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A representation for intransitive indifference relations

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Abstract

Binary relations representable by utility functions with multiplicative error are considered. It is proved that if the error is a power of utility then the underlying binary relation is either an interval order, or a semiorder. Moreover, semiorders can be characterized among interval orders by the magnitude of the power of utility that is used in the form of error function.

Keywords: Utility function; Error; Just noticeable difference; Interval order; Semiorder

JEL classification: D6

1. Introduction

The concept of rationality in decision making has been the building block of many different disciplines that study human behavior. The pioneering work of Samuelson (1938) pointed out the importance and usefulness of numerical representation of the act of decision making as a tool to develop models for human behavior.

Arrow (1959) made the well-known connection between choice and preference. His choice axiom made it equivalent to make any behavioral analysis by considering either the preference or the choice as the primitive of the analysis. He also showed that the chosen

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alternatives are the ones which are undominated among the available alternatives with respect to a weak order.

Although weak orders have been accepted in economics for modelling preference, the transitivity of the indifference relation assumption of weak order was found to be too strong in some behavioral setups. Indeed, Fechner (1860) argued, on the basis of the psychophysics experiments, that ability to discriminate between stimuli is generally not transitive.

Luce (1956) also questioned transitivity of the indifference relation assumption[1] by giving his famous coffee/sugar example. Let us consider several cups of coffee with different amounts of sugar in them. One can be indifferent between 1 mg of sugar and 2 mg of sugar; 2 mg of sugar and 3 mg of sugar, and so forth. Yet, he will distinguish between a cup of coffee with 1 mg of sugar and N mg of sugar provided that N is large enough, and hence is likely to prefer one of them. However, this will violate transitivity of the indifference relation. To capture the idea of utility functions with just noticeable differences, Luce (1956) introduced[2] semiorders[3] and provided the desired characterization.

The concept of semiorders was subsequently simplified in Scott and Suppes (1958). They showed that any semiorder can be represented numerically with a positive constant error level. More precisely, the idea of error in preference requires one to prefer one alternative over another if the difference between the satisfactions from the former and the latter is at least as much as the error level. In other words, the decision maker is not able to distinguish the alternatives which are in the error neighborhood of each other.

The representation of Luce (1956) was able to capture Fechner's law. However, Luce and Edwards (1958), supported by Krantz (1971), theoretically showed the fallacies in the conclusion of Fechner's law.[4] Moreover, Stevens (1957) refuted Fechner's law by showing that the subjective sensation magnitudes are related to the physical intensity of stimuli by a power law, i.e. $S=kI^n$ where S is the sensation magnitude, I is the physical intensity of the stimulus, k is a positive constant, and n is the power which depends on the stimulus. For example, $n=3.5$ for electric shock (current through fingers), $n=1$ for cold (metal contact on arm), $n=0.6$ for smell (heptane), (see Stevens (1957, 1975) for the complete list).

Similar in spirit to Stevens (1957), we consider the multiplicative error functions depending on the value of the utility function of the compared alternatives, where the error is a power function of the utility function. Moreover, our representation will be a generalization of Aleskerov and Masatlioglu (2003) and Scott and Suppes (1958). Additionally, we will show a broader class of semiorders. More precise discussion and characterization of the forms of the error function studied in the literature can be found in Section 2. Section 3 has the main results of this paper; while Section 4 concludes.

[1] Before Luce (1956), Armstrong (1939, 1948, 1950, 1951) interrogated intensively the transitivity of the indifference relation.

[2] Indeed, the idea of semiorders is originated by Wiener (1914) (see also, Fishburn and Monjardet, 1992 and Monjardet, 1988).

[3] The concept of semiorders attracted the attention of many researchers (see, e.g. Pirlot and Vincke, 1997; Roubens and Vincke, 1985).

[4] See also Luce and Suppes (2002) for history of measurement.

2. Preliminaries

Let A be a finite set of alternatives. A binary relation P is a subset of $A \times A$.

Definition 1. A binary relation P is said to have *numerical representation via utility function with error* if there exist a function $u(\cdot)$: $A \rightarrow \mathbb{R}_+$ and an insensitivity threshold (utility discrimination threshold, utility measurement error) ε such that

$$xPy \Leftrightarrow u(x) - u(y) > \varepsilon. \tag{1}$$

Thus, in order to discriminate two alternatives, their utilities must differ by at least ε.

Now we define binary relations which are used in the sequel; for details see, e.g., Luce (1956), Fishburn (1985), Doignon et al. (1986), Aizerman and Aleskerov (1995).

Definition 2. A *biorder* is a binary relation P satisfying the strong intervality condition: If xPy and $zPw \Rightarrow xPw \vee zPy$ for all $x, y, z, w \in A$.

An *interval order* is a biorder P which is irreflexive: xPx for no $x \in A$.

A *semiorder* is an interval order which is semi-transitive: xPy and $yPz \Rightarrow xPw \vee wPz$ for all $x, y, z, w \in A$.

The binary relation P with a numerical representation as in Eq. (1) has been characterized in regard of the form of error function ε in the literature, and these existing results can be summarized as follows[5]:

(1) Scott and Suppes (1958) established that P is a semiorder if and only if $\varepsilon = \text{const} \geq 0$;
(2) Mirkin (1974) and Fishburn (1985) showed independently that P is an interval order if and only if $\varepsilon = \varepsilon_1(x) \geq 0$ where $\varepsilon_1 : A \rightarrow \mathbb{R}_+$.
(3) Doignon et al. (1986), by not requiring the error function to be non-negative, derived semi-transitivity and strong intervality condition.
(4) Aleskerov and Vol'skiy (1987), Agaev and Aleskerov (1993), and Aizerman and Aleskerov (1995) assumed that ε in Eq. (1) depends on both x and y. This makes Eq. (1) look like

$$xPy \Leftrightarrow u(x) - u(y) > \varepsilon_2(x, y) \text{ where } \varepsilon_2 : A \times A \rightarrow \mathbb{R}. \tag{2}$$

It has been proved that

(a) P is acyclic if and only if P is representable as in Eq. (2) with $\varepsilon_2(x, y) \geq 0$;
(b) every binary relation is representable as in Eq. (2) (with no special restriction on $\varepsilon_2(x, y)$);
(c) if the error function is additive, that is,

$$\varepsilon_2(x, y) = \varepsilon_1(x) + \varepsilon_1(y), \tag{3}$$

then P is a biorder, and
(d) if in Eq. (3) it is assumed that $\varepsilon_2(x, y) \geq 0$, then the induced P is an interval order; see Aizerman and Aleskerov (1995).

[5] A more detailed literature review of these results can be found in Aleskerov and Monjardet (2002).

(5) Abbas (1994) introduced a multiplicative error function

$$\varepsilon_2(x,y) = \varepsilon_1(x) \cdot \varepsilon_1(y), \tag{4}$$

(see also Fodor and Roubens, 1997; Abbas et al., 1996). Aleskerov and Masatlioglu (2003) distinguished two cases of the multiplicative error function:

(a) (direct proportion)

$$\varepsilon_1(x) = \alpha \cdot u(x), \qquad \alpha>0, \tag{5}$$

meaning that the relative error in utility measurements $\alpha = \varepsilon_1(x)/u(x)$ is constant (as in most physical measurements),

(b) (inverse proportion)

$$\varepsilon_1(x) = \frac{\alpha}{u(x)}, \qquad \alpha>0, \tag{6}$$

meaning that the accuracy of utility measurement is inversely proportional to utility, and the relative error is hyperbolic $\varepsilon_1(x)/u(x) = \alpha/u^2(x)$, i.e. the lower the utilities, the lower the utility discrimination (as in most human evaluations, e.g. in artistic comparisons).

Aleskerov and Masatlioglu (2003) have investigated binary relations, that are representable as in Eq. (2). They have shown that Eqs. (4) and (6) hold if and only if P is an interval order; and if Eqs. (4) and (5) hold then P is a semiorder and any regular semiorder[6] can be representable as in Eq. (2) where Eqs. (4) and (5) hold. The representability of any semiorder as in Eq. (2) where Eqs. (4) and (5) hold was left as an open question in Aleskerov and Masatlioglu (2003).

In this paper, we consider binary relations that are representable as in Eqs. (2) and (4) under the assumption that the error is a power of utility:

$$\varepsilon_1(x) = \alpha u(x)^{\beta}, \qquad \alpha>0. \tag{7}$$

We are interested in the binary relation's properties determined by values of β. Observe that this approach generalizes the cases of Aleskerov and Masatlioglu (2003) and Scott and Suppes (1958). More precisely, direct proportion and inverse proportion cases of Aleskerov and Masatlioglu (2003) are the cases of $\beta=1$, and $\beta=-1$, respectively. The constant error model of Scott and Suppes (1958) is the case of $\beta=0$.

3. General multiplicative error function

Theorem 1. *If a binary relation P can be numerically represented as in Eqs. (2), (4), and (7) with utility function $u(\cdot)>0$, then P is an interval order.*

Proof. Let a binary relation P be numerically represented as in Eqs. (2), (4), and (7) for some $u(\cdot)>0, \varepsilon(.,.) : A \times A \rightarrow \mathbb{R}$ such that $\varepsilon_2(x,y) = \alpha u(x)^{\beta} \alpha u(y)^{\beta}$ where $\alpha>0$ and $\beta \in \mathbb{R}$. To

[6] The definition of regular semiorder will be given later.

show that P is an interval order, we need to show that (i) P is irreflexive, and (ii) P satisfies strong intervality.

i) P is irreflexive. Since $\varepsilon_2(x,\ x)=\varepsilon_1(x)\cdot\varepsilon_1(x)=\alpha^2 u(x)^{2\beta}>0$, then $\varepsilon_2(x,\ x)>0=u(x)-u(x)$. Thus $x\bar{P}x$.

ii) P satisfies strong intervality. Assume the contrary $xPy\wedge zPw\wedge x\bar{P}w\wedge z\bar{P}y$ for some x, y, z, $w\in A$. Then

$$u(x) - u(y) > \alpha^2 u(x)^\beta u(y)^\beta, \tag{8}$$

$$u(z) - u(w) > \alpha^2 u(z)^\beta u(w)^\beta, \tag{9}$$

$$u(z) - u(w) \leq \alpha^2 u(x)^\beta u(w)^\beta, \tag{10}$$

$$u(z) - u(y) \leq \alpha^2 u(z)^\beta u(y)^\beta. \tag{11}$$

There are three cases:

Case 1: $\beta>0$. Inequalities (8) and (10) imply

$$u(w) - u(y) > \alpha^2 u(x)^\beta \left[u(y)^\beta - u(w)^\beta\right]. \tag{12}$$

Inequalities (9) and (11) imply

$$u(y) - u(w) > \alpha^2 u(z)^\beta \left[u(w)^\beta - u(y)^\beta\right]. \tag{13}$$

(1) If $u(w)=u(y)$ then from Inequality (12) we have

$$u(w) - u(y) = 0 > \alpha^2 u(x)^\beta \left[u(y)^\beta - u(w)^\beta\right] = 0,$$

which is a contradiction.

(2) If $u(w)>u(y)$ then $u(y)-u(w)<0$ and $u(w)^\beta - u(y)^\beta>0$ because $\beta>0$. From Eq. (13) we get

$$0 > u(y) - u(w) > \alpha^2 u(z)^\beta \left[u(w)^\beta - u(y)^\beta\right] > 0,$$

which is a contradiction.

(3) If $u(w)<u(y)$ then $u(w)-u(y)<0$ and $u(y)^\beta-u(w)^\beta>0$ because $\beta>0$. From Inequality (12) we get

$$0 > u(w) - u(y) > \alpha^2 u(x)^\beta \left[u(y)^\beta - u(w)^\beta\right] > 0,$$

which is a contradiction.

Case 2: $\beta<0$. Inequalities (8) and (11) imply

$$u(x) - u(z) > \alpha^2 u(y)^\beta \left[u(x)^\beta - u(z)^\beta\right], \tag{14}$$

and inequalities (9) and (10) imply

$$u(z) - u(x) > \alpha^2 u(w)^\beta \left[u(z)^\beta - u(x)^\beta\right]. \tag{15}$$

Then, a similar analysis as in $\beta>0$ will give a contradiction.

Case 3: $\beta=0$. As $\beta=0$, the error function assigns a positive constant to all values of x. This case is considered by Luce (1956).[7]

Hence, P is an interval order. □

The following theorem is the main theorem of the paper. It not only provides a utility representation with multiplicative error function as in Eqs. (4), and (7) for semiorders, but also classifies semiorders among interval orders with respect to magnitude of β.

Theorem 2. *Let $0\leq\beta\leq 1$. A binary relation P can be numerically represented as in Eqs. (2), (4), and (7) with utility function $u(\cdot)>0$ if and only if P is a semiorder.*

Proof. Let $\beta\in\mathbb{R}$ be in [0, 1] interval. Suppose that a binary relation P has a numerical representation as stated in the formulation of Theorem 2. We prove that P is a semiorder.

By Theorem 1, for all β, the relation P is an interval order. Therefore, it suffices to show that P satisfies the semi-transitivity condition.

Assume the contrary, $xPy\wedge yPz\wedge x\bar{P}t\wedge t\bar{P}z$. Then

$$u(x)-u(y)>\alpha^2 u(x)^\beta u(y)^\beta, \tag{16}$$

$$u(y)-u(z)>\alpha^2 u(y)^\beta u(z)^\beta, \tag{17}$$

$$u(x)-u(t)\leq\alpha^2 u(x)^\beta u(t)^\beta, \tag{18}$$

$$u(t)-u(z)\leq\alpha^2 u(t)^\beta u(z)^\beta. \tag{19}$$

Inequalities (16) and (18) imply

$$\frac{u(x)-u(t)}{u(x)^\beta u(t)^\beta}\leq\alpha^2<\frac{u(x)-u(y)}{u(x)^\beta u(y)^\beta},$$

whence

$$u(y)^{1-\beta}-\frac{u(x)}{u(y)^\beta}<u(t)^{1-\beta}-\frac{u(x)}{u(t)^\beta}.$$

Since $f:\mathbb{R}_{++}\to\mathbb{R}, f(w)=w^{1-\beta}-\frac{c}{w^\beta}$ for some $0\leq\beta\leq 1, c>0$ is a strictly increasing function,

$$u(y)<u(t). \tag{20}$$

On the other hand, Inequalities (17) and (19) imply

$$\frac{u(t)-u(z)}{u(t)^\beta u(z)^\beta}\leq\alpha^2<\frac{u(y)-u(z)}{u(y)^\beta u(z)^\beta},$$

whence

$$u(t)^{1-\beta}-\frac{u(z)}{u(t)^\beta}<u(y)^{1-\beta}-\frac{u(z)}{u(y)^\beta}.$$

[7]See also Aizerman and Aleskerov (1995), Theorem 3.1, p. 84.

Again since f: $\mathbb{R}_{++}\rightarrow\mathbb{R}, f(w) = w^{1-\beta} - \dfrac{c}{w^{\beta}}$ for some $0\leq\beta\leq 1, c>0$ is a strictly increasing function,

$$u(t)<u(y), \tag{21}$$

so Eqs. (20) and (21) will give a contradiction. Hence, P is a semiorder.

Conversely, suppose that P is a semiorder and now we need to prove that there exists the numerical representation required.

Let us construct the partitions which defines the structure of semiorder. If P is non-trivial[8], then there exist n different elements, namely $a_1,a_2,\ldots,a_n$, such that

$$L(a_1)\subset L(a_2)\subset\ldots\subset L(a_n),$$

where $L(x)$ stands for the lower contour set of x, i.e. $L(x)=\{y\in A\text{: } xPy\}$. Note that $L(a_1)=\emptyset$ since P is irreflexive.[9]

Then, any non-trivial semiorder P can be represented as

$$P = \cup_{k=2}^{n}\left[I_k \times \cup_{m=1}^{k-1} J_m\right],$$

where $n\geq 2$, $I_k=\{a\in A\text{: } L(a)=\mathrm{L}(a_k)\}$, $k=1,\ldots,n$, and $J_m=L(a_{m+1})\backslash L(a_m)$, $m=1,\ldots,n-1$, $J_n=A\backslash L(a_n)$, for details see, e.g., Mirkin (1974), Fishburn (1985).

Now define $\{Z_m\}_2^{2n}$ such that

$$Z_m = \begin{cases} I_{m/2}\cap J_{m/2} & \text{if } m \text{ is even} \\ I_{(m-1)/2}\cap J_{(m+1)/2} & \text{otherwise} \end{cases}$$

Since $J_k\cap J_m=\emptyset$ for all $|k-m|\geq 1$, $Z_k\cap Z_l=\emptyset$ as $k\neq l$, and $\cup_{m=2}^{2n} Z_m=A$.

Then, the preference structure of the semiorder is $x_{2n-r}Px_{2n-r-j}$ for all odd r and $j>2$, and $x_{2n-r}Px_{2n-r-j}$ for all even r and $j\geq 2$, where $x_m\in Z_m$ for all $m=2,3,\ldots,2n$. (for details see, e.g., Mirkin, 1974; Fishburn, 1985).

Now construct the numerical function $u(\cdot)$, $\forall x$, $y\in Z_m u(x)=u(y)$ and denoted by $u(x_m)$, and

$$u(x_{2n-r}) = \begin{cases} 1-\frac{r}{2}\left(\dfrac{4n+1}{4n}\right)\alpha^2 & \text{if } r \text{ is even} \\ 1-\left(\dfrac{r-1}{2}\,\dfrac{4n+1}{4n}\alpha^2\right)-(4n-1-r)\dfrac{\alpha^2}{4n} & \text{otherwise,} \end{cases}$$

where $0\leq r\leq 2n-2$, and $\alpha^2 = \frac{1}{8n(n-1)}$.

Note that there may be no element in some Z_m, but this will not affect the representation.

[8] P is non-trivial if $\exists x,y,z\in A$ such that $xPyPz$. In Aleskerov and Masatlioglu (2003) terminology, non-trivial semiorder means regular semiorder.

[9] More precisely, $L(a_1)$ is the smallest lower counter set (the smallest one exists due to the finiteness of A). If there exists $x\in L(a_1)$, then $L(x)=L(a_1)$ since $L(a_1)$ is the smallest lower contour set. This means that $x\in L(x)$, i.e. xPx, which contradicts with irreflexivity of P.

First of all we show that $u(\cdot)$ decreases as r increases, and the utility values are positive. Secondly, we prove two lemmas. Finally, since the preference structure of the semiorder is $x_{2n-r}Px_{2n-r-j}$ for all odd r and $j>2$, and $x_{2n-r}Px_{2n-r-j}$ for all even r and $j\geq 2$, we show:

$u(x_{2n-r})-u(x_{2n-r-j})>\varepsilon_2(x_{2n-r}, x_{2n-r-j})$, for odd r and $j>2$;
$u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r}, x_{2n-r-j})$, for odd r and $j\leq 2$;
$u(x_{2n-r})-u(x_{2n-r-j})>\varepsilon_2(x_{2n-r}, x_{2n-r-j})$, for even r and $j\geq 2$;
$u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r}, x_{2n-r-j})$, for even r and $j<2$.

Let us show that u decreases as r increases, i.e. $u(x_{2n-r})>u(x_{2n-r-1})$ for every $r=0,\ldots, 2n-3$. Assume the contrary, $u(x_{2n-r})\leq u(x_{2n-r-1})$. Consider the two cases:

Case 1: r is odd, i.e.

$$1-\left(\frac{r-1}{2}\cdot\frac{4n+1}{4n}\alpha^2\right)-(4n-1-r)\frac{\alpha^2}{4n}\leq 1-\frac{r+1}{2}\left(\frac{4n+1}{4n}\right)\alpha^2.$$

Then $\alpha^2((2+r)/4n)\leq 0$, contradicting the fact that both α^2 and $((2+r)/4n)$ are positive.

Case 2: r is even, i.e.

$$1-\frac{r}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\leq 1-\left(\frac{r}{2}\frac{4n+1}{4n}\alpha^2\right)-(4n-1-r-1)\frac{\alpha^2}{4n}.$$

Then $0\leq(2+r-4n)\alpha^2/4n$, contradicting the fact that $\alpha^2/4n>0$ and $(2+r-4n)<0$, since $2n-2\geq r$ implies $4n\geq 2r+4>r+2$.

Therefore, $u(\cdot)$ decreases as r increases.

Since $u(\cdot)$ decreases as r and $n>1$ increase, $u(x_{2n})=1$ (the case when r is the least, and hence the utility takes the largest value), and

$$u(x_2)=\left(1-\frac{1}{8n}-\frac{1}{32n^2}\right)=\frac{32n^2-4n-1}{32n^2}>0,$$

(the case when r has the largest, and hence the utility has the least value).

Therefore, $0<u(x_{2n-r})\leq 1$ for $0<r\leq 2n-2$.

Lemma 1. *$u(x_{2n-r})-u(x_{2n-r-1})\leq\alpha^2u(x_2)^2$, where r is even, $0\leq r<2n-2$, and $n>1$.*

Proof of Lemma 1. Assume the contrary that $u(x_{2n-r})-u(x_{2n-r-1})>\alpha^2u(x_2)^2$.

$$\begin{aligned}u(x_{2n-r})-u(x_{2n-r-1})&=\left[1-\frac{r}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right]\\&\quad-\left[1-\left(\frac{r}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right)-(4n-1-r-1)\frac{\alpha^2}{4n}\right]\end{aligned}$$

$$=(4n-2-r)\frac{\alpha^2}{4n}=\left(1-\frac{1}{2n}-\frac{r}{4n}\right)\alpha^2>\alpha^2\left(\frac{32n^2-4n-1}{32n^2}\right)^2=\alpha^2u(x_2)^2.$$

Hence $8n(-32(1+r)n^2+6n-1)>1$ which is a contradiction, since $8n>0$ and $-32(1+r)n^2+6n-1\leq 0$. Indeed, if $-32(1+r)n^2+6n-1>0$ then $6n-1>32(1+r)n^2\geq 32n^2$, implying $1>n(32n-6)$ against $n>1$.

Consequently, $u(x_{2n-r})-u(x_{2n-r-1})\leq\alpha^2u(x_2)^2$ for even r, $0\leq r<2n-2$, and $n>1$. □

Lemma 2. *$u(x_{2n-r})-u(x_{2n-r-2})\leq\alpha^2u(x_2)u(x_3)$, where r is odd, $0<r<2n-2$ and $n>1$.*

Proof of Lemma 2. Assume the contrary that $u(x_{2n-r})-u(x_{2n-r-2})>\alpha^2u(x_2)u(x_3)$.

$$\begin{aligned}u(x_{2n-r})-u(x_{2n-r-2}) &= \left[1-\left(\frac{r-1}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right)-(4n-1-r)\frac{\alpha^2}{4n}\right]\\ &\quad-\left[1-\left(\frac{r+1}{2}\frac{4n+1}{4n}\alpha^2\right)-(4n-1-r-2)\frac{\alpha^2}{4n}\right]\\ &=\frac{\alpha^2}{4n}(4n-1)>\alpha^2\left(\frac{32n^2-4n-1}{32n^2}\right)\left(\frac{32n^2-36n+5}{32n(n-1)}\right).\end{aligned}$$

Then $0>16n^2+16n-5$, which is a contradiction, since $16n^2>0$ and $16n>5$.

Therefore, $u(x_{2n-r})-u(x_{2n-r-2})\leq\alpha^2u(x_2)u(x_3)$, where r is odd, $0<r<2n-2$, and $n>1$. □

Let us complete the proof of the second half of Theorem 2, i.e. show that the constructed utility function with multiplicative error as a power function of utility, represents any non-trivial semiorder.

(a) r is even. We show

$$u(x_{2n-r})-u(x_{2n-r-j})>\varepsilon_2(x_{2n-r},x_{2n-r-j}) \text{ for all } j\geq 2.$$

For $j=2$ we have, since $\beta\in[0,1]$ and $0<u(\cdot)\leq 1$:

$$\begin{aligned}u(x_{2n-r})-u(x_{2n-r-2}) &= \left[1-\frac{r}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right]-\left[1-\frac{r+2}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right]\\ &=\frac{4n+1}{4n}\alpha^2>\alpha^2\geq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-2})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-2}).\end{aligned}$$

For $j>2$, $u(x_{2n-r-j})<u(x_{2n-r-2})$, then we have:

$$\begin{aligned}&u(x_{2n-r})-u(x_{2n-r-j})>u(x_{2n-r})-u(x_{2n-r-2})\\ &\quad>\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-2})]^\beta\geq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-j})]^\beta=\varepsilon^2(x_{2n-r},x_{2n-r-j}).\end{aligned}$$

Now we show

$$u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r},x_{2n-r-j}) \text{ for any } j<2.$$

For $j=1$ by Lemma 1, we get $u(x_{2n-r})-u(x_{2n-r-1})\leq\alpha^2[u(x_2)]^2$. Also,

$$\alpha^2[u(x_2)]^2<\alpha^2u(x_{2n-r})u(x_{2n-r-1})\leq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-1})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-1}).$$

Hence, $u(x_{2n-r})-u(x_{2n-r-1})\leq\varepsilon_2(x_{2n-r},x_{2n-r-1})$.

For $j<1$, i.e. $j=0$ we obtain

$$u(x_{2n-r})-u(x_{2n-r-j})=0\leq\alpha_2[u(x_{2n-r})]^\beta[u(x_{2n-r})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-j}).$$

Thus for any $j\geq 2$,

$$u(x_{2n-r})-u(x_{2n-r-j})>\varepsilon_2(x_{2n-r},x_{2n-r-j})=\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-j})]^\beta,$$

and for any $j<2$,

$$u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r},x_{2n-r-j})=\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-j})]^\beta.$$

(b) Let r be odd. Now we show

$$u(x_{2n-r})-u(x_{2n-r-j})>\varepsilon_2(x_{2n-r},x_{2n-r-j})\text{ for any } j>2.$$

For $j=3$, since $\beta\in[0,1]$ and $0<u(\cdot)\leq 1$, we obtain

$$\begin{aligned}u(x_{2n-r})u(x_{2n-r-3})&=\left[1-\left(\frac{r-1}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right)-(4n-1-r)\frac{\alpha^2}{4n}\right]\\&\quad-\left[1-\frac{r+3}{2}\left(\frac{4n+1}{4n}\right)\alpha^2\right]\\&=\alpha^2\frac{1}{2n}\left(\frac{3}{2}+2n+\frac{r}{2}\right)>\alpha^2\geq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-3})]^\beta\\&=\varepsilon_2(x_{2n-r},x_{2n-r-3}).\end{aligned}$$

For $j>3$ we have

$$\begin{aligned}&u(x_{2n-r})-u(x_{2n-r-j})>u(x_{2n-r})-u(x_{2n-r-3})\geq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-3})]^\beta\\&\quad\geq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-j})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-j}).\end{aligned}$$

Now we show that

$$u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r},x_{2n-r-j})\text{ for any } j\leq 2.$$

For $j=2$ we have by Lemma 2:

$$\begin{aligned}&u(x_{2n-r})-u(x_{2n-r-2})\leq\alpha^2u(x_2)u(x_3)<\alpha^2u(x_{2n-r})u(x_{2n-r-2})\\&\quad\leq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-2})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-2}).\end{aligned}$$

Thus $u(x_{2n-r})-u(x_{2n-r-j})\leq\varepsilon_2(x_{2n-r},x_{2n-r-j})$.
For $j<2$ we get

$$\begin{aligned}&u(x_{2n-r})-u(x_{2n-r-j})<u(x_{2n-r})-u(x_{2n-r-2})\leq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-2})]^\beta\\&\quad\leq\alpha^2[u(x_{2n-r})]^\beta[u(x_{2n-r-j})]^\beta=\varepsilon_2(x_{2n-r},x_{2n-r-j}),\end{aligned}$$

as required.

Now, consider any trivial semiorder, where $\nexists$ x, y, $z \in A$ s.t. $xPyPz$, i.e. the semitranisitivity condition is trivially satisfied. Then, P can be represented as $P=Z_k \times Z_l$ s.t. $k-l \geq n$, and $I_k \cap J_m \neq \emptyset$ for $k \neq l$ and $m \neq n$. Now construct the numerical function $u(\cdot)$, $\forall x, y \in Z_m u(x)=u(y)$ and denoted by $u(x_m)$, and

$$u(x_r) = 1 - \frac{2n-r}{2^n(2n-2)},$$

where $2 \leq r \leq 2n$ and $\alpha^2 = \frac{2^{n-1}}{(2^n-1)^2}$.

Clearly, $u(\cdot)$ increases as r increases, and $0 < u(x_r) \leq 1$.

i) Let $k-l \geq n$. Then,

$$\begin{aligned} u(x_k) - u(x_l) &= 1 - \frac{2n-k}{2^n(2n-2)} - \left[1 - \frac{2n-l}{2^n(2n-2)}\right] \\ &= \frac{k-l}{2^n(2n-2)} \geq \frac{n}{2^n(2n-2)} > \frac{2^{n-1}}{(2^n-1)^2} = \alpha^2 \geq \alpha^2 u(x_k)^\beta u(x_l)^\beta \\ &= \varepsilon_2(x_k, x_l) \text{ for any } \beta \in [0,1]. \end{aligned}$$

ii) Let $k-l \leq n-1$. Then,

$$\begin{aligned} u(x_k) - u(x_l) &= 1 - \frac{2n-k}{2^n(2n-2)} - \left[1 - \frac{2n-l}{2^n(2n-2)}\right] = \frac{k-l}{2^n(2n-2)} \leq \frac{n-1}{2^n(2n-2)} \\ &= \frac{1}{2^{n+1}} < \frac{2^{n-1}}{(2^n-1)^2}\left[1 - \frac{1}{2^n}\right] \leq \alpha^2 u(x_2)^2 < \alpha^2 u(x_k) u(x_l) \leq \alpha^2 u(x_k)^\beta u(x_l)^\beta \\ &= \varepsilon_2(x_k, x_l) \text{ for any } \beta \in [0,1]. \end{aligned}$$

as required. □

The following examples show that P with a numerical representation as in Theorem 2 is not generally a semiorder if $\beta > 1$ or $\beta < 0$.

Example 3. Let $\beta > 1$, $u(x)=3$, $u(y)=2$, $u(z)=1$, $\alpha^2 = 1/10^\beta$ and choose $u(t)$ such that $u(t) - 1 \leq 1/10^\beta u(t)^\beta$ and $u(t) > 3$. Since $\beta > 1$, such $u(t)$ exists (to see it, divide both sides by $u(t)$ and send $u(t)$ to ∞). Then

$$u(x) - u(y) = 1 > \left(\frac{6}{10}\right)^\beta = \alpha^2 u(x)^\beta u(y)^\beta,$$

$$u(y) - u(z) = 1 > \left(\frac{2}{10}\right)^\beta = \alpha^2 u(y)^\beta u(z)^\beta,$$

$$u(x) - u(t) = 3 - u(t) < 0 < \alpha^2 u(x)^\beta u(t)^\beta \text{ (since } u(t) > 3),$$

$$u(t) - u(z) = u(t) - 1 \leq \frac{1}{10^\beta} u(t)^\beta = \alpha^2 u(t)^\beta u(z)^\beta \left(\text{since } u(t) - 1 \leq \frac{1}{10^\beta} u(t)^\beta\right).$$

The above inequalities imply that xPy, yPz, $x\bar{P}t$, and $t\bar{P}z$. Therefore, the induced binary relation P is not a semiorder.

Example 4. Let $\beta<0$, $u(x)=3$, $u(y)=2$, $u(z)=1$, $\alpha^2=1$ and choose $u(t)$ such that $3<3^\beta u(t)^\beta$ and $u(t)<1$. Since $\beta<0$, such $u(t)$ exists (take $u(t)$ close to 0). Then

$$u(x)-u(y)=1>6^\beta=\alpha^2 u(x)^\beta u(y)^\beta,$$

$$u(y)-u(z)=1>2^\beta=\alpha^2 u(y)^\beta u(z)^\beta,$$

$$u(x)-u(t)=3-u(t)<3<3^\beta u(t)^\beta=\alpha^2 u(x)^\beta u(t)^\beta\ \left(\text{since } 3<3^\beta u(t)^\beta\right),$$

$$u(t)-u(z)=u(t)-1<0<\alpha^2 u(t)^\beta u(z)^\beta\ (\text{since } u(t)<1).$$

The above inequalities imply that xPy, yPz, $x\bar{P}t$, and $t\bar{P}z$. Therefore, the induced binary relation P is not a semiorder.

4. Conclusion

The paper contributes to axiomatization of preference modelling. We consider binary relations represented via utility functions and multiplicative error functions, depending on both compared alternatives. We showed that for the error function constructed as a β power of utility, the properties of the induced binary relation depend on the magnitude of the power β. The representation in this paper generalizes the cases of Aleskerov and Masatlioglu (2003) and Scott and Suppes (1958). More precisely, direct proportion and inverse proportion cases of Aleskerov and Masatlioglu (2003) are the cases of $\beta=1$, and $\beta=-1$, respectively. The constant error model of Scott and Suppes (1958) is the case of $\beta=0$.

To sum up, the form of the error function is used to characterize interval and semiorders. Indeed, the representation with *multiplicative* error function that is a power of utility implies that the binary relation is an interval order. It is a semiorder for $0\leq\beta\leq 1$; any semiorder can be represented as in Eqs. (2), (4), and (7) for $0\leq\beta\leq 1$. Moreover, if $\beta>1$ or $\beta<0$, then the representation does not necessarily induce a semiorder relation.

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