

A COMPARISON OF ALTERNATIVE METHODS FOR THE ESTIMATION OF DYNAMIC FACTOR DEMAND MODELS UNDER NON-STATIC EXPECTATIONS*

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Several approaches to the formulation and estimation of dynamic factor demand systems under non-static expectations on the exogenous variables in the firm's decision process have been suggested. Among those approaches there are trade-offs in terms of statistical and computational efficiency, the generality with which the technology and the expectation formation process can be specified, and in terms of informational requirements. This paper analyzes the trade-offs among three alternative approaches in terms of their statistical and computational efficiency within the context of a Monte Carlo experiment.

1. Introduction

In the face of adjustment costs the rational firm's optimal input decisions are intertemporally related. The firm's temporary equilibrium position at each point in time is described by a set of dynamic factor demand equations. Several approaches to the formulation and estimation of dynamic factor demand systems under non-static expectations on the exogenous variables in the firm's decision process have been suggested. In this paper, we compare three of these approaches.

The first, developed by Hansen and Sargent (1980,1981) and Epstein and Yatchew (1985), assumes that the firm sets inputs according to a stochastic closed-loop feedback control policy. It is based on an explicit analytic solution of the firm's intertemporal optimization problem and is restricted to linear-quadratic technologies. The approach assumes an infinite planning horizon and requires a full specification of the expectational model. Expectations on the exogenous variables in the firm's decision process are based on an

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autoregressive model. Hansen and Sargent (1981) and Epstein and Yatchew (1985) suggest different methods for estimating the technology and expectations parameters. Both yield the same parameter estimates. The estimation method suggested by Epstein and Yatchew (1985) hinges on a reparameterization of the production function and is less general since it allows only for first-order changes of factor inputs in the representation of adjustment costs. It is, however, computationally simpler and comparatively more accessible. Nevertheless, the corresponding estimating equations can be quite complex, especially for more than two quasi-fixed factors. For an empirical application of the method by Epstein and Yatchew, see, e.g., Mohnen, Nadiri and Prucha (1985) and Nadiri and Prucha (1985).

The second approach is due to Kennan (1979), Hansen (1982), Hansen and Singleton (1982) and has been implemented in Pindyck and Rotemberg (1983a, b).¹ It also assumes that inputs are set according to a stochastic closed-loop feedback control policy. Expectations are assumed to be rational. The model's parameters are estimated from the set of Euler equations corresponding to the initial planning period only. This is accomplished by replacing all expectations on future variables by their observed values in those future periods and applying an instrumental variable estimation technique. Since the approach does not require an explicit solution of the Euler equations, it allows for considerable flexibility in the choice of the functional form of the production function. Also, an explicit specification of the process that generates the variables exogenous in the firm's decision process or specific assumptions concerning the planning horizon are not required. Furthermore, the approach is easy to implement. However, it is generally not fully efficient, since it ignores information from the remaining Euler equations and, in case of an infinite planning horizon, from the transversality condition.

The third approach, suggested by Prucha and Nadiri (1984), considers a firm with a finite but shifting planning horizon. Contrary to the first two approaches, where the firm is assumed to set its inputs according to a stochastic closed-loop feedback control policy, this approach assumes that input decisions are based on a certainty equivalence feedback control policy. We note that for linear-quadratic technologies the certainty equivalence feedback and closed-loop feedback control policy (based on a certain planning horizon) yield exactly the same input decisions; for general technologies input decisions corresponding to the former policy may be viewed as first-order approximations to those corresponding to the latter policy.² This approach also utilizes the full solution to the firm's intertemporal (finite horizon)

¹For a discussion of this approach in the case where some of the exogenous variables in the decision process are unobserved, see Garber and King (1983).

²See, e.g., Simon (1956), Theil (1957) and Malinvaud (1969) on the principle of certainty and first-order certainty equivalence.

optimization problem. For parameter estimation, Prucha and Nadiri suggest an algorithm that numerically solves the firm's optimization problem at each iteration step, therefore avoiding the need for an explicit analytic solution. As a consequence, considerable flexibility is allowed in both the choice of functional form of the technology and the expectation formation process.

Computationally, the third approach is more involved. Suppose a firm must determine the optimal levels of G quasi-fixed factors over a planning horizon that is $T + 1$ periods long. To evaluate the statistical objective function corresponding to a particular set of (trial) parameter values, the estimation algorithm must solve a system of $(T + 1) \times G$ first-order conditions for each data point in the sample. Depending on the sample size and the values of T and G this can involve a non-trivial computational cost. Consequently, to keep the computational burden comparatively low, Prucha and Nadiri suggest a method to evaluate the gradient of the statistical objective function from analytic expressions without requiring knowledge of an explicit analytic solution to the firm's optimizing problem.

In cases where the planning horizon is infinite, the finite horizon model may serve as an approximation to the infinite horizon model. Prucha and Nadiri offer results suggesting that their finite horizon specification (characterized by a particular endogenous determination of the stocks of the quasi-fixed factors at the end of the planning horizon) can approximate the infinite horizon model quite closely even for moderate sizes of the planning horizon. Since such a feature is important for computational efficiency a further exploration of this issue seems of interest.³

The above discussion implies that among the three approaches, there are trade-offs in terms of statistical and computational efficiency, the generality with which the technology and the expectation formation process can be specified and in terms of informational requirements. This paper analyzes in particular the trade-offs among the three approaches in terms of their statistical and computational efficiency within the context of a Monte Carlo experiment.

We consider a linear-quadratic model with an infinite planning horizon in which expectations on the variables exogenous in the firm's decision process are formed rationally from an autoregressive model. Under this scenario, an explicit analytic solution exists and all three approaches are (at least in an

³The approach of Prucha and Nadiri (1984) is in some respects related to the estimation method suggested by Fair and Taylor (1983) for dynamic non-linear rational expectations models. The method of Fair and Taylor is also based on 'subiterations' at each primary iteration step. However, in terms of the actual design of the estimation algorithm, the two approaches differ substantially. Consequently, there may be substantial differences in terms of computational cost between the two approaches. A comparison in terms of computational efficiency is beyond the scope of the present paper. Chow (1980,1981) has proposed another approach based on the dynamic programming solution. For non-linear systems Chow suggests to approximate that system by a linearized version.

approximative sense) applicable. Consequently, under this scenario we can compare the statistical and computational efficiency of all three approaches with each other. With respect to the third approach we will also investigate how well the finite horizon model approximates the infinite horizon model for different lengths of the planning horizon. In the following we will refer to the first, second and third approach as, respectively, approach I, II and III.

The paper is organized as follows: Section 2 defines the firm's objective function for the finite and infinite horizon model and discusses the corresponding optimal control solutions. The empirical factor demand equations corresponding to the three approaches are given in section 3. In section 4 we discuss the parameter and data design of the Monte Carlo experiment and present the actual Monte Carlo results. Concluding remarks and suggestions for future research are offered in section 5.

2. Technology and optimal control solution

The three approaches considered in this paper allow for the estimation of dynamic factor demand models with two or more non-separable quasi-fixed factors. Since additional estimation problems arise in this case relative to the case of one or several separable quasi-fixed factors, we explore the former, more general case.⁴ Furthermore, we adopt a cost minimization framework to be consistent with much of the recent empirical literature on dynamic factor demand models. Recent empirical applications of dynamic factor demand models include papers of Berndt, Morrison and Watkins (1981), Denny, Fuss and Waverman (1981), Mohnen, Nadiri and Prucha (1985), Morrison and Berndt (1981), Nadiri and Prucha (1984, 1985), and Pindyck and Rotemberg (1983a, b).

Consider a firm that employs one variable factor and two quasi-fixed factors in producing a single output from a technology with adjustment costs. We assume that the firm's production process is described by the following linear-quadratic factor requirement function:

$$\begin{aligned} V_t &= G(X_{t-1}, \Delta X_t, Y_t) \\ &= \alpha_0 + \alpha_Y Y_t + \frac{1}{2} \alpha_{YY} Y_t^2 + a' X_{t-1} + b' X_{t-1} Y_t \\ &\quad + \frac{1}{2} X_{t-1}' A X_{t-1} + \frac{1}{2} \Delta X_t' B \Delta X_t. \end{aligned} \quad (1)$$

Here $a = [\alpha_1, \alpha_2]'$, $b = [\alpha_{1Y}, \alpha_{2Y}]'$, $A = (\alpha_{ij})$ is a 2×2 symmetric matrix and

⁴For a linear-quadratic technology, the additional difficulty arises from the fact that under non-separability it is generally not possible to find explicit expressions for the accelerator coefficients in terms of the underlying technology parameters. All three approaches are, of course, also applicable under separability.

$B = \text{diag}(\check{\alpha}_{11}, \check{\alpha}_{22})$ is a 2×2 diagonal matrix. With Y_t we denote output, V_t is the variable factor input and $X_t = [X_{1t}, X_{2t}]'$ is the vector of end-of-period stocks of the quasi-fixed factors. The vector $\Delta X_t = X_t - X_{t-1}$ represents the internal adjustment costs. Since there is only one variable factor, the factor requirement function (1) can also be viewed as a normalized restricted cost function. It is a special case of one of the two normalized restricted cost functions considered by Denny, Fuss and Waverman (1981). It is assumed to have standard properties, i.e., $G_X < 0$, $G_{|\Delta X|} > 0$, $G_Y > 0$. Furthermore, we assume that $G(\cdot)$ is convex in X_{t-1} and ΔX_t , implying $\alpha_{11} > 0$, $\alpha_{22} > 0$, $\alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0$, $\check{\alpha}_{11} > 0$, $\check{\alpha}_{22} > 0$.

The firm's after tax cost in period t (normalized by the price of the variable factor) is defined as

$$g(X_t, X_{t-1}, Y_t, Q_t) = G(X_{t-1}, \Delta X_t, Y_t)(1 - u) + Q_t'(\Delta X_t + \delta X_{t-1}), \tag{2}$$

where $Q_t = [Q_{t1}, Q_{t2}]'$ is the vector of (normalized) acquisition prices for the quasi-fixed factors, u is the tax rate and $\delta = \text{diag}(\delta_1, \delta_2)$ is the diagonal matrix of depreciation rates. At each point in time t the firm minimizes the expected present value of future costs,

$$C_t = E_t \sum_{\tau=0}^{\infty} g(X_{t+\tau}, X_{t+\tau-1}, Y_{t+\tau}, Q_{t+\tau})(1 + r)^{-\tau}, \tag{3}$$

for given initial stocks X_{t-1} . Here E_t is the expected value operator conditional on information available at time t , r denotes the real discount rate. Y_t and Q_t are assumed to be known at time t .

The closed-loop feedback control policy requires that the firm sets the input vector X_t optimally in each period. In addition, the firm must devise a strategy for setting X_{t+1}, X_{t+2}, \dots as functions of information it knows will become available in the future such that (3) is minimized. The optimal input sequence for the quasi-fixed factors must satisfy the following set of stochastic Euler equations:⁵

$$\begin{aligned} & -BE_{t+\tau}X_{t+\tau+1} + [A + (2+r)B]X_{t+\tau} - (1+r)BX_{t+\tau-1} \\ & = E_{t+\tau}h_{t+\tau}, \quad \tau = 0, \dots, \infty, \end{aligned} \tag{4}$$

⁵This and the subsequent results follow from standard optimal control theory; see, e.g., Kwakernaak and Sivan (1972).

with

$$h_{t+\tau} = - \{ a + bY_{t+\tau+1} + [(1+r)Q_{t+\tau} - (1-\delta)Q_{t+\tau+1}] / (1-u) \}.$$

Unstable solutions are ruled out by the transversality conditions. The certainty equivalence analog to (3) is given by

$$C_t^* = \sum_{\tau=0}^{\infty} g(X_{t+\tau}, X_{t+\tau-1}, E_t Y_{t+\tau}, E_t Q_{t+\tau}) (1+r)^{-\tau}. \quad (5)$$

The certainty equivalence feedback control policy requires the firm to minimize (5) in each period t and to set the current quasi-fixed inputs according to the plan that minimizes (5). The process is repeated every period as new information becomes available. Since $G(\cdot)$ is linear-quadratic, the closed-loop feedback control policy and the certainty equivalence feedback control policy will yield identical input decisions. The input sequence that optimizes (5) conditional on information available in period t must satisfy the following set of non-stochastic Euler equations:

$$-BX_{t+\tau+1} + [A + (2+r)B]X_{t+\tau} - (1+r)BX_{t+\tau-1} = E_t h_{t+\tau}, \quad (6)$$

$$\tau = 0, \dots, \infty.$$

Unstable solutions are again ruled out by the corresponding transversality condition. We assume that the exogenous variables grow of exponential order less than $(1+r)^{1/2}$. The solution of (6) is well known and given by the accelerator equations

$$X_{t,\tau} = MX_{t,\tau}^* + (I-M)X_{t,\tau-1},$$

$$X_{t,\tau}^* = (1+r)^{-1}M^{-1}(I-M)$$

$$\cdot \sum_{i=0}^{\infty} (1+r)^{-i}(I-M)^i B^{-1}E_t h_{t+\tau+i}.$$
(7a)

The matrix of accelerator coefficients M solves the matrix equation

$$BM^2 + (A+rB)M - A = 0. \quad (7b)$$

Furthermore, $C = -BM$ is symmetric and negative definite.⁶ The implied optimal factor inputs in period t corresponding to the closed-loop feedback control policy or, equivalently, corresponding to the certainty equivalence feedback control policy are given by the accelerator model

$$X_t = MX_{t,0}^* + (I - M)X_{t-1}. \tag{8}$$

The certainty equivalence feedback control policy considered in Prucha and Nadiri (1984) assumes a firm with a finite but shifting planning horizon of $T + 1$ periods. The stocks of the quasi-fixed factors at the end of the planning horizon are determined endogenously subject to the assumptions of static expectations and constant firm size beyond the actual planning horizon. This means that under the finite horizon specification the firm minimizes (5) in each period t subject to the constraints $X_{t+\tau} = X_{t+T}$ and with $E_t Y_{t+\tau} = E_t Y_{t+T}$, $E_t Q_{t+\tau} = E_t Q_{t+T}$ for $\tau \geq T$ and sets the current quasi-fixed inputs according to the optimizing plan. As in the infinite horizon case, the process is repeated every period as new information becomes available. Under the constraints of constant firm size and static expectations beyond the actual planning horizon, minimizing (5) is equivalent to minimizing

$$C_t^{**} = \sum_{\tau=0}^T g(X_{t+\tau}, X_{t+\tau-1}, E_t Y_{t+\tau}, E_t Q_{t+\tau})(1+r)^{-\tau} + g(X_T, X_T, E_t Y_{t+T}, E_t Q_{t+T})/[r(1+r)^T]. \tag{9}$$

The first-order conditions are given by

$$-BX_{t+\tau+1} + [A + (2+r)B]X_{t+\tau} - (1+r)BX_{t+\tau-1} = E_t h_{t+\tau}, \tag{10a}$$

$$\tau = 0, \dots, T-1,$$

$$[A + rB]X_{t+T} - rBX_{t+T-1} = E_t \bar{h}_{t+T}, \tag{10b}$$

$$\bar{h}_{t+T} = -[a + bY_{t+T} + (rI + \delta)Q_{t+T}/(1-u)].$$

⁶In somewhat more detail: Let Δ and Π be matrices such that $B^{-1} = \Delta\Delta$ and $\Pi\Delta A\Delta\Pi' = D$ with Π orthogonal and where D is the diagonal matrix of (positive) eigenvalues of $\Delta A\Delta$; let $S = \Delta\Pi'$ and hence $S^{-1}B^{-1}AS = D$; furthermore, let Λ be the diagonal matrix of the stable (positive) roots of the characteristic equation corresponding to (6) which satisfies $\Lambda^2 - \Lambda[D + (2+r)I] + (1+r)I = 0$; then the matrix of accelerator coefficients can be shown to be given by $M = S(I - \Lambda)S^{-1}$. We also note that the expressions in (7) differ somewhat from those given in Epstein and Yatchew (1985). The reason is that we have not assumed that the quasi-fixed factors become immediately productive. Which of the two specifications is more appropriate will depend on the specific application.

A comparison of (6) and (10a) shows that the first-order conditions for the initial T periods are, in both the infinite and finite horizon model, identical. Expressions (10b) may be interpreted as an approximation to the transversality condition and to the first-order conditions of the infinite horizon model for $\tau \geq T$.

3. Expectations formation and empirical demand functions

In the following it is maintained that the infinite horizon model underlying the first and second approaches is the true model. Hansen and Sargent (1981) and Epstein and Yatchew (1985) assume that expectations on the exogenous variables in the firm's input decision process are formed from a (possibly multivariate) autoregressive model. To keep the number of parameters reasonably small, prices and output are assumed to evolve according to the following simple autoregressive processes:

$$\begin{aligned} Q_{t1} &= \rho_{01} + \rho_1 Q_{t-1,1} + \eta_{t1}, \\ Q_{t2} &= \rho_{02} + \rho_2 Q_{t-1,2} + \eta_{t2}, \\ Y_t &= \rho_{03} + \rho_3 Y_{t-1} + \eta_{t3}. \end{aligned} \quad (11)$$

The disturbances $\eta_t = [\eta_{t1}, \eta_{t2}, \eta_{t3}]'$ are taken to be distributed i.i.d. over time. We assume that expectations are formed rationally, i.e.,

$$E_t Z_{t+\tau, i} = \rho_{0i} / (1 - \rho_i) + \rho_i^{\tau} [Z_{ti} - \rho_{0i} / (1 - \rho_i)],$$

with

$$Z_{t1} = Q_{t1}, \quad Z_{t2} = Q_{t2}, \quad Z_{t3} = Y_t, \quad i = 1, 2, 3.$$

Substitution of these expressions into (8) and making use of the factor requirement function (1) yields the following system of demand equations:

$$\begin{aligned} V_t &= \alpha_0 + \alpha_Y Y_t + \frac{1}{2} \alpha_{YY} Y_t^2 + a' X_{t-1} + b' X_{t-1} Y_t \\ &\quad + \frac{1}{2} X'_{t-1} A X_{t-1} + \frac{1}{2} \Delta X'_t B \Delta X_t + \varepsilon_{t1}, \end{aligned} \quad (12)$$

$$X_t = M X_{t,0}^* - (I - M) X_{t-1} + \varepsilon_{t*},$$

$$M X_{t,0}^* = -(I - M) \sum_{i=0}^3 \frac{\rho_i}{1+r} \left[I - \frac{\rho_i}{1+r} (I - M) \right]^{-1} B^{-1} n_{ti}, \quad (13)$$

with

$$n_{t0} = \begin{bmatrix} \alpha_1 + \alpha_{1Y}\rho_{03}/(1 - \rho_3) + (r + \delta_1)[\rho_{01}/(1 - \rho_1)]/(1 - u) \\ \alpha_2 + \alpha_{2Y}\rho_{03}/(1 - \rho_3) + (r + \delta_2)[\rho_{02}/(1 - \rho_2)]/(1 - u) \end{bmatrix},$$

$$n_{t1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} n_{t^*}, \quad n_{t2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} n_{t^*},$$

$$n_{t3} = \begin{bmatrix} \alpha_{1Y} \\ \alpha_{2Y} \end{bmatrix} [Y_t - \rho_{03}/(1 - \rho_3)],$$

$$n_{t^*} = \begin{bmatrix} [(1 + r)/\rho_1 - (1 - \delta_1)][Q_{t1} - \rho_{01}/(1 - \rho_1)]/(1 - u) \\ [(1 + r)/\rho_2 - (1 - \delta_2)][Q_{t2} - \rho_{02}/(1 - \rho_2)]/(1 - u) \end{bmatrix},$$

and $\rho_0 = 1$. These equations determine together with (11) the variable and quasi-fixed factor inputs V_t , X_{1t} and X_{2t} . Note that a stochastic disturbance term is added to each of the factor demand equations. As in Epstein and Yatchew (1985), we interpret the disturbances $\epsilon_t = [\epsilon_{t1}, \epsilon'_{t^*}]' = [\epsilon_{t1}, \epsilon_{t2}, \epsilon_{t3}]'$ as optimization errors. They are assumed to be distributed i.i.d. over time.⁷ Furthermore, we assume ϵ_t and η_t as well as the elements of η_t to be stochastically independent.⁸ We define $\Sigma_\epsilon = (\sigma_{\epsilon ij}) = E(\epsilon_t \epsilon'_t)$ and $\Sigma_\eta = (\sigma_{\eta ij}) = E(\eta_t \eta'_t)$.

Approach I takes the system (11), (12), and (13) as the fundamental system of equations from which the technology parameters and the expectations parameters are to be estimated. Since the approach is based on a complete solution of the firm's intertemporal optimization problem it utilizes all available a priori information. In general, it is not possible to estimate the model parameters from (11), (12), and (13) using standard estimation algorithms. The reason is that, in general, we cannot solve (7b) explicitly for M in terms of B and A (unless the quasi-fixed factors are separable). Therefore, we adopt the estimation procedure suggested by Epstein and Yatchew (1985) which is based on a reparameterization of the model.⁹ Note that while (7b) cannot be solved

⁷Alternatively we could have interpreted the disturbances ϵ_t as random shocks to the technology which are observed by the firm but not by the researcher. In that case, however, we would expect ϵ_{1t} to be heteroscedastic. Since this would further complicate the estimation of the model, we do not adopt this interpretation in the present study.

⁸Given the assumption of independence between ϵ_t and η_t , we can use in the second approach the current period price and output variables as instruments. This avoids putting the second approach into a possibly unfair disadvantage.

⁹As noted in the introduction, Hansen and Sargent (1981) have suggested an alternative estimation procedure that can also be applied to models with adjustment cost that involve higher-order changes in the factor inputs. We have adopted the procedure suggested by Epstein and Yatchew (1985) in the present study, since it is computationally simpler. Both procedures yield the same parameter estimates.

explicitly for M in terms of B and A it can be solved for A in terms of B and M . This suggests that we estimate the elements of B and M rather than those of B and A as the principal parameters. To impose the symmetry of C , we can also estimate B and C instead of B and M . It is not difficult to see from (7b) and the definition of C that

$$A = C - (1 + r) [B - B(C + B)^{-1}B], \quad M = -B^{-1}C. \quad (14)$$

Upon substitution of (14) into (12) and (13), we can estimate B and C and the remaining technology and expectations parameters from (11), (12), and (13) using standard system methods. Once estimates for B and C are obtained we can estimate A from (14). The actual implementation of the factor demand system (11), (12), and (13) requires that the equations be written in scalar notation. The reader can easily verify that the resulting expressions turn out to be quite complex even for our relatively simple example.¹⁰

The general solution to the first-order conditions (6) corresponding to the infinite horizon model, $\{X_{t,\tau}\}_{\tau=0}^{\infty}$, is defined in (7). For comparison with the other two approaches, we note that (13) can be written (in general notation) equivalently as

$$\begin{aligned} X_t &= X_{t,0} + \varepsilon_{t*} \\ &= [A + (2 + r)B]^{-1} \{ BX_{t,1} + (1 + r)BX_{t-1} + E_t h_t \} + \varepsilon_{t*}. \end{aligned} \quad (15)$$

The empirical demand equations for the quasi-fixed factors corresponding to the approach II are given by

$$X_t = [A + (2 + r)B]^{-1} \{ BX_{t+1} + (1 + r)BX_{t-1} + h_t \} + v_{t*}, \quad (16)$$

where v_{t*} denotes the vector of disturbances. The system is completed by the demand equation for the variable factor (12). Eqs. (16) have been obtained from the initial ($\tau = 0$) set of stochastic Euler equations (4) by replacing the unobserved conditional expectations on X_{t+1} , Y_{t+1} and Q_{t+1} by their actual values. As with (13) or (15), we normalize the equation so that the coefficients of the elements of X_t are equal to one. The approach does not require explicit specification of the process that generates the exogenous variables in the firm's decision process. Consequently, it avoids problems that may arise from (possible) misspecification of that process. It requires, however, that expectations are formed rationally, as we have maintained for the 'true' model (11), (12) and (13). From (11), (13), (15) and (16) it is not difficult to see that the

¹⁰ The following result is useful to simplify the expressions: $(I - M)[I - c(I - M)]^{-1}B^{-1} = -(1/c)\{B^{-1} + (1/c)[C - ((1 - c)/c)B]^{-1}\}$, where $c > 0$ is some constant.

disturbances v_{t*} are homoscedastic and of the form

$$\begin{aligned} v_{t*} &= \varepsilon_{t*} + [A + (2+r)B]^{-1} \{ B[X_{t,1} - X_{t+1}] + E_t h_t - h_t \} \\ &= P_0 \eta_{t+1} + P_1 \varepsilon_{t*} + P_2 \varepsilon_{t+1,*}, \end{aligned}$$

where the P_i 's are matrices of constants.

Hansen (1982) and Hansen and Singleton (1982) have introduced a class of generalized instrumental variable estimators that allow the estimation of the technology parameters from the stochastic Euler equations (16) and (12). These estimators minimize the correlation between any variable known at time t and the residuals of the demand equations. In the case of zero optimization errors, the disturbances in (16) simplify to $v_{t*} = P_0 \eta_{t+1}$ and are hence distributed i.i.d. In this case the procedure suggested by Hansen and Hansen and Singleton reduces to non-linear three-stage least squares. [See also Pindyck and Rotemberg (1983a, b) on this point.] We note that the expected values of X_{t+1} , ΔX_t^2 , X_t , Y_{t+1} , Q_{t+1} , conditional on Y_t , Q_t , X_{t-1} , are linear or linear-quadratic in these variables and a constant. The conditioning set is orthogonal to the disturbances in (12) and (16) and consistent with the possibility of optimization errors. For the present Monte Carlo study, we choose these variables as well as their products and cross-products as instruments. From a practical point of view, the empirical factor demand equations corresponding to approach II are much simpler to implement than those of approach I.

In the present context, we interpret the finite horizon model underlying approach III as an approximation to the infinite horizon model.¹¹ Let $\{X_{t,\tau}^T\}_{\tau=0}^T$ denote the solution to the first-order conditions given by (10). Assuming expectations are formed rationally from (11) and adopting an analogous stochastic specification as in (13) or (15), the empirical quasi-fixed factor demand equations corresponding to approach III are given by

$$\begin{aligned} X_t &= [A + (2+r)B]^{-1} \\ &\times \{ BX_{t,1}^T + (1+r)BX_{t-1} + E_t h_t \} + w_{t*}, \end{aligned} \quad (17)$$

¹¹Prucha and Nadiri (1984) argue that the finite horizon model may yield a more realistic description of empirical data than the infinite horizon model. However, the subsequent results suggest that the discussion which of the two models should be considered the more realistic one is rather academic.

where $w_{t,*}$ denotes the vector of disturbances and $X_{t,1}^T$ is defined implicitly by

$$\begin{aligned}
 -BX_{t,\tau+1}^T + [A + (2+r)B]X_{t,\tau}^T - (1+r)BX_{t,\tau-1}^T &= E_t h_{t+\tau}, \\
 \tau &= 0, 1, \dots, T-1, \\
 [A + rB]X_{t,T}^T - rBX_{t,T-1}^T &= E_t \bar{h}_{t+T},
 \end{aligned} \tag{18}$$

with

$$\begin{aligned}
 E_t h_{t+\tau} &= -[n_{t,0} + \rho_1^{\tau+1}n_{t,1} + \rho_2^{\tau+1}n_{t,2} + \rho_3^{\tau+1}n_{t,3}], \quad \tau = 0, 1, \dots, T-1, \\
 E_t \bar{h}_{t+T} &= -[n_{t,0} + \rho_1^T \bar{n}_{t,1} + \rho_2^T \bar{n}_{t,2} + \rho_3^T n_{t,3}].
 \end{aligned}$$

Expressions for $n_{t,0}$, $n_{t,1}$, $n_{t,2}$ and $n_{t,3}$ are defined in (13). The expressions for $\bar{n}_{t,1}$ and $\bar{n}_{t,2}$ are identical to those for $n_{t,1}$ and $n_{t,2}$ except that the terms $(1+r)/\rho_1$ and $(1+r)/\rho_2$ are replaced by $1+r$. The system is completed by the demand equation for the variable factor (12) and model (11) which generates the exogenous variables in the firm's input decision.¹²

We note that replacing $X_{t,1}^T$ by $X_{t,1}$ in (18) brings us back to the accelerator model (13). This is readily seen from a comparison of (15) and (17) and from the fact that (13) and (15) are equivalent. Such a comparison also shows that $w_{t,*} = \varepsilon_{t,*} + [A + (2+r)B]^{-1}B[X_{t,1} - X_{t,1}^T]$. The disturbances $w_{t,*}$ do not satisfy standard assumptions. This, of course, does not, e.g., preclude questions about the properties of the (pseudo-) maximum likelihood estimator for the unknown model parameters.

Since (18) is linear, we can, in principle, solve explicitly for $X_{t,1}^T$, substitute the explicit solution for $X_{t,1}^T$ into (17) and estimate the unknown technology and expectations parameters from (11), (12) and (17) by standard methods. However, in general this is impractical due to the complexity of the explicit solution for $X_{t,1}^T$. The alternative is to numerically solve (18) N times (where N is the sample size) for $X_{1,1}^T, \dots, X_{N,1}^T$ for each new set of trial parameter values used by the iterative algorithm employed in optimizing the statistical objective function. Various numerical algorithms for the optimization of non-linear objective functions are available. The estimation of non-linear econometric models is generally quite expensive even in standard applications.¹³ Consequently, numerical efficiency of the algorithm is important. To calculate the gradient of the statistical objective function we need to calculate the deriva-

¹²We could have written the quasi-fixed factor demand equations for the finite horizon model as $X_t = X_{t,0}^T + w_{t,*}$. However, we prefer the specification in (17) since it emphasizes the relationship with the second approach. Also, (17) can be used conveniently to find starting values for the estimation algorithm by first estimating (17) with $X_{t,1}^T$ replaced by X_{t+1} .

¹³Details on the numerical costs involved in estimating the dynamic factor demand model that underlies the present study will be given in the next section.

tives of $X_{1,1}^T, \dots, X_{N,1}^T$ with respect to the respective parameters. Suppose $X_{i,1}^T$ depends on L parameters. The numerical calculation of the partial derivatives of $X_{i,1}^T$ with respect to its parameters requires that we evaluate $X_{i,1}^T$ at least $L + 1$ times for slightly different parameter values so that we can approximate the partial derivatives by respective difference quotients; see, e.g., Chow (1983, p. 234). Consequently, an algorithm that computes the gradient numerically would require that we solve the system (18) at each basic iteration step at least $(L + 1) \times N$ times. Since those solutions are expensive, Prucha and Nadiri (1984) designed an algorithm for the estimation of models like (11), (12), (17) and (18) that allow the gradient to be evaluated from analytic expressions in terms of the unsubstituted model.¹⁴ The performance of that algorithm is explored here. In terms of the actual implementation of (11), (12), (17) and (18) we note that (18) may be lengthy but that the expressions involved are not very complicated.

4. Parameter and data design and Monte Carlo results

4.1. Parameter and data design

The design of parameter values and data borrows liberally from an empirical application where we estimated a three-factor demand model, similar to the one considered here, using U.S. manufacturing data. In that application Y represented value added, V denoted hours worked, and X_1 and X_2 corresponded to end-of-period net capital stock of equipment and structures, respectively. Data on Y , V , X_1 and X_2 were normalized by their sample means while prices were constructed conformably.¹⁵ In estimating the model, we imposed restrictions on the technology parameters in (1) such that the long-run elasticities of output with respect to the factor inputs, and hence the long-run scale elasticity, were unity at the point of sample means.

While the design of particular parameter values was guided by the parameter estimates obtained from the empirical analysis, the parameter values chosen are not identical to those estimates. Specifically, changes were implemented so that quite different adjustment speeds for the two quasi-fixed factors were obtained. This allows an analysis of the possible implications of the magnitude of the adjustment speed. Regression parameters and non-zero elements of the disturbance variance-covariance matrix which describe the

¹⁴The algorithm is described exemplarily for the case of full-information maximum likelihood estimation. The algorithm put forward by Berndt et al. (1974) is taken as a starting point.

¹⁵In somewhat more detail: The data on value added, hours worked and total compensation (used to calculate an hourly compensation rate) were taken from the NIPA accounts as published in U.S. Department of Commerce (1981). The data on capital stocks and on current and constant dollar investment (needed to calculate the price variables for the capital goods) were taken from U.S. Department of Commerce (1982).

true structure of the model (11)–(13) were chosen as follows: $\alpha_0 = 0.05$, $\alpha_1 = -0.2$, $\alpha_2 = -0.1$, $\alpha_{11} = 0.4$, $\alpha_{22} = 0.3$, $\alpha_{12} = 0.1$, $\ddot{\alpha}_{11} = 0.7$, $\ddot{\alpha}_{22} = 5.0$, $\alpha_Y = 1.5$, $\alpha_{YY} = 1.0$, $\alpha_{1Y} = -0.5$, $\alpha_{2Y} = -0.4$, $\rho_{01} = 0.15$, $\rho_1 = 0.7$, $\rho_{02} = 0.1$, $\rho_2 = 0.7$, $\rho_{03} = 0.1$, $\rho_3 = 0.98$, $\sigma_{\eta 11} = 0.0003$, $\sigma_{\eta 22} = 0.0003$, $\sigma_{\eta 33} = 0.003$, $\sigma_{\varepsilon 11} = 0.001$, $\sigma_{\varepsilon 21} = 0.0005$, $\sigma_{\varepsilon 22} = 0.0005$, $\sigma_{\varepsilon 31} = 0.00005$, $\sigma_{\varepsilon 32} = 0.00005$, $\sigma_{\varepsilon 33} = 0.00005$. Based on these values for the non-zero elements of the variance–covariance matrix, the disturbances were drawn independently over time from a multivariate normal distribution.¹⁶ For initial values for the quasi-fixed factors, the normalized acquisition prices, and output, respectively, we use the 1950 values employed in the empirical study: $X_{01} = 0.580$, $X_{02} = 0.779$, $Q_{01} = 0.740$, $Q_{02} = 0.506$, $Y_0 = 0.623$. The real discount rate, r , the tax rate, u , and the depreciation rates, δ_1 and δ_2 , were chosen to be 0.05, 0.5, 0.13 and 0.07, respectively. Based on these values, the sample data for V , X_1 , X_2 , Q_1 , Q_2 and Y , for each random drawing of the disturbances, were generated by solving the model dynamically. The implied squared correlation coefficients between fitted and actual observations for ΔV , ΔX_1 , ΔX_2 , Q_1 , Q_2 and Y were found to be approximately 0.9, 0.8, 0.8, 0.85, 0.7 and 0.99, respectively.¹⁷ On average each equation fit quite well. Nevertheless, in some cases the convergence of the estimation procedure required a considerable number of iterations. More details are given below.

The data set corresponding to zero disturbances may be used as a reference set for the Monte Carlo data sets that were actually employed by the Monte Carlo experiment. For that reference set we estimate the following average growth rates for V , X_1 , X_2 , Q_1 , Q_2 and Y , respectively, over fifty sample periods: 3.4, 3.6, 3.0, -0.8 , -0.8 and 3.4 percent. These growth rates are within the range of values observed from empirical data. The values selected for the elements of A and B imply the following values for the elements of the accelerator matrix $M = (m_{ij})$: $m_{11} = 0.50$, $m_{12} = 0.09$, $m_{21} = 0.01$ and $m_{22} = 0.19$. Note that these values imply a considerable spread in the adjustment speeds for the two quasi-fixed factors. The positive cross-adjustment coefficients imply that the two quasi-fixed factors are dynamic substitutes and compensate for each other in the adjustment process. Note that the elements of A and B satisfy all restrictions implied by the convexity of the factor requirement function in X and ΔX . Restrictions on the first derivatives of the factor requirement function have been checked and were comfortably satisfied for all observations in the reference data set. The implied scale elasticities are close to one, both in the neighborhood of sample means and over the entire sample. Of course, this implies long-run output elasticities for V , X_1 and X_2

¹⁶We used subroutine GGNML from the IMSL program library as the principal random number generator.

¹⁷The fitted values for ΔV , ΔX_1 and ΔX_2 were calculated by solving (12) and (13) for given observations on Q_1 , Q_2 and Y .

around unity. The calculation of various short- and long-run price elasticities for V , X_1 and X_2 reveals that all of them possess the correct sign with magnitudes within the range of those reported, e.g., by Morrison and Berndt (1981) and Mohnen, Nadiri and Prucha (1985).

4.2. Monte Carlo results

As noted in section 3, the estimating equations corresponding to approach I are given by (11)–(14), for approach II by (12) and (16) and for approach III by (11), (12), (17) and (18). For both approaches I and III we have used the full-information maximum likelihood estimator (based on a Gaussian log-likelihood function) as our estimation method. We did not impose zero-parameter restrictions on the variance–covariance matrix. The parameter estimates corresponding to approach II have been obtained by applying three-stage least squares to data sets that have been generated identically to those underlying approaches I and III, except that the optimization errors in the quasi-fixed factor demand equations ε_{i*} were set equal to zero. The instruments for the three-stage least squares estimator are listed in the preceding section. As noted previously, the disturbances v_{i*} in (16) are, in this case, distributed i.i.d. and the generalized instrumental variable estimator reduces to the three-stage least squares estimator. We cannot expect to obtain more efficient estimates in the presence of non-zero optimization errors. (Note that ε_t and η_t are independent.) Hence, the results of our comparisons should constitute lower bounds for the gain in efficiency between approaches I and III and approach II. [We note that for the purpose of the present Monte Carlo comparison we have not selected a list of instruments that includes X_t and ΔX_t^2 , since for the basic data set with optimization errors these variables are correlated with the disturbances in (12) and (16).]

For each of the three approaches, samples of size $N = 30$ and $N = 50$ are considered. Since for approach II the bias and dispersion of the estimators remained quite large for $N = 50$, we also considered a sample of size $N = 100$ for that approach. The planning horizons for approach III were chosen as $T + 1 = 5$ and $T + 1 = 10$. Hence a total of nine Monte Carlo experiments were performed. All parameter estimates were calculated using TSP 4.0 with a tolerance for convergence equal to 0.001. The true parameter values were used as starting values. For $N = 30$ the number of iterations needed for convergence ranged from 9 to 84 for approaches I and III and from 10 to 238 iterations for approach II; the median number of iterations needed for convergence was 18 for approach I and III and 24 for approach II. (The number of iterations needed for $N = 50$ or $N = 100$ was typically less than that for $N = 30$.)

Two hundred and fifty Monte Carlo trials were performed for each of the experiments corresponding to approaches I and III and (since there was more variation) five hundred Monte Carlo trials were performed for each of the

Table 1
 Characteristics of the small-sample distribution of the parameter estimators corresponding to approach I.

Parameter	True value	Monte Carlo estimates											
		Sample size $N = 30$					Sample size $N = 50$						
		Median	Int. quant. range	Int. decile range	Median abs. dev.	Tail thickness	Skewness	Median	Int. quant. range	Int. decile range	Median abs. dev.	Tail thickness	Skewness
α_0	0.05	0.045	0.075	0.142	0.037	0.101	1.900	0.055	0.036	0.080	0.018	-0.060	2.220
α_1	-0.2	-0.191	0.045	0.095	0.021	-0.014	2.100	-0.194	0.032	0.058	0.016	-0.007	1.826
α_2	-0.1	-0.090	0.068	0.148	0.033	0.707	2.185	-0.092	0.045	0.084	0.023	0.028	1.850
α_{11}	0.4	0.418	0.111	0.230	0.058	0.510	2.080	0.420	0.084	0.157	0.045	0.027	1.857
α_{22}	0.3	0.329	0.160	0.417	0.082	0.111	2.607	0.314	0.120	0.227	0.055	0.058	1.892
α_{12}	0.1	0.099	0.086	0.181	0.043	-0.004	2.110	0.102	0.060	0.121	0.030	-0.027	2.016
α_{11}	0.7	0.630	0.325	0.596	0.174	0.108	1.833	0.655	0.277	0.494	0.156	0.028	1.782
α_{22}	5.0	5.111	2.124	4.681	0.949	0.093	2.204	4.905	1.659	3.236	0.769	0.144	1.951
α_Y	1.5	1.477	0.167	0.310	0.078	-0.027	1.851	1.482	0.091	0.164	0.051	-0.007	1.811
α_{YY}	1.0	1.067	0.426	0.943	0.199	0.015	2.211	1.043	0.224	0.450	0.103	0.035	2.011
α_{1Y}	-0.5	-0.513	0.171	0.360	0.086	-0.086	2.112	-0.521	0.110	0.242	0.055	-0.003	2.214
α_{2Y}	-0.4	-0.435	0.231	0.516	0.100	-0.066	2.228	-0.420	0.145	0.297	0.068	-0.037	2.044
ρ_{01}	0.15	0.158	0.041	0.080	0.023	-0.037	1.924	0.158	0.038	0.072	0.020	-0.010	1.878
ρ_1	0.7	0.684	0.079	0.151	0.041	0.015	1.906	0.684	0.074	0.139	0.039	0.021	1.893
ρ_{02}	0.1	0.109	0.034	0.068	0.019	0.029	1.966	0.108	0.031	0.054	0.015	0.032	1.745
ρ_2	0.7	0.674	0.104	0.191	0.050	-0.043	1.821	0.676	0.095	0.155	0.043	-0.025	1.627
ρ_{03}	0.1	0.110	0.044	0.078	0.022	-0.017	1.754	0.105	0.031	0.054	0.016	0.002	1.710
ρ_3	0.98	0.975	0.025	0.043	0.013	0.021	1.714	0.977	0.012	0.024	0.007	0.030	2.063

Table 2a
 Characteristics of the sample-sample distribution of the parameter estimators corresponding to approach II.

Parameter	True value	Monte Carlo estimates											
		Sample size $N = 30$						Sample size $N = 50$					
		Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	Tail thickness	Median	Int. quant. Range	Int. decile range	Median abs. dev.	Skewness	Tail thickness
α_0	0.05	0.034	0.084	0.145	0.043	0.034	1.738	0.044	0.046	0.090	0.025	0.063	1.957
α_1	-0.2	-0.240	0.039	0.085	0.041	0.012	2.204	-0.208	0.027	0.054	0.016	0.022	1.961
α_2	-0.1	-0.123	0.068	0.160	0.044	-0.012	2.344	-0.108	0.032	0.067	0.019	0.042	2.077
α_{11}	0.4	0.139	0.144	0.288	0.261	-0.055	2.005	0.292	0.087	0.189	0.109	0.038	2.166
α_{22}	0.3	-0.109	0.196	0.490	0.415	-0.124	2.501	0.076	0.123	0.259	0.225	-0.042	2.111
α_{12}	0.1	-0.055	0.172	0.348	0.167	0.013	2.015	0.040	0.095	0.193	0.073	0.023	2.025
$\tilde{\alpha}_{11}$	0.7	1.432	0.477	1.093	0.732	0.103	2.289	1.257	0.330	0.658	0.557	0.103	1.994
$\tilde{\alpha}_{22}$	5.0	6.216	1.894	5.075	1.349	0.131	2.680	6.162	1.205	2.501	1.193	0.078	2.075
α_{γ}	1.5	1.570	0.160	0.325	0.099	0.007	2.026	1.513	0.075	0.143	0.041	0.016	1.906
$\alpha_{\gamma\gamma}$	1.0	-0.036	0.551	1.125	1.056	-0.050	2.043	0.513	0.318	0.716	0.503	0.066	2.251
$\alpha_{1\gamma}$	-0.5	-0.076	0.209	0.437	0.425	0.091	2.095	-0.324	0.138	0.312	0.180	-0.041	2.261
$\alpha_{2\gamma}$	-0.4	0.173	0.318	0.714	0.586	0.046	2.247	-0.107	0.194	0.401	0.298	-0.014	2.070

Table 2b

Characteristics of the small-sample distribution of the parameter estimators corresponding to approach II.

Parameter	True value	Monte Carlo estimates					Tail thickness
		Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	
		Sample size $N = 100$					
α_0	0.05	0.048	0.030	0.061	0.016	-0.034	2.038
α_1	-0.2	-0.197	0.023	0.045	0.012	-0.002	1.962
α_2	-0.1	-0.100	0.027	0.055	0.013	0.015	2.055
α_{11}	0.4	0.386	0.068	0.134	0.038	0.056	1.958
α_{22}	0.3	0.227	0.094	0.193	0.084	0.107	2.046
α_{12}	0.1	0.092	0.068	0.151	0.036	0.042	2.226
$\ddot{\alpha}_{11}$	0.7	1.004	0.148	0.303	0.304	0.070	2.053
$\ddot{\alpha}_{22}$	5.0	5.849	0.853	1.600	0.854	0.068	1.874
α_γ	1.5	1.498	0.060	0.110	0.029	-0.034	1.844
$\alpha_{\gamma\gamma}$	1.0	0.894	0.292	0.594	0.173	0.115	2.037
$\alpha_{1\gamma}$	-0.5	-0.476	0.129	0.259	0.068	-0.052	2.001
$\alpha_{2\gamma}$	-0.4	-0.311	0.167	0.326	0.113	-0.132	1.945

experiments corresponding to approach II. In tables 1-3 we give various characteristics of the small-sample distribution of the estimators. Results concerning the existence of small-sample moments for the full-information maximum likelihood and the three-stage least squares estimator are, to the best of our knowledge, not available for the models considered in this study.¹⁸ We therefore base our characterization of the small-sample distributions on fractiles rather than moments.

In particular, we report Monte Carlo estimates for the median, the interquantile range, the interdecile range and the median absolute deviation. The median is our measure of location; we refer to the difference between the median and the true parameter value as the (median) bias. The interquantile range and the interdecile range are measures of dispersion. They are defined as, respectively, the difference between the 0.75- and 0.25-quantile and the difference between the 0.9- and 0.1-decile. The median absolute deviation (from the true parameter value) is defined as the median of the absolute difference between the estimator and the true parameter value. It is a combined measure of bias and dispersion. (In case the bias is zero and the distribution is symmetric the median absolute deviation equals one half of the interquantile range.) We report, furthermore, Monte Carlo estimates for

¹⁸For the general linear simultaneous equation system satisfying classical assumptions the full-information maximum likelihood estimator is known to possess no finite integral moments; the three-stage least squares estimator is known to possess finite moments up to the order κ , where κ is the degree of overidentification. For more details, see, e.g., Mariano (1982) and Phillips (1984); compare also Hendry (1984).

Table 3a
 Characteristics of the small-sample distribution of the parameter estimators corresponding to approach III: five-period planning horizon.

Parameter	True value	Monte Carlo estimates											
		Sample size $N = 30$					Sample size $N = 50$						
		Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	Tail thickness	Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	Tail thickness
α_0	0.05	0.046	0.079	0.145	0.038	0.097	1.843	0.055	0.036	0.081	0.019	-0.048	2.245
α_1	-0.2	-0.193	0.045	0.093	0.021	-0.002	2.079	-0.196	0.031	0.059	0.015	-0.033	1.866
α_2	-0.1	-0.106	0.043	0.094	0.021	0.087	2.210	-0.107	0.027	0.050	0.015	-0.032	1.869
α_{11}	0.4	0.415	0.113	0.234	0.058	0.029	2.076	0.417	0.087	0.156	0.044	0.004	1.805
α_{22}	0.3	0.226	0.128	0.323	0.099	0.133	2.523	0.218	0.085	0.167	0.088	0.010	1.966
α_{12}	0.1	0.099	0.086	0.181	0.042	-0.014	2.101	0.101	0.064	0.125	0.032	-0.021	1.941
α_{11}	0.7	0.615	0.316	0.573	0.175	0.120	1.813	0.641	0.269	0.474	0.157	0.031	1.765
α_{22}	5.0	5.125	2.048	4.720	1.013	0.083	2.305	4.924	1.714	3.178	0.785	0.157	1.854
α_{γ}	1.5	1.493	0.147	0.273	0.075	-0.026	1.852	1.500	0.076	0.142	0.039	-0.034	1.865
$\alpha_{\gamma\gamma}$	1.0	0.954	0.386	0.835	0.199	0.030	2.150	0.941	0.215	0.441	0.130	0.010	2.053
$\alpha_{1\gamma}$	-0.5	-0.506	0.169	0.370	0.087	-0.110	2.192	-0.515	0.116	0.241	0.054	-0.008	2.088
$\alpha_{2\gamma}$	-0.4	-0.320	0.203	0.436	0.126	-0.118	2.146	-0.315	0.117	0.242	0.100	-0.061	2.060
ρ_{01}	0.15	0.159	0.042	0.079	0.022	-0.050	1.903	0.158	0.038	0.072	0.020	-0.003	1.888
ρ_1	0.7	0.683	0.082	0.150	0.040	0.018	1.833	0.683	0.074	0.141	0.040	0.039	1.943
ρ_{02}	0.1	0.109	0.034	0.068	0.018	0.034	1.993	0.109	0.030	0.054	0.015	0.043	1.773
ρ_2	0.7	0.672	0.106	0.190	0.049	-0.033	1.796	0.676	0.092	0.153	0.043	-0.022	1.659
ρ_{03}	0.1	0.111	0.045	0.078	0.022	-0.022	1.750	0.106	0.032	0.056	0.016	-0.012	1.736
ρ_3	0.98	0.974	0.025	0.045	0.012	0.030	1.751	0.977	0.012	0.024	0.007	0.010	1.977

Table 3b
 Characteristics of the small-sample distribution of the parameter estimators corresponding to approach III: ten-period planning horizon.

Parameter	True value	Monte Carlo estimates											
		Sample size $N = 30$					Sample size $N = 50$						
		Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	Tail thickness	Median	Int. quant. range	Int. decile range	Median abs. dev.	Skewness	Tail thickness
α_0	0.05	0.045	0.075	0.142	0.037	0.110	1.903	0.055	0.036	0.080	0.018	-0.054	2.231
α_1	-0.2	-0.191	0.045	0.095	0.021	-0.014	2.121	-0.194	0.032	0.058	0.016	-0.005	1.814
α_2	-0.1	-0.093	0.064	0.145	0.033	0.080	2.251	-0.095	0.043	0.081	0.022	0.019	1.867
α_{11}	0.4	0.418	0.110	0.230	0.058	0.049	2.084	0.420	0.085	0.157	0.045	0.030	1.849
α_{22}	0.3	0.319	0.160	0.413	0.080	0.113	2.581	0.303	0.120	0.227	0.057	0.046	1.897
α_{12}	0.1	0.100	0.087	0.183	0.043	-0.005	2.094	0.102	0.061	0.122	0.030	-0.030	2.018
α_{11}	0.7	0.631	0.325	0.596	0.174	0.103	1.832	0.654	0.278	0.495	0.156	0.030	1.785
α_{22}	5.0	5.049	2.106	4.699	0.983	0.097	2.217	4.866	1.629	3.203	0.771	0.136	1.966
α_{γ}	1.5	1.478	0.167	0.304	0.077	-0.022	1.820	1.485	0.088	0.161	0.049	-0.005	1.822
$\alpha_{\gamma\gamma}$	1.0	1.054	0.425	0.932	0.201	0.020	2.196	1.034	0.223	0.448	0.099	0.026	2.009
$\alpha_{1\gamma}$	-0.5	-0.514	0.173	0.360	0.086	-0.081	2.087	-0.521	0.110	0.243	0.055	-0.010	2.217
$\alpha_{2\gamma}$	-0.4	-0.422	0.230	0.511	0.105	-0.078	2.219	-0.408	0.143	0.289	0.068	-0.024	2.023
ρ_{01}	0.15	0.158	0.041	0.080	0.022	-0.037	1.925	0.158	0.038	0.072	0.020	-0.011	1.882
ρ_1	0.7	0.684	0.079	0.151	0.041	0.015	1.905	0.684	0.074	0.139	0.040	0.019	1.889
ρ_{02}	0.1	0.109	0.035	0.068	0.019	0.028	1.955	0.108	0.031	0.054	0.015	0.035	1.760
ρ_2	0.7	0.674	0.105	0.191	0.051	-0.042	1.820	0.676	0.096	0.155	0.043	-0.024	1.622
ρ_3	0.1	0.110	0.044	0.078	0.022	-0.012	1.750	0.105	0.032	0.054	0.016	0.003	1.706
ρ_3	0.98	0.975	0.025	0.043	0.012	0.028	1.711	0.977	0.012	0.024	0.007	0.030	2.037

measures of the skewness and the tail thickness of the small-sample distribution of the parameter estimators. Define the quantile-midpoint as the average between the 0.75- and 0.25-quantile. Then our measure of skewness is defined as the ratio of the difference between the quantile-midpoint and the median to the interquartile range. Our measure for the tail thickness is defined as the ratio of the interdecile range to the interquartile range.¹⁹

We note that from the results reported it is possible to recover the underlying 0.75- and 0.25-quantiles and 0.9- and 0.1-deciles. Our measure of skewness will be zero for symmetric distributions, and positive [negative] for distributions which are more spread out to the right [left] of the median. Comparatively larger values for our measure of the tail thickness indicate that comparatively more mass is concentrated in the tails of the distribution. For a normal distribution with mean μ and variance σ^2 the median will clearly equal μ and our measure of skewness will be zero. The interquartile and interdecile range will be, respectively, 1.35σ and 2.57σ ; our measure for the tail thickness will be 1.9. We note that asymptotically all estimators considered are normally distributed.

As expected, the median absolute difference, the interquartile range and the interdecile range decrease in all cases as the sample size increases. No general observations are apparent for the skewness of the sample distributions. Some are left skewed while others are right skewed. On average the parameter distributions exhibit thinner tails as the sample size increases. For samples of size $N = 30$ the maximum observed value for our measure of the tail thickness is 2.68; for $N = 50$ the maximum observed value is 2.26. We emphasize that all three approaches estimate the parameters determining the dynamics of the system, i.e., the accelerator matrix, and the parameters associated with the output variable, with less precision than the remaining parameters. Multicollinearity due to the underlying growth patterns in output and the quasi-fixed factors may be the source of the difficulties with the estimation of the output parameters. The parameter that seems to be the most problematic for estimation is the adjustment cost coefficient of the slow adjusting quasi-fixed factor, $\bar{\alpha}_{22}$. As a partial explanation we note that, for the parameter values considered, comparatively large changes in $\bar{\alpha}_{22}$ induce only comparatively small

¹⁹In more detail: Consider the estimator, say, \hat{a} for the parameter, say, a^0 . For any number λ , $0 < \lambda < 1$, the λ -fractile of the population is defined as the value a_λ such that $\Pr\{\hat{a} \leq a_\lambda\} = \lambda$. Clearly, the median of \hat{a} corresponds to $a_{0.5}$. The 0.75-, 0.25-quantiles and 0.9-, 0.1-deciles correspond to $a_{0.75}$, $a_{0.25}$, $a_{0.9}$ and $a_{0.1}$, respectively. The median absolute deviation corresponds to the median of $|\hat{a} - a^0|$. The interquartile range and interdecile range are defined as $a_{0.75} - a_{0.25}$ and $a_{0.9} - a_{0.1}$, respectively. Our measures for the skewness and tail thickness are given by $\{(a_{0.75} + a_{0.25})/2 - a_{0.5}\}/(a_{0.75} - a_{0.25})$ and $(a_{0.9} - a_{0.1})/(a_{0.75} - a_{0.25})$, respectively. For a further discussion of the above measures, see, e.g., Elashoff and Elashoff (1978), Greenberg (1978), Keller (1978) and the literature cited therein. The Monte Carlo estimates of the λ -fractile of the population are based on the corresponding order statistics; see, e.g., Greenberg (1978).

changes in the accelerator coefficients m_{ij} ($i, j = 1, 2$). For example, a change in $\ddot{\alpha}_{22}$ from the assumed value of 5 to 6 [4] leads to a change in m_{22} from 0.19 to 0.17 [0.21]. In contrast, a change in $\ddot{\alpha}_{11}$ from the assumed value of 0.7 to 0.8 [0.6] leads to a change in m_{11} from 0.50 to 0.48 [0.53].

A comparison of the results from approaches I and III shows that the respective sampling distributions are nearly identical when a ten-period planning horizon is assumed for the latter approach. Moreover, even when approach III is based on only a five-period planning horizon we observe the respective sampling distributions to be very close. This is offered as further evidence that the finite horizon model underlying approach III approximates the infinite horizon model very closely even for planning horizons of moderate length. A detailed inspection of the Monte Carlo results shows that this close approximation not only holds in terms of sampling distribution but also in terms of the actual parameter estimates. The results also suggest that a longer planning horizon is needed for close approximation when a slow adjusting quasi-fixed factor is included.

In comparing the results for approaches I and II, substantial differences in the respective sampling distributions emerge. We note that for small samples the estimator associated with approach II show considerable biases for some of the parameters. This is particularly true for the parameters that determine the dynamics of the factor demand system and some of the parameters associated with the output variable. The sampling distributions associated with approach II are also, on average, characterized by a considerably larger dispersion than those associated with approach I. This comparison demonstrates that by incorporating the full solution of the firm's optimizing problem substantial improvements in terms of statistical efficiency are possible.

All computations have been performed with TSP 4.0 on an IBM 4341. For approach III we have modified TSP 4.0 somewhat and added FORTRAN subroutines to calculate the plan values $X_{t,1}^T$ and their derivatives with respect to the model parameters. As described in Prucha and Nadiri (1984) those derivatives were calculated from expressions obtained by differentiating (18) implicitly. For each approach we have estimated the CPU seconds required for one iteration. In all cases these estimates correspond to the last iteration before convergence. Results corresponding to samples of size $N = 30$ and $N = 50$ are presented in table 4. As expected, approach II is by far less expensive than the other two approaches. Approach III is the most expensive one. However, its computational cost remains, in comparison to approach I, within reasonable bounds. Computational cost increases quite steeply with the planning horizon. This may be avoided, in part, by obtaining preliminary estimates from a model with a short planning horizon which can then be used as starting values for a model with a longer planning horizon.

Table 4
CPU seconds per iteration.

Approach I	Approach II	Approach III ($T + 1 = 5$)	Approach III ($T + 1 = 10$)
		Sample size $N = 30$	
2.05	0.50	2.25	3.10
		Sample size $N = 50$	
3.30	0.75	3.90	5.80

5. Conclusion and suggestions for future research

This paper has analyzed the differences in statistical and computational efficiency among three alternative approaches to the estimation of dynamic factor demand models. The results provide evidence that: (i) considerable gains in statistical efficiency can be obtained by incorporating a full solution of the firm's optimizing problem, either analytically (approach I) or numerically (approach III); (ii) parameters estimates from approach II may be considerably biased (in small samples); (iii) estimation of the parameters determining the dynamics of the factor demand system is especially difficult; (iv) the finite horizon model of approach III approximates the infinite horizon model very closely even for planning horizons of moderate size; (v) approach II is the easiest to implement and computationally the least involved. As expected, approach III is computationally the most expensive. However, since a moderate planning horizon can be chosen and the gradient of the statistical objective function is evaluated from analytic expressions, the computational cost of approach III remains reasonable. The price that has to be paid by approach III for not requiring the knowledge of the explicit analytic solution of the firm's optimum problem seems quite moderate.

Since differences among the approaches exist also in terms of the generality with which technology and expectations formation can be specified and in terms of informational requirements, our results cannot imply a uniform superiority of one approach over the others. In particular, as remarked earlier, only approaches II and III are applicable in cases where the technology is not a linear-quadratic technology. In such situations, the stochastic closed-loop feedback control policy employed by approach II and the certainty equivalence feedback control policy employed by approach III will not yield identical input decisions. However, because of the results on first-order certainty equivalence by Malinvaud (1969), we expect input decisions to be similar, especially when the uncertainty concerning the exogenous variables and the degree of non-linearity are moderate. This issue as well as the

empirical question of which of the two policies better describes actually observed data warrants further investigation.

We note that our analysis has assumed a correctly specified model. Therefore one avenue of future research concerns the robustness of approaches I and III to misspecification in the model generating the exogenous variables; another issue would be the robustness of approach II to nonrational expectations. For approaches I and III one might also investigate how precisely the technology and expectations parameters can be estimated from the technology and quasi-fixed factor demand equations alone. A further analysis of the small sample properties of the generalized instrumental variables estimators used in approach II also seems of interest.

Finally, although the results of the present study are based on what we believe to be a representative example, the robustness of those results using other specifications, parameter constellations and instruments needs to be checked. In such a future study control variate methods may be used to reduce the computational cost and results may be summarized in terms of response functions [see Hendry (1984)]. Preliminary simulation results based on a small number of iterations suggest that the main conclusions of this paper remain robust against alternative parameter constellations.²⁰

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²⁰It should be noted that the example explored in this paper includes one fast and one slow adjusting quasi-fixed factor.

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