INSTRUMENTAL VARIABLE ESTIMATION OF A SPATIAL AUTOREGRESSIVE MODEL WITH AUTOREGRESSIVE DISTURBANCES: LARGE AND SMALL SAMPLE RESULTS

Harry H. Kelejian, Ingmar R. Prucha
and Yevgeny Yuzefovich

ABSTRACT

The purpose of this paper is two-fold. First, on a theoretical level we introduce a series-type instrumental variable (IV) estimator of the parameters of a spatial first order autoregressive model with first order autoregressive disturbances. We demonstrate that our estimator is asymptotically efficient within the class of IV estimators, and has a lower computational count than an efficient IV estimator that was introduced by Lee (2003). Second, via Monte Carlo techniques we give small sample results relating to our suggested estimator, the maximum likelihood (ML) estimator, and other IV estimators suggested in the literature. Among other things we find that the ML estimator, both of the asymptotically efficient IV estimators, as well as an IV estimator introduced in Kelejian and Prucha (1998), have quite similar small sample properties. Our results also suggest the use of iterated versions of the IV estimators.

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Instrumental Variable Estimation of a Spatial Autorgressive Model with Autoregressive Disturbances: Large and Small Sample Results[1]

Harry H. Kelejian and Ingmar R. Prucha
University of Maryland

and

Yevgeny Yuzefovich

Fannie Mae, Virginia

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Abstract

The purpose of this paper is two-fold. First, on a theoretical level we introduce a series-type instrumental variable (IV) estimator of the parameters of a spatial first order autoregressive model with first order autoregressive disturbances. We demonstrate that our estimator is asymptotically efficient within the class of IV estimators, and has a lower computational count than an efficient IV estimator that was introduced by Lee (2003). Second, via Monte Carlo techniques we give small sample results relating to our suggested estimator, the maximum likelihood (ML) estimator, and other IV estimators suggested in the literature. Among other things we find that the ML estimator, both of the asymptotically efficient IV estimators, as well as an IV estimator introduced in Kelejian and Prucha (1998), have quite similar small sample properties. Our results also suggest the use of iterated versions of the IV estimators.

1 Introduction[1]

Spatial models are important tools in economics, regional science and geography in analyzing a wide range of empirical issues. For example, in recent years these models have been applied to contagion problems relating to bank performance as well as international finance issues, various categories of local public expenditures, vote seeking and tax setting behavior, population and employment growth, and the determinants of welfare expenditures, among others.[2]

By far the most widely used spatial models are variants of the one suggested by Cliff and Ord (1973, 1981) for modeling a single spatial relationship. One method of estimation of these models is maximum likelihood; another is the instrumental variable (IV) procedure suggested by Kelejian and Prucha (1998). The Kelejian and Prucha (1998) procedure relates to the parameters of a spatial first order autoregressive model with first order autoregressive disturbances, or, for short, a SARAR(1,1) model, and is based on a generalized moments (GM) estimator of a parameter in the disturbance process. The GM estimator was suggested by Kelejian and Prucha (1999) in an earlier paper.[3] The Kelejian and Prucha (1998, 1999) procedures do not require specific distributional assumptions. They are also easily extended to a systems framework.

However, the IV estimator in Kelejian and Prucha (1998) is based on an approximation to the ideal instruments and, therefore does not fully attain the asymptotic efficiency bound of IV estimators. In a recent paper Lee (2003) extends their approach in terms of the ideal instruments and gives an asymptotically efficient IV estimator.

The purpose of this paper is two-fold. First, on a theoretical level we introduce a series-type IV estimator for the SARAR(1,1) model. This estimator is a natural extension of the one proposed in Kelejian and Prucha

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[2]For example, see Yuzefovich (2003), Kapoor (2003), Pinkse, Slade, and Brett (2002), Allen and Gale (2000), Bell and Bockstael (2000), Bruecker (1998), Calvo and Reinhart (1997), Bollinger and Ihlanfeld (1997), Kelejian and Robinson (2000), Besley and Case (1995), Shroder (1995), and Case, Hines, and Rosen (1993).

[3]Due to publication lags, Kelejian and Prucha (1999) was published at a later date than Kelejian and Prucha (1998) even though it was written at an earlier point in time.

(1998) concerning the selection of instruments. We show that our series-type IV estimator is asymptotically normal and efficient. It is also computationally simple.

Second, via Monte Carlo techniques we give small sample results relating to our suggested estimator, the IV estimators of Lee (2003) and Kelejian and Prucha (1998), iterated versions of those estimators, as well as the maximum likelihood (ML) estimator for purposes of comparison. Among other things we find that the ML estimator, both of the asymptotically efficient IV estimators, as well as the IV estimator of Kelejian and Prucha (1998), have quite similar small sample properties. We also find that iterated versions of the IV estimators typically do not lead to increases in the root mean squared errors, but for certain parameter values, the root mean squared errors are lower. Therefore, the use of such iterated estimators is suggested.

2 Model

Consider the following (cross sectional) first order autoregressive spatial model with first order autoregressive disturbances ($n \in \mathbf{N}$):

$$\begin{aligned} y_n &= X_n\beta + \lambda W_n y_n + u_n, \qquad &|\lambda| < 1, \\ u_n &= \rho M_n u_n + \varepsilon_n, \qquad &|\rho| < 1, \end{aligned} \tag{1}$$

where y_n is the $n \times 1$ vector of observations on the dependent variable, X_n is the $n \times k$ matrix of observations on k exogenous variables, W_n and M_n are $n \times n$ spatial weighting matrices of known constants and zero diagonal elements, β is the $k \times 1$ vector of regression parameters, λ and ρ are scalar autoregressive parameters, u_n is the $n \times 1$ vector of regression disturbances, and ε_n is an $n \times 1$ vector of innovations. As remarked, and consistent with the terminology of Anselin (1988), we refer to this model as an SARAR(1,1,) model. The variables $W_n y_n$ and $M_n u_n$ are typically referred to as spatial lags of y_n and u_n, respectively. For reasons of generality we permit the elements of y_n, X_n, W_n, M_n, u_n and ε_n to depend on n, i.e., to form triangular arrays. We condition our analysis on the realized values of the exogenous variables and so, henceforth, the matrices X_n will be viewed as a matrices of constants.

With a minor exception, we make the same assumptions as in Kelejian and Prucha (1998), along with an additional (technical) assumption which was also assumed by Lee (2003). For the convenience of the reader these assumptions, labeled Assumptions 1-8, are given in the appendix. For a further discussion of these assumptions see Kelejian and Prucha (1998).

Given Assumption 2, the roots of aW_n and of aM_n are less than one in absolute value for all $|a| < 1$; see, e.g., Horn and Johnson (1985, p. 344). Therefore, for $|\lambda| < 1$ and $|\rho| < 1$ the matrices $I_n - \lambda W_n$ and $I_n - \rho M_n$ are nonsingular and furthermore

$$\begin{aligned} (I_n - \lambda W_n)^{-1} &= \sum_{k=0}^{\infty} \lambda^k W_n^k, \\ (I_n - \rho M_n)^{-1} &= \sum_{k=0}^{\infty} \rho^k M_n^k. \end{aligned} \tag{2}$$

It follows from (1) that

$$\begin{aligned} y_n &= (I_n - \lambda W_n)^{-1} X_n \beta + (I_n - \lambda W_n)^{-1} u_n, \\ u_n &= (I_n - \rho M_n)^{-1} \varepsilon_n. \end{aligned} \tag{3}$$

In light of Assumption 4 we have $E(u_n) = 0$ and therefore

$$E(y_n) = (I_n - \lambda W_n)^{-1} X_n \beta = \sum_{k=0}^{\infty} \lambda^k W_n^k X_n \beta. \tag{4}$$

The variance-covariance matrix of u_n is given by

$$E(u_n u_n') = \sigma_\varepsilon^2 (I_n - \rho M_n)^{-1} (I_n - \rho M_n')^{-1}. \tag{5}$$

We also note from (3) that

$$\begin{aligned} E(y_n u_n') &= \sigma_\varepsilon^2 (I_n - \lambda W_n)^{-1} (I_n - \rho M_n)^{-1} (I_n - \rho M_n')^{-1} \\ &\neq 0 \end{aligned}$$

so that, in general, the elements of the spatially lagged dependent vector, $W_n y_n$, are correlated with those of the disturbance vector. One implication of this is that the parameters of (1) can not generally be consistently estimated by ordinary least squares.[4]

[4] For a set of conditions under which the ordinary least squares estimator is consistent see Lee (2002). Among other things, those conditions specify that the spatial weights tend to zero as the sample size increases.

In the following discussion it is helpful to rewrite (1) more compactly as

$$\begin{aligned} y_n &= Z_n\delta + u_n, \\ u_n &= \rho M_n u_n + \varepsilon_n, \end{aligned} \tag{6}$$

where $Z_n = (X_n, W_n y_n)$ and $\delta = (\beta', \lambda)'$. Applying a Cochrane-Orcutt type transformation to this model yields

$$y_{n*} = Z_{n*}\delta + \varepsilon_n, \tag{7}$$

where $y_{n*} = y_n - \rho M_n y_n$ and $Z_{n*} = Z_n - \rho M_n Z_n$. In the following we may also express y_{n*} and Z_{n*} as $y_{n*}(\rho)$ and $Z_{n*}(\rho)$ to indicate the dependence of the transformed variables on ρ.

3 IV Estimators

3.1 IV Estimators in the Literature

In the following let $\widehat{\rho}_n$ be any consistent estimator for ρ, and let $\widehat{\delta}_n = (\widehat{\beta}'_n, \widehat{\lambda}_n)'$ be any $n^{1/2}$-consistent estimator for δ. As one example, $\widehat{\delta}_n$ could be the two stage least squares estimator of δ, and $\widehat{\rho}_n$ could be the corresponding GM estimator of ρ, which was suggested in the first and second steps of the estimation procedure introduced in Kelejian and Prucha (1998).

Recalling (4), the optimal instruments for estimating δ from (7) are

$$\begin{aligned} \overline{Z}_{n*} &= E(Z_{n*}) = (I_n - \rho M_n)E(Z_n) \\ &= (I_n - \rho M_n)(X_n, W_n E(y_n)) \\ &= (I_n - \rho M_n)\left[X_n, W_n(I_n - \lambda W_n)^{-1}X_n\beta\right]. \end{aligned} \tag{8}$$

In the following we will also express $\overline{Z}_{n*}$ as $\overline{Z}_{n*}(\rho, \delta)$. In light of (4) we have

$$\overline{Z}_{n*} = (I_n - \rho M_n)\left[X_n, \sum_{k=0}^{\infty} \lambda^k W_n^{k+1} X_n \beta\right], \tag{9}$$

which shows that the optimal instruments are linear combinations of the columns of $\{X_n, W_n X_n, W_n^2 X_n, \ldots, M_n X_n, M_n W_n X_n, M_n W_n^2 X_n, \ldots\}$. Motivated by this observation, Kelejian and Prucha (1998) introduced their feasible general spatial two stage least squares estimator ($FGS2SLS$) estimator in terms of an approximation to these optimal instruments. Specifically, their approximation to $\overline{Z}_{n*}$ is in terms of fitted values obtained from regressing $Z_{n*}(\widehat{\rho}_n)$ against a set of instruments H_n, which are taken to be a fixed subset of the linearly independent columns of

$$\{X_n, W_nX_n, W_n^2X_n, \ldots, M_nX_n, M_nW_nX_n, M_nW_n^2X_n, \ldots, M_nW_n^qX_n\}$$

where q is a pre-selected constant, and the subset is required to contain at least the linearly independent columns of $\{X_n, M_nX_n\}$. Typically, one would take $q \leq 2$, see e.g., Rey and Boarnet (1999), and Das, Kelejian, and Prucha (2003). Let $\widehat{Z}_{n*}$ denote those fitted values; then

$$\begin{aligned} \widehat{Z}_{n*} &= P_{H_n}Z_{n*}(\widehat{\rho}_n) \\ &= ([X_n - \widehat{\rho}_nM_nX_n, P_{H_n}(W_ny_n - \widehat{\rho}_nM_nW_ny_n)]) \end{aligned} \tag{10}$$

where $P_{H_n} = H_n(H_nH_N)^{-1}H_n'$ denotes the projection matrix corresponding to H_n. In the following we will also express $\widehat{Z}_{n*}$ as $\widehat{Z}_{n*}(\widehat{\rho}_n)$ to signify its dependence on $\widehat{\rho}_n$. Given this notation, the $FGS2SLS$ estimator of Kelejian and Prucha (1998), say $\widehat{\delta}_{F,n}$, is defined as

$$\begin{aligned} \widehat{\delta}_{F,n} &= \left[\widehat{Z}_{n*}(\widehat{\rho}_n)'Z_{n*}(\widehat{\rho}_n)\right]^{-1}\widehat{Z}_{n*}(\widehat{\rho}_n)'y_{n*}(\widehat{\rho}_n) \\ &= \left[\widehat{Z}_{n*}(\widehat{\rho}_n)'\widehat{Z}_{n*}(\widehat{\rho}_n)\right]^{-1}\widehat{Z}_{n*}(\widehat{\rho}_n)'y_{n*}(\widehat{\rho}_n) \end{aligned} \tag{11}$$

In our discussion below, we will also express $\widehat{\delta}_{F,n}$ as $\widehat{\delta}_{F,n}(\widehat{\rho}_n)$ in order to signify its dependence on $\widehat{\rho}_n$.

Kelejian and Prucha (1998) showed that

$$n^{1/2}(\widehat{\delta}_{F,n} - \delta) \xrightarrow{d} N(0, \Psi) \tag{12}$$

where

$$\Psi = \sigma_\varepsilon^2\left[\lim_{n\to\infty} n^{-1}\widehat{Z}_{n*}(\rho)'\widehat{Z}_{n*}(\rho)\right]^{-1}. \tag{13}$$

The $FGS2SLS$ estimator uses a fixed set of instruments and hence in general $\widehat{Z}_{n*}(\widehat{\rho}_n)$ will not approximate the optimal instruments arbitrarily close even as the sample size tends to infinity. For future reference we note that the computation of $W_n^qX_n$ in their procedure could be be determined recursively as $W_n(W_n^{q-1}X_n)$ and so the operational count of their procedure is $O(n^2)$.

In a recent paper Lee (2003) introduced the following IV estimator

$$\widehat{\delta}_{B,n} = \left[\overline{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' Z_{n*}(\widehat{\rho}_n)\right]_{n*}^{-1} \overline{Z}_n(\widehat{\rho}_n, \widehat{\delta}_n)' y_{n*}(\widehat{\rho}_n) \tag{14}$$

where the optimal instrument is approximated by

$$\overline{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n) = (I_n - \widehat{\rho}_n M_n)\left[X_n, W_n(I_n - \widehat{\lambda}_n W_n)^{-1} X_n \widehat{\beta}_n\right], \tag{15}$$

which is obtained by replacing the true parameters values in (8) by the estimators $\widehat{\rho}_n$ and $\widehat{\delta}_n$. In our discussion below we will also express $\widehat{\delta}_{B,n}$ as $\widehat{\delta}_{B,n}($ $\widehat{\rho}_n, \widehat{\delta}_n)$ in order to signify its dependence on $\widehat{\rho}_n$ and $\widehat{\delta}_n$.

Lee (2003) showed that

$$n^{1/2}(\widehat{\delta}_{B,n} - \delta) \xrightarrow{d} N(0, \Psi) \tag{16}$$

with

$$\Psi = \sigma_\varepsilon^2 \left[\lim_{n\to\infty} n^{-1} \overline{Z}_{n*}(\rho, \delta)' \overline{Z}_{n*}(\rho, \delta)\right]^{-1}. \tag{17}$$

Lee also demonstrated, as is expected from the literature on optimal instruments, that Ψ is a lower bound for the asymptotic variance-covariance matrix of any IV estimator for δ. Lee therefore calls his estimator the Best $FGS2SLS$ estimator.

3.2 A Series-Type Efficient IV Estimator

The computation of Lee's (2003) optimal instrument $\overline{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)$ involves the calculation of $W_n(I_n - \widehat{\lambda}_n W_n)^{-1} X_n \widehat{\beta}_n$. Lee (2003) designed a numerical algorithm which simplifies this computation but, never-the-less, the operational count of his procedure is still $O(n^3)$ and, furthermore, requires special programming of the algorithm. Therefore it seems of interest to have available an alternative optimal IV estimator that is computationally simpler and can be readily computed in standard packages such as TSP without the need of further programming.

Towards this end, let r_n be some sequence of natural numbers with $r_n \uparrow \infty$, and in light of (9) consider the following series estimator for $\overline{Z}_{n*}$:

$$\widetilde{Z}_{n*} = (I_n - \widehat{\rho}_n M_n)\left[X_n, \sum_{k=0}^{r_n} \widehat{\lambda}_n^k W_n^{k+1} X_n \widehat{\beta}_n.\right] \tag{18}$$

In the following we also express $\widetilde{Z}_{n*}$ as $\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)$. Using $\widetilde{Z}_{n*}$ we define the following IV estimator for δ:

$$\widehat{\delta}_{S,n} = \left[\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' Z_{n*}(\widehat{\rho}_n)\right]^{-1} \widetilde{Z}_n(\widehat{\rho}_n, \widehat{\delta}_n)' y_{n*}(\widehat{\rho}_n). \tag{19}$$

We will refer to the estimator in (19) as the Best Series $FGS2SLS$ estimator. In our discussion below we will also express $\widehat{\delta}_{S,n}$ as $\widehat{\delta}_{S,n}(\widehat{\rho}_n, \widehat{\delta}_n, r_n)$ in order to signify its dependence on $\widehat{\rho}_n$, $\widehat{\delta}_n$ and r_n. In the appendix we prove the following theorem.

Theorem 1 *Let r_n be some sequence of natural numbers with $0 \leq r_n \leq n$, $r_n \uparrow \infty$, and $r_n = o(n^{1/2})$. Then under the maintained assumptions*

$$n^{1/2}(\widehat{\delta}_{S,n} - \delta) \xrightarrow{d} N(0, \Psi). \tag{20}$$

The theorem demonstrates that $\widehat{\delta}_{S,n}$ is also an asymptotically efficient estimator within the class of IV estimators. In addition, recalling that $W_n^{k+1} X_n$ can be computed recursively as $W_n(W_n^k X_n)$, and noting that $r_n = o(n^{1/2})$, the operational count of the series estimator is $O(n^2 r_n)$, which is $o(n^{2+1/2})$, and therefore is less than that of Lee's estimator, but will exceed that of the $GS2SLS$ estimator.

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4 Small Sample Properties of IV Estimators

4.1 The Monte Carlo Model

In this section we study the small sample properties of the IV estimators discussed in Section 3, as well as iterated versions of those estimators using Monte Carlo techniques. For purposes of comparison we also consider the maximum likelihood estimator as well as the least squares estimator.

The Monte Carlo model is

$$\begin{aligned} y_n &= X_n \beta + \lambda W_n y_n + u_n, & |\lambda| < 1 \\ u_n &= \rho W_n u_n + \varepsilon_n, & |\rho| < 1 \end{aligned} \tag{21}$$

where y_n is $n \times 1$ vector of observations on the dependent variable, $X_n = [x_{1n}, x_{2n}]$ is an $n \times 2$ matrix of constants containing two $n \times 1$ vectors of observations on the exogenous explanatory variables x_{1n} and x_{2n}, W_n is the $n \times n$ spatial weighting matrix of known constants, $\beta = [\beta_1, \beta_2]'$ is the 2×1 vector of regression parameters, u_n is $n \times 1$ vector of disturbances, and ε_n is an $n \times 1$ stochastic vector of innovations whose elements are i.i.d. as $N(0, \sigma_\varepsilon^2)$. Essentially, (21) is identical to (1) except that $k = 2$ and $W_n = M_n$. Finally, let $Z_n = (X_n, W_n y_n)$ and $\delta' = (\lambda, \beta')$. Then, it will again be convenient to express the first equation in (21) as

$$y_n = Z_n \delta + u_n. \tag{22}$$

For future reference and definiteness we note that, given the normality of ε_n, the log-likelihood of the model in (21) is

$$\begin{aligned} \ln L_n(\beta_1, \beta_2, \lambda, \rho, \sigma_\varepsilon^2) &= const - \frac{1}{2} \ln\left(|\Omega_{y_n}|\right) \\ &\quad - \frac{1}{2}(y_n - [(I - \lambda W_n)^{-1} X_n \beta]' \Omega_{y_n}^{-1} [y_n - (I - \lambda W_n)^{-1} X_n \beta], \\ \Omega_{y_n} &= \sigma_\varepsilon^2 (I - \lambda W_n)^{-1} (I - \rho W_n)^{-1} (I - \rho W_n')^{-1} (I - \lambda W_n')^{-1}. \end{aligned} \tag{23}$$

We note that the expression for the log-likelihood can, of course, be further simplified for purposes of computation. We will denote the maximum likelihood estimators for δ and ρ as $\widehat{\delta}_{MLE,n}$ and $\widehat{\rho}_{MLE,n}$, respectively.

The Monte Carlo experiments in this study are designed in a way that makes their results comparable to the previous studies, and, in particular, to the results given in Das, Kelejian and Prucha (2003). We also extend that study in two aspects. First, we consider more experiments involving "extreme" values of the spatial autoregressive parameters λ and ρ, namely the values of 0.9 and -0.9, and, second, we considered estimators which were not considered in Das, Kelejian, and Prucha (2003).

We consider two values of the sample size, n namely 100 and 400. For each of these values we consider a weighting matrix W_n which Kelejian and Prucha (1999) refer to as "3 ahead and 3 behind". The reason for their designation is that the non-zero elements in the i-th row of W_n are in positions $i+1$, $i+2$, $i+3$, and $i-1$, $i-2$, $i-3$, for $i = 4, ..., n-3$. Thus, e.g., via (21), in these rows the j-th element of u_n is directly related to the three elements of u_n directly after it and to the three directly before. The matrix W_n is defined in a circular world so that, e.g., in the first row the non-zero elements are in positions $2, 3, 4$ (i.e., three ahead) and $n, n-1$, and $n-2$ (i.e., three behind). The positioning of the non-zero elements in rows 2, 3, n, $n-1$, and $n-2$

are determined analogously. Furthermore, W_n is row normalized and all of its non-zero elements are equal. Thus, each non-zero element of W_n is $1/6$.

We consider seven values of the autoregressive parameters λ and ρ, namely $-0.9, -0.8, -0.4, 0.0, 0.4, 0.8, 0.9$. We also consider three values of σ_ε^2, namely $0.25, 0.5, 1.0$. These three values of σ_ε^2, in the following for short σ^2, are related to the values of λ and n, and, thus, are woven into the experimental design in the same fashion as in Das, Kelejian, and Prucha (2003). Table 1 describes these values of σ^2.

Table 1. Design values of σ^2

$n = 100$		$n = 400$	
λ	σ^2	λ	σ^2
-0.9	0.5	-0.9	0.5
-0.8	0.25	-0.8	0.5
-0.4	1.0	-0.4	1.0
0.0	0.5	0.0	0.25
0.4	0.25	0.4	0.5
0.8	1.0	0.8	1.0
0.9	0.5	0.9	0.5

The combinations of λ, n and σ^2 are such that the average squared sample correlation coefficient between y_n and its mean vector, $E[y_n] = (I - \lambda W_n)^{-1} X_n \beta$, over all the experiments corresponding to a given value of λ and n is between 0.60 and 0.90.

The values of the exogenous regressors $X_n = [x_{1n}, x_{2n}]$ are based on data described Kelejian and Robinson (1992) which relate to income per capita and the percent of rental housing in 1980 in 760 counties in the US midwestern states. The 760 observations on the income and rental variables were normalized to have zero mean and unit variance. For experiments in which the sample size is 100 the first 100 observations on these variables were used; the first 400 observations were used in experiments in which the sample size is 400. The same vectors of exogenous variables were used in all the experiments of a given sample size n. Finally, the Monte Carlo experiments are based on the regression parameter values (β_1, β_2) =(1,1).

All in all, the seven values of ρ, the seven values of λ, and the two values of the sample size n lead to a total of 98 experiments since only one form of the weighting matrix is considered, and the values for σ^2 are, in all cases, related to those of λ and n. Each Monte Carlo experiment consists of 5000 trials which are based on 5000 different vectors of innovations. The elements of these vectors of innovations are determined from the normal distribution. The same set of 5000 vectors of innovations is used in all experiments that correspond to the same sample size n. Furthermore, the vectors of innovations for experiments in which the sample size is 100 is taken to be the vector of the first 100 elements of the corresponding vector of innovations for the sample size 400.

4.2 The Considered Estimators

Estimators of the Regression Coefficients

In our Monte Carlo study we will consider the Best Series $FGS2SLS$ estimator $\widehat{\delta}_{S,n}(\widehat{\rho}_n, \widehat{\delta}_n, r_n)$ for different sequences of r_n. More specifically, we will take r_n to be the nearest integer to n^α for different values of α, and signify this dependence by using the notation $\widehat{\delta}_{S,n}(\widehat{\rho}_n, \widehat{\delta}_n, \alpha)$. A list of all the estimators considered in our Monte Carlo Study is given below.

1. **The Least Squares (OLS) Estimator:**

$$\hat{\delta}_{OLS,n} = (Z_n' Z_n)^{-1} Z_n' y_n$$

2. **The Two Stage Least Squares ($2SLS$) Estimator:**

$$\begin{aligned} \hat{\delta}_{2SLS,n} &= (\hat{Z}_n' \hat{Z}_n)^{-1} \hat{Z}_n' y_n \\ \hat{Z}_n &= H_n (H_n' H_n)^{-1} H_n' Z_n \\ H_n &= (X_n, W_n X_n, W_n^2 X_n) \end{aligned}$$

3. **The Maximum Likelihood (ML) Estimator:** $\hat{\delta}_{MLE,n}$ based on (23).

4. **The $FGS2SLS$ Estimator:** $\hat{\delta}_{F,n}(\tilde{\rho}_n)$, where $\tilde{\rho}_n$ is the GM estimator of ρ given in Kelejian and Prucha (1999) based on $2SLS$ residuals, and $\hat{\delta}_{F,n}(.)$ is defined in (11).

5. **The True General Spatial Two Stage Least Squares**

$(GS2SLS)$ **Estimator:** $\hat{\delta}_{F,n}(\rho)$, where $\hat{\delta}_{F,n}(.)$ is described in **4** above.

6. The Iterated $FGS2SLS$ (IF) **Estimator:** $\hat{\delta}_{F,n}(\breve{\rho}_n)$, where $\breve{\rho}_n$ is the GM estimator of ρ based on $FGS2SLS$ residuals, and where the GM estimator and $\hat{\delta}_{F,n}(.)$ are described in **4** above.

7. The Best $FGS2SLS$ (LEE) **Estimator:** $\hat{\delta}_{B,n}(\tilde{\rho}_n, \hat{\delta}_{2SLS,n})$, where $\hat{\delta}_{B,n}(.,.)$ is defined in (14), and $\tilde{\rho}_n$ and $\hat{\delta}_{n2SLS}$ are defined in **2** and **4** above.

8. The Iterated Best $FGS2SLS$ $(ILEE)$ **Estimator :** $\hat{\delta}_{Bn}(\ddot{\rho}_n, \ddot{\delta}_n)$ where $\ddot{\rho}_n$ is the GM estimator of ρ based on Best $FGS2SLS$ residuals, $\ddot{\delta}_n = \hat{\delta}_{B,n}(\tilde{\rho}_n, \hat{\delta}_{n2SLS})$, and where the GM estimator, $\hat{\delta}_{n2SLS}$, and $\hat{\delta}_{B,n}(.,.)$ are described in **2**, **4** and **7** above.

9. The Best Series $FGS2SLS$ $(SERj)$ **Estimators:**

$$\hat{\delta}_{S,n}(\tilde{\rho}_n, \hat{\delta}_{2SLS,n}, \alpha_j), \qquad j = 1, 2, 3,$$

where $\alpha_1 = .25$, $\alpha_2 = .35$, $\alpha_3 = .45$, $\tilde{\rho}_n$ and $\hat{\delta}_{2SLS,n}$ are defined in **2** and **4** above, $\hat{\delta}_{S,n}(.,.,.)$ is defined in (19), and where r_n in (19) is taken to be the nearest integer to n^{α_j}.

10. The Iterated Best Series $FGS2SLS$ $(ISERj)$ **Estimators:**

$$\hat{\delta}_{S,n}(\breve{\rho}_n^{(j)}, \breve{\delta}_n^{(j)}, \alpha_j), \qquad j = 1, 2, 3,$$

where $\alpha_1 = .25$, $\alpha_2 = .35$, $\alpha_3 = .45$, and where $\breve{\rho}_n^{(j)}$ is the GM estimator of ρ defined in **4** above which is based on $\breve{\delta}_n^{(j)}$ residuals where $\breve{\delta}_n^{(j)} = \hat{\delta}_{S,n}(\tilde{\rho}_n, \hat{\delta}_{2SLS,n}, \alpha_j)$, and where $\hat{\delta}_{S,n}(.,.,.)$ is defined in (19).

11. Estimators of ρ**:**
We also report results for seven estimators of ρ. These are the ML estimator which is based on (23), and the GM estimators of ρ based on the residuals obtained from the $2SLS$, $FGS2SLS$, LEE, $SER1$, $SER2$, and $SER3$.

4.3 The Efficiency Measure

Our efficiency measure of the estimators for each experiments is based on the empirical distribution over the 5000 Monte Carlo trials. For each trial the coefficient are estimated, and the empirical distribution is defined with respect to these 5000 trials. Following Kelejian and Prucha (1998), our efficiency measure is a variation of the root mean squared error, specifically,

$$RMSE^{*} = \left[bias^{2} + [IQ/1.35]^{2}\right]^{1/2} \tag{24}$$

where *bias* is an absolute difference between the median of the empirical distribution and the true parameter value, and IQ is an interquantile range. That is $IQ = c_1 - c_2$ where c_1 is the 0.75 quantile and c_2 is the 0.25 quantile. Note that if the distribution is normal, the median is equal to the mean and $IQ/1.35$ is approximately equal to the standard deviation. An important feature of the measure in (24) is that it is based on quantiles which always exist. The standard measure of the root mean square error is based on the first and second moments which, as pointed out by Kelejian and Prucha (1999) among others, may not always exist, and so that measure may not be well defined. However, for simplicity of presentation we will refer to our measure of efficiency as the $RMSE$.

4.4 Monte Carlo Results

Tables 2-5 report the $RMSE$s of the considered estimators of the parameters λ, β_1, β_2, and ρ corresponding to 49 sets of experimental parameter values. All of these tables relate to a sample size of $n = 400$. Corresponding tables for the case in which the sample size is $n = 100$ are not given due to space limitations, although certain results from these tables are discussed. These tables can be obtained by writing to the authors.

As a starting point observe that if the experiments involving the values 0.9 and -0.9 of λ and ρ are omitted, the sets of parameter values of the remaining experiments are identical to those considered by Das, Kelejian and Prucha (2003). Therefore, as a check of programs, we note that the results in our tables below which correspond to this subset of experiments are virtually the same as corresponding results reported in Das, Kelejian and Prucha (2003). The minor differences, which are within the range of the statistical error, stem from differences in the vectors of innovations used in the two Monte

Carlo studies.[5]

Somewhat similar to results given in Das, Kelejian and Prucha (2003), it is clear from the tables relating to the parameters λ, β_1, and β_2 that typically $RMSE$s of the OLS estimator are the largest. This relates to theoretical notions concerning the inconsistency of the OLS estimator. Note that the $RMSE$s of the $2SLS$ estimator are typically lower than those of the OLS estimator, but typically larger than that of the other estimators under consideration. The rationalization of this result is that, although the $2SLS$ estimator is consistent, it does not account for the spatial structure of the error term. The $RMSE$s of the ML estimator for λ are typically somewhat lower than those of the other estimators, but for β_1 and β_2 there is little difference between the $RMSE$s of the ML and those of the IV estimators that account for spatial correlation.

Consider now the results relating to the $FGS2SLS$, Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators. Theory indicates that the latter two estimators are asymptotically more efficient than the $FGS2SLS$ estimator. However, our results suggest that in finite samples efficiency differences between these estimators may be limited. Specifically, in our tables $RMSE$ differences between these estimators average to just 2% for the parameter λ. The $RMSE$s of these estimators of β_1 and β_2 are, on average, virtually the same. These finding are important because of the computational and programming simplicity of $FGS2SLS$ estimator.

In comparing Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators it is clear that their $RMSE$s are virtually the same. On average, the difference between the $RMSE$s of the Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators do not exceed 1% for the parameter λ, and 0.5% for the parameters β_1 and β_2. The performance of these estimators is similar not only in terms of averages but also over the whole parameter space. More specifically, the difference between the $RMSE$s of the Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimator (based on $\alpha = 0.45$) typically does not exceed 5% in any of the experimental sets of parameter values when the sample size is 100 and 3% when the sample size is 400. The result that the differences between the $RMSE$s of these two estimators decrease as the sample size increases is consistent with their asymptotic equivalence.

Another result of interest concerning the Best Series $FGS2SLS$ estimator

[5] For example, we ran our programs using the same vectors of innovations that were used in Das, Kelejian and Prucha (2003). The resulting RMSEs turned out to be identical.

relates to its efficiency as a function of α. Specifically, for the sample sizes considered, the efficiency of the Best Series $FGS2SLS$ estimators does not seem to be sensitive to the considered values of α. For example, the Best Series $FGS2SLS$ estimator based on $\alpha = .45$ does not dominate those based on $\alpha = .25$ and $\alpha = .35$. Similarly, the Best Series $FGS2SLS$ estimator based on $\alpha = .35$ does not dominate that based on $\alpha = .25$. Therefore, one may conjecture that in moderate to reasonably large samples a series estimator based on $\alpha = 0.25$ is adequate.

We now compare the $RMSE$s of the ML estimator and those of the $FGS2SLS$, Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators in more detail. Consider first the set of the experiments that do not contain $\rho = 0.9$. Over these experiments the gain in efficiency of the ML estimator relative to these other estimators averages to just $6 - 7\%$ for the parameter λ. For the parameters β_1 and β_2, the ML and these other estimators are roughly equivalent in terms of average $RMSE$s over this set of experiments. Therefore, the suggestion is that if the value of ρ is not close to 1.0, the loss of efficiency of these spatial IV estimators relative to the ML estimator is generally small or nonexistent.[6]

If all the experiments are considered the difference between $RMSE$ averages of the ML and the $FGS2SLS$, Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ rises up to $16-18\%$ for λ and between $2-4\%$ for β_1 and β_2. The reason for such a disparity is that for certain combinations of the true values of the parameters λ and ρ, namely those involving a negative value of λ and a large and positive value of ρ, the $RMSE$s of spatial IV estimators are considerably larger than those of the ML estimator. One reason for this is that such combinations of parameter values of ρ and λ are associated with large $RMSE$s of the $2SLS$ estimator, whose residuals are used in the $FGS2SLS$, Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ procedures for estimation of ρ. Therefore, it is reasonable to believe that iterating on the spatial IV estimators would improve their performance. In fact, for the parameter λ, the average difference between $RMSE$s of the ML and the iterated $FGS2SLS$ estimators decreases to 14%, and between the ML and the iterated Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators to $11 - 12\%$. For the parameters β_1 and β_2 these differences decrease to just

[6] For purpose of comparison to Das, Kelejian and Prucha (2003) we are also reporting averages over the experiments not involving values 0.9 and -0.9 of λ and ρ. These averages are almost identical to the averages over the experiments not involving $\rho = 0.9$.

$2-3\%$. These results suggest that the advantage of the ML over the spatial IV estimators is relatively small even though the experiments are conducted under the most favorable conditions for the ML procedure, namely normally distributed disturbances.

As a general observation we note that iterating on the spatial IV procedures typically does not reduce the efficiency of the estimators, but it substantially improves that efficiency in cases involving a negative value of λ and a large positive value of ρ. The suggestion therefore is that, in practice, it may be advisable to use the iterated version of the $FGS2SLS$, Lee's Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators.

The average difference between the $RMSE$s of the (true) $GS2SLS$ and $FGS2SLS$ estimators of λ is 13% for $n = 100$ and 8% for $n = 400$. This difference decreases to 5% and 2%, respectively, if the experiments which involve $\rho = 0.9$ are not considered. Furthermore, the $RMSE$s of the iterated $FGS2SLS$ estimators of λ are, on average, higher than those of the $G2SLS$ by 8% for the sample size 100 and by 3% for samples of size 400. For the parameters β_1 and β_2 the $RMSE$s of $FGS2SLS$ and iterated $FGS2SLS$ estimators are, on average, higher than those of the $GS2SLS$ estimator by at most 2.7% if the sample size is 100, and by 2.4% if the sample size is 400. Among other things, these results suggest that the loss in finite sample efficiency resulting from the use of the GM estimator of ρ, as compared to its true value, is small in moderate to large samples.

Tables 8-9 relate to the estimators of ρ. Generally the ML estimator is somewhat better than the others, while the performance of the GM estimators based on the residuals of the $FGS2SLS$, the Best $FGS2SLS$ and the Best Series $FGS2SLS$ estimators are very similar throughout the parameter space. The efficiency of the GM estimator of ρ based on the $2SLS$ residuals is similar to that of the other GM estimators if experiments involving $\rho = 0.9$ are not considered. Over these experiments the GM estimators are, on average, roughly 8% worse than the ML estimator. If all the experiments are considered this average difference increases to roughly 16% for the $2SLS$ estimator, and to 10% for the others.

As a final remark we note that the values of the $RMSE$s of almost all the considered estimators corresponding to λ, β_1 and β_2 generally decrease as the sample size increases. An exception to this is the OLS estimator which is not consistent, and whose $RMSE$s, as a result, often increase with the sample size. These findings are in accordance with the asymptotic properties of these estimators.

5 Conclusion

The focus of this paper was two-fold. First, we introduced a series-type IV estimator of the regression parameters of a SARAR(1,1) model. We showed that this estimator is asymptotically normal and efficient, and referred to it as the Best Series $FGS2SLS$ estimator. These large sample properties do not require a distributional assumption, and our estimator does not require special programming for its implementation. Second, we undertook a Monte Carlo study in order to gain insights concerning the small sample properties of our suggested estimator, the Best $FGS2SLS$ estimator of Lee (2003), the $FGS2SLS$ estimator of Kelejian and Prucha (1998), iterated versions of those estimators, as well as the maximum likelihood (ML) estimator.

Our findings indicate that our Best Series $FGS2SLS$ estimator, Lee's (2003) Best $FGS2SLS$ estimator and Kelejian and Prucha's (1999) $FGS2SLS$ estimator have quite similar small sample properties. We also found that iterations of those estimators rarely lead to a loss of efficiency but, for certain extreme values of the disturbance autoregressive parameter, lead to an increase in efficiency. Therefore the suggestion is to iterate on these estimators. We also found that the ML estimator was generally somewhat more efficient than the IV estimators considered but, except for certain extreme values of the parameters, its benefits were limited. Also, all simulations were based on normally distributed innovations, which favors the ML estimator.

Finally, we explored the finite sample efficiency of the ML and GM estimators of the autoregressive parameter in the disturbance process, ρ. Again, the ML estimator has usually a somewhat smaller $RMSE$ than the GM estimators, but its benefit seems modest especially for, say, $|\rho| \leq .8$. The $RMSE$'s of the GM estimators based on the residuals of the Best Series $FGS2SLS$ estimator, the Best $FGS2SLS$ estimator and the $FGS2SLS$ estimator are generally similar and, as one might expect, are less than those based on $2SLS$ residuals.

Finally, one avenue of possible further research relating to small sample issues that would be of interest relates to the development of (suggested) optimality rules for the number of instruments underlying the $FGS2SLS$ procedure, and to the number of terms in the expansion for the Best Series $FGS2SLS$ estimator. On a theoretical level, formal results relating to the estimation of nonlinear spatial models containing spatial lags in both dependent variable and in the disturbance term should certainly be of interest.

A Appendix

A.1 Assumptions of the Model

Assumption 1 *All diagonal elements of the spatial weighting matrices* W_n *and* M_n *are zero.*

Assumption 2 *The row sums of* W_n *and* M_n *are bounded uniformly in absolute value by one; in addition the column sums of* W_n *and* M_n*, as well as the row and column sums of* $(I - \lambda W_n)^{-1}$ *and* $(I - \rho M_n)^{-1}$ *are bounded uniformly in absolute value by some finite constant,* c*.*

Assumption 3 *The regressor matrices* X_n *have full column rank (for* n *large enough). Furthermore, the elements of the matrices* X_n *are uniformly bounded in absolute value.*

Assumption 4 *The innovations* $\{\varepsilon_{i,n} : 1 \leq i \leq n, n \geq 1\}$ *are distributed identically. Further, the innovations* $\{\varepsilon_{i,n} : 1 \leq i \leq n\}$ *are for each* n *distributed (jointly) independently with* $E(\varepsilon_{i,n}) = 0$*,* $E(\varepsilon_{i,n}^2) = \sigma_\varepsilon^2$*, where* $0 < \sigma_\varepsilon^2 < b$ *with* $b < \infty$*. Additionally the innovations are assumed to possess finite fourth moments.*

Assumption 5 *The instrument matrices* H_n *have full column rank* $p \geq k+1$ *(for all* n *large enough). They are composed of a subset of the linearly independent columns of* $(X_n, W_nX_n, W_n^2X_n, \ldots, M_nX_n, M_nW_nX_n, M_nW_n^2X_n, \ldots)$*, where the subset contains at least the linearly independent columns of* (X_n, M_nX_n)*.*

Assumption 6 *The instrument matrix* H_n *is such that*

(a)

$$Q_{HH} = \lim_{n\to\infty} n^{-1}H_n'H_n$$

where Q_{HH} *is finite, and nonsingular.*

(b)

$$Q_{HZ} = \operatorname*{plim}_{n\to\infty} n^{-1}H_n'Z_n$$

and

$$Q_{HMZ} = \operatorname*{plim}_{n\to\infty} n^{-1}H_n'M_nZ_n$$

where Q_{HZ} and Q_{HMZ} are finite, and have full column rank. Furthermore

$$Q_{HZ} - \rho Q_{HMZ} = \operatorname*{plim}_{n\to\infty} n^{-1} H_n'(I - \rho M_n) Z_n$$

has full column rank for all $|\rho| < 1$.

(c)

$$\Phi = \lim_{n\to\infty} n^{-1} H_n'(I - \rho M_n)^{-1}(I - \rho M_n')^{-1} H_n$$

is finite, and nonsingular for all $|\rho| < 1$.

Assumption 7 *The smallest eigenvalue of $\Gamma_n'\Gamma_n$ is bounded away from zero, i.e., $\lambda_{\min}(\Gamma_n'\Gamma_n) \geq \lambda_* > 0$, where*

$$\Gamma_n = \frac{1}{n}\begin{pmatrix} 2E(u_n'\overline{u}_n) & -E(\overline{u}_n'\overline{u}_n) & 1 \\ 2E(\overline{\overline{u}}_n'\overline{u}_n) & -E(\overline{\overline{u}}_n'\overline{\overline{u}}_n) & tr(M_n'M_n) \\ E(u_n'\overline{\overline{u}}_n + \overline{u}_n'\overline{u}_n) & -E(\overline{u}_n'\overline{\overline{u}}_n) & 0 \end{pmatrix} \quad \text{(A.1)}$$

and $\overline{u}_n = M_n u_n$ and $\overline{\overline{u}}_n = M_n\overline{u}_n = M_n^2 u_n$.

Assumption 8 *Let $\overline{Z}_{n*}(\rho,\delta) = (I_n - \rho M_n)\,[X_n, W_n(I_n - \lambda W_n)^{-1} X_n\beta]$, then*

$$\Xi = p\lim_{n\to\infty} n^{-1}\overline{Z}_{n*}(\rho,\delta)'\overline{Z}_{n*}(\rho,\delta)$$

where Ξ is finite and nonsingular.

A.2 Proof of Theorem 1

The proof of this theorem will be in terms of a sequence of lemmas. For the subsequent discussion observe that

$$\begin{aligned} \overline{Z}_{n*}(\rho,\delta) &= (I_n - \rho M_n)\,[X_n, E\overline{y}_n]\,, \\ \widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n) &= I_n - \widehat{\rho}_n M_n)\left[X_n, \widetilde{\overline{y}}_n\right] \end{aligned} \quad \text{(A.2)}$$

where

$$\begin{aligned} E\overline{y}_n &= W_n E y_n = W_n(I_n - \lambda W_n)^{-1} X_n\beta = \sum_{k=0}^{\infty} \lambda^k W_n^{k+1} X_n\beta, \quad \text{(A.3)} \\ \widetilde{\overline{y}}_n &= \sum_{k=0}^{r_n} \widehat{\lambda}_n^k W_n^{k+1} X_n \widehat{\beta}_n. \end{aligned}$$

The proof will also utilize repeatedly the observations summarized in the subsequent remark.

Remark: Let A_n and B_n be two $n \times n$ matrices whose row and column sums are uniformly bounded in absolute value by finite constants c_A and c_B, respectively. Furthermore, let a_n and b_n be $n \times 1$ vectors whose elements are uniformly bounded in absolute value by finite constants c_a and c_b, respectively. It is then readily seen that the row and column sums of $A_n B_n$ are uniformly bounded in absolute value by the finite constant $c_A c_B$, see, e.g., Kelejian and Prucha (1999). Similarly, the elements of $A_n b_n$ are seen to be uniformly bounded in absolute value by $c_A c_b$. Since via Assumption 2 the row sums of W_n are uniformly bounded in absolute value by one, it follows that the elements of $W_n b_n$ are uniformly bounded in absolute value by c_b. By recursive argumentation it follows further that also the elements of $W_n^k b_n = W_n(W_n^{k-1} b_n)$ are uniformly bounded in absolute value by c_b for $k = 2, 3, \ldots$. Since by Assumption 3 the elements of X_n are uniformly bounded in absolute value it follows further that the elements of $A_n X_n$, $n^{-1} X_n' A_n X_n$, $n^{-1} a_n' W_n^{k+1} X_n \beta$, $A_n W_n^{k+1} X_n \beta$, $n^{-1} a_n' W_n^{k+1} X_n$ and $A_n W_n^{k+1} X_n$ are uniformly bounded in absolute value by some finite constant. Finally, let C_n be some $n \times p$ matrix whose elements are uniformly bounded in absolute value; then $n^{-1} C_n' \varepsilon_n = o_p(1)$, given Assumption 4 holds for ε_n.

Lemma 1 *Let* $p\lim_{n\to\infty} \widehat{\lambda}_n = \lambda$ *with* $|\lambda| < 1$*, let* $\widetilde{\lambda}_n = \widehat{\lambda}_n \mathbf{1}\left(\left|\widehat{\lambda}_n\right| < 1\right)$*, and let* r_n *be some sequence of natural numbers with* $r_n \uparrow \infty$ *as* $n \to \infty$*. Then* $p\lim_{n\to\infty} \widetilde{\lambda}_n = \lambda$*, and for any* $p \geq 0$ *we have* $p\lim_{n\to\infty} r_n^p \left|\widehat{\lambda}_n\right|^{r_n} = p\lim_{n\to\infty} r_n^p \left|\widetilde{\lambda}_n\right|^{r_n} = 0$*.*

Proof: For arbitrary $\varepsilon > 0$

$$\begin{aligned} P\left(\left|\widetilde{\lambda}_n - \lambda\right| > \varepsilon\right) &\leq P\left(\left|\widetilde{\lambda}_n - \widehat{\lambda}_n\right| + \left|\widehat{\lambda}_n - \lambda\right| > \varepsilon\right) \\ &\leq P\left(\left|\widetilde{\lambda}_n - \widehat{\lambda}_n\right| > \varepsilon/2\right) + P\left(\left|\widehat{\lambda}_n - \lambda\right| > \varepsilon/2\right) \\ &\leq P\left(\left|\widehat{\lambda}_n\right| \geq 1\right) + P\left(\left|\widehat{\lambda}_n - \lambda\right| > \varepsilon/2\right). \end{aligned}$$

observing that $\widetilde{\lambda}_n - \widehat{\lambda}_n = 0$ for $\left|\widehat{\lambda}_n\right| < 1$ and thus $\left\{\left|\widetilde{\lambda}_n - \widehat{\lambda}_n\right| > \varepsilon/2\right\} \subseteq \left\{\left|\widehat{\lambda}_n\right| \geq 1\right\}$. Since $p\lim_{n\to\infty} \widehat{\lambda}_n = \lambda$ with $|\lambda| < 1$ it follows that both probabilities on the r.h.s. of the last inequality tend to zero, which establishes that $p\lim_{n\to\infty}\widetilde{\lambda}_n = \lambda$.

Next choose some $\delta = (1-|\lambda|)/2 > 0$, then for any $\varepsilon > 0$

$$\begin{aligned} P\left(r_n^p\left|\widehat{\lambda}_n\right|^{r_n} > \varepsilon\right) &= P\left(r_n^p\left|\widehat{\lambda}_n\right|^{r_n} > \varepsilon, \left|\widehat{\lambda}_n - \lambda\right| \le \delta\right) \\ &\quad + P\left(r_n^p\left|\widehat{\lambda}_n\right|^{r_n} > \varepsilon, \left|\widehat{\lambda}_n - \lambda\right| > \delta\right) \\ &\le P\left(r_n^p\left(|\lambda|+\delta\right)^{r_n} > \varepsilon\right) + P\left(\left|\widehat{\lambda}_n - \lambda\right| > \delta\right). \end{aligned}$$

Since $|\lambda|+\delta < 1$, and since $\lim_{x\to\infty} x^p a^x = 0$ for all $0 \le a < 1$, it follows that $\lim_{n\to\infty} r_n^p\left(|\lambda|+\delta\right)^{r_n} = 0$, and hence $P\left(r_n^p\left(|\lambda|+\delta\right)^{r_n} > \varepsilon\right) \to 0$ as $n \to \infty$. Since $\widehat{\lambda}_n$ is a consistent estimator for λ we also have $P\left(\left|\widehat{\lambda}_n - \lambda\right| > \delta\right) \to 0$ as $n \to \infty$. Hence both terms on the r.h.s. of the last inequality limit to zero as $n \to \infty$, which shows that $p\lim_{n\to\infty} r_n^p\left|\widehat{\lambda}_n\right|^{r_n} = 0$. To show that $p\lim_{n\to\infty} r_n^p\left|\widehat{\lambda}_n\right|^{r_n} = 0$ we have only used that $\widehat{\lambda}_n$ is a consistent estimator for λ. Since $\widetilde{\lambda}_n$ is a consistent estimator for λ it follows that also the last claim holds. ■

Lemma 2 *Suppose* $n^{1/2}\left(\widehat{\lambda}_n - \lambda\right) = O_p(1)$ *with* $|\lambda| < 1$ *and define* $\widetilde{\lambda}_n = \widehat{\lambda}_n \mathbf{1}\left(\left|\widehat{\lambda}_n\right| < 1\right)$. *Then* $n^{1/2}\left(\widetilde{\lambda}_n - \lambda\right) = O_p(1)$, $n^{1/2}\left(\left|\widehat{\lambda}_n\right| - |\lambda|\right) = O_p(1)$ *and* $n^{1/2}\left(\left|\widetilde{\lambda}_n\right| - |\lambda|\right) = O_p(1)$.

Proof: Observe that for every $\varepsilon > 0$

$$P\left(n^{1/2}\left|\widetilde{\lambda}_n - \widehat{\lambda}_n\right| \ge \varepsilon\right) \le P\left(\left|\widehat{\lambda}_n\right| \ge 1\right).$$

Since $\widehat{\lambda}_n$ is a consistent estimator for λ with $|\lambda| < 1$, the probability on the r.h.s. tends to zero as $n \to \infty$ and thus $n^{1/2}\left(\widetilde{\lambda}_n - \widehat{\lambda}_n\right) = o_p(1)$. Hence $n^{1/2}\left(\widetilde{\lambda}_n - \lambda\right) = n^{1/2}\left(\widetilde{\lambda}_n - \widehat{\lambda}_n\right) + n^{1/2}\left(\widehat{\lambda}_n - \lambda\right) = o_p(1) + O_p(1) = O_p(1)$. Since $\left|\left|\widehat{\lambda}_n\right| - |\lambda|\right| \le \left|\widehat{\lambda}_n - \lambda\right|$ and $\left|\left|\widetilde{\lambda}_n\right| - |\lambda|\right| \le \left|\widetilde{\lambda}_n - \lambda\right|$ the other claims follow trivially. ■

Lemma 3 *Let r_n be some sequence of natural numbers with $0 \leq r_n \leq n$, $r_n \uparrow \infty$, and $r_n = o(n^{1/2})$. Suppose $n^{1/2}\left(\widehat{\lambda}_n - \lambda\right) = O_p(1)$ with $|\lambda| < 1$, then $p\lim_{n\to\infty} r_n \sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)^2 = 0$.*

Proof: Define $\widetilde{\lambda}_n = \widehat{\lambda}_n \mathbf{1}\left(\left|\widehat{\lambda}_n\right| < 1\right)$, let $\phi_n = r_n \sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)^2$ and $\psi_n = r_n \sum_{k=0}^{r_n}\left(\widetilde{\lambda}_n^k - \lambda^k\right)^2$. Then for every $\varepsilon > 0$

$$\begin{aligned} P\left(|\phi_n| > \varepsilon\right) &= P\left(|\phi_n| > \varepsilon, \left|\widehat{\lambda}_n\right| < 1\right) + P\left(|\phi_n| > \varepsilon, \left|\widehat{\lambda}_n\right| \geq 1\right) \\ &\leq P\left(|\psi_n| > \varepsilon\right) + P\left(\left|\widehat{\lambda}_n\right| \geq 1\right) \end{aligned}$$

observing that for realizations $\omega \in \Omega$ with $\left|\widehat{\lambda}_n(\omega)\right| < 1$ we have $\phi_n(\omega) = \psi_n(\omega)$. Since $p\lim_{n\to\infty} \widehat{\lambda}_n = \lambda$ with $|\lambda| < 1$ it follows immediately that the second probability on the r.h.s. of the last inequality tends to zero. To complete the proof of the claim we next show that the first probability of that r.h.s. tends to zero, i.e., that $\psi_n = o_p(1)$. Observe that

$$\begin{aligned} \psi_n &= \psi_{1n} + \psi_{2n}, \\ \psi_{1n} &= r_n\left[\frac{1}{1-\widetilde{\lambda}_n^2} + \frac{1}{1-\lambda^2} - \frac{2}{1-\widetilde{\lambda}_n\lambda}\right] \\ &= \frac{r_n}{n^{1/2}}\left[\frac{n^{1/2}\left(\widetilde{\lambda}_n - \lambda\right)\left[\widetilde{\lambda}_n\left(1-\lambda^2\right) - \lambda\left(1-\widetilde{\lambda}_n^2\right)\right]}{\left(1-\widetilde{\lambda}_n^2\right)\left(1-\lambda^2\right)\left(1-\widetilde{\lambda}_n\lambda\right)}\right] \\ \psi_{2n} &= -\frac{r_n\widetilde{\lambda}_n^{2(r_n+1)}}{1-\widetilde{\lambda}_n^2} - \frac{r_n\lambda^{2(r_n+1)}}{1-\lambda^2} + 2\frac{r_n\left(\widetilde{\lambda}_n\lambda\right)^{r_n+1}}{1-\widetilde{\lambda}_n\lambda} \end{aligned}$$

Since $\left|\widetilde{\lambda}_n\right| < 1$ all terms are well defined. By Lemma 2 $n^{1/2}\left(\widetilde{\lambda}_n - \lambda\right) = O_p(1)$ - and thus, of course, $p\lim_{n\to\infty} \widetilde{\lambda}_n = \lambda$. By Lemma 1 we have $p\lim_{n\to\infty} r_n\widetilde{\lambda}_n^{r_n} = 0$ and $\lim_{n\to\infty} r_n\lambda^{r_n} = 0$. Observing that $r_n/n^{1/2} = o(1)$ it is then readily seen that $\psi_{1n} = o_p(1)$ and $\psi_{2n} = o_p(1)$ and thus $\psi_n = o_p(1)$. ∎

Lemma 4 *Suppose* $n^{1/2}\left(\widehat{\lambda}_n - \lambda\right) = O_p(1)$ *with* $|\lambda| < 1$*, then*

$$p\lim_{n\to\infty} n^{\kappa} \sum_{k=0}^{n} \left(\widehat{\lambda}_n^k - \lambda^k\right) = 0$$

and

$$p\lim_{n\to\infty} n^{\kappa} \sum_{k=0}^{n} \left(\left|\widehat{\lambda}_n\right|^k - |\lambda|^k\right) = 0$$

for $0 \le \kappa < 1/2$.

Proof: Define $\widetilde{\lambda}_n = \widehat{\lambda}_n \mathbf{1}\left(\left|\widehat{\lambda}_n\right| < 1\right)$ and $\phi_n = n^{\kappa} \sum_{k=0}^{n} \left(\widehat{\lambda}_n^k - \lambda^k\right)$. Consider the decomposition

$$\begin{aligned}
\phi_n &= \phi_{1n} + \phi_{2n}, \\
\phi_{1n} &= n^{\kappa} \sum_{k=0}^{n} \left(\widetilde{\lambda}_n^k - \lambda^k\right), \\
\phi_{2n} &= n^{\kappa} \sum_{k=0}^{n} \left(\widehat{\lambda}_n^k - \widetilde{\lambda}_n^k\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
\phi_{1n} &= n^{\kappa} \left[\frac{1 - \widetilde{\lambda}_n^{n+1}}{1 - \widetilde{\lambda}_n} - \frac{1 - \lambda^{n+1}}{1 - \lambda}\right] \\
&= n^{\kappa - 1/2} \frac{n^{1/2}(\widetilde{\lambda}_n - \lambda)}{\left(1 - \widetilde{\lambda}_n\right)(1 - \lambda)} - \frac{n^{\kappa}\widetilde{\lambda}_n^{n+1}(1 - \lambda) - n^{\kappa}\lambda^{n+1}\left(1 - \widetilde{\lambda}_n\right)}{\left(1 - \widetilde{\lambda}_n\right)(1 - \lambda)}.
\end{aligned}$$

Since $\left|\widetilde{\lambda}_n\right| < 1$ all expressions on the r.h.s. are well defined. By Lemma 2 we have $n^{1/2}\left(\widetilde{\lambda}_n - \lambda\right) = O_p(1)$ and thus $p\lim_{n\to\infty} \widetilde{\lambda}_n = \lambda$. Hence $1/\left[\left(1 - \widetilde{\lambda}_n\right)(1 - \lambda)\right] = O_p(1)$. Observing that $n^{\kappa - 1/2} = o(1)$ and that in light of Lemma 1 $n^{\kappa}\widetilde{\lambda}_n^{n+1} = o_p(1)$ and $n^{\kappa}\lambda^{n+1} = o(1)$ it follows that $\phi_{1n} = o_p(1)$.

Next observe that for every $\varepsilon > 0$

$$P\left(|\phi_{2n}| > \varepsilon\right) \le P\left(\left|\widehat{\lambda}_n\right| \ge 1\right).$$

Since $p\lim_{n\to\infty} \widehat{\lambda}_n = \lambda$ with $|\lambda| < 1$ it follows that the probability on the r.h.s. tends to zero, which establishes that also $\phi_{2n} = o_p(1)$, and thus $\phi_n = o_p(1)$ as claimed.

By Lemma 2 we have $n^{1/2}\left(\left|\widehat{\lambda}_n\right| - |\lambda|\right) = O_p(1)$, and thus the second claim follows as a special case of the first claim. ■

Lemma 5 *Given the model in (1), suppose Assumptions 1-4 hold and $n^{1/2}(\widehat{\lambda}_n - \lambda) = O_p(1)$ with $|\lambda| < 1$ and $\widehat{\beta}_n - \beta = o_p(1)$. Let r_n be some sequence of natural numbers with $0 \le r_n \le n$, $r_n \uparrow \infty$ and $r_n = o(n^{1/2})$, and let $a_n = (a_{1,n}, \ldots, a_{n,n})'$ be some sequence of $n \times 1$ constant vectors whose elements are uniformly bounded in absolute value. Then*

$$n^{-1}a_n'\left(\widetilde{\overline{y}}_n - E\overline{y}_n\right) = o_p(1).$$

Proof: Recall the expressions for $E\overline{y}_n$ and $\widetilde{\overline{y}}_n$ in (A.3). Define $\phi_n = n^{-1}a_n'\left(\widetilde{\overline{y}}_n - E\overline{y}_n\right)$ and consider the decomposition

$$\begin{aligned}
\phi_n &= \phi_{n1} + \phi_{n2} + \phi_{n3} \qquad\qquad (A.4)\\
\phi_{n1} &= n^{-1}\sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)a_n'W_n^{k+1}X_n\beta = \sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)b_n^{(k)},\\
\phi_{n2} &= n^{-1}\sum_{k=0}^{r_n}\widehat{\lambda}_n^k a_n'W_n^{k+1}X_n\left(\widehat{\beta}_n - \beta\right) = \sum_{k=0}^{r_n}\widehat{\lambda}_n^k\,(c_n^{(k)})'\left(\widehat{\beta}_n - \beta\right),\\
\phi_{n3} &= n^{-1}\sum_{k=r_n+1}^{\infty}\lambda^k a_n'W_n^{k+1}X_n\beta = \sum_{k=r_n+1}^{\infty}\lambda^k b_n^{(k)}.
\end{aligned}$$

where $b_n^{(k)} = n^{-1}a_n'W_n^{k+1}X_n\beta$ and $c_n^{(k)} = \left[n^{-1}a_n'W_n^{k+1}X_n\right]'$. Observe that $b_n^{(k)}$ and the elements of $c_n^{(k)}$ are uniformly bounded by some finite constant, say K, in light of the remarks above Lemma 1. To prove the claim we now show that $\phi_{ni} = o_p(1)$ for $i = 1, 2, 3$. Applying the Cauchy-Schwartz and triangle inequalities to the expression for ϕ_{n1} in (A.4) yields

$$\begin{aligned}
|\phi_{n1}| &\le \left[\sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)^2\right]^{1/2}\left[\sum_{k=0}^{r_n}\left(b_n^{(k)}\right)^2\right]^{1/2}\\
&\le K\left[r_n\sum_{k=0}^{r_n}\left(\widehat{\lambda}_n^k - \lambda^k\right)^2\right]^{1/2}.
\end{aligned}$$

That $\phi_{n1} = o_p(1)$ now follows directly from Lemma 3.

Applying the triangle inequalities to the expression for ϕ_{n2} in (A.4) yields

$$\begin{aligned} |\phi_{n2}| &\leq \sum_{k=0}^{r_n} \left|\widehat{\lambda}_n\right|^k \left|(c_n^{(k)})' \left(\widehat{\beta}_n - \beta\right)\right| \\ &\leq \left[K \sum_{k=0}^{r_n} \left|\widehat{\lambda}_n\right|^k\right] \left[\sum_i \left|\widehat{\beta}_{i,n} - \beta_i\right|\right]. \end{aligned}$$

By Lemma 4 with $\kappa = 0$ we have $p\lim_{n\to\infty} \sum_{k=0}^{r_n} \left|\widehat{\lambda}_n\right|^k = \lim_{n\to\infty} \sum_{k=0}^{r_n} |\lambda|^k = 1/(1-|\lambda|)$. Since $\sum_i \left|\widehat{\beta}_{i,n} - \beta_i\right| = o_p(1)$ it follows that $\phi_{n2} = o_p(1)$.

Applying the triangle inequality to the expression for ϕ_{n3} in (A.4) yields

$$|\phi_{n3}| \leq \sum_{k=r_n+1}^{\infty} |\lambda|^k \left|b_n^{(k)}\right| \leq K \sum_{k=r_n+1}^{\infty} |\lambda|^k = K \frac{|\lambda|^{r_n+1}}{1-|\lambda|}.$$

Since $|\lambda| < 1$ it follows that $|\lambda|^{r_n+1} \to 0$ as $n \to \infty$ and thus $\phi_{n3} = o(1)$, which completes the proof of the lemma. ■

Lemma 6 *Given the model in (1), suppose Assumptions 1-4 hold and $n^{1/2}(\widehat{\lambda}_n - \lambda) = O_p(1)$ with $|\lambda| < 1$ and $\widehat{\beta}_n - \beta = o_p(1)$. Let r_n be some sequence of natural numbers with $0 \leq r_n \leq n$, $r_n \uparrow \infty$ and $r_n = o(n^{1/2})$, and let $A_n = (a_{ij,n})$ be some sequence of constant $n \times n$ matrices whose row and column sums are uniformly bounded in absolute value . Then*

$$n^{-1/2}\varepsilon_n' A_n \left(\widetilde{\overline{y}}_n - E\overline{y}_n\right) = o_p(1).$$

Proof: Recall the expressions for $E\overline{y}_n$ and $\widetilde{\overline{y}}_n$ in (A.3). Define $\phi_n = n^{-1/2}\varepsilon_n' A_n \left(\widetilde{\overline{y}}_n - E\overline{y}_n\right)$ and consider the decomposition

$$\phi_n = \phi_{n1} + \phi_{n2} + \phi_{n3} \tag{A.5}$$

$$\phi_{n1} = n^{-1/2} \sum_{k=0}^{r_n} \left(\widehat{\lambda}_n^k - \lambda^k\right) \varepsilon_n' A_n W_n^{k+1} X_n \beta = \sum_{k=0}^{r_n} \left(\widehat{\lambda}_n^k - \lambda^k\right) b_n^{(k)},$$

$$\phi_{n2} = n^{-1/2} \sum_{k=0}^{r_n} \widehat{\lambda}_n^k \varepsilon_n' A_n W_n^{k+1} X_n \left(\widehat{\beta}_n - \beta\right) = \sum_{k=0}^{r_n} \widehat{\lambda}_n^k (c_n^{(k)})' \left(\widehat{\beta}_n - \beta\right),$$

$$\phi_{n3} = n^{-1/2} \sum_{k=r_n+1}^{\infty} \lambda^k \varepsilon_n' A_n W_n^{k+1} X_n \beta = \sum_{k=r_n+1}^{\infty} \lambda^k b_n^{(k)}.$$

where $b_n^{(k)} = n^{-1/2}\varepsilon_n' A_n W_n^{k+1} X_n \beta$ and $c_n^{(k)} = \left[n^{-1/2}\varepsilon_n' A_n W_n^{k+1} X_n\right]'$. Observe that expected value of $b_n^{(k)}$ and of the elements of $c_n^{(k)}$ is zero. Observe further that the elements of $A_n W_n^{k+1} X_n \beta$ and $A_n W_n^{k+1} X_n$ are uniformly bounded by some finite constant, say K, in light of the remarks above Lemma 1. Since the $\varepsilon_{i,n}$ are distributed i.i.d. $(0, \sigma_\varepsilon^2)$ it follows that the variance of $b_n^{(k)}$ and of the elements of $c_n^{(k)}$ are uniformly bounded by $\sigma_\varepsilon^2 K^2$. To prove the claim we now show that $\phi_{ni} = o_p(1)$ for $i = 1, 2, 3$. Applying the Cauchy-Schwartz and triangle inequalities to the expression for ϕ_{n1} in (A.5) yields

$$
\begin{aligned}
|\phi_{n1}| &\leq \left[\sum_{k=0}^{r_n} \left(\widehat{\lambda}_n^k - \lambda^k\right)^2\right]^{1/2} \left[\sum_{k=0}^{r_n} \left(b_n^{(k)}\right)^2\right]^{1/2} \\
&\leq O_p(1) \left[r_n \sum_{k=0}^{r_n} \left(\widehat{\lambda}_n^k - \lambda^k\right)^2\right]^{1/2}.
\end{aligned}
$$

observing that $r_n^{-1} \sum_{k=0}^{r_n} \left(b_n^{(k)}\right)^2 = O_p(1)$ since $r_n^{-1} \sum_{k=0}^{r_n} E\left(b_n^{(k)}\right)^2 \leq \sigma_\varepsilon^2 K^2$. That $\phi_{n1} = o_p(1)$ now follows directly from Lemma 3.

Applying the Cauchy-Schwartz inequality twice to the expression for ϕ_{n2} in (A.5) yields

$$
\begin{aligned}
|\phi_{n2}| &\leq \left[\sum_{k=0}^{r_n} \left|\widehat{\lambda}_n^2\right|^k\right]^{1/2} \left[\sum_{k=0}^{r_n} \left|\left(c_n^{(k)}\right)' \left(\widehat{\beta}_n - \beta\right)\right|^2\right]^{1/2} \\
&\leq \left[\frac{r_n}{n^{1/2}}\right]^{1/2} \left[\sum_{k=0}^{r_n} \left|\widehat{\lambda}_n^2\right|^k\right]^{1/2} \left[r_n^{-1} \sum_{k=0}^{r_n} \left(c_n^{(k)}\right)' \left(c_n^{(k)}\right)\right]^{1/2} \\
&\quad \left[n^{1/2} \left(\widehat{\beta}_n - \beta\right)' \left(\widehat{\beta}_n - \beta\right)\right]^{1/2}.
\end{aligned}
$$

Since the variances of the elements of $c_n^{(k)}$ are uniformly bounded it follows that $E\left(c_n^{(k)}\right)'\left(c_n^{(k)}\right)$ and hence $r_n^{-1} \sum_{k=0}^{r_n} E\left(c_n^{(k)}\right)'\left(c_n^{(k)}\right)$ is uniformly bounded by some finite constant. Thus $r_n^{-1} \sum_{k=0}^{r_n} \left(c_n^{(k)}\right)'\left(c_n^{(k)}\right) = O_p(1)$. By Lemma 4 with $\kappa = 0$ we have $p\lim_{n\to\infty} \sum_{k=0}^{r_n} \left|\widehat{\lambda}_n^2\right|^k = \lim_{n\to\infty} \sum_{k=0}^{r_n} \left|\lambda^2\right|^k = 1/(1 - \left|\lambda^2\right|)$. Since $n^{1/2}\left(\widehat{\beta}_n - \beta\right) = O_p(1)$ and $r_n/n^{1/2} = o(1)$ it follows that $\phi_{n2} = o_p(1)$.

Next observe that $E\phi_{n3} = 0$ since $Eb_n^{(k)} = 0$. To show that $\phi_{n3} = o_p(1)$ it hence suffices to show that $\lim_{n\to\infty} E\phi_{n3}^2 = 0$. Now

$$\begin{aligned} E\phi_{n3}^2 &\leq \sum_{k=r_n+1}^{\infty}\sum_{l=r_n+1}^{\infty} |\lambda|^{k+l} E\left|b_n^{(k)}\right|\left|b_n^{(l)}\right| \\ &\leq \sigma_\varepsilon^2 K^2 \sum_{k=r_n+1}^{\infty}\sum_{l=r_n+1}^{\infty} |\lambda|^{k+l} \leq \sigma_\varepsilon^2 K^2 \left[\frac{|\lambda|^{r_n+1}}{1-|\lambda|}\right]^2. \end{aligned}$$

since $E\left|b_n^{(k)}\right|\left|b_n^{(l)}\right| \leq \left[E\left|b_n^{(k)}\right|^2\right]^{1/2}\left[E\left|b_n^{(l)}\right|^2\right]^{1/2} \leq \sigma_\varepsilon^2 K^2$. Since $|\lambda| < 1$ it follows that $|\lambda|^{r_n+1} \to 0$ as $n \to \infty$ and thus $\phi_{n3} = o_p(1)$, which completes the proof of the lemma. ■

Proof of Theorem 1: Observe that from (6) and (7)

$$y_{n*}(\widehat{\rho}_n) = Z_{n*}(\widehat{\rho}_n)\delta + u_{n*}(\widehat{\rho}_n)$$

with

$$u_{n*}(\widehat{\rho}_n) = u_n - \widehat{\rho}_n M_n u_n = \varepsilon_n - (\widehat{\rho}_n - \rho)M_n u_n.$$

Substitution of this expression into (19) yields after a standard transformation

$$\begin{aligned} n^{1/2}(\widehat{\delta}_{S,n} - \delta) &= \left[n^{-1}\widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n)'Z_{n*}(\widehat{\rho}_n)\right]^{-1} n^{-1/2}\widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n)'\varepsilon_n \qquad \text{(A.6)} \\ &\quad - \left[n^{-1}\widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n)'Z_{n*}(\widehat{\rho}_n)\right]^{-1} n^{-1/2}(\widehat{\rho}_n - \rho)\widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n)'M_n u_n. \end{aligned}$$

We now prove the result in four steps, utilizing the above decomposition.

(Step 1) As our first step we show that

$$p\lim_{n\to\infty} n^{-1}\widetilde{Z}_{n*}(\widehat{\rho}_n,\widehat{\delta}_n)'Z_{n*}(\widehat{\rho}_n) = p\lim_{n\to\infty} n^{-1}\overline{Z}_{n*}(\rho,\delta)'\overline{Z}_{n*}(\rho,\delta) = \Xi. \qquad \text{(A.7)}$$

Observe that

$$\begin{aligned} E\overline{y}_n &= W_n E y_n = W_n(I_n - \lambda W_n)^{-1}X_n\beta = \sum_{k=0}^{\infty}\lambda^k W_n^{k+1}X_n\beta, \qquad \text{(A.8)} \\ \overline{y}_n - E\overline{y}_n &= W_n(I_n - \lambda W_n)^{-1}u_n = W_n(I_n - \lambda W_n)^{-1}(I_n - \rho M_n)^{-1}\varepsilon_n. \qquad \text{(A.9)} \end{aligned}$$

where $\overline{y}_n = W_n y_n$. Recall that

$$\begin{aligned}
\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n) &= (I_n - \widehat{\rho}_n M_n)\left[X_n, \widetilde{\overline{y}}_n\right], \\
Z_{n*}(\widehat{\rho}_n) &= (I_n - \widehat{\rho}_n M_n)\left[X_n, \overline{y}_n\right], \\
\overline{Z}_{n*}(\rho, \delta) &= (I_n - \rho M_n)\left[X_n, E\overline{y}_n\right],
\end{aligned}$$

where

$$\widetilde{\overline{y}}_n = \sum_{k=0}^{r_n} \widehat{\lambda}_n^k W_n^{k+1} X_n \widehat{\beta}_n.$$

It is then readily seen that

$$n^{-1}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' Z_{n*}(\widehat{\rho}_n) = \begin{bmatrix} G_{11,n} & G_{12,n} \\ G_{21,n} & G_{22,n} \end{bmatrix}$$

with

$$\begin{aligned}
G_{11,n} &= n^{-1} X_n'(I_n - \widehat{\rho}_n M_n')(I_n - \widehat{\rho}_n M_n) X_n, \\
G_{12,n} &= n^{-1} X_n'(I_n - \widehat{\rho}_n M_n')(I_n - \widehat{\rho}_n M_n) \overline{y}_n, \\
G_{21,n} &= n^{-1} \widetilde{\overline{y}}_n'(I_n - \widehat{\rho}_n M_n')(I_n - \widehat{\rho}_n M_n) X_n, \\
G_{22,n} &= n^{-1} \widetilde{\overline{y}}_n'(I_n - \widehat{\rho}_n M_n')(I_n - \widehat{\rho}_n M_n) \overline{y}_n,
\end{aligned}$$

and

$$n^{-1}\overline{Z}_{n*}(\rho, \delta)' \overline{Z}_{n*}(\rho, \delta) = \begin{bmatrix} H_{11,n} & H_{12,n} \\ H_{21,n} & H_{22,n} \end{bmatrix}$$

with

$$\begin{aligned}
H_{11,n} &= n^{-1} X_n'(I_n - \rho M_n')(I_n - \rho M_n) X_n, \\
H_{12,n} &= n^{-1} X_n'(I_n - \rho M_n')(I_n - \rho M_n) E\overline{y}_n, \\
H_{21,n} &= n^{-1} E\overline{y}_n'(I_n - \rho M_n')(I_n - \rho M_n) X_n, \\
H_{22,n} &= n^{-1} E\overline{y}_n'(I_n - \rho M_n')(I_n - \rho M_n) E\overline{y}_n.
\end{aligned}$$

From the above expressions we see that

$$
\begin{aligned}
& G_{11,n} - H_{11,n} \\
= \; & n^{-1} X_n' \left[-(\widehat{\rho}_n - \rho)(M_n' + M_n) + (\widehat{\rho}_n^2 - \rho^2) M_n' M_n \right] X_n, \\
& G_{12,n} - H_{12,n} \\
= \; & n^{-1} X_n' \left[-(\widehat{\rho}_n - \rho)(M_n' + M_n) + (\widehat{\rho}_n^2 - \rho^2) M_n' M_n \right] E\overline{y}_n \\
& + n^{-1} X_n' \left[I_n - \widehat{\rho}_n (M_n' + M_n) + \widehat{\rho}_n^2 M_n' M_n \right] (\overline{y}_n - E\overline{y}_n), \\
& G_{21,n} - H_{21,n} \\
= \; & n^{-1} E\overline{y}_n' \left[-(\widehat{\rho}_n - \rho)(M_n' + M_n) + (\widehat{\rho}_n^2 - \rho^2) M_n' M_n \right] X_n \\
& + n^{-1} (\widetilde{\overline{y}}_n' - E\overline{y}_n') \left[I_n - \widehat{\rho}_n (M_n' + M_n) + \widehat{\rho}_n^2 M_n' M_n \right] X_n, \\
& G_{22,n} - H_{22,n} \\
= \; & n^{-1} E\overline{y}_n' \left[-(\widehat{\rho}_n - \rho)(M_n' + M_n) + (\widehat{\rho}_n^2 - \rho^2) M_n' M_n \right] E\overline{y}_n \\
& + n^{-1} (\widetilde{\overline{y}}_n' - E\overline{y}_n') \left[I_n - \widehat{\rho}_n (M_n' + M_n) + \widehat{\rho}_n^2 M_n' M_n \right] E\overline{y}_n \\
& + n^{-1} (\widetilde{\overline{y}}_n' - E\overline{y}_n') \left[I_n - \widehat{\rho}_n (M_n' + M_n) + \widehat{\rho}_n^2 M_n' M_n \right] (\overline{y}_n - E\overline{y}_n) \\
& + n^{-1} E\overline{y}_n' \left[I_n - \widehat{\rho}_n (M_n' + M_n) + \widehat{\rho}_n^2 M_n' M_n \right] (\overline{y}_n - E\overline{y}_n)
\end{aligned}
$$

Upon close inspection, recalling the remarks before Lemma 1, and utilizing (A.8) and (A.9) shows that the terms on the r.h.s. have all either one of the following basic structures, where A_n is some matrix whose row and column sums are uniformly bounded in absolute value:

$$
\begin{aligned}
P_{1n} &= o_p(1) * \left[n^{-1} X_n' A_n X_n \right], \\
P_{2n} &= o_p(1) * \left[n^{-1} X_n' A_n X_n \beta \right], \\
P_{3n} &= o_p(1) * \left[n^{-1} \beta' X_n' A_n X_n \beta \right], \\
P_{4n} &= O_p(1) * \left[n^{-1} X_n' A_n \varepsilon_n \right], \\
P_{5n} &= O_p(1) * \left[n^{-1} \beta' X_n' A_n \varepsilon_n \right], \\
P_{6n} &= O_p(1) * \left[n^{-1} X_n' A_n (\widetilde{\overline{y}}_n - E\overline{y}_n) \right], \\
P_{7n} &= O_p(1) * \left[n^{-1} \beta' X_n' A_n (\widetilde{\overline{y}}_n - E\overline{y}_n) \right], \\
P_{8n} &= O_p(1) * \left[n^{-1} \varepsilon_n' A_n (\widetilde{\overline{y}}_n - E\overline{y}_n) \right]
\end{aligned}
$$

Since the elements of X_n are uniformly bounded in absolute value, so are the elements of $n^{-1} X_n' A_n X_n$, $n^{-1} X_n' A_n X_n \beta$, $n^{-1} \beta' X_n' A_n X_n \beta$, $n^{-1} X_n' A_n$, $n^{-1} X_n' A_n$ and $n^{-1} \beta' X_n' A_n$. Thus clearly $P_{1n} = o_p(1)$, $P_{2n} = o_p(1)$, and $P_{3n} = o_p(1)$.

Using Chebychev's inequality we see that also $P_{4n} = o_p(1)$ and $P_{5n} = o_p(1)$. From Lemma 5 and 6 it follows further that $P_{6n} = o_p(1)$, $P_{7n} = o_p(1)$ and $P_{8n} = o_p(1)$. Thus $n^{-1}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' Z_{n*}(\widehat{\rho}_n) - n^{-1}\overline{Z}_{n*}(\rho,\delta)'\overline{Z}_{n*}(\rho,\delta) = o_p(1)$. Observing that $p\lim_{n\to\infty} n^{-1}\overline{Z}_{n*}(\rho,\delta)'\overline{Z}_{n*}(\rho,\delta) = \Xi$ by assumption completes this step of the proof.

(Step 2) We next show that

$$n^{-1/2}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)'\varepsilon_n - n^{-1/2}\overline{Z}_{n*}(\rho,\delta)'\varepsilon_n = o_p(1). \tag{A.10}$$

Observe that

$$n^{-1/2}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)'\varepsilon_n = \begin{bmatrix} g_{1n} \\ g_{2n} \end{bmatrix}$$

with

$$\begin{aligned} g_{1n} &= n^{-1/2}X_n'(I_n - \widehat{\rho}_n M_n')\varepsilon_n, \\ g_{2n} &= n^{-1/2}\widetilde{\overline{y}}_n'(I_n - \widehat{\rho}_n M_n')\varepsilon_n, \end{aligned}$$

and

$$n^{-1/2}\overline{Z}_{n*}(\rho,\delta)'\varepsilon_n = \begin{bmatrix} h_{1n} \\ h_{2n} \end{bmatrix}$$

with

$$\begin{aligned} h_{1n} &= n^{-1/2}X_n'(I_n - \rho M_n')\varepsilon_n, \\ h_{2n} &= n^{-1/2}\left(E\overline{y}_n\right)'(I_n - \rho M_n')\varepsilon_n. \end{aligned}$$

Thus

$$\begin{aligned} g_{1n} - h_{1n} &= -(\widehat{\rho}_n - \rho)n^{-1/2}X_n'M_n'\varepsilon_n, \\ g_{2n} - h_{2n} &= -(\widehat{\rho}_n - \rho)n^{-1/2}\left(E\overline{y}_n\right)'M_n'\varepsilon_n \\ &\quad + n^{-1/2}(\widetilde{\overline{y}}_n - E\overline{y}_n)'(I_n - \widehat{\rho}_n M_n')\varepsilon_n. \end{aligned}$$

By arguments analogous to those above it is seen that the elements of $M_n X_n$ and $M_n E\overline{y}_n$ are bounded uniformly in absolute value. Because of this we see that the variances of the elements of $n^{-1/2}X_n'M_n'\varepsilon_n$ and $n^{-1/2}\left(E\overline{y}_n\right)'M_n'\varepsilon_n$ are uniformly bounded, and hence $n^{-1/2}X_n'M_n'\varepsilon_n = O_p(1)$ and $n^{-1/2}\left(E\overline{y}_n\right)'M_n'\varepsilon_n = O_p(1)$. Since $\widehat{\rho}_n - \rho = o_p(1)$ it follows that the first two terms on the r.h.s. of the above equations are $o_p(1)$. The last term is seen to be $o_p(1)$ in light of Lemma 6.

(Step 3) We show further that

$$n^{-1/2}(\widehat{\rho}_n - \rho)\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' M_n u_n = o_p(1). \tag{A.11}$$

Observe that

$$n^{-1/2}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' M_n u_n = \begin{bmatrix} f_{1n} \\ f_{2n} \end{bmatrix}$$

with

$$\begin{aligned} f_{1n} &= n^{-1/2}X_n'(I_n - \widehat{\rho}_n M_n')M_n(I_n - \rho M_n)^{-1}\varepsilon_n, \\ f_{2n} &= n^{-1/2}\widetilde{\overline{y}}_n'(I_n - \widehat{\rho}_n M_n')M_n(I_n - \rho M_n)^{-1}\varepsilon_n \\ &= n^{-1/2}\left(E\overline{y}_n\right)'(I_n - \widehat{\rho}_n M_n')M_n(I_n - \rho M_n)^{-1}\varepsilon_n \\ &\quad + n^{-1/2}\left(\widetilde{\overline{y}}_n - E\overline{y}_n\right)'(I_n - \widehat{\rho}_n M_n')M_n(I_n - \rho M_n)^{-1}\varepsilon_n. \end{aligned}$$

In light of the remarks above Lemma 1 we see that f_{1n} is a sum of terms of the form $O_p(1) * \left[n^{-1/2}A_n\varepsilon_n\right]$ where A_n is a matrix whose elements are bounded uniformly in absolute value. Thus the variances of the elements of $n^{-1/2}A_n\varepsilon_n$ are uniformly bounded, which implies that $n^{-1/2}A_n\varepsilon_n$ and thus f_{1n} are $O_p(1)$. By analogous argument we see that also the first term on the r.h.s. of the last equality for f_{2n} is $O_p(1)$. The second term is composed of expressions of the from $O_p(1) * \left[n^{-1/2}\left(\widetilde{\overline{y}}_n - E\overline{y}_n\right)' A_n\varepsilon_n\right]$, where A_n is now some $n \times n$ matrix whose row and column sums are uniformly bounded in absolute value. By Lemma 6 we have $n^{-1/2}\left(\widetilde{\overline{y}}_n - E\overline{y}_n\right)' A_n\varepsilon_n = o_p(1)$, and thus this second term is $o_p(1)$, and $f_{2n} = O_p(1)$. This shows that $n^{-1/2}\widetilde{Z}_{n*}(\widehat{\rho}_n, \widehat{\delta}_n)' M_n u_n = O_p(1)$, and thus the claim made at the beginning of this step holds observing that $\widehat{\rho}_n - \rho = o_p(1)$.

(Step 4) Given (A.6), (A.7), (A.10), and (A.11) it follows that

$$n^{1/2}(\widehat{\delta}_{S,n} - \delta) = \Xi^{-1}n^{-1/2}\overline{Z}_{n*}(\rho, \delta)'\varepsilon_n + o_p(1)$$

Observing that the elements of X_n are uniformly bounded in absolute value, and that the rows and columns sums of a matrix which is obtained as the product of matrices whose rows and columns sums are uniformly bounded in absolute value have again the same property, it follow that the elements of $\overline{Z}_{n*}(\rho, \delta)$ are uniformly bounded in absolute value. Given the maintained assumptions on the innovations ε_n it then follows immediately from Theorem A.1 in Kelejian and Prucha (1998) that

$$n^{-1/2}\overline{Z}_{n*}(\rho, \delta)'\varepsilon_n \xrightarrow{d} N(0, \sigma_\varepsilon^2 \Xi)$$

and hence $n^{1/2}(\widehat{\delta}_{S,n} - \delta) \xrightarrow{d} N(0, \Psi)$ observing that $\Psi = \sigma_\varepsilon^2 \Xi^{-1}$. ■

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Table 2. Root mean square error of the estimators of λ, $N=400$

λ	ρ	σ^2	ML	GS2SLS	FGS2SLS	LEE	SER1	SER2	SER3	TSLS	OLS	IF	ILEE	ISER1	ISER2	ISER3
-0.9	-0.9	0.50	0.057	0.057	0.057	0.057	0.058	0.057	0.057	0.067	0.450	0.057	0.057	0.058	0.058	0.057
-0.9	-0.8	0.50	0.056	0.056	0.056	0.056	0.057	0.056	0.057	0.064	0.411	0.056	0.056	0.058	0.057	0.056
-0.9	-0.4	0.50	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.059	0.280	0.057	0.056	0.057	0.056	0.057
-0.9	0	0.50	0.059	0.060	0.060	0.060	0.061	0.060	0.059	0.060	0.163	0.060	0.060	0.061	0.060	0.061
-0.9	0.4	0.50	0.068	0.072	0.073	0.073	0.073	0.074	0.070	0.078	0.081	0.073	0.073	0.073	0.073	0.074
-0.9	0.8	0.50	0.077	0.092	0.097	0.096	0.096	0.097	0.105	0.192	0.693	0.094	0.090	0.091	0.091	0.094
-0.9	0.9	0.50	0.076	0.095	0.111	0.108	0.107	0.109	0.118	0.353	1.220	0.099	0.094	0.094	0.094	0.096
-0.8	-0.9	0.50	0.055	0.056	0.056	0.056	0.056	0.055	0.056	0.066	0.435	0.056	0.056	0.056	0.056	0.056
-0.8	-0.8	0.50	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.063	0.395	0.055	0.055	0.056	0.055	0.055
-0.8	-0.4	0.50	0.055	0.056	0.056	0.055	0.055	0.055	0.056	0.058	0.264	0.056	0.055	0.055	0.055	0.056
-0.8	0.0	0.50	0.059	0.060	0.060	0.060	0.060	0.060	0.059	0.060	0.148	0.060	0.060	0.060	0.060	0.060
-0.8	0.4	0.50	0.070	0.073	0.074	0.073	0.074	0.074	0.072	0.079	0.089	0.074	0.074	0.074	0.074	0.074
-0.8	0.8	0.50	0.080	0.096	0.101	0.100	0.100	0.101	0.103	0.196	0.711	0.098	0.095	0.095	0.095	0.096
-0.8	0.9	0.50	0.080	0.100	0.119	0.115	0.114	0.114	0.118	0.358	1.204	0.105	0.099	0.099	0.099	0.100
-0.4	-0.9	1.00	0.066	0.068	0.068	0.069	0.069	0.069	0.069	0.080	0.563	0.068	0.069	0.069	0.069	0.069
-0.4	-0.8	1.00	0.067	0.068	0.068	0.069	0.069	0.069	0.069	0.077	0.508	0.068	0.068	0.068	0.068	0.068
-0.4	-0.4	1.00	0.069	0.070	0.070	0.069	0.069	0.069	0.069	0.072	0.316	0.070	0.069	0.069	0.069	0.069
-0.4	0.0	1.00	0.078	0.078	0.078	0.078	0.078	0.078	0.078	0.078	0.136	0.078	0.078	0.078	0.078	0.078
-0.4	0.4	1.00	0.097	0.101	0.102	0.100	0.101	0.100	0.100	0.107	0.208	0.102	0.102	0.102	0.102	0.102
-0.4	0.8	1.00	0.116	0.152	0.171	0.163	0.163	0.163	0.163	0.268	0.914	0.159	0.151	0.151	0.151	0.151
-0.4	0.9	1.00	0.111	0.166	0.242	0.215	0.215	0.215	0.215	0.470	1.206	0.200	0.174	0.173	0.176	0.177
0.0	-0.9	0.25	0.026	0.027	0.027	0.027	0.027	0.027	0.027	0.030	0.105	0.027	0.027	0.027	0.027	0.027
0.0	-0.8	0.25	0.026	0.027	0.027	0.027	0.027	0.027	0.027	0.029	0.093	0.027	0.027	0.027	0.027	0.027
0.0	-0.4	0.25	0.028	0.028	0.028	0.028	0.028	0.028	0.028	0.029	0.053	0.028	0.028	0.028	0.028	0.028
0.0	0.0	0.25	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032	0.032
0.0	0.4	0.25	0.043	0.043	0.043	0.043	0.043	0.043	0.043	0.046	0.089	0.043	0.043	0.043	0.043	0.043
0.0	0.8	0.25	0.072	0.075	0.079	0.077	0.077	0.077	0.077	0.121	0.418	0.077	0.076	0.076	0.076	0.076
0.0	0.9	0.25	0.076	0.088	0.102	0.097	0.097	0.097	0.097	0.220	0.685	0.093	0.088	0.088	0.088	0.088
0.4	-0.9	0.50	0.023	0.024	0.024	0.024	0.024	0.024	0.024	0.026	0.075	0.024	0.024	0.024	0.024	0.024
0.4	-0.8	0.50	0.024	0.024	0.024	0.024	0.024	0.024	0.024	0.026	0.064	0.024	0.024	0.024	0.024	0.024
0.4	-0.4	0.50	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.026	0.032	0.026	0.026	0.026	0.026	0.026
0.4	0.0	0.50	0.030	0.030	0.030	0.030	0.030	0.030	0.030	0.030	0.041	0.030	0.030	0.030	0.030	0.030
0.4	0.4	0.50	0.043	0.043	0.043	0.043	0.043	0.043	0.043	0.045	0.123	0.043	0.043	0.043	0.043	0.043
0.4	0.8	0.50	0.084	0.090	0.093	0.090	0.090	0.090	0.090	0.117	0.399	0.092	0.091	0.091	0.091	0.091
0.4	0.9	0.50	0.093	0.118	0.142	0.133	0.131	0.133	0.133	0.197	0.528	0.130	0.121	0.120	0.120	0.121
0.8	-0.9	1.00	0.011	0.012	0.012	0.012	0.012	0.012	0.012	0.013	0.016	0.012	0.012	0.012	0.012	0.012
0.8	-0.8	1.00	0.012	0.012	0.012	0.012	0.012	0.012	0.012	0.013	0.014	0.012	0.012	0.012	0.012	0.012
0.8	-0.4	1.00	0.013	0.014	0.014	0.013	0.013	0.013	0.013	0.014	0.016	0.014	0.013	0.013	0.013	0.013
0.8	0.0	1.00	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.016	0.035	0.016	0.016	0.016	0.016	0.016
0.8	0.4	1.00	0.024	0.024	0.024	0.024	0.024	0.024	0.024	0.024	0.079	0.024	0.024	0.024	0.024	0.024
0.8	0.8	1.00	0.059	0.057	0.056	0.056	0.057	0.056	0.056	0.061	0.184	0.058	0.061	0.062	0.061	0.061
0.8	0.9	1.00	0.079	0.092	0.095	0.095	0.098	0.095	0.095	0.099	0.210	0.094	0.098	0.099	0.098	0.098
0.9	-0.9	0.50	0.004	0.005	0.005	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.004	0.004	0.004	0.004
0.9	-0.8	0.50	0.004	0.005	0.005	0.004	0.004	0.004	0.004	0.005	0.005	0.005	0.004	0.004	0.004	0.004
0.9	-0.4	0.50	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.006	0.005	0.005	0.005	0.005	0.005
0.9	0	0.50	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.010	0.006	0.006	0.006	0.006	0.006
0.9	0.4	0.50	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.009	0.023	0.009	0.009	0.009	0.009	0.009
0.9	0.8	0.50	0.024	0.023	0.023	0.024	0.024	0.024	0.024	0.024	0.075	0.023	0.025	0.025	0.025	0.025
0.9	0.9	0.50	0.041	0.039	0.039	0.040	0.043	0.042	0.041	0.040	0.101	0.040	0.043	0.046	0.044	0.043
Column Average			0.050	0.055	0.059	0.057	0.057	0.057	0.058	0.086	0.284	0.056	0.055	0.055	0.055	0.056
Col.Av.w/o $\|\lambda\|,\|\rho\|=0.9$			0.051	0.054	0.055	0.055	0.055	0.055	0.055	0.068	0.214	0.055	0.054	0.054	0.054	0.054
Col. Av. w/o $\rho=0.9$			0.045	0.047	0.048	0.048	0.048	0.048	0.048	0.059	0.208	0.048	0.047	0.048	0.047	0.048

Table 3. Root mean square error of the estimators of *B1* , *N=400*

λ	ρ	σ^2	ML	GS2SLS	FGS2SLS	LEE	SER1	SER2	SER3	TSLS	OLS	IF	ILEE	ISER1	ISER2	ISER3
-0.9	-0.9	0.50	0.034	0.034	0.034	0.034	0.034	0.034	0.035	0.041	0.043	0.034	0.034	0.035	0.035	0.035
-0.9	-0.8	0.50	0.034	0.034	0.034	0.034	0.034	0.034	0.035	0.040	0.042	0.034	0.034	0.035	0.035	0.035
-0.9	-0.4	0.50	0.035	0.035	0.035	0.035	0.035	0.035	0.035	0.037	0.038	0.035	0.035	0.035	0.035	0.035
-0.9	0	0.50	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037
-0.9	0.4	0.50	0.040	0.040	0.040	0.040	0.039	0.040	0.040	0.043	0.043	0.040	0.040	0.040	0.040	0.040
-0.9	0.8	0.50	0.043	0.046	0.046	0.046	0.046	0.047	0.049	0.084	0.076	0.046	0.045	0.046	0.046	0.047
-0.9	0.9	0.50	0.044	0.047	0.050	0.050	0.050	0.051	0.053	0.150	0.099	0.048	0.047	0.047	0.047	0.047
-0.8	-0.9	0.50	0.035	0.034	0.034	0.035	0.035	0.035	0.034	0.042	0.048	0.035	0.035	0.035	0.035	0.035
-0.8	-0.8	0.50	0.035	0.034	0.034	0.034	0.035	0.034	0.034	0.040	0.046	0.035	0.035	0.035	0.035	0.035
-0.8	-0.4	0.50	0.035	0.035	0.035	0.035	0.036	0.035	0.035	0.037	0.040	0.035	0.035	0.036	0.036	0.035
-0.8	0.0	0.50	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.037
-0.8	0.4	0.50	0.039	0.040	0.040	0.039	0.039	0.039	0.039	0.042	0.042	0.040	0.040	0.039	0.039	0.040
-0.8	0.8	0.50	0.043	0.046	0.046	0.046	0.046	0.046	0.047	0.083	0.081	0.046	0.045	0.046	0.045	0.046
-0.8	0.9	0.50	0.043	0.047	0.050	0.051	0.051	0.051	0.051	0.145	0.111	0.048	0.047	0.047	0.047	0.047
-0.4	-0.9	1.00	0.051	0.051	0.051	0.052	0.052	0.052	0.052	0.062	0.098	0.051	0.051	0.051	0.051	0.051
-0.4	-0.8	1.00	0.051	0.051	0.051	0.051	0.051	0.051	0.051	0.060	0.091	0.051	0.051	0.051	0.051	0.051
-0.4	-0.4	1.00	0.052	0.051	0.051	0.052	0.052	0.052	0.052	0.054	0.068	0.052	0.052	0.052	0.052	0.052
-0.4	0.0	1.00	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.055	0.053	0.053	0.053	0.053	0.053
-0.4	0.4	1.00	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.058	0.064	0.055	0.055	0.055	0.055	0.055
-0.4	0.8	1.00	0.060	0.063	0.064	0.064	0.064	0.064	0.064	0.104	0.149	0.063	0.063	0.063	0.063	0.063
-0.4	0.9	1.00	0.059	0.065	0.074	0.076	0.075	0.075	0.075	0.177	0.185	0.068	0.067	0.067	0.068	0.068
0.0	-0.9	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.032	0.041	0.027	0.027	0.027	0.027	0.027
0.0	-0.8	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.031	0.038	0.027	0.027	0.027	0.027	0.027
0.0	-0.4	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.028	0.030	0.027	0.027	0.027	0.027	0.027
0.0	0.0	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027
0.0	0.4	0.25	0.028	0.028	0.028	0.028	0.028	0.028	0.028	0.029	0.035	0.028	0.028	0.028	0.028	0.028
0.0	0.8	0.25	0.030	0.030	0.030	0.030	0.030	0.030	0.030	0.050	0.115	0.030	0.030	0.030	0.030	0.030
0.0	0.9	0.25	0.031	0.032	0.033	0.033	0.033	0.033	0.033	0.084	0.183	0.032	0.032	0.032	0.032	0.032
0.4	-0.9	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.047	0.057	0.039	0.039	0.039	0.039	0.039
0.4	-0.8	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.046	0.053	0.039	0.039	0.039	0.039	0.039
0.4	-0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.041	0.042	0.039	0.039	0.039	0.039	0.039
0.4	0.0	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.042	0.039	0.039	0.039	0.039	0.039
0.4	0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.042	0.070	0.039	0.039	0.039	0.039	0.039
0.4	0.8	0.50	0.040	0.040	0.041	0.040	0.040	0.040	0.040	0.067	0.198	0.040	0.041	0.041	0.041	0.041
0.4	0.9	0.50	0.041	0.043	0.043	0.044	0.044	0.044	0.044	0.112	0.257	0.043	0.043	0.043	0.043	0.044
0.8	-0.9	1.00	0.054	0.055	0.056	0.055	0.055	0.055	0.055	0.067	0.068	0.056	0.055	0.055	0.055	0.055
0.8	-0.8	1.00	0.054	0.056	0.056	0.056	0.056	0.056	0.056	0.065	0.065	0.056	0.055	0.055	0.055	0.055
0.8	-0.4	1.00	0.056	0.056	0.057	0.057	0.057	0.057	0.057	0.059	0.060	0.057	0.056	0.057	0.057	0.056
0.8	0.0	1.00	0.056	0.056	0.057	0.057	0.057	0.057	0.057	0.056	0.073	0.057	0.057	0.057	0.057	0.057
0.8	0.4	1.00	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.060	0.128	0.056	0.056	0.056	0.056	0.056
0.8	0.8	1.00	0.055	0.056	0.055	0.056	0.056	0.055	0.056	0.094	0.272	0.055	0.056	0.056	0.056	0.056
0.8	0.9	1.00	0.055	0.057	0.057	0.059	0.061	0.061	0.061	0.160	0.302	0.056	0.059	0.058	0.059	0.060
0.9	-0.9	0.50	0.038	0.039	0.039	0.038	0.038	0.038	0.038	0.047	0.047	0.039	0.038	0.038	0.038	0.038
0.9	-0.8	0.50	0.038	0.039	0.039	0.039	0.039	0.039	0.039	0.046	0.045	0.039	0.038	0.039	0.038	0.039
0.9	-0.4	0.50	0.040	0.040	0.040	0.040	0.040	0.040	0.040	0.042	0.042	0.040	0.040	0.040	0.040	0.040
0.9	0	0.50	0.040	0.040	0.040	0.040	0.040	0.041	0.041	0.040	0.047	0.040	0.040	0.040	0.041	0.041
0.9	0.4	0.50	0.040	0.040	0.040	0.040	0.040	0.040	0.040	0.043	0.074	0.040	0.040	0.040	0.040	0.040
0.9	0.8	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.067	0.217	0.039	0.039	0.039	0.039	0.039
0.9	0.9	0.50	0.039	0.040	0.039	0.040	0.040	0.040	0.040	0.111	0.280	0.039	0.040	0.039	0.039	0.040
Column Average			0.042	0.042	0.043	0.043	0.043	0.043	0.043	0.062	0.090	0.043	0.043	0.043	0.043	0.043
Col.Av.w/o |λ|,|ρ|=0.9			0.043	0.043	0.043	0.043	0.043	0.043	0.043	0.052	0.077	0.043	0.043	0.043	0.043	0.043
Col. Av. w/o ρ=0.9			0.041	0.042	0.042	0.042	0.042	0.042	0.042	0.050	0.071	0.042	0.042	0.042	0.042	0.042

Table 4. Root mean square error of the estimators of *B2* , *N=400*

λ	ρ	σ^2	**ML**	**GS2SLS**	**FGS2SLS**	**LEE**	**SER1**	**SER2**	**SER3**	**TSLS**	**OLS**	**IF**	**ILEE**	**ISER1**	**ISER2**	**ISER3**
-0.9	-0.9	0.50	0.036	0.036	0.036	0.036	0.036	0.036	0.037	0.043	0.055	0.036	0.036	0.036	0.036	0.037
-0.9	-0.8	0.50	0.036	0.036	0.036	0.036	0.036	0.036	0.037	0.042	0.053	0.036	0.036	0.036	0.036	0.037
-0.9	-0.4	0.50	0.037	0.037	0.037	0.037	0.037	0.037	0.037	0.039	0.045	0.037	0.037	0.037	0.037	0.037
-0.9	0	0.50	0.039	0.039	0.039	0.039	0.040	0.040	0.040	0.039	0.041	0.039	0.040	0.040	0.040	0.040
-0.9	0.4	0.50	0.041	0.042	0.042	0.042	0.042	0.042	0.042	0.044	0.043	0.041	0.041	0.042	0.042	0.042
-0.9	0.8	0.50	0.043	0.046	0.046	0.047	0.046	0.046	0.049	0.084	0.084	0.046	0.045	0.045	0.045	0.046
-0.9	0.9	0.50	0.043	0.046	0.049	0.049	0.049	0.049	0.052	0.143	0.122	0.047	0.046	0.046	0.046	0.047
-0.8	-0.9	0.50	0.036	0.036	0.036	0.036	0.036	0.036	0.036	0.043	0.051	0.036	0.036	0.036	0.036	0.036
-0.8	-0.8	0.50	0.036	0.036	0.037	0.036	0.036	0.037	0.037	0.042	0.049	0.036	0.036	0.036	0.036	0.036
-0.8	-0.4	0.50	0.037	0.037	0.037	0.037	0.037	0.038	0.037	0.039	0.043	0.037	0.038	0.038	0.037	0.037
-0.8	0.0	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.040	0.039	0.039	0.039	0.039	0.039
-0.8	0.4	0.50	0.041	0.041	0.041	0.041	0.042	0.041	0.041	0.043	0.043	0.041	0.041	0.042	0.041	0.042
-0.8	0.8	0.50	0.043	0.046	0.046	0.046	0.046	0.046	0.047	0.083	0.077	0.046	0.045	0.045	0.045	0.046
-0.8	0.9	0.50	0.043	0.047	0.049	0.050	0.050	0.049	0.050	0.140	0.106	0.047	0.046	0.047	0.047	0.047
-0.4	-0.9	1.00	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.063	0.057	0.052	0.052	0.052	0.052	0.052
-0.4	-0.8	1.00	0.052	0.052	0.052	0.052	0.052	0.052	0.052	0.060	0.056	0.052	0.052	0.052	0.052	0.052
-0.4	-0.4	1.00	0.053	0.053	0.053	0.053	0.053	0.053	0.053	0.056	0.055	0.053	0.053	0.053	0.053	0.053
-0.4	0.0	1.00	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.055
-0.4	0.4	1.00	0.057	0.057	0.057	0.057	0.057	0.057	0.057	0.060	0.058	0.057	0.057	0.057	0.057	0.057
-0.4	0.8	1.00	0.059	0.064	0.064	0.064	0.064	0.064	0.064	0.107	0.066	0.064	0.063	0.063	0.063	0.063
-0.4	0.9	1.00	0.058	0.065	0.073	0.074	0.074	0.074	0.074	0.157	0.066	0.069	0.068	0.067	0.068	0.068
0.0	-0.9	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.032	0.032	0.027	0.027	0.027	0.027	0.027
0.0	-0.8	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.031	0.031	0.027	0.027	0.027	0.027	0.027
0.0	-0.4	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.028	0.028	0.027	0.027	0.027	0.027	0.027
0.0	0.0	0.25	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027
0.0	0.4	0.25	0.028	0.028	0.028	0.028	0.028	0.028	0.028	0.029	0.029	0.028	0.028	0.028	0.028	0.028
0.0	0.8	0.25	0.030	0.030	0.030	0.030	0.030	0.030	0.030	0.051	0.042	0.030	0.030	0.030	0.030	0.030
0.0	0.9	0.25	0.031	0.032	0.033	0.033	0.033	0.033	0.033	0.086	0.048	0.032	0.032	0.032	0.032	0.032
0.4	-0.9	0.50	0.038	0.039	0.039	0.039	0.039	0.039	0.039	0.046	0.047	0.039	0.039	0.039	0.039	0.039
0.4	-0.8	0.50	0.038	0.038	0.039	0.039	0.039	0.039	0.039	0.045	0.045	0.039	0.039	0.039	0.039	0.039
0.4	-0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.041	0.041	0.039	0.039	0.039	0.039	0.039
0.4	0.0	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.039
0.4	0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.041	0.044	0.039	0.039	0.039	0.039	0.039
0.4	0.8	0.50	0.040	0.040	0.040	0.040	0.040	0.040	0.040	0.068	0.076	0.040	0.041	0.041	0.041	0.041
0.4	0.9	0.50	0.040	0.043	0.044	0.045	0.044	0.045	0.044	0.106	0.091	0.043	0.043	0.042	0.043	0.043
0.8	-0.9	1.00	0.054	0.055	0.056	0.055	0.055	0.055	0.055	0.066	0.066	0.055	0.055	0.055	0.055	0.055
0.8	-0.8	1.00	0.055	0.055	0.056	0.055	0.055	0.055	0.055	0.064	0.064	0.056	0.055	0.055	0.055	0.055
0.8	-0.4	1.00	0.055	0.056	0.055	0.055	0.055	0.055	0.055	0.058	0.058	0.055	0.055	0.055	0.055	0.055
0.8	0.0	1.00	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.056	0.059	0.056	0.056	0.056	0.056	0.056
0.8	0.4	1.00	0.055	0.055	0.055	0.055	0.055	0.055	0.055	0.057	0.074	0.055	0.055	0.055	0.055	0.055
0.8	0.8	1.00	0.054	0.054	0.053	0.054	0.054	0.054	0.054	0.089	0.127	0.054	0.055	0.054	0.055	0.054
0.8	0.9	1.00	0.054	0.055	0.054	0.058	0.060	0.060	0.059	0.122	0.140	0.054	0.058	0.057	0.058	0.058
0.9	-0.9	0.50	0.038	0.039	0.039	0.038	0.038	0.038	0.038	0.047	0.046	0.039	0.038	0.038	0.038	0.038
0.9	-0.8	0.50	0.038	0.039	0.039	0.038	0.039	0.038	0.038	0.045	0.045	0.039	0.038	0.039	0.038	0.038
0.9	-0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.041	0.041	0.039	0.039	0.039	0.039	0.039
0.9	0	0.50	0.040	0.040	0.040	0.040	0.039	0.039	0.039	0.040	0.041	0.040	0.040	0.039	0.039	0.039
0.9	0.4	0.50	0.039	0.039	0.039	0.039	0.039	0.039	0.039	0.041	0.048	0.039	0.039	0.039	0.039	0.039
0.9	0.8	0.50	0.038	0.038	0.038	0.038	0.038	0.038	0.038	0.065	0.104	0.038	0.038	0.038	0.038	0.038
0.9	0.9	0.50	0.038	0.038	0.038	0.039	0.039	0.039	0.039	0.098	0.131	0.038	0.039	0.038	0.039	0.039
Column Average			0.042	0.043	0.043	0.043	0.043	0.043	0.043	0.061	0.060	0.043	0.043	0.043	0.043	0.043
Col.Av.w/o |λ|,|ρ|=0.9			0.043	0.044	0.044	0.044	0.044	0.044	0.044	0.052	0.053	0.044	0.044	0.044	0.044	0.044
Col. Av. w/o ρ=0.9			0.042	0.042	0.042	0.042	0.042	0.042	0.042	0.051	0.053	0.042	0.042	0.042	0.042	0.042

Table 5. Root mean square error of the estimators of ρ , *N=400*

λ	ρ	σ^2	**ML**	**TSLS**	**FGS2SLS**	**LEE**	**SER1**	**SER2**	**SER3**
-0.9	-0.9	0.50	0.114	0.123	0.124	0.124	0.124	0.124	0.125
-0.9	-0.8	0.50	0.115	0.123	0.124	0.124	0.124	0.124	0.125
-0.9	-0.4	0.50	0.114	0.115	0.118	0.117	0.117	0.117	0.118
-0.9	0	0.50	0.101	0.101	0.103	0.102	0.103	0.103	0.102
-0.9	0.4	0.50	0.075	0.080	0.077	0.077	0.078	0.078	0.076
-0.9	0.8	0.50	0.034	0.064	0.041	0.040	0.040	0.040	0.043
-0.9	0.9	0.50	0.020	0.075	0.027	0.026	0.026	0.026	0.029
-0.8	-0.9	0.50	0.113	0.123	0.123	0.124	0.124	0.124	0.124
-0.8	-0.8	0.50	0.115	0.123	0.123	0.123	0.123	0.123	0.124
-0.8	-0.4	0.50	0.114	0.115	0.117	0.117	0.116	0.117	0.117
-0.8	0.0	0.50	0.101	0.101	0.103	0.103	0.103	0.103	0.103
-0.8	0.4	0.50	0.076	0.080	0.077	0.077	0.078	0.078	0.077
-0.8	0.8	0.50	0.035	0.065	0.042	0.041	0.041	0.041	0.042
-0.8	0.9	0.50	0.020	0.076	0.028	0.028	0.027	0.027	0.028
-0.4	-0.9	1.00	0.119	0.133	0.134	0.133	0.133	0.133	0.133
-0.4	-0.8	1.00	0.122	0.132	0.133	0.133	0.132	0.133	0.133
-0.4	-0.4	1.00	0.122	0.124	0.128	0.127	0.127	0.127	0.127
-0.4	0.0	1.00	0.111	0.112	0.113	0.112	0.112	0.112	0.112
-0.4	0.4	1.00	0.086	0.094	0.090	0.089	0.089	0.089	0.089
-0.4	0.8	1.00	0.041	0.093	0.060	0.058	0.058	0.058	0.058
-0.4	0.9	1.00	0.022	0.125	0.053	0.046	0.046	0.046	0.046
0.0	-0.9	0.25	0.103	0.113	0.115	0.115	0.115	0.115	0.115
0.0	-0.8	0.25	0.104	0.113	0.116	0.115	0.115	0.115	0.115
0.0	-0.4	0.25	0.104	0.108	0.110	0.110	0.110	0.110	0.110
0.0	0.0	0.25	0.093	0.095	0.096	0.096	0.096	0.096	0.096
0.0	0.4	0.25	0.073	0.073	0.073	0.073	0.073	0.073	0.073
0.0	0.8	0.25	0.036	0.054	0.041	0.040	0.040	0.040	0.040
0.0	0.9	0.25	0.022	0.060	0.030	0.029	0.029	0.029	0.029
0.4	-0.9	0.50	0.104	0.116	0.118	0.117	0.117	0.117	0.117
0.4	-0.8	0.50	0.106	0.116	0.118	0.117	0.117	0.117	0.117
0.4	-0.4	0.50	0.106	0.111	0.112	0.112	0.112	0.112	0.112
0.4	0.0	0.50	0.097	0.098	0.099	0.099	0.099	0.099	0.099
0.4	0.4	0.50	0.078	0.079	0.079	0.079	0.079	0.079	0.079
0.4	0.8	0.50	0.047	0.068	0.055	0.053	0.053	0.053	0.053
0.4	0.9	0.50	0.029	0.081	0.053	0.047	0.046	0.047	0.047
0.8	-0.9	1.00	0.102	0.118	0.120	0.118	0.119	0.118	0.118
0.8	-0.8	1.00	0.105	0.117	0.119	0.118	0.119	0.118	0.118
0.8	-0.4	1.00	0.107	0.111	0.114	0.113	0.113	0.113	0.113
0.8	0.0	1.00	0.099	0.101	0.102	0.102	0.102	0.102	0.102
0.8	0.4	1.00	0.082	0.084	0.083	0.083	0.083	0.083	0.084
0.8	0.8	1.00	0.066	0.078	0.069	0.065	0.067	0.065	0.065
0.8	0.9	1.00	0.053	0.104	0.093	0.074	0.075	0.074	0.073
0.9	-0.9	0.50	0.099	0.111	0.113	0.113	0.113	0.113	0.113
0.9	-0.8	0.50	0.101	0.111	0.113	0.113	0.113	0.113	0.113
0.9	-0.4	0.50	0.101	0.107	0.108	0.108	0.108	0.108	0.108
0.9	0	0.50	0.090	0.093	0.094	0.093	0.093	0.093	0.093
0.9	0.4	0.50	0.071	0.073	0.072	0.072	0.072	0.072	0.072
0.9	0.8	0.50	0.049	0.052	0.048	0.048	0.049	0.048	0.048
0.9	0.9	0.50	0.044	0.059	0.052	0.047	0.051	0.049	0.048
Column Average			0.082	0.098	0.091	0.090	0.090	0.090	0.090
Col.Av.w/o |λ|,|ρ|=0.9			0.089	0.098	0.095	0.094	0.094	0.094	0.094
Col. Av. w/o ρ=0.9			0.091	0.100	0.098	0.097	0.098	0.098	0.098