferred to more. Now indifference curves will be upward sloping upward. An increase in price will reduce DM’s satisfaction. In order to keep the level of satisfaction constant, the quality will have to be increased. The direction of preference in this case is upward and to the left as represented in the figure.

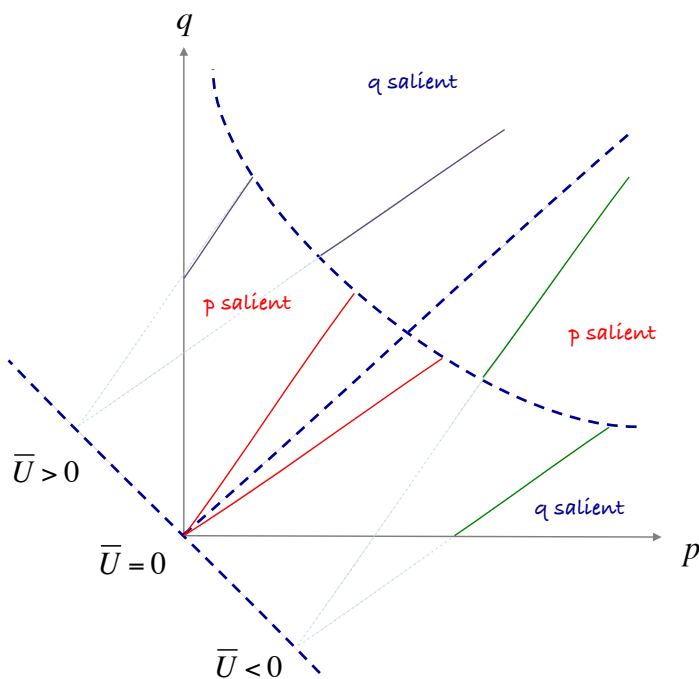


Figure 3: BGS in price-quality domain

Other models

In this subsection, we present the functional forms of the other models of salience we discussed (adapted to be linear, in some cases, to aid comparability), and show that they are not RPM.

- Gabaix and Laibson [2006] assume that consumers choose between $(-p, -p^*)$ and $(-p-e, 0)$ where $(-p, -p^*)$ indicates purchasing an add-on while $(-p-e, 0)$ indicates paying an additional e to substitute away from the add-on. True utility is $u(x, y) = x +$

y . Sophisticated agents and informed myopic agents maximize u , while uninformed myopic agents maximize $u_M(x, y) = x$.

- Kőszegi and Szeidl [2013] and Bushong et al. [2015] assume that the DM maximizes

$$U(x, C) = g_1(C)x_1 + g_2(C)x_2$$

where C is the comparison set, fixed per decision problem, and

$$g_i(C) = g \left(\max_{y \in C} y_i - \min_{y' \in C} y'_i \right).$$

The difference is in the properties that g satisfies. Kőszegi and Szeidl [2013] require $g'(x) > 0$ for all $x > 0$. Bushong et al. [2015] require $g'(x)x + g(x) > 0$ and $g'(x) < 0$ for all $x > 0$.

- Bhatia and Golman [2013] assume that the DM chooses the bundle x that maximizes

$$U(x|r) = \alpha_1(r_1)[V(x_1) - V(r_1)] + \alpha_2(r_2)[V(x_2) - V(r_2)]$$

given that a reference point r , where each α_i is increasing and positive.

- ? assumes a rational DM would maximize $u(a, w)$ but actually maximizes

$$u(a, (w_1 m_1^*, \dots, w_n m_n^*))$$

where

$$m^* \in \arg \min_{m \in [0,1]^n} \frac{1}{2} \sum_{i,j} (1 - m_i) \Lambda_{ij} (1 - m_j) + \kappa \sum_i m_i^\alpha$$

where Λ_{ij} incorporates the “variance” in the marginal utility of dimensions i and j . m_i^* is often zero, and so $(w_1 m_1^*, \dots, w_n m_n^*)$ is a “sparse” vector

All of the above fail to be RPM as the indifference curves have the same slope everywhere for a fixed reference point or reference set; indeed, Gabaix and Laibson do not incorporate a reference point at all.

- **?. AE: I don’t completely follow this model. The following is my interpretation.** The DM maximizes $u(x|(r^t)_{t=1}^T)$ where

$$MRS_{12}(x|(r^t)_{t=1}^T) = \frac{du(x|(r^t)_{t=1}^T)/dx_1}{du(x|(r^t)_{t=1}^T)/dx_2} \leq \frac{du(x|(q^t)_{t=1}^T)/dx_1}{du(x|(q^t)_{t=1}^T)/dx_2} = MRS_{12}(x|(q^t)_{t=1}^T)$$

when $r_1^t > q_1^t$ and $q_j^{t'} = q_j^t$ whenever $(t', j) \neq (t, 1)$.

Proof of Theorem 1

Proof. First, we show the regional affine representation for each reference f . Second, we extend it across frames. To save notation, until Lemma 8, we fix f and write R_i instead of $R_i(f)$ and \succsim instead of \succsim_f .

Lemma 1. *For each R_i , there is an affine and increasing $\hat{v}_i : R_i \rightarrow \mathbb{R}$ so that for $x, y \in R_i$, $x \succsim y \iff \hat{v}_i(x) \geq \hat{v}_i(y)$.*

Proof. For each R_i , pick arbitrary $x_i \in R_i$ and $\epsilon_i > 0$ s.t. $B_{\epsilon_i}(x_i) \subset R_i$. $B_{\epsilon_i}(x_i)$ is a mixture space and \succsim satisfies the mixture space axioms when restricted to it, so let it have the representation v_i , normalized so that $v_i(x_i) = 0$. Define \hat{v}_i by $\hat{v}_i(x) = \frac{1}{\alpha}v_i(x\alpha x_i)$ for any $\alpha \in (0, 1]$ so that $x\alpha x_i \in B_{\epsilon_i}(x_i)$. To see \hat{v}_i is well defined, suppose $x\alpha x_i, x\beta x_i \in B_{\epsilon_i}(x_i)$ and (WLOG) $\beta < \alpha$. Then, $x\beta x_i = (x\alpha x_i)\frac{\beta}{\alpha}x_i$, and since v_i is affine, $\frac{1}{\beta}v_i(x\beta x_i) = \frac{1}{\alpha}v_i(x\alpha x_i)$. \square

Lemma 2. *If $x_i \in R_i$, $x_j \in R_j$, and $x_i \sim x_j$, then there is $\alpha > 0, \beta \in \mathbb{R}$ such that for $x \in R_i$ and $y \in R_j$, $x \succsim y \iff \hat{v}_i(x) \geq \alpha\hat{v}_j(y) + \beta$.*

Proof. WLOG, take $\hat{v}_i(x_i) = 0$. As above, there is ϵ_i such that $B_{\epsilon_i}(x_i) \subset R_i$. By Regional Continuity (RC) and Monotonicity (RM), there is $\epsilon_j > 0$ such that $B_{\epsilon_j}(x_j) \subset R_j$ and for all $y \in B_{\epsilon_j}(x_j)$, $x^* = x_i + \epsilon_i \succ_r y \succ_r x_i - \epsilon_i = x_*$; let $\epsilon = \min \epsilon_k$. For any $y \in R_j$ and α such that $y\alpha x_j \in B_{\epsilon}(x_j)$, there exists $\beta \in (0, 1)$ such that $x^*\beta x_* \sim_r y\alpha x_i$ by RC, RM, and Weak Order. Let $V_j(y) = \alpha^{-1}\hat{v}_i(x^*\beta x_*)$. This is well defined for the same reason as above, and is also affine, increasing, and ranks alternatives in the same way as \hat{v}_j . Thus, $V_j(y) = a\hat{v}_j(y) + b$ for $a > 0$ and $b \in \mathbb{R}$.

For any $x \in R_i$ and $y \in R_j$, pick α such that $x\alpha x_i \in B_{\epsilon}(x_i)$ and $y\alpha x_j \in B_{\epsilon}(x_j)$. By construction, $y\alpha x_j \sim y'$ when $y' \in B_{\epsilon}(x_j)$ and $\hat{v}_i(y') = \alpha V_j(y)$. Thus, $x\alpha x_i \succsim y' \sim y\alpha x_j$ holds if and only if $\hat{v}_i(x) \geq V_j(y)$ and $x \succsim y \iff x\alpha x_i \succsim y\alpha x_j$, completing the proof. \square

Definition 6. A finite sequence Q_1, \dots, Q_m with $Q_i \in \{R_1, \dots, R_n\}$ is an *indifference sequence (IS)* if there exists $x_1, \dots, x_m, y_1, \dots, y_m$ with $x_j \in Q_j$, $y_j \in Q_{j+1}$ and $x_j \sim y_j$. The function v is a *utility for the IS* Q_1, \dots, Q_m if v is affine and increasing on each Q_i and for all i , $x, y \in Q_i \cup Q_{i+1}$: $x \succsim y \iff v(x) \geq v(y)$.

For an IS Q_1, \dots, Q_n with utility v , label $v(Q_i) = (l_i, u_i)$. Note that we allow $Q_i = R_j$ and $Q_k = R_j$ for $i \neq k$.

Lemma 3. *For an indifference sequence Q_1, \dots, Q_n , there is an affine, increasing utility v for it.*

Proof. By Lemma 2, there is α_i, β_i such that $x \succsim y \iff \hat{v}_i(x) \geq \alpha_i\hat{v}_{i+1}(y) + \beta_i$ for all $x \in R_i$ and $y \in R_{i+1}$. For any $x \in Q_i$ for $i = 1, \dots, n$, define

$$v(x) = \prod_{j=1}^{i-1} \alpha_j \hat{v}_i(x) + \sum_{j=1}^{i-1} \prod_{k=1}^{j-1} \alpha_k \beta_j.$$

When $x, y \in R_{i+1} \cup R_i$, Lemma 2 implies that $x \succsim y \iff v(x) \geq v(y)$. \square

Lemma 4. Fix an IS Q_1, \dots, Q_n with utility v . If $x_j \in Q_j$ for $j = i, i+1, i+2$ with $x_i \sim x_{i+1} \sim x_{i+2}$, then $Q_1, \dots, Q_i, Q_{i+2}, \dots, Q_n$ is an IS (after relabeling) with utility v .

Proof. The Lemma is vacuously true for any 1 or 2-element IS. Fix an IS Q_1, \dots, Q_n with $n \geq 3$ and v as above, and suppose $x_j \in Q_j$ for $j = i, i+1, i+2$ with $x_i \sim x_{i+1} \sim x_{i+2}$. By transitivity $x_i \sim x_{i+2}$, so $Q_1, \dots, Q_i, Q_{i+2}, \dots, Q_n$ is an IS; it remains to be shown that v is a utility for it. There is an $\epsilon > 0$ s.t. $B = B_\epsilon(v(x_i)) \subset (l_j, u_j)$ for $j = i, i+1, i+2$. Let $v^{-1}(u) : B \rightarrow Q_{i+1}$ be an arbitrary point in Q_{i+1} such that $v[v^{-1}(u)] = u$. Now, fix $x \in Q_i$ and $y \in Q_{i+2}$. For α small enough, $v(x\alpha x_i), v(y\alpha x_{i+2}) \in B$. Then $x\alpha x_i \sim v^{-1}(v(x\alpha x_i))$ and $y\alpha x_{i+2} \sim v^{-1}(v(y\alpha x_{i+2}))$. So

$$\begin{aligned}
& x \succsim y \\
& \iff x\alpha x_i \succsim y\alpha x_{i+2} \\
& \iff v^{-1}(v(x\alpha x_i)) \succsim v^{-1}(v(y\alpha x_{i+2})) \\
& \iff v[v^{-1}(v(x\alpha x_i))] \geq v[v^{-1}(v(y\alpha x_{i+2}))] \\
& \iff \alpha v(x) + (1 - \alpha)v(x_i) \geq \alpha v(y) + (1 - \alpha)v(x_{i+2}) \\
& \iff v(x) \geq v(y)
\end{aligned}$$

This establishes the Lemma. □

Lemma 5. Fix an IS Q_1, \dots, Q_n with utility v . If $(l_1, u_1) \cap (l_n, u_n) \neq \emptyset$, then there exists i and $x_j \in Q_j$ for $j = i+1, i+2, i+3$ with $x_i \sim x_{i+1} \sim x_{i+2}$.

Proof. If there is i with $(l_i, u_i) \cap (l_{i+2}, u_{i+2}) \neq \emptyset$, then there is $u \in \bigcap_{j=i, i+1, i+2} (l_j, u_j)$ so there exists $x_j \in Q_j$ with $v(x_j) = u$ for $j = i, i+1, i+2$ and thus by hypothesis, $x_i \sim x_{i+1} \sim x_{i+2}$. We show there exists such an i by contradiction. If $l_{i+2} > u_i$ for all i or $l_i > u_{i+2}$ for all i , then $(l_1, u_1) \cap (l_n, u_n) = \emptyset$, a contradiction. So there must exist i such that $[l_{i+2} > u_i$ and $l_{i+2} > u_{i+4}]$ or $[u_{i+2} < l_i$ and $u_{i+2} < l_{i+4}]$. In the first case, $l_{i+2} \in (l_{i+1}, u_{i+1}) \cap (l_{i+3}, u_{i+3})$; in the second, $u_{i+2} \in (l_{i+1}, u_{i+1}) \cap (l_{i+3}, u_{i+3})$. In either case, we have a contradiction. □

Lemma 6. Fix an IS Q_1, \dots, Q_n with utility v . Then for all $x, y \in \bigcup_i Q_i$, $x \succsim y \iff v(x) \geq v(y)$.

Proof. This is clearly true if $n = 1$. (IH) Suppose the claim is true for any IS with $m < n$ elements. Fix an IS Q_1, \dots, Q_n with utility v . If $x \notin Q_1 \cup Q_n$ or $y \notin Q_1 \cup Q_n$, then the claim immediately follows from the IH, and clearly holds if $x, y \in Q_i$ for some i . So it suffices to consider arbitrary $x \in Q_1$ and $y \in Q_n$. By Lemmas 4 and 5, if $(u_1, l_1) \cap (l_n, u_n) \neq \emptyset$, we can form a shorter IS from Q_1 to Q_n and the claim then follows from the IH.

There are two cases to consider: $l_n > u_1$ and $u_n < l_1$. If $l_n > u_1$, then there exists $y' \in Q_{n'}$ for $n' < n$ with $v(y') = l_n$. By the IH, that $Q_1, \dots, Q_{n'}$ is an IS and $v(y') > v(x)$, $y' \succ x$. Similarly, since $Q_{n'}, \dots, Q_n$ is an IS and $v(y) > v(y')$, $y \succ y'$. By transitivity, $y \succ x$ and since $v(y) > l_n > u_1 > v(x)$, the claim holds. Similar arguments obtain the desired conclusion when $u_n < l_1$. □

Define the relation \cong by $x \cong y \iff$ there exists an IS Q_1, \dots, Q_m with $x \in Q_1$ and $y \in Q_m$. It is easy to see that \cong is an equivalence relation (reflexive and transitive). Let $[x]$ denote the \cong equivalence class of x .

Lemma 7. *If $y \notin [x]$ and $x \succ y$, then $x' \succ y'$ for all $x' \in [x]$ and $y' \in [y]$.*

Proof. Fix $x, y \in X$ with $y \notin [x]$ and $x \succ y$, and assume $x \in R_k$. Pick any $y' \in [y]$. By definition, there is an IS Q_1, \dots, Q_m with $y' \in Q_m$ and $y \in Q_1$. Let $i = 1$ any $y_1 = y$. If there exists $y'' \in Q_i$ with $y'' \succsim x$, then $y'' \succsim x \succ y_i$, so by RC and connectedness of Q_i , we can find $z \in Q_i$ with $z \sim x$. If that occurs, then R_k, Q_i, \dots, Q_1 is an IS and $y \in [x]$, a contradiction. Thus $x \succ y''$ for all $y'' \in Q_i$. Moreover, there exists $y_{i+1} \in Q_{i+1}$ with $x \succ y_{i+1}$ by transitivity and definition of IS, so we can apply above logic to Q_{i+1} as well. Inductively, $x \succ y'$. Since y' is arbitrary, this extends to any $y' \in [y]$. Similar arguments show that $x' \succ y$ for any $x' \in [x]$. Combining, $x' \succ y'$ whenever $x' \in [x]$ and $y' \in [y]$. \square

Let A_1, \dots, A_n be the distinct equivalence classes of \cong . By Lemma 7, these sets can be completely ordered by \succ , i.e. $A_i \succ A_j \iff x \succ y$ for all $x \in A_i$ and $y \in A_j$. WLOG, $A_1 \succ A_2 \succ \dots \succ A_n$.

By Lemma 6, there is v_i on A_i so that v_i is affine and increasing on region contained in A_i and $x \succsim y \iff v_i(x) \geq v_i(y)$ for all $x, y \in A_i$. Note that $v_i(A_i)$ is bounded for all $i > 1$. Define $V(x) = v_1(x)$ for all $x \in A_1$. For $x \in A_i$ with $i > 1$, define $V(x)$ recursively by

$$V(x) = v_i(x) - \sup_{y \in A_i} v_i(y) + \inf_{y \in A_{i-1}} V(y) - 1$$

Observe $V(\cdot)$ is a positive affine transformation of $v_i(\cdot)$ on A_i and if $x \in A_i, y \in A_j$ and $i > j$, then $V(x) > V(y)$. Thus V represents \succsim and is affine and increasing when restricted to any given region.

Up to now, we fixed $f \in \mathcal{F}$ and constructed a representation for \succsim_f . Since f is arbitrary, this establishes that each \succsim_f has a representation $V(\cdot|f)$ that is affine and increasing on $R_i(f)$ for $i = 1, \dots, n$. Denoting $u_j(\cdot|f)$ the restriction to $R_j(f)$, we have established the existence of an RPM if $u_j(\cdot|f')$ is a positive affine transformation of $u_j(\cdot|f)$ for any f, f' .

Lemma 8. *For $i = 1, \dots, n$ and all $f, f' \in \mathcal{F}$, there are $\alpha > 0$ and $\beta \in \mathbb{R}$ so that $u_i(\cdot|f') = \alpha u_i(\cdot|f) + \beta$.*

Proof. Write $u \approx v$ if there are $\alpha > 0$ and $\beta \in \mathbb{R}$ so that $u = \alpha v + \beta$. Pick any $f \in \mathcal{F}$ and let

$$E = \{e \in \mathcal{F} : u_i(\cdot|e) \approx u_i(\cdot|f)\}.$$

E is closed, since if $e_n \in E$ and $e_n \rightarrow e \in \mathcal{F}$, then for any open ball $B \subset R_i(e)$, $R_i(e_n) \cap B \neq \emptyset$ for n large enough by lower semicontinuity of R_i . Observe $B' = R_i(e_n) \cap B \subset R_i(e_n) \cap R_i(e)$ is a mixture space and FI plus the usual uniqueness argument for affine utility functions gives that $u_i(\cdot|e_n) \approx u_i(\cdot|e)$. Now, consider $E' = cl(E^c)$. If $E' \cap E \neq \emptyset$, then the same argument as above with a sequence in E^c converging to $e \in E$ yields a contradiction. Hence $E' \cap E = \emptyset$ and $E \cup E' = \mathcal{F}$. Since \mathcal{F} is connected, $E' = \emptyset$ and $E = \mathcal{F}$. \square

Thus $u_i(\cdot|f)$ is an affine transformation of $u_i(\cdot|f')$ for all $f, f' \in \mathcal{F}$ and $\{\succsim_f\}$ conforms to RPM under \mathcal{R} . \square

Proof of Proposition 1

Proof. Note first that if there is no non-empty open set with this property, then the largest set revealed part of the region around x is \emptyset . Otherwise, we show there is a largest set using Zorn's lemma. Let $A_{t \in T}$ be a chain of open sets revealed part of the region around x , ordered by \subseteq . We claim that $\bigcup_{t \in T} A_t = B$ is an upper bound and also revealed part of the region around x . As a union of open sets, it is clearly open. If $a, b, a\alpha c, b\alpha c \in B$, then there are $t_a, t_b, t_{a\alpha c}, t_{b\alpha c}$ so that $a \in A_{t_a}, b \in A_{t_b}$, etc. Let A_{t^*} be the \subseteq -maximum of the four. Then $a, b, a\alpha c, b\alpha c \in A_{t^*}$. It follows that $a \succsim_r b \iff a\alpha c \succsim_r b\alpha c$. Thus there is at least one \subseteq -maximal element.

Clearly $R_i(r)$ is a revealed region around x when $x \in R_i(r)$. Assume $u_i(\cdot|r)$ is not a positive affine transformation of $u_j(\cdot|r)$ whenever $i \neq j$. If the result is false, then A is a revealed region around x and there exists $y \in A \setminus R_i(r)$. There is a region $R_j(r)$, $j \neq i$, such that $y \in R_j(r)$ or, since A is open and $\bigcup_{j=1}^n R_j(r)$ is dense, there is $R_j(r)$ $y' \in A \setminus R_i(r)$. WLOG, assume the former. Then there exists $z \in A \cap R_j(r) \setminus y$ such that $y \sim_r z$. However, for $\beta > 0$ small enough, $y\beta x, z\beta x \in R_i(r)$. But by assumption, the slope of the indifference curves in R_i differs from that of R_j . Thus we have $y\beta x \not\sim_r z\beta x$, a contradiction of A being a revealed region. \square

Proof of Proposition 2

Proof. First, we show (i) \implies (ii). Set $\sigma(a, b) = \max\{a/b, b/a\}$. Clearly σ is a salience function. Fix f and set $A = \{x : \sigma(x_1, e_1) > \sigma(x_2, e_2)\}$. We show $A = R_1(e)$.

Claim $A \cap R_2(e) = \emptyset$. If not, pick $x \in A \cap R_2(e)$. $x \in A$ implies either (a) $x_1/e_1 > x_2/e_2$ and $x_1/e_1 > e_2/x_2$ or (b) $e_1/x_1 > x_2/e_2$ and $e_1/x_1 > e_2/x_2$. If (a) and $x_2 \geq e_2$, then

$$x_1 \geq e_1 e_2 / x_2 > e_1,$$

so there exists $\lambda \in (0, 1]$ such that $(\lambda x_1 + (1 - \lambda)e_1, x_2) = (e_1 e_2 / x_2, x_2) = x'$. If (a) and $x_2 < e_2$, then

$$x_1 > e_1 x_2 / e_2 > e_1,$$

so there exists $\lambda \in (0, 1)$ such that $(\lambda x_1 + (1 - \lambda)e_1, x_2) = (e_1 x_2 / e_2, x_2) = x'$. By moderation and $x \in R_2(e)$, $x' \in R_2(e)$. However, $x'_1 x'_2 = e_1 e_2$ or $x'_1 / x'_2 = e_1 / e_2$ so $x' \notin R_2(e)$ by equal salience, a contradiction. A similar contradiction obtains if (b) holds.

Now, since $A \cap R_2(e) = \emptyset$ and $R_1(e) \cup R_2(e)$ is dense, $A \subset cl(R_1(e))$. By Smooth Regions, $R_1(e) = \int cl(R_1(e))$. Thus $A \subseteq R_1(e)$. Similarly, for $B = \{x : \sigma(x_1, e_1) < \sigma(x_2, e_2)\}$, $B \subseteq R_2(e)$. But

$$(A \cup B)^c = \{x : x_1 x_2 = e_1 e_2 \text{ or } x_1 / x_2 = e_1 / e_2\} \cap (R_1(e) \cup R_2(e)) = \emptyset.$$

Thus $A = R_1(e)$ and $B = R_2(e)$, completing the proof.

Now, we show (ii) implies (i). First, moderation follows from ordering. Second, equal salience follows from symmetry and homogeneity of degree zero. Third, Smooth Regions follows from continuous σ .

Finally, to see (iii) if and only if (ii), fix any salience function s . Observe $s(a, b) > s(c, d)$ iff $s(a/b, 1) > s(c/d, 1)$ by homogeneity iff $s(\max(a/b, b/a), 1) > s(\max(c/d, d/c), 1)$ by symmetry iff $\max(a/b, b/a) > \max(c/d, d/c)$ by ordering. Thus any salience function is equivalent. \square

Proof of Proposition 3

Proof. Necessity is obvious, so assume Axioms 1-4 and SFI. By Theorem 1, there exists a RPM.²⁰ Without loss of generality, normalize so that $u_1(\cdot|r) = u_1(\cdot|r')$ for all r, r' . Observe that $u_i(R_i(r)|r) \cap u_j(R_j(r)|r) \neq \emptyset$ for all r . Suppose $u_i(\cdot|r) \neq u_i(\cdot|r')$ for some r, r' and some i . Then, let $\bar{\epsilon} = d(r, r')$ and pick a sequence $\hat{r}_n \rightarrow \hat{r}$ such that: $u_i(\cdot|\hat{r}_n) \neq u_i(\cdot|r)$ and $d(\hat{r}_n, r) \rightarrow \inf\{d(r', r) : u_i(\cdot|r) \neq u_i(\cdot|r')\}$. Similarly, let r_n be a sequence such that $r_n \rightarrow \hat{r}$ and $u_i(\cdot|r) = u_i(\cdot|r_n)$. By each $R_i(r)$ is open, there exists ϵ , x_i and x_1 such that $B_\epsilon(x_i) \subset R_i(\hat{r})$, $B_\epsilon(x_1) \subset R_1(\hat{r})$, and $x_i \sim_{\hat{r}} x_1$. By continuity of the region functions, $B_\epsilon(x_i) \subseteq R_i(\hat{r}_n) \cap R_i(r_n)$ and $B_\epsilon(x_1) \subseteq R_1(\hat{r}_n) \cap R_1(r_n)$ for n large enough. For z close enough to x_i , there exists $y(z) \in B_\epsilon(x_1)$ such that $z \sim_{\hat{r}} y(z)$. But then by SFI, $z \sim_{r_n} y(z)$ and $z \sim_{\hat{r}_n} y(z)$. Thus $u_i(z|r_n) = u_1(y(z)|r_n) = u_1(y(z)|\hat{r}_n) = u_i(z|\hat{r}_n)$ for all z close enough to x_i , implying that $u_i(\cdot|r_n) = u_i(\cdot|\hat{r}_n)$, a contradiction. Conclude $u_i(\cdot|r) = u_i(\cdot|r')$ for all r, r' . \square

Proof of Theorem 2

Lemma 9. *For any r , there exists $x_i \in R_i(r)$ for $i = 1, 2$ with $x_1 \sim_r x_2$.*

Proof. We first show the result for $r = (z, z)$. Let $R_i^z = \{x \in R_i(a) : x_1 x_2 < z^2\}$, the points in $R_i(r)$ above the Cobb-Douglas line passing through r . Pick $a \in R_1^z$ and $b \in R_2^z$. If $a \succ_r b$, then by reflection and S1-S3, $a' \succ_r b'$ when $a' = (a_2, a_1) \in R_2^z$ and $b' = (b_2, b_1) \in R_1^z$. A simple proof using completeness, transitivity, RC and that both R_1^z, R_2^z open and connected yields $x_i \in R_i(r)$ for $i = 1, 2$ with $x_1 \sim_r x_2$. Similar proofs hold if $a \sim_r b$ or $b \succ_r a$.

Note the above arguments are true for $R_i^{z'}$ when $z' < z$, since $R_i^{z'} \subset R_i^z \subset R_i(r)$. Within each R_i^z , indifference curves are linear, parallel and downward sloping by RL and RM. For z' to be close 0, the x_1, x_2 we find must lie on indifference curves that intersect the boundary without leaving R_i^z . Thus for any r' , we can find x'_1 and x'_2 so that $x_i \sim_r x'_i$ and $x_i, x'_i \in R_i(r) \cap R_i(r')$ for $i = 1, 2$. But then by SFI and transitivity, $x'_1 \sim_{r'} x'_2$. \square

Proof. By the Lemma, we can apply Proposition 3. Hence there are u_1, u_2 such that if $x \in R_i(r)$ and $y \in R_j(r)$, then $x \succ_f y \iff u_i(x) \geq u_j(y)$. Now, $u_i(x) = a^i x_1 + b^i x_2 + c_i$, and $c_1 = 0$ WLOG. RM implies $a^i, b^i > 0$.

²⁰By SFI, there is a point of indifference between all regions so a disconnected region is not problematic.

Fix arbitrary $r = (r_1, r_2)$ and let $r' = (r_2, r_1)$. For any distinct $x, y \in R_1(r)$ such that $x \sim y$, $a^1x_1 + b^1x_2 = a^1y_1 + b^1y_2$. By reflection and the structure of \mathcal{R} , we have $(x_2, x_1), (y_2, y_1) \in R_2(r')$ and $(x_2, x_1) \sim_{r'} (y_2, y_1)$. Hence, $a^2x_2 + b^2x_1 + c^2 = a^2y_2 + b^2y_1 + c^2$ and $b^2/a^2 = (y_1 - x_1)/(x_2 - y_2)$ and $(x_1 - y_1)/(y_2 - x_2) = a^1/b^1$, so there exists $\alpha > 0$ such that $b^2 = \alpha a^1$ and $b_1 = \alpha a_2$.

For contradiction, suppose $\alpha \neq 1$. Pick $(x, y) \in R_1(1, 1)$ and $(a, b) \in R_2(1, 1)$ so that $(x, y) \sim_{1,1} (a, b)$. Note $(b, a) \in R_1(1, 1)$ and $(y, x) \in R_2(1, 1)$. By Reflection, $(y, x) \sim_{1,1} (b, a)$. Then

$$\begin{aligned} a^1x + b^1y &= \alpha[a^1b + b^1a] + c_2 \\ a^1b + b^1a &= \alpha[a^1x + b^1y] + c_2 \\ a^1b + b^1a &= \alpha[\alpha[a^1b + b^1a] + c_2] + c_2 \\ a^1b + b^1a &= \alpha^2[a^1b + b^1a] + (1 + \alpha)c_2 \\ (1 - \alpha^2)[a^1b + b^1a] &= (1 + \alpha)c_2 \end{aligned}$$

Since we can find similar indifferences for any bundles in a small enough neighborhood of (a, b) , this requires $\alpha = 1$. Now, the first two equalities imply

$$a^1b + b^1a = [[a^1b + b^1a] + c^2] + c^2$$

and so $c^2 = 0$.

After a normalization, we have $u_1(x) = wx_1 + (1 - w)x_2$ and $u_2(x) = (1 - w)x_1 + wx_2$. To conclude, we must show $w > \frac{1}{2}$. Pick any $y \in (0, \bar{x})$. Consider the line $L(y) = \{x' \in X : u_2(x') = u_2(y, y)\}$. This is the line with slope $\frac{-w}{1-w}$ that intersects the 45-degree line at (y, y) . For any $x \in L(y)$ such that $x_1 > y > x_2$, we can find r, r' so that $(y, y) \in R_2(r) \cap R_2(r')$, $(x_1, x_2) \in R_2(r) \cap R_1(r')$. By construction $(x_1, x_2) \sim_r (y, y)$, so since $x_1 > x_2$, by SDO we have $(x_1, x_2) \succ_{r'} (y, y)$ and $wx_1 + (1 - w)x_2 > y = (1 - w)x_1 + wx_2$. Letting $(x_1, x_2) \rightarrow (y, y)$ gives that $w > \frac{1}{2}$. \square

Proof of Proposition 4

Proof. By Proposition 3, drop the dependence on r and write $u_i(\cdot)$ instead of $u_i(\cdot|r)$. Let $U_i = \{x \in X : u_i(x) > u_j(x) \forall j \neq i\}$ and $L_i = \{x \in X : u_i(x) < u_j(x) \forall j \neq i\}$. Define $R_{-i}(r) = \bigcup_{j \neq i} R_j(r)$.

Lemma 10. *If \succ_r satisfies Monotonicity and $z \in U_i \cap R_i(r)$, then $R_{-i}(r) \cap \{x : x \gg z\} \cap U_i = \emptyset$.*

Proof. Suppose not, so there is $z \in U_i \cap R_i(r)$ such that $A = R_{-i}(r) \cap \{x : x \gg z\} \cap U_i \neq \emptyset$. Then let y_n be a sequence of points in A approaching as close as possible to z . WLOG, $y_n \rightarrow \bar{y}$ (since y_n must eventually belong to the compact set $\bar{B}_\epsilon(z)$ for some $\epsilon > 0$). Then we can pick

$x_n \in R_i(r) \cap \{y : y \leq \bar{y}\}$ that converges to $\bar{y} \in bd(R_i(r))$.²¹ Noting $u_{-i}(x) = \max_{j \neq i} u_j(x)$ is continuous,

$$\lim u_i(x_n) = u_i(\bar{y}) > u_{-i}(\bar{y}) = \lim u_{-i}(y_n)$$

and so $x_n \succ_r y_m$ for n, m large enough, but $y_m \geq x_n$ by taking n large enough that $d(x_n, \bar{y}) < d(y_m, \bar{y})$. This contradicts Monotonicity. \square

Lemma 11. *If \succsim_r satisfies Monotonicity and $r \in U_i$, then $R_{-i}(r) \cap \{x : x \gg r\} \cap U_i = \emptyset$.*

Proof. Follows from applying Lemma 10 to a sequence x_n in $R_i(r)$ that converges to r . \square

Lemma 12. *If \succsim_r satisfies Monotonicity and $z \in L_i \cap R_i(r)$ then $R_{-i}(r) \cap \{x : x \ll z\} \cap L_i = \emptyset$.*

Proof. Dual to Lemma 10. \square

Lemma 13. *If \succsim_r satisfies Monotonicity and $r \in L_i$, then $R_{-i}(r) \cap \{x : x \ll r\} \cap L_i = \emptyset$.*

Proof. Follows from applying Lemma 12 to a sequence x_n in $R_i(r)$ that converges to r . \square

By salience utilities, there are i and j so that $U_i = L_j = \{(x_1, x_2) : x_2 \leq x_1\}$ and $U_j = L_i = \{(x_1, x_2) : x_2 \geq x_1\}$; without loss of generality, let $i = 1$ and $j = 2$. Pick $r_0 \in X$ so that $u_1(r_0) = u_2(r_0)$; such a point exists because $(\bar{x}, \bar{x}) \in X$. Note that $\{x : u_1(x) = u_2(x)\}$ is an upward sloping line, and for $\epsilon > 0$, $u_1(r_0 - (0, \epsilon)) > u_2(r_0 - (0, \epsilon))$ and $u_2(r_0 + (0, \epsilon)) > u_1(r_0 + (0, \epsilon))$. Noting $U_1 = L_2$ and $U_2 = L_1$, by the Lemmas and that the regions are dense:

$$\begin{aligned} cl(R_1(r_0 - (0, \epsilon))) &\supseteq \{x \in U_1 : x \gg r_0 - (0, \epsilon)\} \\ cl(R_2(r_0 + (0, \epsilon))) &\supseteq \{x \in U_2 : x \gg r_0 + (0, \epsilon)\} \\ cl(R_1(r_0 + (0, \epsilon))) &\supseteq \{x \in L_1 : x \ll r_0 + (0, \epsilon)\} \\ cl(R_2(r_0 - (0, \epsilon))) &\supseteq \{x \in L_2 : x \ll r_0 - (0, \epsilon)\} \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and applying continuity of each R_i , $cl(R_1(r_0)) \supseteq \{x \in U_1 : x \gg r_0\} \cup \{x \in U_2 : x \ll r_0\}$ and $cl(R_2(r_0)) \supseteq \{x \in U_1 : x \ll r_0\} \cup \{x \in U_2 : x \gg r_0\}$.

For all r close enough to r_0 , $R_k(r) \cap U_k \neq \emptyset$ and $R_k(r) \cap L_k \neq \emptyset$ for $k = 1, 2$. Pick such an r that also belongs to U_1 . By the above, $R_i(r)$ intersects U_k and L_k for all $i, k \in \{1, 2\}$. $\supseteq \{x \in U_1 : x \gg r\} \cup \{x \in U_2 : x \ll r\}$.

Pick $y \in R_2(r) \cap U_2$. In \mathbb{R}^n , connected is equivalent to path connected, so there is a continuous function $\theta : [0, 1] \rightarrow R_2(r) \cup \{r\}$ with $\theta(0) = y$ and $\theta(1) = r$. There must exist z_0 such that $\theta(z_0) \in \{x : u_1(x) = u_2(x)\}$. Because $R_2(r)$ is open, there exists $\epsilon > 0$ s.t. $B_\epsilon(\theta(z_0)) \subset R_2(r)$. Observing that $B_\epsilon(\theta(z_0))$ intersects both L_2 and U_2 , by Lemmas 10 and 12, $R_1(r)$ intersects neither $\{(\theta(z_0)_1, z) : z \in \mathbb{R}_+\}$ nor $\{(z, \theta(z_0)_2) : z \in \mathbb{R}_+\}$. This contradicts connectedness of $R_1(r)$. \square

²¹If $\bar{y} = r$, then take $x_n = r$ for all n ; otherwise, $d(x', r) < d(\bar{y}, r)$ implies $x' \in R_i(r)$.

Proof of Proposition 5

Proof. By Theorem 1 and salient thinking, we can apply Proposition 3 to get a RD-RPM representation with $u_i(\cdot|r) = u_i(\cdot)$. There exists $a, b \in \mathbb{R}_{++}^2$ and $c \in \mathbb{R}$ such that $u_1(x) = a \cdot x + b$ and $u_2(x) = b \cdot x$, perhaps after a normalization. Let r be arbitrary and then pick $x, x', x'' \in R_1(r)$ and $y, y', y'' \in R_2(r)$ as in the definition of salient thinking. Then, $u_1(x) = u_2(y)$. Hence, $a_1\epsilon_1 = u_1(x') - u_1(x) > u_2(y') - u_2(y) = b_1\epsilon_1$, and $b_2\epsilon_2 = u_2(y'') - u_2(y) > u_1(x'') - u_1(x) = a_2\epsilon_2$. Conclude $a_1 > b_1$ and $a_2 > b_2$.

If there exists x^* where $u_1(x^*) = u_2(x^*)$, then a violation of Monotonicity follows from replacing (\bar{x}, \bar{x}) with x^* in the same arguments establishing Proposition 4. Since X is connected and $u_1(\cdot) - u_2(\cdot)$ is a continuous function, either $u_1(x) > u_2(x)$ for all $x \in X$ or $u_2(x) > u_1(x)$ for all $x \in X$. Set i such that $u_i(x) > u_{-i}(x)$ for some $x \in X$. Then Lemmas 10 and 12 establish that $cl(R_i(r)) = \bigcup_{x \in cl(R_i(r))} \{y : y \geq x\}$ and $cl(R_j(r)) = \bigcup_{x \in cl(R_j(r))} \{y : y \leq x\}$. \square