Journal of Econometrics 60 (1994) 23–63. North-Holland

Generic uniform convergence and equicontinuity concepts for random functions

An exploration of the basic structure*

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Received July 1990, final version received June 1992

Equicontinuity-type concepts for random functions, which are important for establishing convergence results for such functions, have increasingly been used in the econometrics literature. In this paper we define and discuss various equicontinuity-type concepts for random functions and employ those concepts to provide sufficient conditions for uniform convergence and, in particular, for uniform laws of large numbers. Furthermore, we clarify the differences and similarities between uniform laws of large numbers based on pointwise and local laws of large numbers given in the recent literature as they relate to differences in the employed equicontinuity-type concepts.

Key words: Uniform convergence; Uniform laws of large numbers; Stochastic equicontinuity

1. Introduction

Equicontinuity-type concepts for random functions are basic notions that facilitate convergence results for such functions. Those concepts have been used widely in the statistics and probability literature [see, e.g., Pollard (1984, 1989) and Alexander (1987) for some recent references]. Equicontinuity-type concepts for random functions have recently also been utilized more widely in the

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*We would like to thank the editor Arnold Zellner and the referees for helpful comments. Any remaining errors are our responsibility.

econometrics literature. In particular, and as will be explained in more detail below, all of the recent uniform convergence results for random functions in Andrews (1987, 1989c), Bierens (1981, 1989), Newey (1989), and Pötscher and Prucha (1986a, b, 1989a, b) can be viewed as having been obtained by verifying implicitly or explicitly some equicontinuity-type conditions for the underlying random functions. Furthermore, equicontinuity-type concepts for random functions have also been used in Andrews (1989a, b) and Pötscher and Prucha (1991a, b).

One goal of this paper is to clarify some aspects of these recent uniform convergence results as they relate to differences in the employed equicontinuity-type conditions, and to extend and consolidate these results. The uniform convergence results in Andrews (1987), Bierens (1981, 1989), and Pötscher and Prucha (1986a, b, 1989a, b) were obtained from a verification of the so-called first-moment continuity condition, which is actually a first-moment equicontinuity-type condition, and from local laws of large numbers. (The term local laws of large numbers refers to laws of large numbers that hold for certain local bracketing functions.) Alternatively, Newey (1989) and Andrews (1989c) derived uniform convergence results from the verification of a 'stochastic' equicontinuity-type condition, which is actually a 'stochastic' *uniform* equicontinuity-type condition, and from pointwise laws of large numbers.[1] These results essentially use a stochastic version of Ascoli–Arzelà's theorem. We show below that given a standard domination condition the first-moment equicontinuity-type condition used by the approach based on local laws of large numbers and a suitably defined 'stochastic' equicontinuity-type condition are in fact equivalent. We show furthermore that, given a standard domination condition, a suitably defined first-moment *uniform* equicontinuity-type condition and the 'stochastic' *uniform* equicontinuity-type conditions used by the approach based on pointwise laws of large numbers are in fact again equivalent.[2] Therefore it is the difference in the degree of the *uniformity* in the employed equicontinuity-type conditions that represents the essential difference between the two approaches.[3]

Except for Andrews (1989c) all uniform convergence results in the literature cited above use a compact parameter space. The latter paper shows that totally boundedness of the parameter space suffices. (For the approach based on local laws of large numbers the maintained assumptions have to be appropriately

[1]The term pointwise laws of large numbers refers here to laws of large numbers that hold at all points of the parameter space.

[2]Actually, also the former approach requires some degree of uniformity (which however need not be postulated explicitly if the parameter space is compact). However, as discussed in more detail later, the degree of uniformity required by the former approach is much less than that required by the latter approach.

[3]The uniform equicontinuity-type conditions used in Andrews (1989c) and Newey (1989) only differ in inessential details in regard to the derivation of uniform convergence results.

modified.) While a weakening of the compactness assumption may be useful from a practical point of view, we also show below that the uniform convergence result on a totally bounded parameter space in Andrews (1989c) is only apparently more general as *any* uniform convergence result on a totally bounded parameter space can be deduced from a uniform convergence result on a compact parameter space by an extension argument.

Equicontinuity-type concepts are not only of interest for establishing uniform convergence results. A simple but important application of equicontinuity-type concepts arises if we want to establish that the difference between the random functions evaluated at some estimator and at the probability limit of that estimator converges to zero. To illustrate this let Q_n denote some sequence of random functions which are indexed by some parameter $\theta \in \Theta$, let $\hat{\theta}_n$ denote some estimator for $\bar{\theta}_n$ with $\hat{\theta}_n - \bar{\theta}_n \to 0$ i.p. as $n \to \infty$, and let P denote the probability law. (For simplicity assume for this illustration that Θ is a subset of Euclidean space.) Clearly for every $\varepsilon > 0$ and $\delta > 0$ we have

$$\begin{aligned} & \mathrm{P}(|Q_n(\hat{\theta}_n) - Q_n(\bar{\theta}_n)| > \varepsilon) \\ & \leq \mathrm{P}(|Q_n(\hat{\theta}_n) - Q_n(\bar{\theta}_n)| > \varepsilon, |\hat{\theta}_n - \bar{\theta}_n| < \delta) \\ & \quad + \mathrm{P}(|Q_n(\hat{\theta}_n) - Q_n(\bar{\theta}_n)| > \varepsilon, |\hat{\theta}_n - \bar{\theta}_n| \geq \delta) \\ & \leq \mathrm{P}(\sup_{\theta' \in \Theta} \sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) + \mathrm{P}(|\hat{\theta}_n - \bar{\theta}_n| \geq \delta). \end{aligned}$$

Consequently, $Q_n(\hat{\theta}_n) - Q_n(\bar{\theta}_n) \to 0$ i.p. as $n \to \infty$, given Q_n satisfies the equicontinuity-type condition $\overline{\lim}_{n\to\infty} \mathrm{P}(\sup_{\theta' \in \Theta} \sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) \to 0$ as $\delta \to 0$ for every $\varepsilon > 0$. If $\bar{\theta}_n \equiv \bar{\theta}$, the less stringent equicontinuity-type condition $\overline{\lim}_{n\to\infty} \mathrm{P}(\sup_{|\theta - \theta'| < \delta} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) \to 0$ as $\delta \to 0$ for every $\varepsilon > 0$ and $\theta' = \bar{\theta}$ suffices, as is easily seen. Clearly, if we are concerned with a.s. or L_p convergence, then we need corresponding a.s. or L_p equicontinuity-type concepts.

The above discussion indicates that we are confronted with a manifold of equicontinuity-type concepts, which are useful in different contexts and which differ in their degree of uniformity and whether those concepts are defined as i.p., a.s., or L_p statements. Given this manifold of different but related equicontinuity-type conditions, it seems of interest to explore their relationships and differences in more detail. Hence, another goal of this paper is to carefully define and distinguish between different equicontinuity-type concepts for random functions and to analyze the relationship between those concepts.

Section 2 defines respective equicontinuity-type concepts for random functions and establishes various implications and certain equivalencies between these notions. In section 3 we give two theorems that provide basic conditions under which pointwise and local convergence results can be transferred into uniform ones. One of those theorems is a stochastic version of Ascoli–Arzelà's Theorem. Section 4 applies the results of sections 2 and 3 to the derivation of uniform laws of large numbers, i.e., to the important special case of sample averages of random functions. In that section we also present several sets of sufficient conditions for the existence of uniform laws of large numbers and discuss their relationship to the results in Andrews (1987, 1989c), Newey (1989), and Pötscher and Prucha (1986b, 1989a, b). Section 5 shows that any uniform convergence result on a totally bounded parameter space can always be reduced to a uniform convergence result on a compact parameter space. Section 6 contains some illustrative counterexamples concerning uniform laws of large numbers and the relationship of the respective equicontinuity-type concepts. Proofs are given in the appendix.

2. Equicontinuity concepts for random functions

2.1. Definitions of and relationships between equicontinuity concepts

Let (Θ, ρ) be a (nonempty) metric space, let $(\Omega, \mathscr{A}, P)$ be a probability space and let $Q_n\colon \Omega \times \Theta \to \mathbb{R}$ be a sequence of functions that are measurable in their first argument. The dependence of Q_n on $\omega \in \Omega$ will frequently be suppressed in the notation below. All suprema and infima over subsets of Θ of random functions used below are assumed to be (P-a.s.) measurable.[4] With $B(\theta', \delta)$ we denote the open ball $\{\theta \in \Theta\colon \rho(\theta, \theta') < \delta\}$. We now define various equicontinuity-type concepts for Q_n. Definition 2.1 presents equicontinuity-type concepts for a sequence of random functions at a given parameter value. As mentioned in the Introduction, uniform versions of equicontinuity-type concepts of a sequence of random functions are needed to establish certain uniform convergence results. Definitions 2.2 and 2.3 present two alternative formulations of such uniform versions of equicontinuity-type concepts, which facilitate two alternative approaches to uniform convergence results.

Definition 2.1. Q_n is asymptotically $\mathrm{L_p}$ equicontinuous ($\mathrm{AL_pEC}$) at $\theta' \in \Theta$ for $p > 0$ iff

$$\overline{\lim_{n\to\infty}}\ \mathrm{E} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.1a}$$

[4] For sufficient conditions see, e.g., Pollard (1984, app. C) and Pötscher and Prucha (1989b, lemma A2). We note that some of the results below can also be shown to hold without this measurability condition, given their proper formulation in terms of outer probabilities.

Q_n is asymptotically L_0 equicontinuous (AL_0EC) at $\theta' \in \Theta$ iff for every $\varepsilon > 0$

$$\overline{\lim_{n \to \infty}} \, P\left(\sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon \right) \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.1b}$$

Q_n is a.s. asymptotically equicontinuous (a.s.AEC) at $\theta' \in \Theta$ iff

$$\overline{\lim_{n \to \infty}} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{2.1c}$$

If (2.1a) [(2.1b)] {(2.1c)} holds with $\overline{\lim}_{n \to \infty}$ replaced by $\sup_n$, then Q_n is said to be L_p equicontinuous (L_pEC) at θ' [L_0 equicontinuous (L_0EC) at θ'] {a.s. equicontinuous (a.s.EC) at θ'}. If any of the above properties holds for all $\theta' \in \Theta$ (with a *common* exceptional null set for the a.s. case), then we say that this property holds on Θ.

Definition 2.2. Q_n is uniformly asymptotically L_p equicontinuous (UAL_pEC) on Θ for $p > 0$ iff

$$\sup_{\theta' \in \Theta} \overline{\lim_{n \to \infty}} \, E \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.2a}$$

Q_n is uniformly asymptotically L_0 equicontinuous (UAL_0EC) on Θ iff for every $\varepsilon > 0$

$$\sup_{\theta' \in \Theta} \overline{\lim_{n \to \infty}} \, P\left(\sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon \right) \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.2b}$$

Q_n is a.s. uniformly asymptotically equicontinuous (a.s.UAEC) on Θ iff

$$\sup_{\theta' \in \Theta} \overline{\lim_{n \to \infty}} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{2.2c}$$

If (2.2a) [(2.2b)] {(2.2c)} holds with $\overline{\lim}_{n \to \infty}$ replaced by $\sup_n$, then Q_n is said to be uniformly L_p equicontinuous (UL_pEC) on Θ [uniformly L_0 equicontinuous (UL_0EC) on Θ] {a.s. uniformly equicontinuous (a.s.UEC) on Θ}.

Definition 2.3. Q_n is asymptotically L_p uniformly equicontinuous (AL_pUEC) on Θ for $p > 0$ iff

$$\overline{\lim_{n \to \infty}} \, E \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.3a}$$

Q_n is asymptotically L_0 uniformly equicontinuous (AL_0UEC) on Θ iff for every $\varepsilon > 0$

$$\overline{\lim_{n\to\infty}}\ P\left(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon\right) \to 0 \quad \text{as} \quad \delta \to 0; \tag{2.3b}$$

Q_n is a.s. asymptotically uniformly equicontinuous (a.s.AUEC) on Θ iff

$$\overline{\lim_{n\to\infty}}\ \sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)} |Q_n(\theta) - Q_n(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{2.3c}$$

If (2.3a) [(2.3b)] {(2.3c)} holds with $\overline{\lim}_{n\to\infty}$ replaced by $\sup_n$, then Q_n is said to be L_p uniformly equicontinuous (L_pUEC) on Θ [L_0 uniformly equicontinuous (L_0UEC) on Θ] {a.s. uniformly equicontinuous (a.s.UEC) on Θ}.[5]

In the literature some of the above distinct concepts have been referred to by one and the same name: for example, AL_0EC is called stochastic equicontinuity in Andrews (1989a, b). Andrews (1989c) and Pollard (1989) on the other hand use stochastic equicontinuity to refer to AL_0UEC, which – as shown below – is much stronger than AL_0EC even for compact Θ. For compact Θ, a variant of AL_0UEC was called uniform stochastic equicontinuity by Newey (1989). By introducing the above definitions for equicontinuity of random functions we hope to avoid this clash of terminology in the literature. Furthermore, by distinguishing between asymptotic and nonasymptotic equicontinuity concepts we achieve that the definitions adopted here are in case the functions are not random in accordance with standard definitions of equicontinuity and uniform equicontinuity.

The above equicontinuity-type conditions essentially control the size of the modulus of continuity or uniform continuity. The ability to control the size of such moduli has proven to be essential for deriving convergence results for stochastic processes as it basically implies tightness of the sequence of stochastic processes; see, e.g., Billingsley (1968, ch. 2, 3) and Pollard (1984).

The following remarks explore positive and negative results regarding the existence of implications between the respective equicontinuity concepts. The negative results are based on counterexamples collected in section 6. For nonrandom functions it is well-known that equicontinuity and uniform equicontinuity coincide if Θ is compact. The remarks explore, among other things, to what extent a generalization of this result is possible for random functions.

[5]Of course, if $\overline{\lim}_{n\to\infty}$ is replaced by $\sup_n$, then conditions (2.2c) and (2.3c) coincide.

Remark 2.1. (i) The following implications among the equicontinuity concepts are obvious:[6]

$$\begin{array}{ccccc}
\mathrm{AL_pEC}\ [\mathrm{L_pEC}] & \Rightarrow & \mathrm{AL_0EC}\ [\mathrm{L_0EC}] & \Leftarrow & \mathrm{a.s.AEC}\ [\mathrm{a.s.EC}], \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{UAL_pEC}\ [\mathrm{UL_pEC}] & \Rightarrow & \mathrm{UAL_0EC}\ [\mathrm{UL_0EC}] & \Leftarrow & \mathrm{a.s.UAEC}\ [\mathrm{a.s.UEC}], \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathrm{AL_pUEC}\ [\mathrm{L_pUEC}] & \Rightarrow & \mathrm{AL_0UEC}\ [\mathrm{L_0UEC}] & \Leftarrow & \mathrm{a.s.AUEC}\ [\mathrm{a.s.UEC}].
\end{array}$$

(ii) If Θ is compact, we have furthermore: $\mathrm{AL_pEC}$ [$\mathrm{L_pEC}$] on $\Theta \Leftrightarrow \mathrm{UAL_pEC}$ [$\mathrm{UL_pEC}$], $p \geq 0$, and a.s.AEC [a.s.EC] on $\Theta \Leftrightarrow$ a.s.UAEC [a.s.UEC] $\Leftrightarrow$ a.s.AUEC [a.s.UEC]. If (Θ, ρ) is totally bounded, we only have: a.s.UAEC [a.s.UEC] $\Leftrightarrow$ a.s.AUEC [a.s.UEC].[7] For a proof of these results see Lemma A.2.

(iii) We emphasize that – in contrast to the a.s. case – the implications $\mathrm{UAL_pEC}$ [$\mathrm{UL_pEC}$] $\Rightarrow \mathrm{AL_pUEC}$ [$\mathrm{L_pUEC}$] (and hence the implications $\mathrm{AL_pEC}$ [$\mathrm{L_pEC}$] on $\Theta \Rightarrow \mathrm{AL_pUEC}$ [$\mathrm{L_pUEC}$]), $p \geq 0$, do *not* hold in general, *even if* Θ *is compact* (and $Q_n \equiv Q$); see Example 1 in section 6.[8]

(iv) If Θ is not compact, then the implications $\mathrm{AL_pEC}$ [$\mathrm{L_pEC}$] on $\Theta \Rightarrow \mathrm{UAL_pEC}$ [$\mathrm{UL_pEC}$], $p \geq 0$, as well as a.s.AEC [a.s.EC] on $\Theta \Rightarrow$ a.s.UAEC [a.s.UEC] clearly do not hold in general. This is readily seen by choosing $Q_n \equiv Q$ nonrandom and observing that continuity and uniform continuity do not necessarily coincide if Θ is not compact.

[6] The implications in the first line of the diagram hold whether the equicontinuity properties in this line are all interpreted to hold at a given $\theta' \in \Theta$ or on Θ. Note also that a.s.AEC at θ' for all $\theta' \in \Theta$ implies $\mathrm{AL_0EC}$ on Θ. Furthermore, to establish the implications indicated in the diagram by $\Leftarrow$, observe that $\overline{\lim}_{n\to\infty} \mathrm{P}(|X_n| > \varepsilon) \leq \mathrm{P}(\overline{\lim}_{n\to\infty} |X_n| > \varepsilon/2)$ holds for any sequence of random variables X_n.

[7] A metric space (Θ, ρ) is totally bounded if for every $\delta > 0$ there exist finitely many θ_i, $1 \leq i \leq M(\delta)$, such that the open balls $B(\theta_i, \delta)$ cover Θ. Note that total boundedness is not a topological concept as it is possible to have two metrics ρ and σ on Θ that induce the same topology, but where (Θ, ρ) is totally bounded while (Θ, σ) is not. However, if Θ is compact, then (Θ, ρ) is totally bounded for any ρ generating the given topology. If Θ is a subset of Euclidean space and ρ is the Euclidean metric, then (Θ, ρ) is totally bounded iff Θ is bounded as a subset of Euclidean space.

[8] If (Θ, ρ) is totally bounded, $\mathrm{L_0UEC}$ can be implied from $\mathrm{UL_0EC}$ if we impose a rate of convergence on $\sup_{\theta' \in \Theta} \sup_n \mathrm{P}(\sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta') > \varepsilon)$: More specifically, let $\mathcal{N}(\delta)$ denote the smallest number of open balls $B(\theta_i', \delta)$ necessary to cover Θ, i.e., $\mathcal{N}(\delta)$ is the covering number of Θ. Then, if $\mathcal{N}(\delta) \sup_{\theta' \in \Theta} \sup_n \mathrm{P}(\sup_{\theta \in B(\theta', 2\delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) \to 0$ as $\delta \to 0$, it follows that Q_n is $\mathrm{L_0UEC}$, since $\mathrm{P}(\sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) \leq \mathrm{P}(\max_{1 \leq i \leq N(\delta)} \sup_{\theta \in B(\theta_i', 2\delta)} |Q_n(\theta) - Q_n(\theta_i')| > \varepsilon/2) \leq \mathcal{N}(\delta) \sup_{\theta' \in \Theta} \mathrm{P}(\sup_{\theta \in B(\theta', 2\delta)} |Q_n(\theta) - Q_n(\theta'| > \varepsilon/2)$. Compare Billingsley (1968, theorem 8.3) for a result that is similar in spirit.

(v) If (Θ, ρ) is not totally bounded, then the implication a.s.UAEC $\Rightarrow$ a.s.AUEC does not hold in general (even if Q_n is nonrandom); see Example 2 in section 6.

(vi) In general, even on compact Θ, also the following implications do not hold: $\mathrm{AL_pEC}$ [$\mathrm{L_pEC}$] $\Rightarrow$ a.s.AEC [a.s.EC], $\mathrm{UAL_pEC}$ [$\mathrm{UL_pEC}$] $\Rightarrow$ a.s.UAEC [a.s.UEC], $\mathrm{AL_pUEC}$[$\mathrm{L_pUEC}$] $\Rightarrow$ a.s.AUEC [a.s.UEC]. In fact, even the implication $\mathrm{L_pUEC}$ $\Rightarrow$ a.s.AEC does not hold as shown in Example 3 in section 6.

(vii) If Q_n is nonrandom, the equicontinuity concepts in each row of the above diagram coincide, as can be readily seen. Therefore, it follows from (ii) that if Q_n is nonrandom and if furthermore Θ is compact, all asymptotic equicontinuity concepts coincide and also all nonasymptotic equicontinuity concepts coincide.

Remark 2.2. (i) If $Q_n \equiv Q$, the equicontinuity concepts given in Definitions 2.1–2.3 reduce to corresponding continuity concepts. (The term 'equicontinuity' is then to be replaced with 'continuity' in the above definitions.) We note that $\mathrm{L_0}$ continuity at θ' $\Leftrightarrow$ a.s. continuity at θ', and $\mathrm{L_0}$ uniform continuity $\Leftrightarrow$ a.s. uniform continuity, since $\sup_{\theta\in B(\theta',\delta)} |Q(\theta) - Q(\theta')|$ and $\sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} |Q(\theta) - Q(\theta')|$ are monotone in δ. However, $\mathrm{L_0}$ continuity on Θ does not imply a.s. continuity on Θ; even uniform L_0 continuity does not imply a.s. continuity on Θ, as is readily seen from Example 1 in section 6, recalling that a.s. continuity on Θ requires a common exceptional null set.

(ii) It should be noted that $\mathrm{L_0}$ continuity is a stronger concept than continuity in probability, where the latter is defined as $Q(\theta) \to Q(\theta')$ in probability as $\theta \to \theta'$, or equivalently, $\sup_{\theta\in B(\theta',\delta)} \mathrm{P}(|Q(\theta) - Q(\theta')| > \varepsilon) \to 0$ as $\delta \to 0$.

We next discuss conditions, apart from the trivial case $Q_n \equiv Q$, under which the respective asymptotic and nonasymptotic versions of Definitions 2.1–2.3 coincide.

Remark 2.3. (i) If each Q_n is $\mathrm{L_p}$ continuous at θ' [on Θ], $p \geq 0$, then $\mathrm{AL_pEC}$ at θ' [on Θ] $\Leftrightarrow$ $\mathrm{L_pEC}$ at θ' [on Θ]. If each Q_n is $\mathrm{L_p}$ uniformly continuous, $p \geq 0$, then $\mathrm{AL_pUEC}$ $\Leftrightarrow$ $\mathrm{L_pUEC}$. However, if each Q_n is uniformly $\mathrm{L_p}$ continuous (or even $\mathrm{L_p}$ uniformly continuous), $p \geq 0$, then $\mathrm{UAL_pEC}$ does not in general imply $\mathrm{UL_pEC}$ unless, e.g., Θ is compact; cp. Example 2 in section 6 and Remark 2.1(ii).

(ii) If each Q_n is a.s. continuous at θ' [on Θ], then a.s.AEC at θ' [on Θ] $\Leftrightarrow$ a.s.EC at θ' [on Θ]. If each Q_n is a.s. uniformly continuous, then a.s.AUEC $\Leftrightarrow$ a.s.UEC. However, if each Q_n is a.s. uniformly continuous, a.s.UAEC does not in general imply a.s.UEC unless, e.g., (Θ, ρ) is totally bounded; cp. Example 2 in section 6 and Lemma A.2.

In Remark 2.1(i) we noted the obvious fact that the respective $\mathrm{L_p}$ equicontinuity concepts imply the corresponding $\mathrm{L_0}$ equicontinuity concepts. We next show that also the reverse implication holds under the following uniform integrability type conditions for $p > 0$:

$$\overline{\lim_{n\to\infty}}\ \mathrm{E}(D_n^p \mathbf{1}(D_n > M)) \to 0 \quad \text{as} \quad M \to \infty, \tag{2.4a}$$

$$\sup_n \mathrm{E}(D_n^p \mathbf{1}(D_n > M)) \to 0 \quad \text{as} \quad M \to \infty, \tag{2.4b}$$

where $D_n = \sup_{\theta\in\Theta} |Q_n(\theta)|$. Note that (2.4a) and (2.4b) are equivalent if $\mathrm{E}D_n^p < \infty$ for all $n \geq 1$. The following result is of importance as it will allow us to demonstrate the similarities in the different approaches taken in the literature to establish uniform convergence results.

Theorem 2.1.[9] (a) *For* $0 \leq r \leq p$:

$$AL_pEC[L_pEC] \text{ at } \theta'\in\Theta \quad \Rightarrow \quad AL_rEC[L_rEC] \text{ at } \theta'\in\Theta, \tag{2.5a}$$

$$UAL_pEC[UL_pEC] \quad \Rightarrow \quad UAL_rEC[UL_rEC], \tag{2.5b}$$

$$AL_pUEC[L_pUEC] \quad \Rightarrow \quad AL_rUEC\ [L_rUEC]. \tag{2.5c}$$

(b) *Under* (2.4a) [(2.4b)] *with* $p > 0$ *also the reverse implications in* (2.5a)–(2.5c) *hold for* $0 \leq r \leq p$.

A simple sufficient condition for (2.4a) or (2.4b) is clearly given by $\overline{\lim}_{n\to\infty}$ $\mathrm{E}(D_n^s) < \infty$ or $\sup_n \mathrm{E}(D_n^s) < \infty$, respectively, for some $s > p$. Theorem 2.1 is similar in spirit to Theorem 6.1 of Pötscher and Prucha (1991a).

2.2. Some sufficient conditions

In the following we discuss several sufficient conditions for the respective equicontinuity-type conditions. A further discussion of such conditions for the important special case where Q_n is an average is given in section 4.

Simple sufficient conditions are provided by Lipshitz-type conditions [cp. Andrews (1987, 1989c)]. First consider the following global Lipshitz-type condition: There exists an $\eta > 0$ and a null set N such that for all $\theta, \theta'\in\Theta$ with $\rho(\theta, \theta') < \eta$ and all $\omega\in\Omega - N$ we have

$$|Q_n(\theta) - Q_n(\theta')| \leq B_n h(\rho(\theta, \theta')), \tag{2.6}$$

[9]For the reverse implication of (2.5a) and (2.5b) in part (b), we could replace D_n with $D_n(\theta') = \sup_{\theta\in B(\theta',\delta)} |Q_n(\theta)|$ in (2.4); however for the reverse of (2.5b), we then have to modify (2.4) by also taking the supremum over Θ in the expressions in (2.4).

where $B_n\colon \Omega \to [0,\infty)$, $h\colon [0,\infty) \to [0,\infty)$ with $h(x) \downarrow 0$ as $x \downarrow 0$, and where the Lipshitz bounds B_n do not depend on θ or θ' and satisfy either[10]

$$\overline{\lim_{n\to\infty}}\ \mathrm{E}B_n^p < \infty \quad \Big[\sup_n \mathrm{E}B_n^p < \infty\Big] \quad \text{for some } p > 0, \quad \text{or} \tag{2.7a}$$

$$\sup_n \mathrm{P}(B_n > M) \to 0 \quad \text{as } M \to \infty, \quad \text{or} \tag{2.7b}$$

$$\overline{\lim_{n\to\infty}}\ B_n < \infty \quad \text{a.s.} \quad \Big[\sup_n B_n < \infty \quad \text{a.s.}\Big]. \tag{2.7c}$$

Then (2.6) and (2.7a) imply that the random functions Q_n are $\mathrm{AL_pUEC}$ [$\mathrm{L_pUEC}$], (2.6) and (2.7b) imply that the Q_n are $\mathrm{L_0UEC}$, and (2.6) and (2.7c) imply that the Q_n are a.s.AUEC [a.s.UEC].

Next consider the following local rather than global Lipshitz-type condition: For each $\theta' \in \Theta$ there exists an $\eta = \eta(\theta') > 0$ and a null set $N(\theta')$ such that for all θ with $\rho(\theta, \theta') < \eta$ and all $\omega \in \Omega - N(\theta')$ we have that

$$|Q_n(\theta) - Q_n(\theta')| \le B_n h(\rho(\theta, \theta')) \tag{2.8}$$

holds, where now the Lipshitz bounds $B_n = B_n(\theta')$ are allowed to depend on θ' and satisfy

$$\overline{\lim_{n\to\infty}}\ \mathrm{E}B_n^p(\theta') < \infty \quad \Big[\sup_n \mathrm{E}B_n^p(\theta') < \infty\Big] \quad \text{for some} \quad p > 0, \quad \text{or} \tag{2.9a}$$

$$\sup_n \mathrm{P}(B_n(\theta') > M) \to 0 \quad \text{as} \quad M \to \infty, \quad \text{or} \tag{2.9b}$$

$$\overline{\lim_{n\to\infty}}\ B_n(\theta') < \infty \quad \text{a.s.} \quad \Big[\sup_n B_n(\theta') < \infty \quad \text{a.s.}\Big]. \tag{2.9c}$$

Then (2.8) and (2.9a) imply that the random functions Q_n are $\mathrm{AL_pEC}$ [$\mathrm{L_pEC}$], (2.8) and (2.9b) imply that the Q_n are $\mathrm{L_0EC}$; furthermore (2.8) and (2.9c) imply that the Q_n are a.s.AEC [a.s.EC] on Θ if the null set $N(\theta')$ and the exceptional null set in (2.9c) do not depend on θ'.

[10]Note that $\sup_n \mathrm{P}(B_n > M) \to 0$ as $M \to \infty$ is equivalent to $\overline{\lim}_{n\to\infty} \mathrm{P}(B_n > M) \to 0$ as $M \to \infty$.

Finally, let

$$\sup_{\theta' \in \Theta} \overline{\lim_{n\to\infty}} \, \mathrm{E}B_n^p(\theta') < \infty \quad \left[\sup_{\theta' \in \Theta} \sup_n \mathrm{E}B_n^p(\theta') < \infty\right] \text{ for some } p > 0, \text{ or} \tag{2.10a}$$

$$\sup_{\theta' \in \Theta} \overline{\lim_{n\to\infty}} \, \mathrm{P}(B_n(\theta') > M) \to 0 \quad \left[\sup_{\theta' \in \Theta} \sup_n \mathrm{P}(B_n(\theta') > M) \to 0\right] \quad \text{as } M \to \infty, \text{ or} \tag{2.10b}$$

$$\sup_{\theta' \in \Theta} \overline{\lim_{n\to\infty}} \, B_n(\theta') < \infty \quad \text{a.s} \quad \left[\sup_{\theta' \in \Theta} \sup_n B_n(\theta') < \infty \quad \text{a.s.}\right]. \tag{2.10c}$$

If η in the definition of the local Lipshitz-type condition (2.8) does not depend on θ', then (2.8) and (2.10a) imply that the random functions Q_n are $\mathrm{UAL_pEC}$ [$\mathrm{UL_pEC}$], (2.8) and (2.10b) imply that the Q_n are $\mathrm{UAL_0EC}$ [$\mathrm{UL_0EC}$], and if additionally $N(\theta')$ does not depend on θ', then (2.8) and (2.10c) imply that the Q_n are a.s.UAEC [a.s.UEC].

The verification of any of the equicontinuity conditions $\mathrm{L_0EC}$, $\mathrm{UL_0EC}$, or $\mathrm{L_0UEC}$ or the asymptotic counter parts involves the establishing of a maximal inequality. The Lipshitz-type condition discussed above essentially allows one to imply this maximal inequality from bounds on $\mathrm{P}(B_n > \varepsilon/h(\rho(\theta, \theta')) \geq \mathrm{P}(|Q_n(\theta) - Q_n(\theta')| > \varepsilon)$. For further techniques for verifying $\mathrm{L_0EC}$, $\mathrm{UL_0EC}$, or $\mathrm{L_0UEC}$ see, e.g., the Chaining Lemma in Pollard (1984).

3. Approaches to uniform convergence and Ascoli–Arzelà's theorem

In this section we compare two basic approaches for the derivation of uniform convergence results. The first approach utilizes a stochastic variant of Ascoli–Arzelà's Theorem and is based on asymptotic $\mathrm{L_0}$ uniform equicontinuity, i.e., $\mathrm{AL_0UEC}$, and pointwise convergence i.p. of Q_n [or a.s. asymptotic uniform equicontinuity, i.e., a.s.AUEC, and pointwise a.s. convergence of Q_n]. The second approach adopts Wald's (1949) bracketing idea and is based on uniform asymptotic $\mathrm{L_1}$ equicontinuity, i.e., $\mathrm{UAL_1EC}$, of Q_n and convergence i.p. [a.s. convergence] of certain local bracketing functions derived from Q_n. (If Θ is compact, only $\mathrm{AL_1EC}$ of Q_n has to be verified for the second approach since in this case $\mathrm{AL_1EC}$ and $\mathrm{UAL_1EC}$ are equivalent.) We give two basic theorems that describe these two approaches. The first one of these results is a slightly generalized version of results in Andrews (1989c) and Newey (1989) and only requires pointwise convergence on a dense subset of Θ. The second result essentially only reformulates the strategy used to prove ULLNs in, e.g., Andrews (1987) and Pötscher and Prucha (1986a, b, 1989a, b).

We shall need the following asymptotic variant of Ascoli–Arzelà's Theorem:[11]

Ascoli–Arzela's Theorem. *Let f_n: $\Theta \to \mathbb{R}$ and $\bar{f}_n$: $\Theta \to \mathbb{R}$ be sequences of functions and assume $\bar{f}_n$ to be asymptotically uniformly equicontinuous.*

(a) *If (Θ, ρ) is totally bounded, if $f_n(\theta) - \bar{f}_n(\theta) \to 0$ as $n \to \infty$ for all $\theta \in \Theta_0$, where Θ_0 is a dense subset of Θ, and if f_n is asymptotically uniformly equicontinuous, then $\sup_{\theta\in\Theta}|f_n(\theta) - \bar{f}_n(\theta)| \to 0$ as $n \to \infty$.*
(b) *If $\sup_{\theta\in\Theta}|f_n(\theta) - \bar{f}_n(\theta)| \to 0$ as $n \to \infty$, then the sequence f_n is asymptotically uniformly equicontinuous and $f_n(\theta) - \bar{f}_n(\theta) \to 0$ as $n \to \infty$ for all $\theta \in \Theta$.*

If $\bar{f}_n \equiv \bar{f}$, and if $\Theta_0 \equiv \Theta$ or $\bar{f}$ is continuous, then for part (a) of the theorem the assumption that $\bar{f}_n$ is asymptotically uniformly equicontinuous (i.e., $\bar{f}$ is uniformly continuous) can be dropped; in fact, uniform continuity of $\bar{f}$ then follows as a conclusion of part (a). We now present the first basic uniform convergence result for random functions. The proof of the a.s. part crucially utilizes the feature that in part (a) of the above Ascoli–Arzelà Theorem pointwise convergence is only required to hold on a dense subset of Θ and the fact that a totally bounded metric space is separable. The i.p. part of the following result with $\Theta_0 \equiv \Theta$ has been given in Newey (1989, theorem 1) for compact Θ and Andrews (1989c, theorem 1) for totally bounded Θ; also, the i.p. part is a special case of Theorem 10.2 in Pollard (1989).

Theorem 3.1.[12] *Let $\bar{Q}_n$: $\Theta \to \mathbb{R}$ be an asymptotically uniformly equicontinuous sequence of nonrandom functions.*

(a) *If (Θ, ρ) is totally bounded, if $Q_n(\theta) - \bar{Q}_n(\theta) \to 0$ a.s. [i.p.] as $n \to \infty$ for all $\theta \in \Theta_0$, where Θ_0 is a dense subset of Θ, and if Q_n is a.s.AUEC [AL_0UEC], then $\sup_{\theta\in\Theta}|Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.] as $n \to \infty$.*
(b) *If $\sup_{\theta\in\Theta}|Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.] as $n \to \infty$, then the sequence Q_n is a.s.AUEC [AL_0UEC] and $Q_n(\theta) - \bar{Q}_n(\theta) \to 0$ a.s. [i.p.] as $n \to \infty$ for all $\theta \in \Theta$.*

Obviously, the above theorem also covers the case $\Theta_0 = \Theta$. Recall also that a.s.AUEC implies AL_0UEC, and hence the i.p. part of the theorem clearly also

[11]Ascoli–Arzelà's Theorem is typically stated for an equicontinuous sequence of functions on a compact space; see, e.g., Dunford and Schwartz (1957, theorem IV.6.7). Of course, for sequences of [uniformly] continuous functions the properties of asymptotic [uniform] equicontinuity and [uniform] equicontinuity coincide; cp. Remark 2.3(ii). Furthermore, if Θ is compact, [asymptotic] uniform equicontinuity and [asymptotic] equicontinuity coincide; cp. Remark 2.1(ii).

[12]Of course, in Theorem 3.1 we could have absorbed $\bar{Q}_n$ into Q_n without loss of generality. However, this is not the case in Theorem 3.2 given below. We have chosen the above formulation of Theorem 3.1 for reasons of comparability.

holds under the stronger a.s.AUEC assumption. Furthermore note that, in view of Lemma A.2, the assumptions that Q_n is a.s.AUEC and that $\bar{Q}_n$ is AUEC in Theorem 3.1(a) could be replaced, respectively, by a.s.UAEC and UAEC (or even by a.s.AEC on Θ and AEC on Θ if Θ is compact).

The second basic uniform convergence result for random functions is modeled on the method of proof used in Wald (1949), Andrews (1987), and Pötscher and Prucha (1986a, b, 1989a, b), and represents the 'first-moment equicontinuity' approach. In the next theorem the expectations are assumed to be finite.

Theorem 3.2. *Let $\bar{Q}_n = \mathrm{E}Q_n$, let (Θ, ρ) be totally bounded, and assume that*

$$\sup_{\theta \in B(\theta', \delta_k)} Q_n(\theta) - \mathrm{E} \sup_{\theta \in B(\theta', \delta_k)} Q_n(\theta) \to 0 \quad \textit{a.s. [i.p.]} \quad \textit{as} \quad n \to \infty, \tag{3.1a}$$

$$\inf_{\theta \in B(\theta', \delta_k)} Q_n(\theta) - \mathrm{E} \inf_{\theta \in B(\theta', \delta_k)} Q_n(\theta) \to 0 \quad \textit{a.s. [i.p.]} \quad \textit{as} \quad n \to \infty, \tag{3.1b}$$

for all $k \geq 1$ and all $\theta' \in \Theta$, where δ_k is some sequence of positive numbers converging to zero. Let Q_n be UAL_1EC, then $\sup_{\theta \in \Theta} |Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.] as $n \to \infty$.

Recall that if Θ is compact, $\mathrm{UAL_1EC}$ reduces to $\mathrm{AL_1EC}$. Furthermore, if Θ is compact, the sequence δ_k in (3.1) can be allowed to depend on θ', as can be seen from the proof of the theorem.

Before discussing Theorems 3.1 and 3.2 in more detail, we give a result which shows that $\mathrm{L_p}$ equicontinuity-type conditions on Q_n with $p \geq 1$ already imply equicontinuity-type conditions for $\mathrm{E}Q_n$.

Theorem 3.3. *Let $\bar{Q}_n = \mathrm{E}Q_n$, which is assumed to be finite, and let $p \geq 1$. If Q_n is AL_pEC [L_pEC], then $\bar{Q}_n$ is AEC [EC]. If Q_n is UAL_pEC [UL_pEC], then $\bar{Q}_n$ is UAEC [UEC]. If Q_n is AL_pUEC [L_pUEC], then $\bar{Q}_n$ is AUEC [UEC].*

Given the uniform integrability type condition (2.4a) [(2.4b)] holds, Theorem 3.3 also applies [in view of Remark 2.1(i) and Theorem 2.1] if Q_n satisfies asymptotic [nonasymptotic] a.s. or $\mathrm{L_r}$, $r < 1$, equicontinuity-type conditions.

If in Theorem 3.1 $\bar{Q}_n = \mathrm{E}Q_n$, then the condition that $\bar{Q}_n$ is asymptotically uniformly equicontinuous is already implied by the assumption that Q_n is a.s.AUEC [$\mathrm{AL_0UEC}$] in Theorem 3.1(a), given the uniform integrability-type condition (2.4a) with $p \geq 1$ is satisfied; this follows from Theorem 2.1(b), which then implies that Q_n is $\mathrm{AL_pUEC}$, and Theorem 3.3. Similarly, the assumption in Theorem 3.2 that Q_n is $\mathrm{UAL_1EC}$ implies that $\bar{Q}_n = \mathrm{E}Q_n$ is UAEC in view of Theorem 3.3; in view of Lemma A.2, $\bar{Q}_n = \mathrm{E}Q_n$ is then even AUEC.

Sufficient conditions for the equicontinuity-type conditions employed in Theorems 3.1–3.3 have been given in section 2.2. For the special case of uniform laws of large numbers further sufficient conditions will be discussed in section 4.

Comparing the approaches corresponding to Theorems 3.1 and 3.2 in the context of convergence in probability we see that the first approach, which transforms pointwise convergence into uniform convergence, requires more *uniformity* in the equicontinuity-type condition than the second approach, which transforms local convergence into uniform convergence and which only assumes UAL_1EC rather than the stronger AL_1UEC condition. [Since a uniform integrability-type condition like (2.4) will typically hold in a given application, the fact that Theorems 3.1 and 3.2 use L_0 and L_1 equicontinuity concepts, respectively, seems rather immaterial since under (2.4) AL_0UEC is equivalent to AL_1UEC in view of Theorem 2.1.] Comparing the approaches corresponding to Theorems 3.1 and 3.2 in the context of a.s. convergence we see again that the condition that Q_n is a.s.AUEC maintained by Theorem 3.1 is – given a uniform integrability type condition – stronger and again requires more uniformity than the condition UAL_1EC maintained in Theorem 3.2. [Note that despite the equivalence of a.s.AUEC with a.s.UAEC for totally bounded (Θ, ρ) and its equivalence even with a.s.AEC on Θ for compact Θ, a.s.AUEC still not only implies UAL_1EC but even AL_1UEC, given a uniform integrability type condition!] Example 4 in section 6 shows that the equicontinuity-type condition UAL_1EC maintained in Theorem 3.2 (and even UL_pEC with arbitrarily large p) is in general not sufficient to allow the transformation of pointwise convergence into uniform convergence.

For a compact parameter space, the condition in Theorem 3.2 that the sequence Q_n is UAL_1EC reduces even to AL_1EC. The former condition represents the appropriate assumption needed to cover the case of a totally bounded parameter space. In contrast, in the convergence i.p. part of Theorem 3.1 the condition that Q_n is AL_0UEC has to be assumed even if Θ is compact [and also represents the appropriate assumption for the case of totally bounded (Θ, ρ)].

In comparing the two approaches it is furthermore important to observe that, given the uniform integrability-type condition (2.4a) holds with $p = 1$, the assumptions of Theorem 3.2 deliver – via its conclusion and Theorem 3.1(b) – the assumptions maintained by Theorem 3.1(a).[13] (In particular, the assumptions of the a.s. [i.p.] convergence part of Theorem 3.2, which include the condition UAL_1EC, imply the even stronger equicontinuity condition a.s.AUEC [AL_1UEC].) Conversely, given that (2.4a) holds with $p = 1$, the assumptions of

[13]This can be seen as follows: Since Q_n is UAL_1EC, it follows from Theorem 3.3 that $\bar{Q}_n$ is UAEC, and hence is AUEC in view of Lemma A.2. Since total boundedness is assumed in Theorem 3.2, the remaining conditions in Theorem 3.1(a) follow immediately from uniform convergence in view of Theorem 3.1(b). That Q_n is even AL_1UEC follows then from Theorem 2.1. Note that (2.4a) was actually only used in the last step to imply AL_1UEC of Q_n.

Theorem 3.1(a) (with $\bar{Q}_n = EQ_n$ assumed finite for $n \geq 1$) deliver – via its conclusion and Lemma A.3 – the assumptions maintained by Theorem 3.2.[14] Thus, given the uniform integrability-type condition (2.4a) holds with $p = 1$, the two approaches are equivalent in that they cover the same class of problems.[15]

As discussed above, if we use Theorem 3.2, we only have to verify UAL_1EC (or AL_1EC for compact Θ) rather than a.s.AUEC or AL_0UEC. This may be advantageous especially in situations where verifying local convergence is easy (or at least not more difficult than verifying pointwise convergence). We also note that a potential advantage of Theorem 3.2 in an application may be that one only has to verify UAL_1EC for both a.s. and i.p. uniform convergence results.

Andrews (1989c) defined a further stochastic equicontinuity-type concept which he labeled 'strong stochastic equicontinuity' to derive a strong uniform convergence result. In light of Theorem 2 in Andrews (1989c) and Theorem 3.1 we see that a.s.AUEC and 'strong stochastic equicontinuity' are equivalent given (Θ, ρ) is totally bounded and $Q_n(\theta) \to 0$ a.s. as $n \to \infty$ for all $\theta \in \Theta$. [As in Andrews (1989c) we assume here without loss of generality $\bar{Q}_n = 0$.][16]

4. Uniform laws of large numbers

In this section we consider uniform convergence for the special case where $Q_n(\theta) = n^{-1} \sum_{t=1}^{n} q_t(\omega, \theta)$ and $\bar{Q}_n(\theta) = EQ_n(\theta)$, i.e., we consider uniform laws of large numbers (ULLNs), as an important application of equicontinuity-type concepts for random functions. We maintain throughout this section that the functions q_t: $\Omega \times \Theta \to \mathbb{R}$ are measurable in their first argument and integrable for each $\theta \in \Theta$ and $t \geq 1$. Again, we shall frequently suppress the dependence of q_t on ω in the notation.

4.1. Cesàro equicontinuity-type concepts

Of course, for the above choice for Q_n and $\bar{Q}_n$ Theorems 3.1 and 3.2 represent ULLNs. However, in applications it is often more natural to imply the conditions on Q_n from conditions on q_t. This can be accomplished in different

[14]This can be seen as follows: The conditions a.s.AUEC [AL_0UEC] clearly imply UAL_1EC in view of Remark 2.1(i) and Theorem 2.1. The local convergence conditions (3.1) follow from Lemma A.3.

[15]Implicitly the discussion has also established the following partial converse to Theorem 3.2: If $\sup_{\theta \in \Theta} |Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.] as $n \to \infty$ with $\bar{Q}_n = EQ_n$ (assumed to be finite for $n \geq 1$) and if (2.4a) with $p = 1$ holds, then $Q_n - \bar{Q}_n$ is UAL_1EC (and even AL_1UEC) and the local convergence conditions (3.1) are satisfied.

[16]It is readily seen that in general a.s.AUEC implies strong stochastic equicontinuity, but not conversely. [Note that in order to be well-defined the definition of strong stochastic equicontinuity in Andrews (1989c) has to be amended by, in our notation, the condition $\sup_{n \geq m} |Q_n(\theta)| < \infty$ for all $\theta \in \Theta$ a.s.]

ways. One route of verifying the equicontinuity-type conditions employed in Theorems 3.1 and 3.2 that proves useful is to introduce intermediate equicontinuity-type concepts for q_t which can be used to imply the equicontinuity-type conditions for Q_n. Sufficient conditions for these intermediate equicontinuity-type concepts for q_t will be discussed in more detail below. We now introduce such intermediate equicontinuity-type conditions for q_t as analogs to Definitions 2.1–2.3.

Definition 4.1. q_t is asymptotically Cesàro $\mathrm{L_p}$ equicontinuous ($\mathrm{ACL_pEC}$) at $\theta' \in \Theta$ for $p > 0$ iff

$$\overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \mathrm{E} \sup_{\theta \in B(\theta',\delta)} |q_t(\theta) - q_t(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{4.1a}$$

q_t is asymptotically Cesàro $\mathrm{L_0}$ equicontinuous ($\mathrm{ACL_0EC}$) at $\theta' \in \Theta$ iff for every $\varepsilon > 0$

$$\overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \mathrm{P}\left(\sup_{\theta \in B(\theta',\delta)} |q_t(\theta) - q_t(\theta')| > \varepsilon \right) \to 0 \quad \text{as} \quad \delta \to 0; \tag{4.1b}$$

q_t is a.s. asymptotically Cesàro equicontinuous (a.s.ACEC) at $\theta' \in \Theta$ iff

$$\overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \sup_{\theta \in B(\theta',\delta)} |q_t(\theta) - q_t(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{4.1c}$$

If (4.1a) [(4.1b)] {(4.1c)} holds with $\overline{\lim}_{n\to\infty}$ replaced by $\sup_n$, then q_t is said to be Cesàro $\mathrm{L_p}$ equicontinuous ($\mathrm{CL_pEC}$) at θ' [Cesàro $\mathrm{L_0}$ equicontinuous ($\mathrm{CL_0EC}$) at θ'] {a.s. Cesàro equicontinuous (a.s.CEC) at θ'}. If any of the above properties holds for all $\theta' \in \Theta$ (with a *common* exceptional null set for the a.s. case), then we say that this property holds on Θ.

Definition 4.2. q_t is uniformly asymptotically Cesàro $\mathrm{L_p}$ equicontinuous ($\mathrm{UACL_pEC}$) on Θ for $p > 0$ iff

$$\sup_{\theta' \in \Theta} \overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \mathrm{E} \sup_{\theta \in B(\theta',\delta)} |q_t(\theta) - q_t(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{4.2a}$$

q_t is uniformly asymptotically Cesàro $\mathrm{L_0}$ equicontinuous ($\mathrm{UACL_0EC}$) on Θ iff for every $\varepsilon > 0$

$$\sup_{\theta' \in \Theta} \overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \mathrm{P}\left(\sup_{\theta \in B(\theta',\delta)} |q_t(\theta) - q_t(\theta')| > \varepsilon \right) \to 0 \text{ as } \delta \to 0; \tag{4.2b}$$

q_t is a.s. uniformly asymptotically Cesàro equicontinuous (a.s.UACEC) on Θ iff

$$\sup_{\theta' \in \Theta} \overline{\lim_{n \to \infty}}\, n^{-1} \sum_{t=1}^{n} \sup_{\theta \in B(\theta', \delta)} |q_t(\theta) - q_t(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{4.2c}$$

If (4.2a) [(4.2b)] {(4.2c)} holds with $\overline{\lim}_{n \to \infty}$ replaced by $\sup_n$, then q_t is said to be uniformly Cesàro L_p equicontinuous (UCL_pEC) on Θ [uniformly Cesàro L_0 equicontinuous (UCL_0EC) on Θ] {a.s. uniformly Cesàro equicontinuous (a.s.UCEC) on Θ}.

Definition 4.3. q_t is asymptotically Cesàro L_p uniformly equicontinuous (ACL_pUEC) on Θ for $p > 0$ iff

$$\overline{\lim_{n \to \infty}}\, n^{-1} \sum_{t=1}^{n} E \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |q_t(\theta) - q_t(\theta')|^p \to 0 \quad \text{as} \quad \delta \to 0; \tag{4.3a}$$

q_t is asymptotically Cesàro L_0 uniformly equicontinuous (ACL_0UEC) on Θ iff for every $\varepsilon > 0$

$$\overline{\lim_{n \to \infty}}\, n^{-1} \sum_{t=1}^{n} P\left(\sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |q_t(\theta) - q_t(\theta')| > \varepsilon \right) \to 0 \text{ as } \delta \to 0; \tag{4.3b}$$

q_t is a.s. asymptotically Cesàro uniformly equicontinuous (a.s.ACUEC) on Θ iff

$$\overline{\lim_{n \to \infty}}\, n^{-1} \sum_{t=1}^{n} \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |q_t(\theta) - q_t(\theta')| \to 0 \quad \text{a.s.} \quad \text{as} \quad \delta \to 0. \tag{4.3c}$$

If (4.3a) [(4.3b)] {(4.3c)} holds with $\overline{\lim}_{n \to \infty}$ replaced by $\sup_n$, then q_t is said to be Cesàro L_p uniformly equicontinuous (CL_pUEC) on Θ [Cesàro L_0 uniformly equicontinuous (CL_0UEC) on Θ] {a.s. Cesàro uniformly equicontinuous (a.s. CUEC) on Θ}.

The concept of ACL_0UEC was introduced in Andrews (1989c) under the name of 'termwise stochastic equicontinuity'. We note that *most but not all* of the implications discussed in Remarks 2.1 and 2.3 also hold for the corresponding Cesàro equicontinuity concepts.[17] Certain of these implications are collected in Lemmata A.2 and A.4. For later use we note that in particular ACL_pUEC [CL_pUEC] $\Rightarrow$ $UACL_pEC$ [UCL_pEC] for $p \geq 0$.

[17] E.g., a.s.UACEC does in general not imply a.s.ACUEC even for compact Θ.

Similar as in section 2 we now give a theorem that shows that the various Cesàro L_p equicontinuity concepts coincide for different values of p under the following uniform integrability-type condition:

$$\overline{\lim_{n\to\infty}}\, n^{-1}\sum_{t=1}^{n} \mathrm{E}(d_t^p \mathbf{1}(d_t > M)) \to 0 \quad \text{as} \quad M \to\infty, \tag{4.4}$$

where $d_t = \sup_{\theta\in\Theta}|q_t(\omega,\theta)|$. Note that (4.4) implies that $\mathrm{E}d_t^p < \infty$ for all $t \geq 1$ and hence (4.4) is equivalent to $\sup_n n^{-1}\sum_{t=1}^{n}\mathrm{E}(d_t^p\mathbf{1}(d_t > M)) \to 0$ as $M \to \infty$.

Theorem 4.1. (a) *For* $0 \leq r \leq p$:

$$ACL_pEC[CL_pEC] \text{ at } \theta'\in\Theta \quad\Rightarrow\quad ACL_rEC\ [CL_rEC] \text{ at } \theta'\in\Theta, \tag{4.5a}$$

$$UACL_pEC[UCL_pEC] \quad\Rightarrow\quad UACL_rEC[UCL_rEC], \tag{4.5b}$$

$$ACL_pUEC[CL_pUEC] \quad\Rightarrow\quad ACL_rUEC[CL_rUEC]. \tag{4.5c}$$

(b) *Under (4.4) with* $p > 0$ *also the reverse implications in (4.5a)–(4.5c) hold for* $0 \leq r \leq p$.

The next theorem relates Cesàro equicontinuity-type concepts for q_t to corresponding equicontinuity-type concepts for $Q_n = n^{-1}\sum_{t=1}^{n} q_t$.

Theorem 4.2. (a) *If* q_t *is a.s.ACEC [a.s.CEC], then* Q_n *is a.s.AEC [a.s.EC]. If* q_t *is a.s.UACEC [a.s.UCEC], then* Q_n *is a.s.UAEC [a.s.UEC]. If* q_t *is a.s.ACUEC [a.s.CUEC], then* Q_n *is a.s.AUEC [a.s.UEC].*

(b) *Suppose that* $r \geq 1$, *or suppose that (4.4) holds for some* $p \geq 1$ *and* $0 \leq r \leq p$. *If* q_t *is* ACL_rEC $[CL_rEC]$, *then* Q_n *is* AL_rEC $[L_rEC]$. *If* q_t *is* $UACL_rEC$ $[UCL_rEC]$, *then* Q_n *is* UAL_rEC $[UL_rEC]$. *If* q_t *is* ACL_rUEC $[CL_rUEC]$, *then* Q_n *is* AL_rUEC $[L_rUEC]$.

4.2. ULLNs based on Cesàro equicontinuity-type conditions

In this subsection we give, as corollaries to Theorems 3.1 and 3.2, two ULLNs that utilize the above-defined Cesàro equicontinuity-type concepts. We then give results concerning sufficient conditions for the assumptions of those corollaries that are easier to verify in applications.

The in probability part of the following ULLN with $\Theta_0 \equiv \Theta$ corresponds to Theorem 4 in Andrews (1989c). The a.s. part of the following ULLN differs from Theorem 6 in Andrews (1989c); in particular, the a.s. part of the following ULLN only requires strong pointwise laws of large numbers, whereas Andrews' Theorem 6 assumes a strong law of large numbers for certain suprema.

Corollary 4.3. Let (Θ, ρ) be totally bounded, let

$$n^{-1}\sum_{t=1}^{n}[q_t(\omega,\theta)-\mathrm{E}q_t(\omega,\theta)]\to 0 \quad a.s.\ [i.p.] \quad as \quad n\to\infty, \tag{4.6}$$

for all $\theta\in\Theta_0$, where Θ_0 is a dense subset of Θ. Let q_t be a.s.ACUEC [ACL_0UEC] and assume that (4.4) holds for some $p\geq 1$. Then (a) $\sup_{\theta\in\Theta}|n^{-1}\sum_{t=1}^{n}[q_t(\omega,\theta)-\mathrm{E}q_t(\omega,\theta)]|\to 0$ *a.s. [i.p.] as $n\to\infty$, and* (b) $n^{-1}\sum_{t=1}^{n}\mathrm{E}q_t$ *is asymptotically uniformly equicontinuous.*

The above ULLN is readily obtained from Theorem 3.1, utilizing the building blocks provided by Theorems 2.1, 3.3, 4.1, 4.2 and Lemma A.4. Since a.s.ACUEC implies $\mathrm{ACL_0UEC}$ by Lemma A.4, the i.p. part of the theorem clearly also holds under the a.s.ACUEC assumption.

Assuming q_t to be $\mathrm{UACL_1EC}$ and combining Theorems 3.2, 3.3, and 4.2 immediately yields a ULLN, but the convergence conditions (3.1) are then not in the form of a law of large numbers. It turns out, however, that the proof of Theorem 3.2 can be readily modified to yield the following ULLN, where now the convergence conditions take the form of laws of large numbers. In the following corollary the expectations are assumed to be finite.

Corollary 4.4. Let (Θ, ρ) be totally bounded and assume that

$$n^{-1}\sum_{t=1}^{n}\left\{\sup_{\theta\in B(\theta',\delta_k)}q_t(\omega,\theta)-\mathrm{E}\sup_{\theta\in B(\theta',\delta_k)}q_t(\omega,\theta)\right\}\to 0 \quad a.s.\ [i.p.] \quad as \quad n\to\infty, \tag{4.7a}$$

$$n^{-1}\sum_{t=1}^{n}\left\{\inf_{\theta\in B(\theta',\delta_k)}q_t(\omega,\theta)-\mathrm{E}\inf_{\theta\in B(\theta',\delta_k)}q_t(\omega,\theta)\right\}\to 0 \quad a.s.\ [i.p.] \quad as \quad n\to\infty, \tag{4.7b}$$

for all $k\geq 1$ and all $\theta'\in\Theta$, where δ_k is some sequence of positive numbers converging to zero. Let q_t be $UACL_1EC$, then (a) $\sup_{\theta\in\Theta}|n^{-1}\sum_{t=1}^{n}[q_t(\omega,\theta)-\mathrm{E}q_t(\omega,\theta)]|\to 0$ *a.s. [i.p.] as $n\to\infty$, and* (b) $n^{-1}\sum_{t=1}^{n}\mathrm{E}q_t$ *is asymptotically uniformly equicontinuous.*

We note that, analogously to the comments after Theorem 3.2, for compact Θ the condition $\mathrm{UACL_1EC}$ reduces to $\mathrm{ACL_1EC}$; cp. Lemma A.2. Furthermore, for compact Θ the sequence δ_k in (4.7) can be allowed to depend on θ' as can be seen from the proof of the corollary. We emphasize that (4.7) does *not* represent a uniform convergence condition, but simply laws of large numbers for certain suprema and infima of q_t.

The following remarks discuss modifications of Corollaries 4.3 and 4.4 and relate those corollaries to ULLNs introduced recently in the literature.

Remark 4.1. (i) If in Corollary 4.3 the uniform integrability-type condition (4.4) is replaced by (2.4a), then the part based on a.s.ACUEC still holds. This is seen as follows: q_t is a.s.ACUEC $\Rightarrow$ Q_n is a.s.AUEC $\Rightarrow$ Q_n is $\mathrm{AL_0UEC}$ $\Rightarrow$ Q_n is $\mathrm{AL_1UEC}$ by (2.4a) and Theorem 2.1(b), and hence $\bar{Q}_n = \mathrm{E}Q_n$ is AUEC by Theorem 3.3, observing that $\mathrm{E}|q_t| < \infty$ for all $t \geq 1$ is maintained throughout this section. Consequently Theorem 3.1 applies.

(ii) If q_t is assumed to be $\mathrm{ACL_1UEC}$ rather than $\mathrm{ACL_0UEC}$ in Corollary 4.3, then the i.p. part holds without condition (4.4), since it then follows immediately from Theorems 4.2 and 3.3 that Q_n is $\mathrm{AL_1UEC}$ and $\bar{Q}_n$ is AUEC, observing that $\mathrm{E}|q_t| < \infty$. Consequently Theorem 3.1 applies.

(iii) If in Corollary 4.3 the condition that q_t is a.s.ACUEC [$\mathrm{ACL_0UEC}$] is strengthened to a.s.CUEC [$\mathrm{CL_0UEC}$], then in part (b) of the corollary $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ is even uniformly equicontinuous. A similar remark applies to the modifications of Corollary 4.3 discussed in (i) and (ii) [if in (i) also (2.4a) is strengthened to (2.4b)].

(iv) If in Corollary 4.4 the condition that q_t is $\mathrm{UACL_1EC}$ is strengthened to $\mathrm{UCL_1EC}$, then in part (b) of the corollary $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ is even uniformly equicontinuous; cp. Theorem 3.3.

Remark 4.2. (i) The ULLNs in Andrews (1987) and Pötscher and Prucha (1986a, 1989a) have been derived by the approach outlined in Corollary 4.4. These ULLNs transform strong [weak] local laws of large numbers, i.e., (4.7), into strong [weak] ULLNs. The proofs in both papers proceed by verifying the so-called first-moment continuity condition, which is actually, as remarked earlier, a first-moment equicontinuity-type condition. In the present terminology this condition amounts to the property that the q_t are $\mathrm{CL_1EC}$, which is equivalent to $\mathrm{UCL_1EC}$, as Θ is assumed to be compact in those papers. Since $\mathrm{UCL_1EC}$ and not only $\mathrm{UACL_1EC}$ is verified in those papers, equicontinuity (which coincides with uniform equicontinuity since Θ is compact) and not only asymptotic equicontinuity of $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ is obtained.

(ii) By essentially following the approach outlined in Corollary 4.3, Newey (1989) as well as Andrews (1989) obtained versions of the ULLNs in Andrews (1987) and Pötscher and Prucha (1989a); those versions transform *weak* pointwise laws of large numbers, i.e., the weak version of (4.6), into *weak* ULLNs. The proofs in Andrews (1989) and Newey (1989) proceed by verifying explicitly or implicitly that the q_t are $\mathrm{ACL_0UEC}$.

Comparing the approaches corresponding to Corollaries 4.3 and 4.4, we see that the first approach, which transforms pointwise laws of large numbers into ULLNs, requires more *uniformity* in the Cesàro equicontinuity-type condition than the second approach, which transforms local laws of large numbers into ULLNs; cp. the corresponding discussion after Theorems 3.1 and 3.2. Example 4 in section 6 shows that the Cesàro equicontinuity-type condition $\mathrm{UACL}_1\mathrm{EC}$ maintained in Corollary 4.4 (and even $\mathrm{UCL}_p\mathrm{EC}$ with arbitrarily large p) is in general not sufficient to allow the transformation of pointwise laws of large numbers into a ULLN.

For a compact parameter space, the condition in Corollary 4.4 that the sequence q_t is $\mathrm{UACL}_1\mathrm{EC}$ reduces even to $\mathrm{ACL}_1\mathrm{EC}$. The former condition represents the appropriate assumption needed to cover the case of a totally bounded parameter space. In contrast, in the convergence i.p. part of Corollary 4.3 the condition that q_t is $\mathrm{ACL}_0\mathrm{UEC}$ has to be assumed even if Θ is compact [and this condition also represents the appropriate assumption for the case of totally bounded (Θ, ρ)]; cp. the corresponding discussion after Theorems 3.1 and 3.2.

The assumptions of Corollary 4.3 that q_t is a.s.ACUEC or $\mathrm{ACL}_0\mathrm{UEC}$ imply that q_t is $\mathrm{ACL}_1\mathrm{UEC}$ [since (4.4) is assumed to hold]. A potential advantage of the approach given in Corollary 4.4 is that it only requires the weaker Cesàro equicontinuity-type condition $\mathrm{UACL}_1\mathrm{EC}$ (or $\mathrm{ACL}_1\mathrm{EC}$ for compact Θ) for q_t. While various sufficient conditions are available to imply $\mathrm{UACL}_1\mathrm{EC}$ or even $\mathrm{ACL}_1\mathrm{UEC}$, sufficient conditions for a.s.ACUEC seem to be scarce as discussed in section 4.3 below. Hence, especially in order to derive strong ULLNs, it seems that the approach of Corollary 4.4 is more flexible than the approach of Corollary 4.3.

The assumption of pointwise rather than local laws of large numbers might be considered an advantage of the approach of Corollary 4.3. However, mixing type conditions on q_t, which are usually used to imply pointwise laws of large numbers, will typically carry over to mixing type conditions for the local bracketing functions $\sup_{\theta \in B(\theta', \delta)} q_t$ and $\inf_{\theta \in B(\theta', \delta)} q_t$; cp., e.g., Andrews (1987) and Pötscher and Prucha (1989a) for results regarding ergodic, α-mixing or ϕ-mixing processes, and Pötscher and Prucha (1991a) for results regarding L_p-approximable processes and near-epoch-dependent processes.

The above discussion of the different degrees of uniformity in the Cesàro equicontinuity-type conditions maintained by Corollaries 4.3 and 4.4 explains why Newey (1989) had to sharpen Andrews' (1987) local Lipshitz-type condition to hold globally in order to obtain a ULLN which is based on pointwise laws of large numbers rather than local laws of large numbers. (Recall from section 2.2 that local Lipshitz-type conditions are in general only sufficient to imply $\mathrm{UAL}_1\mathrm{EC}$ but not $\mathrm{AL}_1\mathrm{UEC}$ for $Q_n = n^{-1} \sum_{t=1}^{n} q_t$; see also section 4.3 and the discussion in Example 4 in section 6 below.)

The above discussion also helps to clarify the relationships of the ULLNs in Pötscher and Prucha (1989a) and Newey (1989). Pötscher and Prucha (1989a) verify from their catalogue of assumptions that q_t is $\mathrm{UCL}_1\mathrm{EC}$ which allows, in light of Corollary 4.4, the transformation of strong and weak local laws of large numbers into strong and weak ULLNs. Newey (1989) showed that the same catalogue of assumptions also allows the transformation of pointwise *weak* laws of large numbers into *weak* ULLNs; cp. also Andrews (1989). The latter result is possible since the catalogue in Pötscher and Prucha (1989a) happens to be such that it not only implies that the q_t are UCL$_1$EC but even CL$_1$UEC, as can, e.g., be seen from a simple modification of the proof in Pötscher and Prucha (1989a); see also Theorem 4.5 below. Hence also the assumptions for the convergence in probability part of Corollary 4.3 can be implied from the catalogue in Pötscher and Prucha (1989a). (It is less than obvious how one would imply the assumptions of the a.s. part of Corollary 4.3 from that catalogue of assumptions.) The ULLN given in Pötscher and Prucha (1989a) is essentially a special case of the ULLN in Pötscher and Prucha (1989b). It is therefore interesting to note that Example 4 in section 6 shows that under the weakened assumptions of the latter ULLN the assumption of the existence of local laws of large numbers can now no longer be replaced by that of pointwise laws of large numbers.

4.3. *Sufficient conditions for Cesàro equicontinuity and ULLNs*

Various sets of sufficient conditions are available to imply that q_t is UACL$_1$EC or ACL$_0$UEC; cp. Andrews (1987, 1989c), Newey (1989), Pötscher and Prucha (1986a, 1989a, b). In light of Corollaries 4.3 and 4.4 those conditions then permit the derivation of weak and strong ULLNs based on local laws of large numbers or weak ULLNs based on pointwise laws of large numbers. In contrast, the only simple and useful sufficient condition implying that q_t is a.s.ACUEC (or more directly that $n^{-1}\sum_{t=1}^{n} q_t$ is a.s.AUEC), which – in light of Corollary 4.3 (or Theorem 3.1) – then permit strong ULLNs based on strong pointwise laws of large numbers, seems to be a Lipshitz-type condition as will be discussed later in this section.

In the following we now discuss several sets of sufficient conditions for the assumptions of Corollaries 4.3 and 4.4. We introduce the following assumption.[18]

Assumption 4.1. Let $(z_t)_{t\in\mathbb{N}}$ be a stochastic process on $(\Omega,\mathscr{A},P)$ taking its values in Z, where $(Z,\mathscr{Z})$ is a measurable space.

(a) Let $q_t(\theta)=\sum_{k=1}^{K} r_{kt}(z_t)s_{kt}(z_t,\theta)$, where the r_{kt} are real functions on Z which are $\mathscr{Z}$-measurable and satisfy $\sup_n n^{-1}\sum_{t=1}^{n}\mathrm{E}|r_{kt}(z_t)|<\infty$ for all $1\le k\le K$. The s_{kt} are real functions on $Z\times\Theta$ which are $\mathscr{Z}$-measurable for each

[18]As remarked above all suprema and infima as, e.g., $\sup_{\theta\in B(\theta',\delta_k)} q_t(\omega,\theta)$ are assumed to be (P-a.s.) measurable functions on Ω. If $q_t(\omega,\theta)$ is of the form $s_t(z_t,\theta)$, it is often useful to know under which conditions such suprema and infima are $\mathscr{Z}$-measurable functions of z_t (or coincide with such functions a.s.), e.g., to know that $\sup_{\theta\in B(\theta',\delta_k)} s_t(z,\theta)$ is $\mathscr{Z}$-measurable. This is often helpful for the transfer of mixing properties of the process z_t to such suprema and infima, which then allows straightforward verification of local laws of large numbers. Cp., e.g., Pötscher and Prucha (1989a, p. 676) and Lemma A2 and A5 in Pötscher and Prucha (1989b).

$\theta \in \Theta$, and for a sequence of sets (K_m) with $K_m \in \mathscr{Z}$ the families $\{s_{kt}(z,.): z \in K_m, t \geq 1\}$, $1 \leq k \leq K$, satisfy the following uniform asymptotic equicontinuity-type condition:

$$\sup_{\theta' \in \Theta} \overline{\lim_{t \to \infty}} \sup_{z \in K_m} \sup_{\theta \in B(\theta', \delta)} |s_{kt}(z, \theta) - s_{kt}(z, \theta')| \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.8}$$

(b) The sequence (K_m) also satisfies

$$\lim_{m \to \infty} \left\{ \overline{\lim_{n \to \infty}} \, n^{-1} \sum_{t=1}^{n} \mathrm{P}(z_t \notin K_m) \right\} = 0. \tag{4.9}$$

The following theorem is deduced from Corollaries 4.3 and 4.4 by showing that under its assumptions q_t is $\mathrm{ACL_1UEC}$. As a result we obtain both weak and strong ULLNs from Corollary 4.4, but only a weak ULLN from Corollary 4.3.

Theorem 4.5. Assume that (Θ, ρ) is totally bounded, that Assumption 4.1 holds, and that (4.4) is satisfied for some $p \geq 1$. If the weak pointwise laws of large numbers defined in (4.6) or the weak local laws of large numbers defined in (4.7) hold [If the strong local laws of large numbers defined in (4.7) hold], then:[19]

(a) $\sup_{\theta \in \Theta} |n^{-1} \sum_{t=1}^{n} [q_t(\omega, \theta) - \mathrm{E}q_t(\omega, \theta)]| \to$ *i.p. [a.s.] as* $n \to \infty$,
(b) $n^{-1} \sum_{t=1}^{n} \mathrm{E}q_t$ *is asymptotically uniformly equicontinuous. [If* $\overline{\lim}$ *in (4.8) and (4.9) is replaced with* sup, *then* $n^{-1} \sum_{t=1}^{n} \mathrm{E}q_t$ *is uniformly equicontinuous.]*

Condition (4.8) has appeared in the literature in different guises: It is a generalization of condition (Ia) in Pötscher and Prucha (1989b) for noncompact Θ. A version of (4.8) is also verified in the proof of Pötscher and Prucha's (1989a) ULLN; cp. Lemma A1 in that paper. In order to generalize Pötscher and Prucha's (1989a) ULLN to noncompact Θ, Andrews (1989c) introduced a close relative of Assumption 4.1, which he labeled TSE-2. However, as shown in Example 6 in section 6 below, Andrew's (1989c) condition TSE-2 is not sufficient to allow the derivation of a ULLN, and hence the parts of his Lemma 4 and Theorem 5 corresponding to TSE-2 are not valid.

Given (Θ, ρ) is totally bounded it follows from Lemma A.1 that (4.8) is equivalent to the (formally stronger) condition

$$\overline{\lim_{t \to \infty}} \sup_{z \in K_m} \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |s_{kt}(z, \theta) - s_{kt}(z, \theta')| \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.8'}$$

[19] As in Corollary 4.3 the pointwise laws of large numbers only have to hold for θ in a dense subset of Θ. As in Corollary 4.4 the local laws of large numbers are assumed to hold for all $\theta' \in \Theta$ and a sequence δ_k as in that corollary.

If Θ is compact, Lemma A.1 implies further that (4.8) as well as (4.8′) are each equivalent to

$$\overline{\lim_{t\to\infty}} \sup_{z\in K_m} \sup_{\theta\in B(\theta',\delta)} |s_{kt}(z,\theta)-s_{kt}(z,\theta')| \to 0 \quad \text{as} \quad \delta\to 0, \ \forall\theta'\in\Theta. \tag{4.8″}$$

The equivalence of (4.8) and (4.8′) for totally bounded (Θ, ρ) explains why it is possible to establish that q_t is ACL$_1$UEC and not only UACL$_1$EC in the proof of Theorem 4.5. (This observation is closely related to the discussion in the last paragraph of section 4.2.)

In the following remark we discuss several sufficient conditions for Assumption 4.1.

Remark 4.3. (i) Assumption 4.1(b) can usually be implied by weak moment conditions on the marginal distributions of z_t or asymptotic stationarity assumptions on z_t; see Pötscher and Prucha (1989a, b) for details. If the sets K_m can be chosen to be compact, then Assumption 4.1(b) becomes an asymptotic tightness condition for the average of the marginal distributions of z_t.

(ii) Let (Θ, ρ) be a totally bounded metric space and (Z, ν) a metric space. Define the distance between two points (z, θ) and (z', θ') in $Z\times\Theta$ by $\max\{\nu(z,z'), \rho(\theta,\theta')\}$. (Of course, this metric induces the product topology on $Z\times\Theta$.) Let $s_{kt|K_m\times\Theta}$ denote the restriction of s_{kt} to $K_m\times\Theta$. A sufficient condition for (4.8) is that the families $\{s_{kt|K_m\times\Theta}\colon t\geq 1\}$ are asymptotically uniformly equicontinuous on $K_m\times\Theta$. (Of course, this condition is in turn implied if the families $\{s_{kt}\colon t\geq 1\}$ are asymptotically uniformly equicontinuous on $Z\times\Theta$. However, except for, e.g., compact Z, this latter condition is rather restrictive.) For the important case where the sets K_m are compact, a sufficient condition for the families $\{s_{kt|K_m\times\Theta}\colon t\geq 1\}$ to be asymptotically uniformly equicontinuous on $K_m\times\Theta$ (which coincides with uniformly asymptotically equicontinuous since $K_m\times\Theta$ is totally bounded w.r.t. the above metric) is that for all $z'\in Z$

$$\sup_{\theta'\in\Theta} \overline{\lim_{t\to\infty}} \sup_{(z,\theta)\in B((z',\theta'),\delta)} |s_{kt}(z,\theta)-s_{kt}(z',\theta')| \to 0 \quad \text{as} \quad \delta\to 0,$$

where $B((z',\theta'),\delta)$ is the open ball with center (z',θ') and radius δ in $Z\times\Theta$; cp. Lemma A.5.[20]

[20]Lemma A.5 actually shows that the sufficient condition can be slightly weakened to: for all $z'\in K_m$

$$\sup_{\theta'\in\Theta} \overline{\lim_{t\to\infty}} \sup_{(z,\theta)\in B^*((z',\theta'),\delta)} |s_{kt}(z,\theta)-s_{kt}(z',\theta')| \to 0 \quad \text{as} \quad \delta\to 0,$$

where $B^*((z',\theta'),\delta)$ is the open ball with center (z',θ') and radius δ in $K_m\times\Theta$.

(iii) Let Θ be compact and let (Z, v) be a metric space. Then condition (4.8) reduces to condition (4.8″). Suppose further that the sets K_m are compact: Then similar as in (ii) a sufficient condition for (4.8″), and hence for (4.8), is in view of Lemma A.1 that the families $\{s_{kt|K_m \times \Theta}: t \geq 1\}$ are asymptotically equicontinuous on $K_m \times \Theta$. This in turn is clearly implied by the condition that $\{s_{kt}: t \geq 1\}$ is asymptotically equicontinuous on $Z \times \Theta$, i.e., for all $(z', \theta') \in Z \times \Theta$

$$\overline{\lim_{t \to \infty}} \sup_{(z,\theta) \in B((z',\theta'),\delta)} |s_{kt}(z, \theta) - s_{kt}(z', \theta')| \to 0 \quad \text{as} \quad \delta \to 0.$$

Pötscher and Prucha (1989a) used the slightly stronger condition that $\{s_{kt}: t \geq 1\}$ is equicontinuous on $Z \times \Theta$ as a basic assumption of their ULLN.

(iv) Clearly, if the averages of the marginal distributions of z_t are tight and if (4.8) – or any of the sufficient conditions given in (ii) and (iii) – holds for any compact set K_m, then Assumption 4.1 holds.

A further sufficient condition for the basic condition in Corollary 4.3, namely that q_t is $\mathrm{ACL_0UEC}$, is Andrews' (1989c) condition TSE-1. This condition may be useful for certain processes but only applies to processes with a limited degree of heterogeneity. To see this, consider the following example: Suppose the process z_t satisfies $\mathrm{P}(z_t = e_t) \geq \alpha > 0$ for all $t \geq 1$, where e_t is a sequence such that $e_t \neq e_s$ for $t \neq s$. Then TSE-1 is violated as is easily seen by choosing $A_m = \{e_m, e_{m+1}, \ldots\}$ in TSE-1.[21] This example also shows that TSE-1 cannot be inferred from simple moment conditions on z_t.

Next we discuss how Lipshitz-type conditions can be employed to establish ULLNs. This discussion draws on section 2.2, which shows how Lipshitz-type conditions can be used to imply equicontinuity-type conditions for random functions.

Observe that, whenever q_t satisfies a global or local Lipshitz-type condition with Lipshitz bound b_t, then clearly $Q_n = n^{-1} \sum_{t=1}^{n} q_t$ satisfies the global or local Lipshitz-type condition (2.6) or (2.8), respectively, with Lipshitz bound $B_n = n^{-1} \sum_{t=1}^{n} b_t$. The equicontinuity-type conditions on Q_n in Theorem 3.1, i.e., a.s.AUEC or $\mathrm{AL_0UEC}$, then follows if q_t satisfies a global Lipshitz-type condition with Lipshitz bound b_t and if $B_n = n^{-1} \sum_{t=1}^{n} b_t$ satisfies (2.7c) or (2.7b). A ULLN based on pointwise laws of large numbers can now be obtained directly from Theorem 3.1. [Alternatively, the Cesàro equicontinuity-type conditions on q_t in Corollary 4.3, i.e., a.s.ACUEC or $\mathrm{ACL_0UEC}$, follow if q_t satisfies a global Lipshitz-type condition with Lipshitz bound b_t and $B_n = n^{-1} \sum_{t=1}^{n} b_t$

[21] Of course, we assume that $\{e_t\} \in Z$. More generally TSE-1 is violated if z_t is such that $\mathrm{P}(z_t \in E_t) \geq \alpha > 0$ for all $t \geq 1$ where $E_t \in \mathscr{Z}$ is a pairwise disjoint sequence; to see this put $A_m = \bigcup_{t \geq m} E_t$ in TSE-1.

satisfies (2.7c) or $\overline{\lim}_{n\to\infty} n^{-1}\sum_{t=1}^{n} \mathrm{P}(b_t > M) \to 0$ as $M \to \infty$. A ULLN based on pointwise laws of large numbers can then be obtained directly from Corollary 4.3.]

Furthermore, the Cesàro equicontinuity-type condition on q_t in Corollary 4.4, i.e., UACL$_1$EC, follows if q_t satisfies a local Lipshitz-type condition of the form (2.8) with Lipshitz bound b_t, where η does not depend on θ', and $B_n(\theta') = n^{-1}\sum_{t=1}^{n} b_t(\theta')$ satisfies (2.10a) with $p = 1$. [As noted in Lemma A.2, if Θ is compact, UACL$_1$EC reduces to ACL$_1$EC, and then it suffices to verify a Lipshitz-type condition of the form (2.8), where η may now depend on θ', and the simpler condition (2.9a) for $B_n(\theta') = n^{-1}\sum_{t=1}^{n} b_t(\theta')$.] A ULLN based on local laws of large numbers can then be obtained directly from Corollary 4.4.

Global Lipshitz-type conditions have been used in Andrews (1989c) and Newey (1989) and local Lipshitz-type conditions have been used in Andrews (1987), respectively, to derive ULLNs.

4.4. *ULLNs based on a truncation approach*

Apart from Lipshitz-type conditions another sufficient condition for $Q_n = n^{-1}\sum_{t=1}^{n} q_t$ to be a.s.AUEC would be Hoadley's (1971) assumption that q_t is a.s. uniformly equicontinuous (which for compact Θ coincides with a.s. equicontinuity on Θ). But, as discussed in Andrews (1987) and Pötscher and Prucha (1986b, 1989a), this condition is very restrictive for typical applications. (Observe that the condition that the sequence q_t is a.s. uniformly equicontinuous is far more restrictive than the condition that q_t is a.s. Cesàro uniformly equicontinuous or that $Q_n = n^{-1}\sum_{t=1}^{n} q_t$ is a.s. uniformly equicontinuous.) However, this is not necessarily the case if the a.s. uniform equicontinuity assumption is made for suitably truncated versions of the q_t. Pötscher and Prucha (1986b, 1989b), motivated by this observation, introduced a general truncation device that gives conditions under which ULLNs for truncated versions of q_t imply a ULLN for the functions q_t themselves. We emphasize that the truncation device depends only on the *existence* of a ULLN for the truncated versions of the q_t (and not on the particular catalogue of sufficient conditions from which it may have been derived); cp. Pötscher and Prucha (1989b, lemma 1).

The ULLN given as Theorem 2 in Pötscher and Prucha (1989b) assumes that Θ is compact and that the truncated versions of q_t are a.s. equicontinuous on Θ.[22] The proof of that ULLN proceeded by first verifying a ULLN for the truncated versions of q_t along the lines of Corollary 4.4 and then by applying the truncation device. The truncation device only assumes that Θ is a metric space and hence does not rely on the compactness of Θ. Therefore we can use the

[22]The following discussion relates to the version of that theorem which maintains Assumption 2′ of that paper.

truncation device and Corollary 4.4 to obtain a version of Theorem 2 in Pötscher and Prucha (1989b) for totally bounded Θ, if we assume that the truncated versions of q_t are a.s. uniformly equicontinuous on Θ. In the following we now develop variants of that theorem based on pointwise and local laws of large numbers using the truncation device and Corollaries 4.3 and 4.4.

More specifically, assume that $(z_t)_{t\in\mathbb{N}}$ is a stochastic process on $(\Omega, \mathscr{A}, P)$ taking its values in Z, where $(Z, \mathscr{Z})$ is a measurable space. Furthermore, let $q_t(\theta) = s_t(z_t, \theta)$, where each s_t is a real function on $Z \times \Theta$ which is $\mathscr{Z}$-measurable for each $\theta \in \Theta$. For a sequence of sets $(K_m)_{m\in\mathbb{N}}$ with $K_m \in \mathscr{Z}$ let $s_{t,m}(z, \theta) = s_t(z, \theta)\mathbf{1}_{K_m}(z)$, let $d_{t,m} = \sup_{\theta\in\Theta}|s_t(z_t, \theta)\mathbf{1}_{K_m}(z_t)|$, and let $d_{t,m,c} = \sup_{\theta\in\Theta}|s_t(z_t, \theta)\mathbf{1}_{Z-K_m}(z_t)|$.

Assumption 4.2. For a sequence of sets $(K_m)_{m\in\mathbb{N}}$ with $K_m \in \mathscr{Z}$ let for each $m\in\mathbb{N}$ the sequence of random functions $s_{t,m}(z_t, \theta)$ be a.s.UAEC. Furthermore let

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}}\, n^{-1} \sum_{t=1}^{n} \mathrm{E}d_{t,m,c} = 0, \tag{4.10}$$

and let $d_{t,m,c}$ satisfy a weak [strong] law of large numbers for each $m\in\mathbb{N}$.

Theorem 4.6. Assume that (Θ, ρ) is totally bounded, that Assumption 4.2 holds, and that (4.4) is satisfied for some $p \geq 1$. Given that for each $m\in\mathbb{N}$ the sequence $s_{t,m}(z_t, \theta)$ satisfies weak [strong] pointwise laws of large numbers or weak [strong] local laws of large numbers, then:[23]

(a) $\sup_{\theta\in\Theta}|n^{-1}\sum_{t=1}^{n}[q_t(\omega, \theta) - \mathrm{E}q_t(\omega, \theta)]| \to 0$ *i.p. [a.s.] as* $n \to \infty$,

(b) $n^{-1}\sum_{t=1}^{n}\mathrm{E}q_t$ *is asymptotically uniformly equicontinuous. [If* $\overline{\lim}$ *in (4.10) is replaced with* sup, *and if a.s.UAEC in Assumption 4.2 is replaced by a.s.UEC, then* $n^{-1}\sum_{t=1}^{n}\mathrm{E}q_t$ *is uniformly equicontinuous.]*

Remark 4.4. (i) The part of Theorem 4.6 based on pointwise laws of large numbers also holds if condition (4.4) is replaced by (2.4a) and if $\mathrm{E}d_t < \infty$ for all $t\in\mathbb{N}$ is assumed. (This can be shown by using Theorem 3.1 rather than Corollary 4.3.)

(ii) In view of Lemma A.2 the condition that $s_{t,m}(z_t, \theta)$ is a.s.UAEC clearly is equivalent to a.s.AUEC since (Θ, ρ) is assumed to be totally bounded. For compact Θ it is even equivalent to a.s.AEC on Θ; cp. Assumption 2′ in Pötscher and Prucha (1989b). If Θ is compact, the part of Theorem 4.6 based on local

[23]Cp. footnote 19. We also note that the dense subsets on which the pointwise laws of large numbers for $s_{t,m}$ are assumed to hold may depend on m.

laws of large numbers also holds if a.s.AEC on Θ is weakened to a.s.AEC at θ' for all $\theta' \in \Theta$; cp. also Assumption 2 in Pötscher and Prucha (1989b).

(iii) Condition (4.8) applied to $s_t(z, \theta)$ is sufficient for $s_{t,m}(z_t, \theta)$ to be a.s.UAEC.

(iv) If only asymptotic uniform equicontinuity of $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ has been deduced from Theorem 4.6, uniform equicontinuity can be obtained by showing that $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ is continuous for each $n \in \mathbb{N}$; cp. Remark 2.3. (Of course, this continuity follows if $n^{-1}\sum_{t=1}^{n} q_t$ is assumed to be continuous and a uniform integrability condition holds.)

We note that Theorem 4.6 maintains the assumption that a law of large numbers holds for $d_{t,m,c}$. That is, similarly as in Theorem 6 of Andrews (1989c), we need in the above theorem the assumption that a law of large numbers holds for certain suprema, even if the theorem is based on pointwise laws of large numbers.

5. Compactness versus total boundedness

Uniform convergence results formulated for totally bounded and not only for compact parameter spaces are clearly convenient, as in applications parameter spaces of interest may, e.g., not be closed (as subsets of Euclidean space). In this section we show, however, that from a mathematical point of view uniform convergence results on a totally bounded parameter space are not really more general than those on a compact parameter space. More precisely, recall from Theorem 3.1 that (given $\bar{Q}_n$ is AUEC) for totally bounded (Θ, ρ) a.s. [i.p.] pointwise convergence of Q_n–$\bar{Q}_n$ to zero on a dense subset of Θ plus a.s.AUEC [$\mathrm{AL_0UEC}$] of Q_n is equivalent to a.s. [i.p.] uniform convergence of $Q_n - \bar{Q}_n$ to zero.[24] In the following we show that for a totally bounded parameter space it is always possible to extend the given functions Q_n and $\bar{Q}_n$ to a larger compact space in such a way that these equicontinuity-type conditions as well as the pointwise convergence property carry over to the extended functions on the larger and compact parameter space. That is, whereas the formulation of uniform convergence results in terms of a totally bounded parameter space is convenient, such results do not really cover a wider class of problems than uniform convergence results that assume a compact parameter space.

Recall the following elementary facts about metric spaces [see, e.g., Royden (1968)]: Every metric space (Θ, ρ) can be isometrically embedded into a complete metric space (Θ^*, ρ^*) as a dense subspace. (Θ^*, ρ^*) is unique up to isometries and is called the completion of (Θ, ρ). If we identify Θ with $i(\Theta)$,

[24]The assumption that $\bar{Q}_n$ is AUEC is no restriction of generality, as we can always replace Q_n by $Q_n - \bar{Q}_n$ and set $\bar{Q}_n$ equal to zero.

where i: $\Theta \to \Theta^*$ is the isometric embedding, then we can view Θ as a subspace of Θ^*. If (Θ, ρ) is totally bounded, then (Θ^*, ρ^*) is compact.

Lemma 5.1.[25] *Let (Θ, ρ) be a totally bounded metric space and let Q_n be a.s.AUEC on Θ [AL$_p$UEC on Θ for some $p \geq 0$]. Then there exists an extension Q_n^*: $\Omega \times \Theta^* \to \mathbb{R}$ which is a.s.AUEC on Θ^* [AL$_p$UEC on Θ^*], and is $\mathscr{A}$-measurable for each $\theta \in \Theta^*$.*

Clearly, it follows from the above lemma (as a nonstochastic special case) that if $\bar{Q}_n$ is asymptotically uniformly equicontinuous on the totally bounded space (Θ, ρ), then there exists an extension $\bar{Q}_n^*$: $\Theta^* \to \mathbb{R}$ which is asymptotically uniformly equicontinuous on Θ^*. Also the convergence of $Q_n(\theta) - \bar{Q}_n(\theta) \to 0$ a.s. [i.p.] for θ belonging to a dense subset Θ_0 of Θ automatically implies that $Q_n^*(\theta) - \bar{Q}_n^*(\theta) \to 0$ a.s. [i.p.] on a dense subset of Θ^*, since Θ_0 is also dense in Θ^*. Hence all assumptions maintained by Theorem 3.1(a) for Q_n (and $\bar{Q}_n$) on Θ also hold for the extended functions Q_n^* (and $\bar{Q}_n^*$) on Θ^*. Thus, whenever $\sup_{\theta\in\Theta} |Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.], then also $\sup_{\theta\in\Theta^*} |Q_n^*(\theta) - \bar{Q}_n^*(\theta)| \to 0$ a.s. [i.p]. Hence uniform convergence on a totally bounded parameter space can in principle always be reduced to uniform convergence on a compact parameter space.

Lemma 5.1 clearly is similar in spirit to the well-known fact that any uniformly continuous function on a metric space can be extended to the completion of the metric space as a uniformly continuous function.

6. Counter examples

Example 1: Let $\Theta = \Omega = [0, 1]$, let P be the Lebesgue measure, and let $Q_n(\omega, \theta) = \mathbf{1}_{\{\theta\}}(\omega)$. Then Q_n is UL$_p$EC (for all $p \geq 0$) as $\sup_{\theta' \in \Theta} \sup_n E \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p \leq 2\delta$. Although Θ is compact, Q_n is not AL$_p$UEC for any $p \geq 0$ since we have $\sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| = 1$. Furthermore note that $Q_n(\theta) \to 0$ a.s. for each $\theta \in \Theta$, since $Q_n(\theta) = 0$ a.s. for each $\theta \in \Theta$, but $\sup_{\theta\in\Theta} |Q_n(\theta)| = 1$, and hence no uniform convergence result holds.

Also the following a.s. continuous version of the above example is UL$_p$EC but not AL$_p$UEC: Let $f(x) = 1 - |x|$ for $|x| \leq 1$ and $f(x) = 0$ else and choose $Q_n(\omega, \theta) = f((\omega - \theta)n)$. ▮

Example 2: Choose $\Theta = \mathbb{R}$ with ρ as the usual metric, Q_n nonrandom, $Q_n(\theta) = nf(\theta - n)$, where f is defined as in Example 1 above. Then Q_n is not a.s.AUEC (and hence not UL$_p$EC since Q_n is nonrandom), but Q_n is a.s.UAEC

[25] Q_n^* is an extension of Q_n in the sense that $Q_n^*(\omega, \theta) = Q_n(\omega, \theta)$ holds for all $(\omega, \theta) \in \Omega \times \Theta$.

(which coincides here with UAL_pEC since Q_n is nonrandom). Furthermore each Q_n is a.s. uniformly continuous (which coincides here with L_p uniform continuity since Q_n is nonrandom). The fact that (Θ, ρ) is not totally bounded is essential in this example in view of Remark 2.1(ii). ▮

Example 3: Let Z_n be bounded in probability and satisfy $\overline{\lim}_{n\to\infty}|Z_n| = \infty$ a.s. [e.g., Z_n is i.i.d. N(0, 1)], $\Theta = [a, b]$, $Q_n(\theta) = \theta Z_n$. Then Q_n is not a.s.AEC at any θ', but Q_n is L_0UEC on Θ (and even L_pUEC if $\sup_n E|Z_n|^p < \infty$) since $\sup_n P(\sup_{\theta'\in\Theta}\sup_{\theta\in B(\theta',\delta)}|Q_n(\theta) - Q_n(\theta')| > \varepsilon) \leq \sup_n P(|Z_n| > \varepsilon/\delta) \to 0$ as $\delta \to 0$ in view of the assumed boundedness in probability. (Note that each Q_n is of course continuous in θ for all $\omega\in\Omega$.) ▮

Example 4: Let $\Theta = Z = \Omega = [0, 1]$, let P be the Lebesgue measure, put $z_t(\omega) = \omega$, and $q_t(z_t, \theta) = \mathbf{1}_{\{\theta\}}(\omega)$. Hence $Q_n(\theta) = n^{-1}\sum_{t=1}^n q_t(z_t, \theta) = \mathbf{1}_{\{\theta\}}(\omega)$. From Example 1 we know that Q_n is UL_pEC for all $p \geq 0$, but not AL_pUEC. (Since q_t is independent of t, it follows that q_t is UCL_pEC but not ACL_pUEC.) As noted in Example 1, $Q_n(\theta) = n^{-1}\sum_{t=1}^n q_t(z_t, \theta) \to 0$ a.s. as $n \to \infty$ for each $\theta\in\Theta$, i.e., the q_t satisfy pointwise laws of large numbers since $Eq_t(z_t, \theta) = 0$. However, as pointed out in Example 1, neither a weak nor a strong ULLN holds since $\sup_{\theta\in\Theta}|Q_n(\theta) - EQ_n(\theta)| = \sup_{\theta\in\Theta}|Q_n(\theta)| = 1$ for all $\omega\in\Omega$.

The given example clearly satisfies Assumptions 1, 2, 3 and 4(b) in Pötscher and Prucha (1989a) with $K_m = Z = [0, 1]$. Since no ULLN holds it follows from Theorem 2 in Pötscher and Prucha (1989a) that the bracketing functions q_t^* and q_{t*} do not satisfy (weak or strong) laws of large numbers. This example hence shows that in Theorem 2 in Pötscher and Prucha (1989a) the assumption of the existence of local laws of large numbers cannot be replaced by the assumption of the existence of pointwise laws of large numbers. The example also satisfies all assumptions of Andrews' (1987) ULLN based on Lipshitz-type conditions [with, e.g., $h(x) = x^{1/2}$] except the local laws of large numbers. This shows that the Lipshitz-type conditions have to be assumed to hold globally in order to imply a ULLN from pointwise laws of large numbers.

The example exploits the fact that Assumption 2′ (and even an asymptotic version of this assumption) but not Assumption 2 in Pötscher and Prucha (1989a) is violated, i.e., the null sets, on which equicontinuity fails, depend on θ.[26] ▮

The next example shows that it is possible that the assumptions for the i.p. part of Corollary 4.4 are satisfied (and hence that a weak ULLN holds), but that

[26]If $q_t(z_t, \theta)$ is defined as $a_t\mathbf{1}_{\{\theta\}}(\omega)$ with $a_t\in\mathbb{R}$, $a_t > 0$, $a_t \to 0$, then Assumption 2′ of Pötscher and Prucha (1989a) still fails. However, in this case a strong ULLN holds, since Assumptions 1, 2, 3, 4(a), 4(b) in Pötscher and Prucha (1989a) are satisfied. Note that in this modified example an 'asymptotic' version of Assumption 2′ holds.

no strong ULLN holds despite the existence of strong pointwise laws of large numbers.

Example 5: Let $\Theta = \Omega = [0, 1]$, let P be the Lebesgue measure, $q_t(\omega, \theta) = a_t(\omega)\mathbf{1}_{\{\theta\}}(\omega)$, where a_t satisfy $0 \le a_t \le 1$ and $n^{-1}\sum_{t=1}^n a_t \to 0$ i.p. but not a.s.; clearly such a sequence exists. Since $\mathrm{E}q_t(\omega, \theta) = 0$ and $\sup_{\theta\in\Theta}|n^{-1}\sum_{t=1}^n q_t(\omega, \theta)| = (n^{-1}\sum_{t=1}^n a_t(\omega))\sup_{\theta\in\Theta}\mathbf{1}_{\{\theta\}}(\omega) = n^{-1}\sum_{t=1}^n a_t(\omega)$ it follows immediately that in this example a weak but not a strong ULLN holds. Clearly $q_t(\omega, \theta)$ satisfies a strong law of large numbers for each θ, as $q_t(\omega, \theta)$ is a.s. equal to zero for each θ. Furthermore $q_t(\omega, \theta)$ satisfies weak local laws of large numbers since $\sup_{\theta\in(\theta'-\delta,\theta'+\delta)} q_t(\omega, \theta) = a_t(\omega)\mathbf{1}_{(\theta'-\delta,\theta'+\delta)}(\omega) \le a_t(\omega)$ and $\inf_{\theta\in(\theta'-\delta,\theta'+\delta)} q_t(\omega, \theta) = 0$. Clearly, $q_t(\omega, \theta)$ is also $\mathrm{UCL_pEC}$ for all $p \ge 0$ and hence all assumptions of the i.p. part of Corollary 4.4 are satisfied. ▌

The following example shows that Theorem 5 in Andrews (1989c) is incorrect; cp. the discussion after Theorem 4.5.

Example 6: Choose $Z = \{0\} \cup \{a^{-i}: i\in\mathbb{N}\} \cup \{-a^{-i}: i\in\mathbb{N}\}$ and $\Theta = \{ca^{-i}: i\in\mathbb{N}\} \cup \{-ca^{-i}: i\in\mathbb{N}\}$, with $a > 2$ and $c = (a+1)/(2a)$. Let $z_t = a^{-t}\xi_t$, where ξ_t is i.i.d. with $\mathrm{P}(\xi_t = 1) = \mathrm{P}(\xi_t = -1) = 1/2$. Define $q(z, \theta) = \mathrm{sign}(z)/(|z| - \theta)$ for $z \ne 0$ and $q(0, \theta) = 0$. Observe that the points in Θ are the midpoints of adjacent points in Z. Hence Θ and Z are disjoint and $q(z, \theta)$ is well-defined on $Z \times \Theta$. Clearly, Θ (with the standard metric) is totally bounded. Furthermore, $\mathrm{E}q(z_t, \theta) = 0$ for all $\theta\in\Theta$ since ξ_t is symmetrically distributed and $q(.,\theta)$ is antisymmetric. Observe that for each $\theta = \pm ca^{-i}$ we have $||z_t| - \theta| \ge (a-1)/(2a^{i+1}) > 0$ for all t. Hence the variance of $q(z_t, \theta)$ is bounded in t for each $\theta\in\Theta$. Therefore $q(z_t, \theta)$ satisfy strong pointwise laws of large numbers as the conditions of Kolmogorov's strong law of large numbers are satisfied. Next observe that $\sup_{\theta\in\Theta}|q(z, \theta)| = 2a^{j+1}/(a-1)$ if $z = \pm a^{-j}$ and $\sup_{\theta\in\Theta}|q(0, \theta)| = 0$. Therefore, $\overline{\lim}_{n\to\infty} n^{-1}\sum_{t=1}^n \mathrm{E}d_t\mathbf{1}(d_t > M) = \overline{\lim}_{n\to\infty} n^{-1}\sum_{t=1}^{L(M)} 2a^{t+1}/(a-1) = 0$, where $L(M)$ is the smallest integer such that $2a^{L(M)+2}/(a-1) > M$. This shows that the domination condition DM in Andrews (1989c) is satisfied. Next we show that also Assumption TSE-2 in Andrews (1989c) is satisfied: Put $K = 1$, $r_{kt} \equiv 1$, $s_{kt} = q$, and choose $C_j = \{0\} \cup \{a^{-i}: i \le j\} \cup \{-a^{-i}: i \le j\}$, which are clearly compact, nondecreasing, and whose union is Z. By construction of Z and Θ we have for each $z = \pm a^{-i}$ that $\inf_{\theta\in\Theta}||z| - \theta| \ge (a-1)/(2a^{i+1}) > 0$. Hence $q(z, .)$ is uniformly continuous on Θ for any given $z \ne 0$. [Of course $q(0, .)$ is also uniformly continuous on Θ.] Since C_j is finite, $q(z, .)$ is continuous in θ uniformly over $\theta\in\Theta$ and $z\in C_j$. Hence Assumption TSE-2(a) is satisfied. Assumption TSE-2(b) holds trivially. Assumption TSE-2(c) is satisfied if we choose $\tilde{C}_j \equiv Z$ since Z is compact. Consequently, all assumptions in Theorem 5 in Andrews (1989c) are satisfied but a ULLN does not hold as is seen from the following argument:

$$A_n = \sup_{\theta\in\Theta}\left| n^{-1}\sum_{t=1}^{n}[q(z_t,\theta)-\mathrm{E}q(z_t,\theta)]\right| = \sup_{\theta\in\Theta}\left| n^{-1}\sum_{t=1}^{n}\xi_t/(a^{-t}-\theta)\right|$$

$$\geq \left| n^{-1}\sum_{t=1}^{n}\xi_t a^t\right|,$$

since for any $n \geq 1$ the map $x \to n^{-1}\sum_{t=1}^{n}\xi_t/(a^{-t}-x)$ is continuous in a neighborhood of zero and since zero is a limiting point of Θ. From $a > 2$ we have that $a^n - \sum_{t=1}^{n-1} a^t = a^n(a-2)/(a-1) + a/(a-1) > 0$. Hence $A_n \geq n^{-1}|a^n - |\sum_{t=1}^{n-1}\xi_t a^t||$ $\geq n^{-1}|a^n - \sum_{t=1}^{n-1}a^t| = n^{-1}\{a^n(a-2)/(a-1) + a/(a-1)\} \to\infty$ as $n\to\infty$. ∎

Appendix

Lemma A.1. *Let (Y,d) be a metric space, let X be a set, let f_j: $X\times Y\to\mathbb{R}$ for $j\geq 1$ be a sequence of functions, and let $B(y',\delta)=\{y\in Y\text{: } d(y,y')<\delta\}$. Consider the following conditions*:

(1) $\overline{\lim}_{j\to\infty}\sup_{x\in X}\sup_{y'\in Y}\sup_{y\in B(y',\delta)}|f_j(x,y)-f_j(x,y')| \to 0$ *as* $\delta\to 0$,

(2) $\sup_{y'\in Y}\overline{\lim}_{j\to\infty}\sup_{x\in X}\sup_{y\in B(y',\delta)}|f_j(x,y)-f_j(x,y')| \to 0$ *as* $\delta\to 0$,

(3) $\overline{\lim}_{j\to\infty}\sup_{x\in X}\sup_{y\in B(y',\delta)}|f_j(x,y)-f_j(x,y')| \to 0$ *as* $\delta\to 0$ *for all* $y'\in Y$.

We then have:

(a) *(1) ⇒ (2) ⇒ (3).*

(b) *If (Y,d) is totally bounded, then (1) ⇔ (2) ⇒ (3).*

(c) *If Y is compact, then (1) ⇔ (2) ⇔ (3).*

(If $\overline{\lim}$ *is replaced by* sup, *then the analogous implications hold also, and (1) and (2) coincide trivially.)*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial. We first show (3) ⇒ (1) for compact Y. From condition (3) we have that for every $\eta>0$ and $y'\in Y$ there exists a $\delta(\eta,y')>0$ such that for $0<\delta\leq\delta(\eta,y')$ we have $\overline{\lim}_{j\to\infty}\sup_{x\in X}\sup_{y\in B(y',\delta)}|f_j(x,y)-f_j(x,y')|<\eta$. Hence there exists an index $m(y',\delta(\eta,y'))$ such that for each $y'\in Y$, $m\geq m(y',\delta(\eta,y'))$, and $0<\delta\leq\delta(\eta,y')$

$$\sup_{j\geq m}\sup_{x\in X}\sup_{y\in B(y',\delta)}|f_j(x,y)-f_j(x,y')|<\eta, \tag{A.1}$$

observing that the expression on the l.h.s. is monotone in δ. By compactness of Y we can find finitely many open balls $B(y_i', \delta(\eta, y_i')/2)$, $1 \le i \le K = K(\eta)$, covering Y. Define $\delta(\eta) = \min\{\delta(\eta, y_i')\colon 1 \le i \le K\}$. Then we have for all $j \in \mathbb{N}$ and all $x \in X$, $y' \in Y$:

$$\sup_{y \in B(y', \delta(\eta)/2)} |f_j(x, y) - f_j(x, y')| \le 2 \sup_{y \in B(y_i', \delta(\eta, y_i'))} |f_j(x, y) - f_j(x, y_i')|,$$

where i is an index for which $d(y_i', y') < \delta(\eta, y_i')/2$, observing that $d(y, y') < \delta(\eta)/2$ and $d(y_i', y') < \delta(\eta, y_i')/2$ imply $d(y, y_i') < \delta(\eta, y_i')$. Hence for all $j \in \mathbb{N}$,

$$\sup_{y' \in Y} \sup_{x \in X} \sup_{y \in B(y', \delta(\eta)/2)} |f_j(x, y) - f_j(x, y')|$$

$$\le 2 \sup_{x \in X} \max_{1 \le i \le K} \sup_{y \in B(y_i', \delta(\eta, y_i'))} |f_j(x, y) - f_j(x, y_i')|.$$

Now choose $m \ge \max\{m(y_i', \delta(\eta, y_i'))\colon 1 \le i \le K(\eta)\}$. It then follows from (A.1) that

$$\overline{\lim_{j \to \infty}} \sup_{y' \in Y} \sup_{x \in X} \sup_{y \in B(y', \delta(\eta)/2)} |f_j(x, y) - f_j(x, y')|$$

$$\le 2 \sup_{j \ge m} \sup_{x \in X} \max_{1 \le i \le K} \sup_{y \in B(y_i', \delta(\eta, y_i'))} |f_j(x, y) - f_j(x, y_i')| \le 2\eta.$$

This establishes condition (1) observing that the l.h.s. of the last inequality does not increase if $\delta(\eta)/2$ in that expression is replaced by some $\delta \le \delta(\eta)/2$. This proves the claim for Y compact. That the implication (2) $\Rightarrow$ (1) also holds for (Y, d) totally bounded can be shown analogously, observing that $\delta(\eta, y')$ can now be chosen independently of y' and hence a finite cover of balls $B(y_i', \delta(\eta)/2)$ exists. The proof for the claim in the parenthesis is analogous. ∎

Lemma A.2. (a) *Let (Θ, ρ) be a totally bounded metric space. Then Q_n is a.s.UAEC on Θ $\Leftrightarrow$ Q_n is a.s.AUEC on Θ.*
(b) *Let (Θ, ρ) be a compact metric space and $p \ge 0$, then:*

(b1) *Q_n is AL_pEC [L_pEC] on Θ $\Leftrightarrow$ Q_n is UAL_pEC [UL_pEC] on Θ.*
(b2) *Q_n is a.s.AEC [a.s.EC] on Θ $\Leftrightarrow$ Q_n is a.s.UAEC [a.s.UEC] on Θ $\Leftrightarrow$ Q_n is a.s.AUEC [a.s.UEC] on Θ.*
(b3) *q_t is ACL_pEC [CL_pEC] on Θ $\Leftrightarrow$ q_t is $UACL_pEC$ [UCL_pEC] on Θ.*
(b4) *q_t is a.s.ACEC [a.s.CEC] on Θ $\Leftrightarrow$ q_t is a.s.UACEC [a.s.UCEC] on Θ.*

Proof. Parts (a) and (b2) follow easily from Lemma A.1 choosing X as a set containing exactly one element. To prove (b1) we first show that

$\mathrm{AL_pEC} \Rightarrow \mathrm{UAL_pEC}$ for $p > 0$. Given Q_n is $\mathrm{AL_pEC}$, then for every $\eta > 0$ and $\theta' \in \Theta$ there exists a $\delta(\eta, \theta') > 0$ and an index $m(\theta', \delta(\eta, \theta'))$ such that for each $\theta' \in \Theta$, $m \geq m(\theta', \delta(\eta, \theta'))$, and $0 < \delta \leq \delta(\eta, \theta')$

$$\sup_{n \geq m} \mathrm{E} \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p < \eta, \tag{A.2}$$

observing that the expression on the l.h.s. is monotone in δ. By compactness of Θ there are finitely many open balls $B(\theta_i', \delta(\eta, \theta_i')/2)$, $1 \leq i \leq K = K(\eta)$, covering Θ. Define $\delta(\eta) = \min\{\delta(\eta, \theta_i')\colon 1 \leq i \leq K\}$. Then we have for all $n \in \mathbb{N}$ and all $\theta' \in \Theta$:

$$\sup_{\theta \in B(\theta', \delta(\eta)/2)} |Q_n(\theta) - Q_n(\theta')|^p \leq 2^{p+1} \sup_{\theta \in B(\theta_i', \delta(\eta, \theta_i'))} |Q_n(\theta) - Q_n(\theta_i')|^p,$$

where i is an index for which $\rho(\theta_i', \theta') < \delta(\eta, \theta_i')/2$. Hence for all $n \in \mathbb{N}$ and $\theta' \in \Theta$,

$$\mathrm{E} \sup_{\theta \in B(\theta', \delta(\eta)/2)} |Q_n(\theta) - Q_n(\theta')|^p$$

$$\leq 2^{p+1} \max_{1 \leq i \leq K} \mathrm{E} \sup_{\theta \in B(\theta_i', \delta(\eta, \theta_i'))} |Q_n(\theta) - Q_n(\theta_i')|^p, \tag{A.3}$$

which implies that for all $\theta' \in \Theta$ and all $m \geq 1$,

$$\overline{\lim_{n \to \infty}} \mathrm{E} \sup_{\theta \in B(\theta', \delta(\eta)/2)} |Q_n(\theta) - Q_n(\theta')|^p$$

$$\leq 2^{p+1} \max_{1 \leq i \leq K} \sup_{n \geq m} \mathrm{E} \sup_{\theta \in B(\theta_i', \delta(\eta, \theta_i'))} |Q_n(\theta) - Q_n(\theta_i')|^p. \tag{A.4}$$

Now choose $m \geq \max\{m(\theta_i', \delta(\eta, \theta_i'))\colon 1 \leq i \leq K(\eta)\}$. It then follows from (A.2) and (A.4) that $\sup_{\theta' \in \Theta} \overline{\lim}_{n \to \infty} \mathrm{E}\sup_{\theta \in B(\theta', \delta(\eta)/2)} |Q_n(\theta) - Q_n(\theta')|^p \leq 2^{p+1}\eta$. Since the latter inequality obviously holds also with $\delta(\eta)/2$ replaced by any smaller δ we have established $\mathrm{UAL_pEC}$. The proof of the implication $\mathrm{L_pEC} \Rightarrow \mathrm{UL_pEC}$ is identical except that (A.2) now holds for all $m \geq 1$. The reverse implications $\mathrm{UAL_pEC} \Rightarrow \mathrm{AL_pEC}$ and $\mathrm{UL_pEC} \Rightarrow \mathrm{L_pEC}$ hold trivially. The case $p = 0$ as well as parts (b3) and (b4) can be shown analogously. ▮

Proof of Theorem 2.1. (a) Follows from Lyapunov's and Markov's inequality. (b) It suffices to give the proof for $r = 0$. Assume that Q_n is $\mathrm{AL_0EC}$ at θ'. Choose $\varepsilon > 0$ and M such that $\overline{\lim}_{n \to \infty} \mathrm{E}(D_n^p \mathbf{1}(D_n > M)) < 2^{-p-1}\varepsilon$ and define $Z_n(\delta) = \sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p$. Then $\overline{\lim}_{n \to \infty} \mathrm{E}\sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')|^p \leq \overline{\lim}_{n \to \infty} \mathrm{E}Z_n(\delta)\mathbf{1}(Z_n(\delta) \leq \varepsilon) + \overline{\lim}_{n \to \infty} \mathrm{E}Z_n(\delta)\ \mathbf{1}(Z_n(\delta) > \varepsilon) \leq \varepsilon + 2^{p+1}\overline{\lim}_{n \to \infty} \mathrm{E}D_n^p \times \mathbf{1}(Z_n(\delta) > \varepsilon, D_n > M) + 2^{p+1} \overline{\lim}_{n \to \infty} \mathrm{E}D_n^p \mathbf{1}(Z_n(\delta) > \varepsilon, D_n \leq M) \leq 2\varepsilon + 2^{p+1} M^p \times \overline{\lim}_{n \to \infty} \mathrm{P}(\sup_{\theta \in B(\theta', \delta)} |Q_n(\theta) - Q_n(\theta')| > \varepsilon) < 3\varepsilon$ if δ is small enough. Hence Q_n is $\mathrm{AL_pEC}$ at θ'. The proof for all other cases is analogous. ▮

Proof of Ascoli–Arzelà's Theorem. Without loss of generality we may put $\bar{f}_n \equiv 0$. (a) Choose $\varepsilon > 0$. Then there exists a $\delta(\varepsilon) > 0$ such that for $0 < \delta \leq \delta(\varepsilon)$ we have $\overline{\lim}_{n\to\infty} \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} |f_n(\theta) - f_n(\theta')| < \varepsilon$. Let θ'_i, $1 \leq i \leq K = K(\varepsilon)$, be such that the open balls $B(\theta'_i, \delta(\varepsilon)/2)$ cover the totally bounded space Θ. Find $\tilde{\theta}_i \in \Theta_0$ such that $\rho(\theta'_i, \tilde{\theta}_i) < \delta(\varepsilon)/2$. Then the open balls $B(\tilde{\theta}_i, \delta(\varepsilon))$ also cover Θ. Now for every $\theta\in\Theta$ there exists a $\tilde{\theta}_i$ such that for all $n\in\mathbb{N}$: $|f_n(\theta)| \leq \sup_{\theta\in B(\tilde{\theta}_i,\delta(\varepsilon))} |f_n(\theta) - f_n(\tilde{\theta}_i)| + |f_n(\tilde{\theta}_i)|$ and hence $\sup_{\theta\in\Theta} |f_n(\theta)| \leq \max_{1\leq i\leq K} \sup_{\theta\in B(\tilde{\theta}_i,\delta(\varepsilon))} |f_n(\theta) - f_n(\tilde{\theta}_i)| + \max_{1\leq i\leq K} |f_n(\tilde{\theta}_i)| \leq \sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta(\varepsilon))} |f_n(\theta) - f_n(\theta')| + \max_{1\leq i\leq K} |f_n(\tilde{\theta}_i)|$. This proves $0 \leq \overline{\lim}_{n\to\infty} \sup_{\theta\in\Theta} |f_n(\theta)| < \varepsilon$ since $f_n(\tilde{\theta}_i)$ converges to zero by assumption. (b) Asymptotic uniform equicontinuity follows from $\sup_{\theta'\in\Theta} \sup_{\theta\in B(\theta',\delta)} |f_n(\theta) - f_n(\theta')| \leq 2 \sup_{\theta\in\Theta} |f_n(\theta)|$, the rest is trivial. ∎

Proof of Theorem 3.1. For the a.s. part of (a) observe that, since (Θ, ρ) is separable and metric, we can find a countable subset Θ_1 of Θ_0 which is also dense in Θ. Since Θ_1 is countable we can, after exclusion of a common exceptional null set, assume that for each ω outside this null set $Q_n(\omega, \theta) - \bar{Q}_n(\theta)$ satisfies all the assumptions of Ascoli–Arzelà's Theorem with Θ_1 in place of Θ_0, and hence the result follows from that theorem. Also the a.s. result in (b) follows immediately from Ascoli–Arzelà's Theorem. The i.p. part can be proved similarly; cp. also Andrews (1989c, proof of theorem 1), observing that θ_j can be chosen to belong to Θ_0. ∎

Proof of Theorem 3.2. The proof is similar to the argument given in Pötscher and Prucha (1989a, p. 681) and Andrews (1987, pp. 1469–1470). Since Q_n is $\mathrm{UAL_1EC}$, for every $\eta > 0$ and $\theta'\in\Theta$ we can find a $\delta(\eta) > 0$ and an $n(\theta', \eta)\in\mathbb{N}$ such that $\mathrm{E}\sup_{\theta\in B(\theta',\delta)} |Q_n(\theta) - Q_n(\theta')| < \eta$ for all $0 < \delta \leq \delta(\eta)$ and $n \geq n(\theta', \eta)$. Choose $\delta_k \leq \delta(\eta)$ such that (3.1) holds. Since Θ is totally bounded there exist finitely many θ'_i, $1 \leq i \leq K = K(\eta)$, such that the open balls $B(\theta'_i, \delta_k)$ cover Θ. For each $\theta'\in\Theta$ choose θ'_i such that $\theta'\in B(\theta'_i, \delta_k)$, then we have for all $n\in\mathbb{N}$: $\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) + \mathrm{E}\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) \leq Q_n(\theta') - \mathrm{E}Q_n(\theta') \leq \sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) + \mathrm{E}\sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta)$. For $n \geq n_0(\eta) = \max\{n(\theta'_i, \eta)\colon 1 \leq i \leq K\}$ and all $\theta'\in\Theta$ it now follows: $\min_{1\leq i\leq K} \{\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\inf_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta)\} - 2\eta \leq Q_n(\theta') - \mathrm{E}Q_n(\theta') \leq \max_{1\leq i\leq K} \{\sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta) - \mathrm{E}\sup_{\theta\in B(\theta'_i,\delta_k)} Q_n(\theta)\} + 2\eta$. For $n \geq n_0(\eta)$ we hence have $\sup_{\theta\in\Theta} |Q_n(\theta) - \mathrm{E}Q_n(\theta)| \leq A_n + 2\eta$, where $A_n = A_n(\eta)$ converges to zero a.s. [i.p.] as $n \to \infty$ as a consequence of (3.1). The claim now follows since η was arbitrary. ∎

Proof of Theorem 3.3. Obvious. ∎

Lemma A.3. Let (2.4a) hold for some $p \geq 1$ and let $\sup_{\theta\in\Theta} |Q_n(\theta) - \bar{Q}_n(\theta)| \to 0$ a.s. [i.p.] as $n \to \infty$, where $\bar{Q}_n = \mathrm{E}Q_n$. Then for any nonempty subset $B \subseteq \Theta$

we have

$$\sup_{\theta\in B} Q_n(\theta) - \mathrm{E}\sup_{\theta\in B} Q_n(\theta) \to 0 \quad a.s.\ [i.p.] \quad as \quad n\to\infty, \tag{A.5a}$$

$$\inf_{\theta\in B} Q_n(\theta) - \mathrm{E}\inf_{\theta\in B} Q_n(\theta) \to 0 \quad a.s.\ [i.p.] \quad as \quad n\to\infty. \tag{A.5b}$$

Proof. Note that in view of (2.4a) the expectations $\mathrm{E}\sup_{\theta\in B}Q_n$ and $\mathrm{E}\inf_{\theta\in B}Q_n$ are finite and hence $\sup_{\theta\in B}Q_n$ and $\inf_{\theta\in B}Q_n$ are a.s. finite, except possibly for finitely many n. Clearly, $|\sup_{\theta\in B}Q_n - \sup_{\theta\in B}\bar{Q}_n| \le \sup_{\theta\in B}|Q_n - \bar{Q}_n| \le \sup_{\theta\in\Theta}|Q_n - \bar{Q}_n|$ and $|\inf_{\theta\in B}Q_n - \inf_{\theta\in B}\bar{Q}_n| \le \sup_{\theta\in B}|Q_n - \bar{Q}_n| \le \sup_{\theta\in\Theta}|Q_n - \bar{Q}_n|$. Consequently, $|\mathrm{E}\sup_{\theta\in B}Q_n - \sup_{\theta\in B}\bar{Q}_n| \le \mathrm{E}|\sup_{\theta\in B}Q_n - \sup_{\theta\in B}\bar{Q}_n| \le \mathrm{E}\sup_{\theta\in\Theta}|Q_n - \bar{Q}_n|$ and $|\mathrm{E}\inf_{\theta\in B}Q_n - \inf_{\theta\in B}\bar{Q}_n| \le \mathrm{E}|\inf_{\theta\in B}Q_n - \inf_{\theta\in B}\bar{Q}_n| \le \mathrm{E}\sup_{\theta\in\Theta}|Q_n - \bar{Q}_n|$. The conclusion of the lemma now follows from the above inequalities and the triangle inequality if we can show that $\mathrm{E}\sup_{\theta\in\Theta}|Q_n - \bar{Q}_n| \to 0$ as $n\to\infty$. Since $\sup_{\theta\in\Theta}|Q_n - \bar{Q}_n| \to 0$ a.s. [i.p.] as $n\to\infty$, this is the case if we can establish that $\overline{\lim}_{n\to\infty}\ \mathrm{E}C_n\mathbf{1}(C_n > M) \to 0$ as $M\to\infty$, where $C_n = \sup_{\theta\in\Theta}|Q_n - \bar{Q}_n|$. Now observe that (2.4a) implies $\overline{\lim}_{n\to\infty}\mathrm{E}D_n < \infty$. Also $\overline{\lim}_{n\to\infty}\ \mathrm{E}C_n\mathbf{1}(C_n > M) \le \overline{\lim}_{n\to\infty}\mathrm{E}[(D_n + \mathrm{E}D_n)\mathbf{1}(D_n + \mathrm{E}D_n > M)] \le \overline{\lim}_{n\to\infty}\mathrm{E}[(D_n + \mathrm{E}D_n)\times \mathbf{1}(D_n > M')]$ for any $M' < M - \overline{\lim}_{n\to\infty}\mathrm{E}D_n$. Hence $0 \le \overline{\lim}_{n\to\infty}\mathrm{E}C_n \times \mathbf{1}(C_n > M) \le \overline{\lim}_{n\to\infty}\mathrm{E}D_n\ \ \mathbf{1}(D_n > M') + \overline{\lim}_{n\to\infty}\mathrm{E}D_n\mathrm{E}\mathbf{1}(D_n > M') \to 0$ for $M'\to\infty$ because of (2.4a). ∎

Lemma A.4. (a) *q_t is a.s.ACEC [a.s.CEC] on Θ $\Rightarrow$ q_t is ACL_0EC [CL_0EC] on Θ*, (b) *q_t is a.s.UACEC [a.s.UCEC] on Θ $\Rightarrow$ q_t is $UACL_0EC$ [UCL_0EC] on Θ*, (c) *q_t is a.s.ACUEC [a.s.CUEC] on Θ $\Rightarrow$ q_t is ACL_0UEC [CL_0UEC] on Θ.*

Proof. (a) Let $R_t(\theta',\delta) = \sup_{\theta\in B(\theta',\delta)}|q_t(\theta) - q_t(\theta')|$ and for $\varepsilon > 0$ let $\phi_\varepsilon(x) = x/\varepsilon$ for $0\le x\le\varepsilon$ and $\phi_\varepsilon(x) = 1$ for $\varepsilon < x \le \infty$. Then

$$\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ n^{-1}\sum_{t=1}^{n} \mathrm{P}(R_t(\theta',\delta) > \varepsilon) = \overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ n^{-1}\sum_{t=1}^{n}\mathrm{E}\mathbf{1}_{(\varepsilon,\infty]}(R_t(\theta',\delta))$$

$$\le \lim_{\delta\to 0}\ \overline{\lim_{n\to\infty}}\ n^{-1}\sum_{t=1}^{n}\mathrm{E}\phi_\varepsilon(R_t(\theta',\delta)) \le \overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ \mathrm{E}\phi_\varepsilon\left(n^{-1}\sum_{t=1}^{n}R_t(\theta',\delta)\right)$$

$$\le \mathrm{E}\phi_\varepsilon\left(\overline{\lim_{\delta\to 0}}\ \overline{\lim_{n\to\infty}}\ n^{-1}\sum_{t=1}^{n}R_t(\theta',\delta)\right),$$

by Jensen's inequality, since ϕ_ε is concave, and by dominated convergence observing that ϕ_ε is monotone, bounded, and continuous. The last expression in the above inequality is zero since $\overline{\lim}_{\delta\to 0}\,\overline{\lim}_{n\to\infty}\, n^{-1}\sum_{t=1}^{n} R_t(\theta',\delta)=0$ a.s. in view of (4.1c) and since $\phi_\varepsilon(0)=0$. (b) and (c) are proved analogously. ∎

Note that Lemma A.4(a) also holds if q_t is only a.s.ACEC [a.s.CEC] at θ' for all $\theta'\in\Theta$.

Proof of Theorem 4.1. Analogous to the proof of Theorem 2.1. ∎

Proof of Theorem 4.2. (a) Obvious from the triangle inequality. (b) If $r\geq 1$, the claim follows from Jensen's inequality. Otherwise, if q_t is ACL$_r$EC, it follows from Theorem 4.1(b) that q_t is ACL$_p$EC. Since $p\geq 1$, it follows by the previous argument that Q_n is AL$_p$EC and by Theorem 2.1(a) that it is AL$_r$EC. The proof of the remaining claims is analogous. ∎

Proof of Corollary 4.3. q_t is a.s.ACUEC $\Rightarrow q_t$ is ACL$_0$UEC $\Rightarrow q_t$ is ACL$_1$UEC $\Rightarrow Q_n$ is AL$_1$UEC, where the implications follow from Lemma A.4, Theorem 4.1(b) and condition (4.4), and Theorem 4.2(b), respectively. Hence under both sets of assumptions on q_t in the corollary, $\bar{Q}_n=\mathrm{E}Q_n$ is AUEC by Theorem 3.3. (Note that $\mathrm{E}|q_t|<\infty$ for all $t\geq 1$ is maintained.) Furthermore, if q_t is a.s.ACUEC, then Q_n is a.s.AUEC by Theorem 4.2(a), and if q_t is ACL$_0$UEC, then Q_n is AL$_0$UEC by Theorem 2.1(a), as it is even AL$_1$UEC as shown above. The corollary now follows from Theorem 3.1. ∎

Proof of Corollary 4.4. The proof of part (a) is similar to the argument in Pötscher and Prucha (1989a, p. 681) and Andrews (1987, pp. 1469–1470) with modifications as in the proof of Theorem 3.2. To prove part (b) observe that Q_n is UAL$_1$EC in view of Theorem 4.2(b), and hence $\bar{Q}_n$ is UAEC in view of Theorem 3.3. Since (Θ,ρ) is totally bounded, it follows furthermore from Lemma A.2 that $\bar{Q}_n$ is even AUEC. ∎

Proof of Theorem 4.5. We first verify that the q_t are ACL$_0$UEC, i.e., that for any $\varepsilon>0$

$$\overline{\lim_{n\to\infty}}\, n^{-1}\sum_{t=1}^{n}\mathrm{P}\left(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)}|q_t(\theta)-q_t(\theta')|>\varepsilon\right)\to 0 \quad\text{as}\quad \delta\to 0. \tag{A.6}$$

Clearly the expression in (A.6) is bounded by $\overline{\lim}_{n\to\infty}A^1_{nm}(\delta)+\overline{\lim}_{n\to\infty}A^2_{nm}(\delta)$, with

$$A^1_{nm}(\delta)=n^{-1}\sum_{t=1}^{n}\mathrm{P}\left(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)}|q_t(\theta)-q_t(\theta')|\,\mathbf{1}_{K_m}(z_t)>\varepsilon/2\right),$$

$$A^2_{nm}(\delta)=n^{-1}\sum_{t=1}^{n}\mathrm{P}\left(\sup_{\theta'\in\Theta}\ \sup_{\theta\in B(\theta',\delta)}|q_t(\theta)-q_t(\theta')|\,\mathbf{1}_{Z-K_m}(z_t)>\varepsilon/2\right),$$

where we use the convention $\infty.0 = 0$. From Lemma A.1 it follows that (4.8) is equivalent to (4.8′), and hence we have that for each $\eta > 0$, for each $m \in \mathbb{N}$, and all $1 \leq k \leq K$ there is a $\delta_0(\eta, m) > 0$ and a $t_0(\eta, m) \geq 1$ such that for $t \geq t_0$, $0 < \delta \leq \delta_0$, $m \in \mathbb{N}$, and $1 \leq k \leq K$,

$$\sup_{z \in K_m} \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |s_{kt}(z, \theta) - s_{kt}(z, \theta')| < \eta. \tag{A.7}$$

Now

$$A^1_{nm}(\delta) \leq$$

$$\sum_{k=1}^{K} n^{-1} \sum_{t=1}^{n} \mathrm{P}\left(|r_{kt}(z_t)| \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |s_{kt}(z_t, \theta) - s_{kt}(z_t, \theta')| 1_{K_m}(z_t) > \varepsilon/2K \right)$$

$$\leq \sum_{k=1}^{K} n^{-1} \sum_{t=1}^{n} \mathrm{P}\left(|r_{kt}(z_t)| \sup_{z \in K_m} \sup_{\theta' \in \Theta} \sup_{\theta \in B(\theta', \delta)} |s_{kt}(z, \theta) - s_{kt}(z, \theta')| > \varepsilon/2K \right).$$

Now for $0 < \delta \leq \delta_0$ and $n \geq t_0$ we have from (A.7) that $A^1_{nm}(\delta) \leq \sum_{k=1}^{K} [(t_0 - 1)n^{-1} + n^{-1} \sum_{t=t_0}^{n} \mathrm{P}(|r_{kt}(z_t)|\eta > \varepsilon/2K)]$ which implies that $\overline{\lim}_{n\to\infty} A^1_{nm}(\delta) \leq \eta(2K/\varepsilon) \sum_{k=1}^{K} \overline{\lim}_{n\to\infty} n^{-1} \sum_{t=1}^{n} \mathrm{E}|r_{kt}(z_t)|$. Since $\sup_n n^{-1} \times \sum_{t=1}^{n} \mathrm{E}|r_{kt}(z_t)| < \infty$ by Assumption 4.1(a) and η was arbitrary, we have hence shown for each $m \in \mathbb{N}$ that $\overline{\lim}_{\delta\to 0} \overline{\lim}_{n\to\infty} A^1_{nm}(\delta) = 0$. Next observe that $A^2_{nm}(\delta) \leq n^{-1} \sum_{t=1}^{n} \mathrm{P}(z_t \notin K_m)$ holds for all $\delta > 0$. By Assumption 4.1(b) we hence have $\lim_{m\to\infty} \overline{\lim}_{\delta\to 0} \overline{\lim}_{n\to\infty} A^2_{nm}(\delta) = 0$. This establishes (A.6). Since (4.4) is assumed in Theorem 4.5, it follows furthermore from Theorem 4.1(b) that q_t is also $\mathrm{ACL_1UEC}$, and hence clearly $\mathrm{UACL_1EC}$. Theorem 4.5 now follows from Corollaries 4.3 and 4.4. The remaining claims follow analogously in light of Remark 4.1(iii). ∎

Lemma A.5. Let (X, d_x) and (Y, d_y) be metric spaces, with X compact, and f_t: $X \times Y \to \mathbb{R}$. Let $X \times Y$ be endowed with the metric $d = \max(d_x, d_y)$. If for each $x' \in X$

$$\sup_{y' \in Y} \overline{\lim_{t\to\infty}} \sup_{(x,y) \in B((x',y'),\delta)} |f_t(x, y) - f_t(x', y')| \to 0 \quad \text{as} \quad \delta \to 0,$$

where $B((x', y'), \delta)$ is the open ball with center (x', y') and radius δ in $X \times Y$, then $\{f_t\colon t \geq 1\}$ is uniformly asymptotically equicontinuous on $X \times Y$. Furthermore, if (Y, d_y) is also totally bounded, then $\{f_t\colon t \geq 1\}$ is even asymptotically uniformly equicontinuous on $X \times Y$.

Proof. To prove the first claim observe that by assumption for every $\eta > 0$ and $x' \in X$ there exists a $\delta(\eta, x') > 0$ such that for all $y' \in Y$ and all $0 < \delta \leq \delta(\eta, x')$ we have $\overline{\lim}_{t\to\infty} \sup_{(x,y)\in B((x',y'),\delta)} |f_t(x,y) - f_t(x',y')| < \eta$. Hence there exists an index $m(x', y', \delta(\eta, x'))$ such that for each $(x', y')\in X \times Y$, all $t \geq m(x', y', \delta(\eta, x'))$, and $0 < \delta \leq \delta(\eta, x')$,

$$\sup_{(x,y)\in B((x',y'),\delta)} |f_t(x,y) - f_t(x',y')| < \eta, \tag{A.8}$$

observing that the expression on the l.h.s. is monotone in δ. By compactness of X we can find finitely many open balls $B(x_i', \delta(\eta, x_i')/2)$, $1 \leq i \leq K = K(\eta)$, covering X. Define $\delta(\eta) = \min\{\delta(\eta, x_i')\colon 1 \leq i \leq K\}$. Now for every (x', y') there exists an index i, $1 \leq i \leq K$, such that $d((x', y'), (x_i', y')) \leq \delta(\eta, x_i')/2$. Hence for all $t\in\mathbb{N}$ and any $y'\in Y$:

$$\sup_{x'\in X} \sup_{(x,y)\in B((x',y'),\delta(\eta)/2)} |f_t(x,y) - f_t(x',y')|$$

$$\leq 2 \max_{1\leq i\leq K} \sup_{(x,y)\in B((x_i',y'),\delta(\eta,x_i'))} |f_t(x,y) - f_t(x_i',y')|.$$

Observing that for all $y'\in Y$ and all $t \geq \max\{m(x_i', y', \delta(\eta, x_i'))\colon 1 \leq i \leq K(\eta)\}$ the r.h.s. of the above inequality is, in view of (A.8), not larger than 2η, we have for all $y'\in Y$:

$$\overline{\lim_{t\to\infty}} \sup_{x'\in X} \sup_{(x,y)\in B((x',y'),\delta(\eta)/2)} |f_t(x,y) - f_t(x',y')| \leq 2\eta.$$

Hence clearly,

$$\sup_{y'\in Y} \sup_{x'\in X} \overline{\lim_{t\to\infty}} \sup_{(x,y)\in B((x',y'),\delta(\eta)/2)} |f_t(x,y) - f_t(x',y')| \leq 2\eta.$$

Since η was arbitrary and the l.h.s. of the above inequality does not increase if $\delta(\eta)/2$ is replaced by some $\delta \leq \delta(\eta)/2$, this establishes that f_t is UAEC on $X \times Y$. The second claim follows now immediately from Lemma A.2. observing that $(X \times Y, d)$ is totally bounded. ▮

Lemma A.6. Let $h_n(\theta)$ and $h_{m,n}(\theta)$ be real functions on the metric space (Θ, ρ) for $m, n\in\mathbb{N}$. If the family $\{h_{m,n}\colon n\in\mathbb{N}\}$ is asymptotically uniformly equicontinuous [uniformly equicontinuous] for each $m\in\mathbb{N}$ and $\lim_{m\to\infty} \overline{\lim}_{n\to\infty} \sup_{\theta\in\Theta} |h_{m,n}(\theta) - h_n(\theta)| = 0$ [$\lim_{m\to\infty} \sup_n \sup_{\theta\in\Theta} |h_{m,n}(\theta) - h_n(\theta)| = 0$], then $\{h_n\colon n\in\mathbb{N}\}$ is asymptotically uniformly equicontinuous [uniformly equicontinuous].

Proof. For $\eta > 0$ there exists an index $m_0 = m_0(\eta)$ such that $\overline{\lim}_{n\to\infty} \sup_{\theta\in\Theta} |h_{m_0,n}(\theta) - h_n(\theta)| < \eta$. Hence for some $n_0 = n_0(\eta, m_0)$ we have

$\sup_{\theta\in\Theta}|h_{m_0,n}(\theta) - h_n(\theta)| < \eta$ for $n \geq n_0$. Furthermore there exists a $\delta = \delta(\eta, m_0) > 0$ such that $\rho(\theta, \theta') < \delta$ implies $\overline{\lim}_{n\to\infty}|h_{m_0,n}(\theta) - h_{m_0,n}(\theta')| < \eta$. Therefore there exists an index $n_1 = n_1(\eta, \delta, m_0)$ such that for $n \geq n_1$ and $\rho(\theta, \theta') < \delta$ we have $|h_{m_0,n}(\theta) - h_{m_0,n}(\theta')| < \eta$. Let $n_2 = \max\{n_0, n_1\}$. Then if $\rho(\theta, \theta') < \delta$ and $n \geq n_2$ we get $|h_n(\theta) - h_n(\theta')| \leq |h_n(\theta) - h_{m_0,n}(\theta)| + |h_{m_0,n}(\theta) - h_{m_0,n}(\theta')| + |h_{m_0,n}(\theta') - h_n(\theta')| < 3\eta$. This proves that $\{h_n: n\in\mathbb{N}\}$ is asymptotically uniformly equicontinuous. The proof for the second claim is analogous. ▮

Proof of Theorem 4.6. Observe that in light of Lemma A.2 the sequence $s_{t,m}(z_t, \theta)$ is a.s.AUEC since (Θ, ρ) is totally bounded. Since condition (4.4) implies that $d_t < \infty$ a.s. for all $t\in\mathbb{N}$, it follows that $s_{t,m}(z_t, \theta)$ is a.s.ACUEC. If $s_{t,m}(z_t, \theta)$ satisfies weak [strong] pointwise laws of large numbers, then it follows from Corollary 4.3 that $s_{t,m}(z_t, \theta)$ satisfies for each $m\in\mathbb{N}$ a weak [strong] ULLN and $n^{-1}\sum_{t=1}^{n} \mathrm{E}s_{t,m}(z_t, \theta)$ is AUEC. Observe that a.s.ACUEC implies $\mathrm{ACL_0UEC}$ by Lemma A.4. Since (4.4) holds with $p \geq 1$, Theorem 4.1(b) shows that $s_{t,m}(z_t, \theta)$ is $\mathrm{ACL_1UEC}$ for all $m\in\mathbb{N}$ and hence is $\mathrm{UACL_1EC}$. If $s_{t,m}(z_t, \theta)$ satisfies weak [strong] local laws of large numbers, then it follows from Corollary 4.4 that $s_{t,m}(z_t, \theta)$ satisfies for each $m\in\mathbb{N}$ a weak [strong] ULLN and $n^{-1}\sum_{t=1}^{n} \mathrm{E}s_{t,m}(z_t, \theta)$ is AUEC. Therefore all assumptions of Lemma 1(a) in Pötscher and Prucha (1989b) are satisfied. (An inspection of the proof of that lemma shows that the lemma also holds if a.s. convergence is replaced by i.p. convergence.) This proves part (a) of the theorem.

The claim in part (b) of the theorem that $n^{-1}\sum_{t=1}^{n} \mathrm{E}q_t$ is AUEC follows from Lemma A.6 observing $\lim_{m\to\infty} \overline{\lim}_{n\to\infty} \sup_{\theta\in\Theta}|n^{-1}\sum_{t=1}^{n} [\mathrm{E}s_{t,m}(z_t, \theta) - \mathrm{E}s_t(z_t, \theta)]| \leq \lim_{m\to\infty} \overline{\lim}_{n\to\infty} n^{-1}\sum_{t=1}^{n} \mathrm{E}d_{t,m,c} = 0$. The claim in parenthesis in part (b) follows in view of Remark 4.1(iii) and Lemma A.6. ▮

Proof of Lemma 5.1. Since (Θ, ρ) is totally bounded, there exists a countable dense subset Θ_0 of Θ. Observe that Θ_0 is also dense in Θ^*. Choose $\delta_n > 0$, $\delta_n \to 0$ as $n \to \infty$. For $\theta\in\Theta^* - \Theta$ we define $Q_n^*(\omega, \theta) = \inf\{Q_n(\omega, \tilde{\theta}): \tilde{\theta}\in\Theta_0, \rho^*(\tilde{\theta}, \theta) < \delta_n\}$ if $\inf\{Q_n(\omega, \tilde{\theta}): \tilde{\theta}\in\Theta_0, \rho^*(\tilde{\theta}, \theta) < \delta_n\} > -\infty$ and $Q_n^*(\omega, \theta) = 0$ if the infimum equals $-\infty$. For $\theta\in\Theta$ we put $Q_n^*(\omega, \theta) = Q_n(\omega, \theta)$ for all $\omega\in\Omega$. Let $\delta > 0$ be arbitrary. Then for $n \geq n_0 = n_0(\delta)$ we have $\delta_n < \delta/3$. Now, for any $\theta, \theta'\in\Theta^*$ satisfying $\rho^*(\theta, \theta') < \delta/3$ we have from the very definition of the extension that $|Q_n^*(\theta) - Q_n^*(\theta')| \leq \sup_{\tilde{\theta}'\in\Theta}\sup_{\tilde{\theta}\in B(\tilde{\theta}',\delta)}|Q_n(\tilde{\theta}) - Q_n(\tilde{\theta}')|$ [where $B(\tilde{\theta}', \delta)$ denotes the open ball in Θ]. Hence $\sup_{\theta'\in\Theta^*}\sup_{\theta\in B^*(\theta',\delta/3)} |Q_n^*(\theta) - Q_n^*(\theta')| \leq \sup_{\tilde{\theta}'\in\Theta}\sup_{\tilde{\theta}\in B(\tilde{\theta}',\delta)}|Q_n(\tilde{\theta}) - Q_n(\tilde{\theta}')|$ holds for $n \geq n_0(\delta)$ [where $B^*(\theta', \delta/3)$ denotes the open ball in Θ^*]. Hence we have bounded the modulus of uniform continuity of Q_n^* by that of Q_n. This establishes the lemma. ▮

References

Alexander, K.S., 1987, The central limit theorem for empirical processes on Vapnik–Červonenkis classes, Annals of Probability 15, 178–203.

Andrews, D.W.K., 1987, Consistency in nonlinear econometric models: A generic uniform law of large numbers, Econometrica 55, 1465–1471.
Andrews, D.W.K., 1989a, Asymptotics for semiparametric econometric models: I. Estimation, Cowles Foundation discussion paper no. 908 (Yale University, New Haven, CT).
Andrews, D.W.K., 1989b, Asymptotics for semiparametric econometric models: II. Stochastic equicontinuity, Cowles Foundation discussion paper no. 909 (Yale University, New Haven, CT).
Andrews, D.W.K., 1989c, Generic uniform convergence, Cowles Foundation unpublished manuscript (Yale University, New Haven, CT).
Bierens, H.J., 1981, Robust methods and asymptotic theory in nonlinear econometrics, Lecture notes in economics and mathematical systems 192 (Springer-Verlag, Berlin).
Bierens, H.J., 1989, Least squares estimation of linear and nonlinear ARMAX models under data heterogeneity, Department of Econometrics unpublished manuscript (Free University, Amsterdam).
Billingsley, P., 1968, Convergence of probability measures (Wiley, New York, NY).
Dunford, N. and J.T. Schwartz, 1957, Linear operators, Part I: General theory (Interscience, New York, NY).
Hoadley, B., 1971, Asymptotic properties of maximum likelihood estimators for the independent not identically distributed case, Annals of Mathematical Statistics 42, 1977–1991.
Newey, W.K., 1989, Uniform convergence in probability and stochastic equicontinuity, Department of Economics unpublished manuscript (Princeton University, Princeton, NJ).
Pollard, D., 1984, Convergence of stochastic processes (Springer-Verlag, New York, NY).
Pollard, D., 1989, Empirical processes: Theory and applications, Regional conference series in applied mathematics (Society for Industrial and Applied Mathematics, Philadelphia, PA) forthcoming.
Pötscher, B.M. and I.R. Prucha, 1986a, A class of partially adaptive one-step M-estimators for the nonlinear regression model with dependent observations, Journal of Econometrics 32, 219–251.
Pötscher, B.M. and I.R. Prucha, 1986b, Consistency in nonlinear econometrics: A generic uniform law of large numbers and some comments on recent results, Department of Economics working paper no. 86–9 (University of Maryland, College Park, MD).
Pötscher, B.M. and I.R. Prucha, 1989a, A uniform law of large numbers for dependent and heterogeneous data processes, Econometrica 57, 675–683.
Pötscher, B.M. and I.R. Prucha, 1989b, On the formulation of uniform laws of large numbers: A truncation approach, Department of Economics unpublished manuscript (University of Maryland, College Park, MD).
Pötscher, B.M. and I.R. Prucha, 1991a, Basic structure of the asymptotic theory in dynamic nonlinear econometric models, Part I: Consistency and approximation concepts, Econometric Reviews 10, 125–216.
Pötscher, B.M. and I.R. Prucha, 1991b, Basic structure of the asymptotic theory in dynamic nonlinear econometric models, Part II: Asymptotic normality, Econometric Reviews 10, 253–325.
Royden, H.L., 1968, Real analysis (Macmillan, London).
Wald, A., 1949, Note on the consistency of the maximum likelihood estimate, Annals of Mathematical Statistics 20, 595–600.