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NOTES AND COMMENTS

THE VARIANCE-COVARIANCE MATRIX OF THE MAXIMUM LIKELIHOOD ESTIMATOR IN TRIANGULAR STRUCTURAL SYSTEMS: CONSISTENT ESTIMATION

BY INGMAR R. PRUCHA[1]

1. INTRODUCTION

IN A RECENT ARTICLE Lahiri and Schmidt (1978) analyzed the properties of various estimators in triangular structural systems. In triangular structural systems the determinant of the Jacobian is unity; therefore the likelihood function for such systems is the same as for seemingly unrelated regression (SUR) systems. As a consequence, Lahiri and Schmidt (1978) point out that the full information maximum likelihood (FIML) estimator and the iterative SUR estimator are identical in triangular structural systems.

The algebraic equality between the FIML and iterative SUR estimator in triangular structural systems might be thought to imply that in such systems the variance-covariance (VC) matrix estimator typically associated with the iterative SUR estimator is a consistent estimator for the asymptotic VC matrix of the FIML estimator. The purpose of this note is to show that this is generally not the case. This issue is not addressed in Lahiri and Schmidt (1978).

2. A SIMPLE EXAMPLE

It is sufficient to demonstrate the above point in terms of a simple example. Consider the following two equation triangular structural system:

$$y_1 = e + u_1, \qquad y_2 = y_1\alpha + u_2. \tag{1}$$

Here y_1 and y_2 denote the $T\times 1$ vectors of observations on the endogenous variables, e is a $T\times 1$ vector of unit elements, u_1 and u_2 are the $T\times 1$ vectors of disturbances. The vectors of contemporaneous disturbances are assumed to be i.i.d. normal with zero mean and positive definite VC matrix $\Sigma = (\sigma_{ij})$ where $i, j = 1, 2$; that is, $[u_1', u_2']' \sim N(0, \Sigma\otimes I_T)$. The unknown parameters of interest are α, σ_{11}, σ_{12}, and σ_{22}.

The above model can be written equivalently as:

$$y_1 = e + u_1, \qquad y_2 = y_1\alpha + (y_1 - e)\tau + u_2^*, \tag{2}$$

where $\tau = \sigma_{12}/\sigma_{11}$ and $u_2^* = u_2 - \tau u_1$. The second equation in (2) corresponds to a decomposition of y_2 into $E(y_2|y_1)$ and $y_2 - E(y_2|y_1)$. Note that the disturbance vector u_2^* is independent of u_1, has zero mean and VC matrix $\sigma_{22}^* I_T$ with $\sigma_{22}^* = \sigma_{22} - \sigma_{12}^2/\sigma_{11}$. Furthermore note that the parameter transformation involved is one-to-one. Hence the FIML estimator for α, say $\hat\alpha$, can be obtained from a least squares (LS) regression of y_2 on y_1 and $y_1 - e$. This yields: $\hat\alpha = [y_1'y_2 - \hat\tau y_1'(y_1 - e)]/[y_1'y_1]$ with $\hat\tau = [\hat u_1'\hat u_2]/[\hat u_1'\hat u_1]$ and $\hat u_1 = y_1 - e$, $\hat u_2 = y_2 - y_1\hat\alpha$. The second equation in (2) satisfies the assumptions of the classical LS model with stochastic i.i.d. regressors. Standard LS theory then implies that $\sqrt{T}(\hat\alpha - \alpha) \to^{\text{i.d.}} N(0, \Phi)$ with $\Phi = \sigma_{22}^*$. The usual formula for the VC matrix of the LS estimator yields the following consistent estimator for Φ: $\hat\Phi = \hat\sigma_{22}^*/[y_1'y_1(1-\hat\rho^2)/T]$ where $\hat\sigma_{22}^* = \hat u_2^{*\prime}\hat u_2^*/T$ with $\hat u_2^* = \hat u_2 - \hat\tau\hat u_1$ and where $\hat\rho^2 = [y_1'u_1]^2/[y_1'y_1u_1'u_1]$.

Let $\bar\Sigma = (\bar\sigma_{ij})$ be some estimator for Σ. The corresponding feasible SUR estimator for α, say $\bar\alpha$, can then be obtained from a LS regression of y_2 on y_1 with τ constrained to

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$\bar{\tau} = \bar{\sigma}_{12}/\bar{\sigma}_{11}$. This yields: $\bar{\alpha} = [y_1'y_2 - \bar{\tau}y_1'(y_1 - e)]/[y_1'y_1]$. The iterative feasible SUR procedure constrains τ at each iteration to its estimate based on residuals from the previous iteration. At convergence the estimate for τ equals the FIML estimate $\hat{\tau}$ and we get the FIML estimate for α. The estimator for Φ typically associated with the iterative SUR estimator is $\tilde{\Phi} = \hat{\sigma}_{22}^*/[y_1'y_1/T]$. For the present example it is readily seen that plim $\tilde{\Phi} = \sigma_{22}^*/(1+\sigma_{11})$ and consequently

$$\text{(3)} \qquad \text{plim } \tilde{\Phi} < \Phi.$$

Thus, $\tilde{\Phi}$ is generally an inconsistent estimator for Φ.[2] We note that the above arguments and, in particular, the inequality in (3) generalize; it is readily seen that for general linear or nonlinear triangular structural systems the probability limit of the difference between the VC matrix estimator typically associated with the iterative SUR estimator and the true asymptotic VC matrix of the FIML estimator is negative semi-definite.

3. CONCLUSION

The iterative SUR estimator is (because of computational advantages) often used to calculate FIML estimates in triangular structural systems. This note demonstrates by means of an example that the VC matrix estimator $\tilde{\Phi}$ typically associated with the iterative SUR estimator is inconsistent for the asymptotic VC matrix Φ of the FIML estimator, despite the fact that the iterative SUR and FIML estimators are identical. This point is simple. However, unless we are careful, an error may be committed: In most econometric packages the iterative SUR routine will not check if the structure of the model is triangular, or truly seemingly unrelated. Therefore, an iterative SUR routine will typically report estimates of Φ based on $\tilde{\Phi}$ even for triangular structural systems.[3]

Department of Economics, University of Maryland, College Park, MD 20742, U.S.A.

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REFERENCES

LAHIRI, L., AND P. SCHMIDT (1978): "On the Estimation of Triangular Structural Systems," *Econometrica*, 46, 1217-1221.

[2] Clearly, this result remains true, even if the model is recursive (i.e. Σ is diagonal) unless $\sigma_{12} = 0$ is explicitly imposed. If the second equation in (1) contained an exogenous x_1 instead of y_1 with lim $x_1'x_1/T = m_{xx}$ finite, then $\Phi = \text{plim } \hat{\Phi} = \sigma_{22}^*/m_{xx}$ observing that plim $x_1'u_1/T = 0$. In this case $\tilde{\Phi} = \hat{\sigma}_{22}^*/[x_1'x_1/T]$ would be consistent.

[3] To obtain consistent estimates of Φ we may take the FIML estimates for the regression parameters obtained with the iterative SUR routine and start the FIML routine from those estimates. Convergence of the FIML routine should be immediate. Upon convergence the FIML routine will typically compute and report consistent estimates of Φ.