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On the $I^2(q)$ Test Statistic for Spatial Dependence: Finite Sample Standardization and Properties

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ABSTRACT *One of the most widely used tests for spatial dependence is Moran's (1950) I test. The power of the test will depend on the extent to which the spatial-weights matrix employed in computing the Moran I test statistic properly specifies existing interaction links between spatial units. Empirical researchers are often unsure about the use of a particular spatial-weights matrix. In light of this Prucha (2011) introduced the $I^2(q)$ test statistic. This test statistic combines quadratic forms based on several, say q, spatial-weights matrices, while at the same time allows for a proper controlling of the size of the test. In this paper, we first introduce a finite-sample standardized version of the $I^2(q)$ test. We then perform a Monte Carlo study to explore the finite-sample performance of the $I^2(q)$ tests. For comparison, the Monte Carlo study also reports on the finite-sample performance of Moran I tests as well as on Moran I tests performed in sequence.*

Des statistiques du test $I^2(q)$ pour la dépendance spatiale: harmonisation des échantillons finis et propriétés

RÉSUMÉ *un des tests les plus répandus de la dépendance spatiale est celui de Moran. La puissance de ce test est tributaire de la mesure dans laquelle la matrice de pondération spatiale, employée pour calculer correctement les statistiques du test, spécifie correctement les liens d'interaction existants entre unités spatiales. Les chercheurs empiriques éprouvent souvent des incertitudes en ce qui concerne l'emploi d'une certaine matrice de pondération spatiale. Pour cette raison, Prucha (2011) a introduit la statistique de test $I^2(q)$, assurant la combinaison de formes quadratiques sur plusieurs matrices de pondération spatiale, par exemple q. Dans la présente communication, nous introduisons une version harmonisée aux éléments finis de ce test, et nous présentons un compte rendu sur une étude Monte Carlo correspondante.*

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On the $I^2(q)$ Test Statistic for Spatial Dependence: Finite Sample Standardization and Properties

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Abstract

One of the most widely used tests for spatial dependence is Moran's (1950) I test. The power of the test will depend on the extent to which the spatial-weights matrix employed in computing the Moran I test statistic properly specifies existing interaction links between spatial units. Empirical researchers are often unsure about the use of a particular spatial-weights matrix. In light of this Prucha (2011) introduced the $I^2(q)$ test statistic. This test statistic combines quadratic forms based on several, say q, spatial-weights matrices, while at the same time allows for a proper controlling of the size of the test. In this paper, we first introduce a finite-sample standardized version of the $I^2(q)$ test. We then perform a Monte Carlo study to explore the finite-sample performance of the $I^2(q)$ tests. For comparison, the Monte Carlo study also reports on the finite-sample performance of Moran I tests as well as on Moran I tests performed in sequence.

Key Words: Moran I test, $I^2(q)$ test, spatial dependence, Cliff-Ord-type spatial models

JEL Classification: C21, C31

1 Introduction [1]

One of the most widely used tests for spatial dependence is Moran's (1950) I test.[2] Moran's I test statistic is formulated in terms of a normalized quadratic form of the variables to be tested for spatial dependence, with the elements of a spatial-weights matrix serving as the weights in the quadratic form. The power of the test will depend on the extent to which the employed spatial-weights matrix properly specifies existing interaction links between spatial units.

One problem with the use of the Moran I test statistic is that researchers are often not sure about their specification of the spatial-weights matrix. For example, a researcher may not be sure whether spatial interactions are best modeled via a contiguity-type matrix or an inverse-distance matrix. Let I_C and I_I denote, respectively, the Moran I test statistic corresponding to the contiguity-type matrix and the inverse-distance matrix. Under the null hypothesis H_0 of zero spatial correlation I_C and I_I are both distributed asymptotically normal $(0, 1)$, given some regularity conditions; see Kelejian and Prucha (2001) for details. Now suppose the researcher uses I_C to decide whether or not to accept H_0. That is, if the desired significance level is 5 percent she/he would accept H_0 if $|I_C| \leq 1.96$. This test has the correct asymptotic size of 0.05 under H_0. However the test may have low power under the alternative hypothesis H_1, if spatial interactions are better modeled by an inverse-distance matrix than by a contiguity-type matrix. Consequently, the researcher may be unsure as to whether she/he should accept H_0 even if $|I_C| \leq 1.96$, and the researcher may decide to perform a second test using the I_I test statistic, and only accept H_0 if additionally $|I_I| \leq 1.96$. The outcome of such a sequential testing procedure does not depend on the order in which the tests are preformed because H_0 is accepted if and only if $|I_C| \leq 1.96$ and $|I_I| \leq 1.96$. Alternatively stated, H_0 is rejected if and only if at least one of the two test statistics exceeds in absolute value 1.96. We thus refer to this testing procedure as a union of rejections (UR) test, consistent with terminology adopted by Harvey, Leybounre and Taylor (2009).

The significance level of the UR testing procedure, which is given by $1 - P[|I_C| \leq 1.96, |I_I| \leq 1.96]$, will in general exceed the desired level of 0.05 because[3]

$$P[|I_C| \leq 1.96, |I_I| \leq 1.96] \leq P[|I_C| \leq 1.96] = P[|I_I| \leq 1.96] = 0.95.$$

Of course, the problem that a sequence of tests leads to an overall significance level that exceeds the desired one is not specific to the Moran I test, and is a

[1] We would like to that Harry Kelejian, Gianfranco Piras, and two anonymous referees for their helpful comments and suggestions. We would also like to thank the participants of the 5th World Conference of the Spatial Econometrics Association, Toulouse, 2011, for helpful discussions.

[2] There is a large literature on the Moran I test, including Burridge (1980), Cliff and Ord (1981), Anselin (1988), Anselin and Florax (1995), Pinkse (1999), and Kelejian and Prucha (2001).

[3] Formally the statement $P[|I_C| \leq 1.96] = P[|I_I| \leq 1.96] = 0.95$ on the r.h.s. of the above inequality should be understood as to hold asymptotically.

well known problem in statistics; see, e.g. Lehmann and Romano (2010, Ch. 9).[4]

The above example illustrates the desirability of a test that checks for spatial correlation based on several possible specifications of spatial-weights matrices, but can be performed at the desired significance level. To this effect Prucha (2011) introduced the $I^2(q)$ test statistic which combines quadratic forms based on several, say q, spatial-weights matrices.

In essence, the $I^2(q)$ test statistic is a quadratic form of q quadratic forms of the variables to be tested for spatial dependence, normalized by the inverse of the variance-covariance matrix. For $q = 1$ the $I^2(q)$ test statistic is just the square of the Moran I test statistic. Prucha (2011) shows that the $I^2(q)$ can be viewed as a Lagrange Multiplier test statistic corresponding to a spatial-autoregressive or moving-average model of order q, which generalizes a corresponding result by Burridge (1980) for the Moran I test statistic.[5]

This paper offers two main contributions. First, we introduce a standardized version of the $I^2(q)$ statistic, say $I_S^2(q)$, which modifies each of the quadratic forms such that under the null hypothesis they are standardized $(0, 1)$ in the case where the variables under consideration are normally distributed. For $q = 1$ the $I_S^2(q)$ statistic is just the square of the standardized Moran I test statistic, say I_S, as given in Cliff and Ord (1981), which seems to have performed well in small samples. Furthermore, Prucha (2011) showed that under the null hypothesis of zero spatial correlation the $I^2(q)$ statistic is asymptotically distributed $\chi^2(q)$. As an additional part of the first contribution, we also show that the $I^2(q)$ and $I_S^2(q)$ test statistics have the same asymptotic distribution. The $I^2(q)$ and $I_S^2(q)$ tests with, say, an asymptotic nominal size of 5 percent will thus reject H_0 if their value exceeds the .95 fractile of the $\chi^2(q)$ distribution.

The purpose of deriving the asymptotic distribution of a test statistic is to provide an approximation for the actual finite-sample distribution, while maintaining general assumptions on the data-generating process. Therefore it is important to explore the quality of the obtained approximation. The second contribution of this paper is an analysis of the finite-sample properties of the $I^2(q)$ and $I_S^2(q)$ tests via a Monte Carlo study, and a comparison of those properties with those of the Moran I and I_S tests. Under H_0, we explore the actual finite-sample significance levels of these tests and we also explore the magnitude of the size distortions when using a UR test, as described above.

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[4] In principle it is, of course, possible to adjust the significance levels of the respective tests such that the overall test has a significance level of 0.05. However we are not aware of formal results to this affect for the specific problem at hand.

[5] Martellosio (2010, 2012) provides interesting results on the power of Lagrange Multiplier tests within the context of a linear regression model. In essence, he shows that for nearly all spatial-weights matrices there exist regressors (on a space with positive measure) such that, when holding the sample size fixed, the power of the test tends to zero as the spatial-autoregressive parameter tends to a singular point. We note that this result should not be interpreted as to establish the inconsistency of Lagrange Multiplier tests, where consistency is as usual defined that for any(!) given parameter value that is admissible under the alternative the power of the test goes to one as the sample size increases to infinity. In fact, Kelejian and Prucha (2001) give results that establish the consistency of the Moran I test under fairly general conditions.

Under H_1, we report on the power of the considered tests. More specifically, under H_1 our Monte Carlo study explores the case where spatial dependence is generated from a spatial-autoregressive process of order 2, for short an SAR(2) process.[6] We consider various sets of values for the spatial-autoregressive parameters associated with two spatial-weights matrices.

When one of the spatial-autoregressive parameters is zero, the data are generated by a spatial-autoregressive process of order 1, for short an SAR(1) process. When the data are generated by a SAR(1) process, we expect the Moran I test based on the correct spatial-weights matrix to outperform the $I^2(2)$ test that uses the correct spatial-weights matrix and some another spatial-weights matrix. In contrast, when the data are generated by a SAR(1) process, we expect a $I^2(2)$ test that uses the correct spatial-weights matrix and some other spatial-weights matrix to outperform a Moran I test that does not use the correct spatial-weights matrix. Of course, we would also expect the Moran I test based on the true spatial-weights matrix to outperform a Moran I test based on a misspecified spatial-weights matrix. Our study explores the extent of these power gains/losses under the above describes scenarios.

We note that the Moran I test is a special $I^2(q)$ test with $q = 1$. Thus our study also provides information on the performance of the class of $I^2(q)$ tests when the weight matrices are misspecified.

Allowing for the researcher to be unsure about the proper choice of the spatial-weights matrix is similar in spirit to the specification of the spatial HAC estimator introduced in Kelejian and Prucha (2007). Of course, space does not have to be geographic space, and thus the $I^2(q)$ and $I_S^2(q)$ test statistics have wider applications in testing for cross-sectional dependencies arising from social interactions and other types of cross-sectional interactions.

2 Model and Test Statistics

2.1 Model

Consider the model

$$y = X\beta + u, \tag{1}$$

where $y = [y_1, \ldots, y_n]'$ is the $n \times 1$ vector of endogenous variables, $X = (x_{ik})$ is the $n \times K$ matrix of nonstochastic regressors, β is the $K \times 1$ vector of regression parameters, $u = [u_1, \ldots, u_n]'$ is the $n \times 1$ vector of disturbances. We assume that the u_i are identically distributed with $Eu_i = 0$ and $Eu_i^2 = \sigma^2$ and finite $2 + \delta$ moments for some $\delta > 0$. We want to test the hypothesis that the u_i are

[6] As pointed out by one of the referees, a clearer terminology would be to refer to the process as a SAR(W_1, W_2) process rather than a SAR(2) process, where W_1 and W_2 denote the underlying spatial-weights matrices. Our abbreviated terminology is consistent with that developed by Anselin and Florax (1995). Since this terminology is already used widely we also maintain this terminology in this paper

uncorrelated, i.e., we want to test

$$H_0 : E[uu'] = \sigma^2 I_n. \tag{2}$$

We assume furthermore that the elements of X are uniformly bounded in absolute value, that $n^{-1}X'X \to Q$, where Q is finite and positive definite. Additionally, let A be some $n \times n$ matrix whose row and column sums are bounded in absolute value, then $n^{-1}X'AX \to M_{XAX}$, where M_{XAX} is finite. Now let $\widetilde{\beta}_n = (X'X)^{-1}X'y$ denote the ordinary least squares (OLS) estimator, then it is readily seen that under H_0 we have $n^{1/2}(\widetilde{\beta}_n - \beta) \xrightarrow{d} N(0, \sigma^2 Q^{-1})$.

Also in the following we denote with $W_1 = (w_{1,ij})$ and $W_2 = (w_{2,ij})$ two $n \times n$ nonstochastic spatial-weights matrices. Each spatial-weights matrix has diagonal elements that are zero and has row and column sums of the absolute elements that are uniformly bounded by some finite constant.

2.2 Test Statistics

Now consider the case where the researcher believes that W_1 provides a proper representation of potential links for spatial interdependencies between the u_i. In this case, the researcher could test H_0 using the standard Moran I test statistic:

$$I_n = \frac{\widetilde{u}'W\widetilde{u}}{\widetilde{\sigma}^2 \left[tr\left((W' + W)W\right)\right]^{1/2}} \tag{3}$$

based on $W = W_1$, and where $\widetilde{u} = y - X\widetilde{\beta}$ denotes the OLS residuals and $\widetilde{\sigma}^2 = n^{-1}\widetilde{u}'\widetilde{u}$ is the corresponding estimator for σ^2. Under the above assumptions it follows from Kelejian and Prucha (2001) that $I_n \xrightarrow{d} N(0,1)$, and thus $I_n^2 \xrightarrow{d} \chi^2(1)$.[7] Burridge (1980) established that I_n^2 is identical to the Lagrange Multiplier test statistic for testing $\rho = 0$ if the disturbance process is assumed to be a first-order spatial-autoregressive or spatial-moving-average process, i.e., $u = \rho W u + \varepsilon$ or $u = \varepsilon + \rho W \varepsilon$, based on a Gaussian likelihood.

Cliff and Ord (1981) introduced the following finite-sample standardized version of the Moran I test statistic:

$$I_{S,n} = \frac{\widetilde{u}'W\widetilde{u}/\widetilde{\sigma}^2 - \mu_S}{\phi_S^{1/2}} \tag{4}$$

with

$$\begin{aligned} \mu_S &= \frac{-n}{n-K} tr(P_x W), \qquad (5) \\ \phi_S &= \frac{n^2}{(n-K)(n-K+2)} \left\{ tr\left[M_x W M_x W + M_x W M_x W'\right]\right\} \\ &\quad - \frac{2n^2}{(n-K)^2(n-K+2)} \left[tr\left(P_x W\right)\right]^2 \end{aligned}$$

[7] We note that this result was obtained without assuming that the disturbances are normally distributed.

and where $P_x = X(X'X)^{-1}X'$ and $M_x = I - P_x$. We note that μ_S and ϕ_S represent the mean and variance of $\widetilde{u}'W\widetilde{u}/\widetilde{\sigma}^2$ under normality. Still, the $I_{S,n}$ test statistic remains well defined even if the disturbances are not normally distributed, and the statistic was found to perform well also for non-normal distributions.

Next consider the case where the researcher is not sure whether W_1, W_2 or both properly model the spatial interdependencies between the u_i. In this case the researcher could test H_0 using the $I^2(q)$ test statistic with $q = 2$:

$$I_n^2(2) = \begin{bmatrix} \widetilde{u}'W_1\widetilde{u}/\widetilde{\sigma}^2 \\ \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 \end{bmatrix}' \Phi^{-1} \begin{bmatrix} \widetilde{u}'W_1\widetilde{u}/\widetilde{\sigma}^2 \\ \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 \end{bmatrix}. \tag{6}$$

where $\Phi = (\phi_{rs})$ and for $r, s = 1, 2$:

$$\phi_{rs} = \frac{1}{2} tr\left[(W_r + W_r')(W_s + W_s')\right], \qquad r, s = 1, 2. \tag{7}$$

Under the above assumptions, it follows from Prucha (2011) that $I_n^2(2) \xrightarrow{d} \chi^2(2)$ under H_0. That paper also establishes that $I_n^2(2)$ is identical to the Lagrange Multiplier test statistic for testing $\rho_1 = \rho_2 = 0$ if the disturbance process is assumed to be a second-order spatial-autoregressive or spatial-moving-average process, i.e., $u = \rho_1 W_1 u + \rho_2 W_2 u + \varepsilon$ or $u = \varepsilon + \rho_1 W_1 \varepsilon + \rho_2 W_2 \varepsilon$, based on a Gaussian likelihood.

Towards developing a finite-sample standardized version of the Moran I test statistic we prove the following theorem in the appendix.

Theorem 1 *Suppose the assumptions given in Section 2.1 above hold, suppose furthermore that H_0 is true and that additionally $u \sim N(0, \sigma^2 I_n)$. Then for $r, s = 1, 2$*

$$\begin{aligned} \mu_{S,r} &= E(\widetilde{u}'W_r\widetilde{u}/\widetilde{\sigma}^2) = \frac{-n}{n-K} tr\left(P_x W_r\right) \qquad (8) \\ \phi_{S,rs} &= cov(\widetilde{u}'W_r\widetilde{u}/\widetilde{\sigma}^2, \widetilde{u}'W_s\widetilde{u}/\widetilde{\sigma}^2) \\ &= \frac{n^2}{(n-K)(n-K+2)} \left\{tr\left[M_x W_r M_x W_s + M_x W_r M_x W_s'\right]\right\} \\ &\quad - \frac{2n^2}{(n-K)^2(n-K+2)} \left[tr\left(P_x W_r\right)\right]\left[tr\left(P_x W_s\right)\right]. \end{aligned}$$

Of course, defining $\Phi_S = (\phi_{S,rs})$, this implies that for $u \sim N(0, \sigma^2 I_n)$ we have

$$\Phi_S^{-1/2} \begin{bmatrix} \widetilde{u}'W_1\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,1} \\ \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,2} \end{bmatrix} \sim (0, I_2)$$

In analogy to the finite-sample standardized version of the Moran I test statistic, we now introduce the following finite-sample standardized test statistic:

$$I_{S,n}^2(2) = \begin{bmatrix} \widetilde{u}'W_1\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,1} \\ \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,2} \end{bmatrix}' \Phi_S^{-1} \begin{bmatrix} \widetilde{u}'W_1\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,1} \\ \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 - \mu_{S,2} \end{bmatrix}. \tag{9}$$

In the appendix we prove the following theorem.

Theorem 2 *Suppose the assumptions of Theorem 1 hold, and that* $\lim_{n\to\infty} n^{-1}\Phi$ *is finite and nonsingular, then*

$$I_n^2(2) - I_{S,n}^2(2) \xrightarrow{p} 0.$$

The above theorem establishes that under the assumptions of Theorem 1 the test statistics $I_n^2(2)$ and $I_{S,n}^2(2)$ are asymptotically equivalent, and consequently $I_{S,n}^2(2) \xrightarrow{d} \chi^2(2)$. The theorem was given for the case $q = 2$. However, the setup and theorem readily extends to the case where the researcher considers $q \geq 2$ spatial-weights matrices $W_1, \ldots, W_q$ in that the expressions for $\mu_{S,r}$ and $\phi_{S,rs}$ continue to hold for $r, s = 1, \ldots, q$.

Remark: Of course (1) implies that y and u have the same variance-covariance (VC) matrix. Thus testing the hypothesis H_0: $Euu' = \sigma^2 I_n$ is equivalent to testing for the absence of spatial correlation in y, i.e., testing the hypothesis $H_0 : VC(y) = \sigma^2 I_n$. If $\beta = 0$, then $y = u$ and we can use the Moran I_n and $I_n^2(2)$ test statistics with $\widehat{u} = y$ to test $H_0 : VC(y) = \sigma^2 I_n$. If X only contains an intercept, i.e., $X = e$ with $e = [1, \ldots, 1]'$, then (1) simplifies to

$$y_i = \beta + u_i$$

and $\widehat{u}_i = y_i - \overline{y}$, where $\overline{y}$ denotes the sample mean of the y_i. Of course, in this case $P_x = n^{-1}ee'$ and $M_x = I - n^{-1}ee'$.

3 Monte Carlo Model

We now describe the Monte Carlo design employed for our explorations of the finite-sample properties of tests for spatial autocorrelation based on the I_n, $I_{S,n}$, $I_n^2(2)$, and $I_{S,n}^2(2)$ test statistics, as well as for a UR test. We generated the data for our Monte Carlo study from model (1) with $X = [x_1, x_2]$ and correspondingly $\beta = (\beta_1, \beta_2)'$, i.e., the Monte Carlo study is based on the model

$$y = x_1\beta_1 + x_2\beta_2 + u. \tag{10}$$

The disturbances are assumed to be generated by a second-order spatial-autoregressive process of the form

$$u = \rho_1 W_1 u + \rho_2 W_2 u + \varepsilon. \tag{11}$$

The Monte Carlo design is an adaptation and extension of the design used in Arraiz et al. (2010). The two $n \times 1$ regressors x_1 and x_2 are normalized versions of income per-capita and the proportion of housing units which are rentals in 1980, in 760 counties in US mid-western states. These data were taken from Kelejian and Robinson (1995). We normalized the 760 observations on these variables by subtracting from each observation the corresponding sample average, and then dividing that result by the sample standard deviation. The first n values of these normalized variables were used in our Monte Carlo experiments for sample sizes n less than 760. The same set of observations on these variables were used in all Monte Carlo repetitions.

Our spatial-weights matrices correspond to different patterns of locations in space. In particular, in addition to considering situations where all units are located on an equally spaced grid, we also consider situations where units which are located in the northeast portion of that space are closer to each other, and have more neighbors than the units corresponding to other quadrants of that space. Abstractly, the design is motivated by the states located in the northeastern portion of the US, as compared to western and southern states. To define these various patters of locations underlying our Monte Carlo experiments consider a square grid with both the x and y coordinates only taking on the values 1, 1.5, 2, 2.5, ..., $\bar{m}$. Let the units in the northeast quadrant of the grid be at the indicated discrete coordinates: $m + 1 \leq x \leq \bar{m}$ and $m + 1 \leq y \leq \bar{m}$ with $0 \leq m \leq \bar{m}$. Let the remaining units be located only at integer values of the coordinates: $x = 1, 2, ..., m$ and $y = 1, 2, ..., m$. The next figure illustrates the implied location patters for $m = 2$ and $\bar{m} = 4$.

Figure 1: Example of a North-East Modified-Rook Matrix: $m = 2$ and $\overline{m} = 4$

4.0	*		*		*	*	*
3.5					*	*	*
3.0	*		*		*	*	*
2.5							
2.0	*		*		*		*
1.5							
1.0	*		*		*		*
	1.0	1.5	2.0	2.5	3.0	3.5	4.0

We note that for a given $\bar{m}$, the number of units located in the northeast quadrant is inversely related to m. The case where all units are located on an equally spaced grid with a minimum distance of one corresponds to $m = \overline{m}$. We consider the following location patterns:

Table Location Patterns

Location Pattern	m	$\bar{m}$	Sample Size n	Percent of Units in NE
1	2	7	105	77
2	5	15	486	74
3	6	9	97	26
4	14	20	485	25
5	10	10	100	0
6	22	22	484	0

On the space described above we define distance in terms of the Euclidean distance. That is for any two units, i_1 and i_2, which have coordinates (x_1, y_1) and (x_2, y_2), respectively, we define the distance between them as

$$d(i_1, i_2) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2}.$$

We now use this distance measure to define different spatial-weights matrices. We note that all spatial-weights matrices employed in the Monte Carlo experiments have been normalized by their spectral radius; see Kelejian and Prucha (2010) for a further discussion of this normalization.[8] For simplicity of presentation we will ignore the normalization in the subsequent definitions of the spatial-weights matrices.

In conjunction with location patterns 1-4 we consider two spatial-weights matrices. We will refer to those matrices, respectively, as an inverse-distance matrix and a generalized-contiguity matrix, and denote them as W_I and W_C, respectively. More specifically, the elements of W_I are defined as $w_{I,ij} = 1/d(i,j)^2$, i.e., the weights are assumed to decline proportionally to the square distance. The elements of the W_C matrix are defined as

$$w_{C,ij} = \begin{cases} 1 & \text{if} \quad 0 < d(i,j) \leq 1 \\ 0 & \text{else} \end{cases}.$$

In the Monte Carlo experiments corresponding to location patterns 1-4 we take $W_1 = W_I$ and $W_2 = W_C$.

In conjunction with location patterns 5-6 we consider a further set of two spatial-weights matrices. We will refer to those matrices as a bishop and a rook matrix, and denote them with W_B and W_R, respectively. The elements of the W_B matrix are defined as

$$w_{B,ij} = \begin{cases} 1 & \text{if} \quad d(i,j) = \sqrt{2} \\ 0 & \text{else} \end{cases},$$

the elements of the W_R matrix are defined as

$$w_{R,ij} = \begin{cases} 1 & \text{if} \quad d(i,j) = 1 \\ 0 & \text{else} \end{cases}.$$

[8] Let A be some $n \times n$ matrix, and let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A. The spectral radius of A is defined as $\max_{i=1,\ldots,n} |\lambda_i|$.

In Monte Carlo experiments corresponding to location patterns 5-6 we take $W_1 = W_B$ and $W_2 = W_R$.

The scalar spatial autoregressive parameters ρ_1 and ρ_2 are taken to be $-.9, -.5. -2., 0, .2, .5, .9$. To ensure that the data generating process is well defined we checked that $I_n - \rho_1 W_1 - \rho_2 W_2$ is nonsingular for all considered parameter constellations.[9] The values for the regression parameters are $\beta_1 = \beta_2 = 1$. The elements of $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_n]'$ are generated as i.i.d. $N(0, \sigma^2)$, where we take w.o.l.o.g. $\sigma^2 = 1$.

Each Monte Carlo experiment is based on 10,000 repetitions.

4 Monte Carlo Results

The results of the Monte Carlo experiments are reported in tables 1–6. Each table corresponds to one of the 6 location patterns described above. As an overview, in columns 1–2 of each table we specify the values of ρ_1 and ρ_2. In columns 3–8 we report on the rejection rates of the 6 considered tests, which are described in more detail below. The last column reports on a measure of the correlation induced by that data-generating process.

In more detail, the first and second test statistics we report are the Moran I test statistic I_n and its finite-sample standardized version $I_{S,n}$ based on W_1, where the test statistics are defined in (3) and (4), respectively. The third test we report is the following UR test:

Step 1 Compute the Moran I test statistic I_n based on W_1. Reject H_0 if the test statistic exceeds the critical value. Otherwise perform step 2.

Step 2 Compute the Moran I test statistic I_n based on W_2. Reject H_0 if the test statistic exceeds the critical value. Otherwise accept H_1.

The fourth reported test applies the same logic as the third test, but using $I_{S,n}$ instead of I_n. (Recall that for UR tests the order in which the individual tests are performed does not matter.) The fifth and sixth tests are the $I_n^2(2)$ test statistic and its finite-sample standardized version $I_{S,n}^2(2)$ based on W_1 and W_2, where the test statistics are defined in (6) and (9).

In all cases, the critical values used in performing the tests were such that the asymptotic nominal size of the test was .05 under the null hypothesis of zero spatial correlation. For the UR tests, each of the Moran I tests was performed with an a priori asymptotic nominal size of 0.05.

To provide some insight into the extent of correlation generated by the different parameter constellations and weight matrices we also compute a correlation measure. From (11) we have $u = [I - \rho_1 W_1 - \rho_2 W_2]^{-1} \varepsilon$ and thus the variance-covariance matrix of the u is given by

$$\Sigma = (\sigma_{ij}) = [I - \rho_1 W_1 - \rho_2 W_2]^{-1} [I - \rho_1 W_1' - \rho_2 W_2']^{-1} . \qquad (12)$$

[9] We note that the computation of the $I^2(q)$ and $I_S^2(q)$ does not involve the estimation of ρ_1 and ρ_2, and hence for this study we have not been concerned with defining a contiguous parameter space for which $I_n - \rho_1 W_1 - \rho_2 W_2$ is non-singular.

Let $R = (r_{ij})$ denote the corresponding correlation matrix, then $r_{ij} = \sigma_{ij} / [\sigma_{ii}\sigma_{jj}]^{1/2}$. Our correlation measure, denoted in the tables with "Corr. Measure" is the average of the largest 5 absolute correlations above the main diagonal.

We next discuss some of the major findings of the Monte Carlo study discernible from tables 1-6. We start with a discussion of the results as they pertain to the size (or the significance levels) of the respective tests. By definition the size of the tests is the rejection rate under the null hypothesis of zero spatial correlation, and thus the lines in the tables corresponding to $\rho_1 = \rho_2 = 0$ report on the actual size of the tests. Ideally we would like to see the actual size of the tests to be close to the asymptotic nominal size of 0.05. An inspection of the tables shows that for both Moran I and both $I^2(2)$ tests the actual size is indeed close to the nominal size. This is in contrast to the UR tests, which exhibit in part quite serious size distortions, leading us to over-reject H_0. In particular for the experiments reported in tables 5-6 the actual size is typically larger than 0.09.

We next discuss the results as they pertain to the power of the tests under the alternative $H_1 : \rho_1 \neq 0$ and/or $\rho_2 \neq 0$. Given that the UR tests were found to be not correctly sized we focus our discussion on the Moran I and $I^2(2)$ tests. Overall the power of the tests is seen to increase with the sample size, although for some experiments, especially with $|\rho_1|$ and $|\rho_2|$ small, the increase is slow. Furthermore, for the most part, the tables show that the statistics are consistent with power going to 1 as the amount of spatial correlation increases.

We emphasize that the amount of correlation generated by the spatial autoregressive model (11) is complex, as is evident from the expression for Σ on the r.h.s. of (12). Given the complexity and nonlinearity of the expression one would not necessarily expect for the correlation to uniformly increase in $|\rho_1|$ and $|\rho_2|$. Indeed this seems confirmed by an inspection of the correlation measure we report in our tables. In fact, it seems that combinations of negative ρ_1 and positive ρ_2 can lead to relatively small values for our correlation measure, which in turn then seems to be reflected in the power of the tests for those parameter constellations.

We now consider the case where $\rho_2 = 0$ and $\rho_1 \neq 0$ in more detail. In this case we expect the Moran I tests, which are based on W_1, to outperform the $I^2(2)$ tests given that by design the former incorporate the information that $\rho_2 = 0$, while the latter do not. Indeed, we see some relative loss in power from the $I^2(2)$ tests, but the results suggest that the loss is mostly modest. For example, consider the results for location pattern 5 with sample size 100 reported in table 5 : For $\rho_1 = 0.2$ the power of the Moran I_S and $I_S^2(2)$ test is 0.29 and 0.26, respectively. For $\rho_1 = 0.5$ the power is .94 and .90, respectively, and for $\rho_1 = .9$ the power is 1 and 1, respectively. As the sample size increases the power of the Moran I_S and $I_S^2(2)$ tests shifts toward one, which makes any relative loss in power less consequential.

We next consider the case where $\rho_2 \neq 0$. In this case we expect the $I^2(2)$ tests to outperform the Moran I tests in that the the $I^2(2)$ tests incorporate information about W_2. Indeed, an inspection of the tables shows that the results are in essence consistent with this conjecture. Importantly, we note that for certain parameter constellations and weight matrices the improvement in power can be quite dramatic. For example, consider again results for location pattern 5 with sample size 100 reported in table 5 : When $\rho_1 = -.5$ and $\rho_2 = -.9$ the power increases from .38 to .99, and when $\rho_1 = 0$ and $\rho_2 = 0.5$ the power increases from about 0.31 to about .88.

Recall that the Moran I test is a special case of a $I^2(q)$ test with $q = 1$. In looking at the results when $\rho_1 = 0$ and $\rho_2 \neq 0$ we can thus also obtain some limited insights into the performance of $I^2(q)$ tests for situations where the weight matrices are misspecified. (Note that our Moran I test is based on W_1, while the data are in this case generated from $u = \rho_2 W_2 u + \varepsilon$.) The results suggest that for the weight matrices considered the power still increases to one, although there is loss of power relative to the $I^2(2)$ test which incorporates W_2.

An inspection of the tables suggest furthermore that the finite-sample standardized tests perform somewhat better than the non-standardized counterparts.

5 Concluding Remarks

This paper offers two main contributions: The first contribution is the introduction of a standardized version of the $I^2(q)$ test which modifies each of the quadratic forms such that under the null hypothesis they are standardized $(0, 1)$ in the case where the variables under consideration are normally distributed.

The second contribution is a Monte Carlo investigation of the finite-sample performance of the $I^2(q)$ tests. The Monte Carlo study also reports on the finite-sample performance of Moran I as well as UR tests. Overall, the results suggest that the $I^2(q)$ tests perform well. The penalty for using more weight matrices than needed seems modest. Thus, unless the researcher is very sure which weights matrix is the relevant one, it may be prudent to apply a more robust $I^2(q)$ test, which combines several weights matrices while preserving the proper size, rather than the original Moran I test. We note that the $I^2(q)$ test has the interpretation of being a Lagrange Multiplier test if the underlying data generating process is a spatial AR(q) or MA(q) process.

A Appendix

The following lemma is due to Pitman (1937) and Koopmans (1942); cp. Cliff and Ord (1981), p. 43:

Lemma A.1 *: Suppose $\eta_1, \ldots, \eta_p$ are distributed i.i.d. $N(0,1)$, then any scale free function of them, $h(\eta_1, \ldots, \eta_p)$ is distributed independently of $\sum_{i=1}^{p} \eta_i^2$.*

Proof of Theorem 1: For $r = 1, 2$ define

$$Q_r = \widetilde{V}_r / \widetilde{\sigma}^2 = \frac{\widetilde{u}' W_r \widetilde{u}}{\widetilde{\sigma}^2} = n \frac{\widetilde{u}' W_r \widetilde{u}}{\widetilde{u}' \widetilde{u}} = n \frac{(\widetilde{u}/\sigma)' W_r (\widetilde{u}/\sigma)}{(\widetilde{u}/\sigma)'(\widetilde{u}/\sigma)} \tag{A.1}$$

and observe that

$$\widetilde{u}/\sigma = M_x(u/\sigma)$$

with

$$M_x = I - P_x, \qquad P_x = X(X'X)^{-1}X',$$

and where $u/\sigma \sim N(0, I)$. Hence in deriving the moment of Q_r we can proceed w.o.l.o.g. under the assumption that $u \sim N(0, I)$.

Recall that M_x is idempotent with rank $n - K$. Hence there exist an orthogonal matrix S such that

$$S M_x S' = \begin{bmatrix} I_{n-K} & 0 \\ 0 & 0 \end{bmatrix}. \tag{A.2}$$

Consider the partition

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

where S_1 is $n - K \times n$ and S_2 is $K \times n$. Observe that because of orthogonality $S'S = SS' = I_n$. Thus

$$\begin{bmatrix} I_{n-K} & 0 \\ 0 & I_K \end{bmatrix} = \begin{bmatrix} S_1 S_1' & S_1 S_2' \\ S_2 S_1' & S_2 S_2' \end{bmatrix}$$

and, in particular, $S_1 S_1' = I_{n-K}$ and $S_2 S_1' = 0$. Observe furthermore that in light of (A.2) we have $M_x = S_1' S_1$. Now define

$$\psi = S\widetilde{u} = S M_x u = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

where

$$\begin{aligned} \psi_1 &= S_1 M_x u = S_1 S_1' S_1 u = S_1 u \sim N(0, I_{n-K}) \\ \psi_2 &= S_2 M_x u = S_2 S_1' S_1 u = 0. \end{aligned}$$

Next define

$$A_r = \begin{bmatrix} A_{r,11} & A_{r,12} \\ A_{r,21} & A_{r,22} \end{bmatrix} = S'W_rS,$$

then

$$Q_r = n\frac{\widetilde{u}'W_r\widetilde{u}}{\widetilde{u}'\widetilde{u}} = n\frac{\widetilde{u}'SS'W_rSS'\widetilde{u}}{\widetilde{u}'SS'\widetilde{u}} = n\frac{\psi_1'A_{r,11}\psi_1}{\psi_1'\psi_1}.$$

Since Q_r is scale free, Q_r and $\psi_1'\psi_1$ are independent in light of Lemma A.1. Thus since for $p,q = 0,1$

$$Q_r^pQ_s^q\left(\psi_1'\psi_1\right)^{p+q} = n^{p+q}\left[\psi_1'A_{r,11}\psi_1\right]^p\left[\psi_1'A_{s,11}\psi_1\right]^q$$

we have

$$E\left(Q_r^pQ_s^q\right)E\left[\left(\psi_1'\psi_1\right)^{p+q}\right] = n^{p+q}E\left\{\left[\psi_1'A_{r,11}\psi_1\right]^p\left[\psi_1'A_{s,11}\psi_1\right]^q\right\}$$

which in turn implies that

$$E\left(Q_r^pQ_s^q\right) = n^{p+q}\frac{E\left[\left(\widetilde{u}'W_r\widetilde{u}\right)^p\left(\widetilde{u}'W_s\widetilde{u}\right)^q\right]}{E\left[\left(\widetilde{u}'\widetilde{u}\right)^{p+q}\right]}. \tag{A.3}$$

Next observe that

$$\begin{aligned} E\left(\widetilde{u}'\widetilde{u}\right) &= E\left[u'M_xM_xu\right] \\ &= tr\left[M_x^2\right] = tr\left[M_x\right] = n-K \\ E\left(\widetilde{u}'W_r\widetilde{u}\right) &= E\left[u'M_xW_rM_xu\right] \\ &= tr\left[M_xW_rM_x\right] = tr\left[M_x^2W_r\right] = tr\left[M_xW_r\right] \\ &= tr\left(W_r\right) - tr\left(P_xW_r\right) \\ &= -tr\left(P_xW_r\right) \text{ since } diag(W_r) = 0. \end{aligned}$$

Hence

$$\mu_{S,r} = E\left(\widetilde{V}_r/\widetilde{\sigma}^2\right) = E\left(Q_r\right) = n\frac{E\left(\widetilde{u}'W_r\widetilde{u}\right)}{E\left(\widetilde{u}'\widetilde{u}\right)} = -n\frac{tr\left(P_xW_r\right)}{n-K} = \frac{-n}{n-K}tr\left(P_xW_r\right). \tag{A.4}$$

Next observe that $\widetilde{u}'W_r\widetilde{u} = u'M_xW_rM_xu = u'B_ru$ with

$$B_r = (1/2)\left[M_x(W_r' + W_r)M_x\right]$$

Hence, utilizing the expressions for the variance and co-variances of quadratic forms given, e.g., in Lemma A.1 in Kelejian and Prucha (2010) we have

$$\begin{aligned} E\left[\left(\widetilde{u}'W_r\widetilde{u}\right)\left(\widetilde{u}'W_s\widetilde{u}\right)\right] &= E\left[\left(u'B_ru\right)\left(u'B_su\right)\right] \\ &= cov(u'B_ru, u'B_su) + \left[E(u'B_ru)\right]\left[E(u'B_su)\right] \\ &= 2tr(B_rB_s) + tr(B_r)tr(B_s), \end{aligned}$$

where

$$\begin{aligned}
tr(B_r B_s) &= \frac{1}{4} tr\left[M_x(W_r' + W_r)M_x M_x(W_s' + W_s)M_x\right] \\
&= \frac{1}{4} tr\left[M_x(W_r' + W_r)M_x(W_s' + W_s)\right] \\
&= \frac{1}{4} tr\left[M_x W_r' M_x W_s' + M_x W_r' M_x W_s + M_x W_r M_x W_s' + M_x W_r M_x W_s\right] \\
&= \frac{1}{2} tr\left[M_x W_r M_x W_s + M_x W_r M_x W_s'\right], \\
tr(B_r) &= \frac{1}{2} tr\left[M_x(W_r' + W_r)M_x\right] = \frac{1}{2} tr\left[M_x(W_r' + W_r)\right] = tr\left[M_x W_r\right] \\
&= -tr(P_x W_r).
\end{aligned}$$

Further

$$\begin{aligned}
E\left[(\widetilde{u}'\widetilde{u})^2\right] &= E\left[(u' M_x u)^2\right] = var(u' M_x u) + \left[E(u' M_x u)\right]^2 \\
&= 2tr(M_x M_x) + \left[tr(M_x)\right]^2 \\
&= \left[tr(M_x)\right]\left[2 + tr(M_x)\right] \\
&= (n-K)(n-K+2),
\end{aligned}$$

and thus

$$\begin{aligned}
E\left(Q_r^2\right) &= n^2 \frac{E\left[(\widetilde{u}' W_r \widetilde{u})^2\right]}{E\left[(\widetilde{u}'\widetilde{u})^2\right]} \\
&= \frac{n^2}{(n-K)(n-K+2)} \left\{ tr\left[M_x W_r M_x W_r + M_x W_r M_x W_r'\right] + \left[tr\left(P_x W_r\right)\right]^2 \right\}, \\
E\left(Q_r Q_s\right) &= n^2 \frac{E\left[(\widetilde{u}' W_r \widetilde{u})\left(\widetilde{u}' W_s \widetilde{u}\right)\right]}{E\left[(\widetilde{u}'\widetilde{u})^2\right]} \\
&= \frac{n^2}{(n-K)(n-K+2)} \left\{ tr\left[M_x W_r M_x W_s + M_x W_r M_x W_s'\right] + \left[tr\left(P_x W_r\right)\right]\left[tr\left(P_x W_s\right)\right] \right\}.
\end{aligned}$$

From this it follows furthermore that

$$\begin{aligned}
\phi_{S,rs} &= cov\left(\widetilde{V}_r/\widetilde{\sigma}^2, \widetilde{V}_s/\widetilde{\sigma}^2\right) = E\left(Q_r Q_s\right) - (EQ_r)(EQ_s) \qquad \text{(A.5)} \\
&= \frac{n^2}{(n-K)(n-K+2)} \left\{ tr\left[M_x W_r M_x W_s + M_x W_r M_x W_s'\right] \right\} \\
&\quad - \frac{2n^2}{(n-K)^2(n-K+2)} \left[tr\left(P_x W_r\right)\right]\left[tr\left(P_x W_s\right)\right],
\end{aligned}$$

which completes the proof of the theorem. ■■

Proof of Theorem 2: Observe that for $r, s = 1, 2$

$$\begin{aligned}
tr\,(P_x W_r) &= tr\left[(n^{-1}X'X)^{-1}n^{-1}X'W_rX\right] \to tr(M_{XX}^{-1}M_{XW_rX}), \\
tr(P_xW_rW_s) &= tr\left[(n^{-1}X'X)^{-1}n^{-1}X'W_rW_sX\right] \to tr(M_{XX}^{-1}M_{XW_rW_sX}) \\
tr(P_xW_rP_xW_s) &= tr\left[(n^{-1}X'X)^{-1}n^{-1}X'W_rX(n^{-1}X'X)^{-1}n^{-1}X'W_sX\right] \\
&\to tr(M_{XX}^{-1}M_{XW_rX}M_{XX}^{-1}M_{XW_sX})
\end{aligned}$$

where $M_{XX} = \lim n^{-1}X'X$, and $M_{XAX} = \lim n^{-1}X'AX$ for $A = W_r,\ W_rW_s$. Since M_{XX}^{-1} and M_{XAX} are finite by assumption it follows that

$$\begin{aligned}
tr\,(P_xW_r) &= O(1) \\
tr\,[M_xW_rM_xW_s] &= tr\,[(I-P_x)W_r(I-P_x)W_s] \\
&= tr\,[W_rW_s] - tr(P_xW_rW_s) - tr(W_rP_xW_s) + tr(P_xW_rP_xW_s) \\
&= tr\,[W_rW_s] + O(1) \\
tr\,[M_xW_rM_xW_s'] &= tr\,[W_rW_s'] + O(1).
\end{aligned}$$

Hence by (A.4)

$$n^{-1/2}\left[\widetilde{V_r}/\widetilde{\sigma}^2 - E(\widetilde{V}_r/\widetilde{\sigma}^2)\right] = n^{-1/2}\widetilde{V}_r/\widetilde{\sigma}^2 + o(1)$$

recalling that $\mu_{S,r} = E(\widetilde{V}_r/\widetilde{\sigma}^2) = \frac{-n}{n-K}tr\,(P_xW_r)$, and by (A.5)

$$n^{-1}\phi_{S,rs} = n^{-1}cov\left(\widetilde{V}_r/\widetilde{\sigma}^2, \widetilde{V}_s/\widetilde{\sigma}^2\right) = n^{-1}tr\,[W_r(W_s + W_s')] + o(1).$$

Recalling further that $\phi_{rs} = \frac{1}{2}tr\,[(W_r + W_r')(W_s + W_s')] = tr\,[W_r(W_s + W_s')]$ we see that

$$n^{-1}\Phi_S - n^{-1}\Phi = o(1) \text{ and } \left[n^{-1}\Phi_S\right]^{-1} - \left[n^{-1}\Phi\right]^{-1} = o(1).$$

Observing furthermore that $n^{-1/2}\widetilde{V}/\widetilde{\sigma}^2 = O_p(1)$ it follows from the above that

$$\begin{aligned}
I_S^2(q) &= \left[\widetilde{V}/\widetilde{\sigma}^2 - E\widetilde{V}/\widetilde{\sigma}^2\right]'\Phi_S^{-1}\left[\widetilde{V}/\widetilde{\sigma}^2 - E\widetilde{V}/\widetilde{\sigma}^2\right]' \\
&= n^{-1/2}\left[\widetilde{V}/\widetilde{\sigma}^2 - E\widetilde{V}/\widetilde{\sigma}^2\right]'\left[n^{-1}\Phi_S\right]^{-1}n^{-1/2}\left[\widetilde{V}/\widetilde{\sigma}^2 - E\widetilde{V}/\widetilde{\sigma}^2\right] \\
&= \left[n^{-1/2}\widetilde{V}/\widetilde{\sigma}^2 + o(1)\right]'\left\{\left[n^{-1}\Phi\right]^{-1} + o(1)\right\}\left[n^{-1/2}\widetilde{V}/\widetilde{\sigma}^2 + o(1)\right] \\
&= \left[n^{-1/2}\widetilde{V}/\widetilde{\sigma}^2\right]'\left[n^{-1}\Phi\right]^{-1}n^{-1/2}\widetilde{V}/\widetilde{\sigma}^2] + o_p(1) \\
&= I^2(q) + o_p(1).
\end{aligned}$$

It now follows from Theorem 1 that under H_0 we have $I_S^2(q) \xrightarrow{d} \chi^2(2)$. ■■

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Table 1: Power and Size of Tests: Location Pattern 1, $n = 105$, $W_1 = W_I$, $W_2 = W_C$, %NE= 77

ρ_1	ρ_2	Rejection Rate of Tests						Corr.
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	Measure
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	.844	.8461	.9789	.9785	.8926	.895	.2293
-.9	-.5	.5996	.6023	.8145	.8133	.5388	.5454	.194
-.9	-.2	.3923	.3947	.5237	.5242	.2697	.2768	.1653
-.9	0	.2495	.2504	.3197	.3199	.1867	.1909	.1445
-.9	.2	.164	.1652	.1907	.1916	.1805	.184	.1216
-.9	.5	.0677	.0694	.0898	.0952	.2296	.2347	.091
-.9	.9	.1178	.1312	.3679	.3913	.5326	.5418	.1452
-.5	-.9	.6329	.6356	.9282	.9279	.7844	.7856	.1941
-.5	-.5	.3275	.3296	.6096	.6084	.3512	.3561	.1447
-.5	-.2	.152	.1536	.2709	.2699	.1315	.1351	.1105
-.5	0	.0793	.0808	.1267	.1273	.0791	.0813	.0847
-.5	.2	.0418	.0431	.0625	.0649	.0838	.0858	.0561
-.5	.5	.0776	.0871	.1635	.1774	.2091	.2174	.0735
-.5	.9	.549	.5709	.7656	.78	.7503	.762	.1979
-.2	-.9	.4254	.4285	.8609	.8599	.6804	.6825	.1833
-.2	-.5	.1575	.16	.4344	.4332	.2451	.248	.1056
-.2	-.2	.0548	.0556	.1339	.1338	.0644	.0661	.0661
-.2	0	.0281	.0298	.0536	.0558	.0418	.0439	.0357
-.2	.2	.0558	.0615	.0853	.0943	.0807	.0878	.0284
-.2	.5	.3015	.3207	.4246	.4475	.3781	.3965	.1018
-.2	.9	.9073	.9153	.9635	.9667	.9489	.9535	.3686
0	-.9	.2977	.3013	.7929	.7917	.6104	.6114	.1765
0	-.5	.0845	.0856	.3199	.3194	.182	.1839	.0946
0	-.2	.028	.0302	.0818	.0834	.0481	.0496	.0373
0	0	.0408	.0465	.0614	.0687	.0516	.0559	0
0	.2	.1526	.1658	.196	.2111	.1566	.1695	.0449
0	.5	.5738	.5926	.6747	.6918	.6003	.6172	.1655
0	.9	.9915	.9921	.9963	.9965	.9949	.9953	.7828
.2	-.9	.1915	.1933	.7045	.7033	.5451	.5466	.1693
.2	-.5	.0377	.0383	.2195	.2183	.1384	.1396	.0869
.2	-.2	.0354	.0392	.0725	.0759	.0537	.0579	.0284
.2	0	.1168	.1292	.1427	.1556	.1052	.1161	.0419
.2	.2	.3587	.3799	.4151	.435	.3363	.3556	.1069
.2	.5	.8369	.848	.8833	.8914	.8389	.848	.3255
.2	.9	.9994	.9994	.9995	.9995	.9991	.9991	.8191
.5	-.9	.0806	.0819	.5423	.5402	.4364	.4384	.1573
.5	-.5	.0263	.0298	.1192	.122	.1069	.1106	.073
.5	-.2	.16	.1745	.1808	.1951	.1479	.1613	.0622
.5	0	.4326	.4517	.4611	.48	.3799	.402	.1403
.5	.2	.7773	.7919	.8084	.8208	.7451	.7593	.3016
.5	.5	.9995	.9995	.9996	.9996	.9993	.9993	.9992
.5	.9	.9997	.9997	.9997	.9997	.9997	.9997	.9014
.9	-.9	.0284	.0301	.3079	.3073	.3335	.338	.1377
.9	-.5	.2072	.2204	.2331	.2473	.2465	.2606	.0807
.9	-.2	.6834	.6989	.6949	.7095	.6419	.6579	.2763
.9	0	.9416	.9455	.9481	.9508	.9288	.9323	.7452
.9	.2	.985	.9863	.9885	.9893	.982	.9834	.7655
.9	.5	.9999	.9999	1	1	.9999	.9999	.9862
.9	.9	1	1	1	1	1	1	.9678

Table 2: Power and Size of Tests: Location Pattern 2, $n = 486$, $W_1 = W_I$, $W_2 = W_C$, %NE= 74

ρ_1	ρ_2	Rejection Rate of Tests						Corr. Measure
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	1	1	1	1	1	1	.1932
-.9	-.5	.9999	.9999	1	1	.9996	.9996	.1571
-.9	-.2	.9747	.9738	.9882	.9873	.9451	.9434	.1271
-.9	0	.8343	.8309	.8614	.8581	.7022	.7006	.1049
-.9	.2	.4833	.4784	.4942	.4891	.4573	.4567	.0805
-.9	.5	.0682	.0687	.2291	.2376	.6049	.6053	.0634
-.9	.9	.6886	.7063	.992	.9926	.9964	.9965	.1414
-.5	-.9	1	1	1	1	1	1	.1736
-.5	-.5	.9862	.9857	.9985	.9985	.9928	.9928	.121
-.5	-.2	.7487	.7442	.8573	.8543	.6717	.6682	.0867
-.5	0	.3517	.3469	.4101	.4049	.245	.2428	.0609
-.5	.2	.073	.0723	.0962	.0963	.1544	.1548	.0324
-.5	.5	.2868	.3054	.6663	.6788	.6968	.7024	.0732
-.5	.9	.994	.9944	.9999	.9999	.9999	.9999	.2133
-.2	-.9	.9995	.9994	1	1	1	1	.1662
-.2	-.5	.9053	.9023	.9884	.9876	.9528	.9518	.0926
-.2	-.2	.3611	.3573	.5587	.5525	.3511	.3471	.0546
-.2	0	.0802	.0792	.1194	.1178	.0686	.0683	.0253
-.2	.2	.0898	.0978	.1817	.195	.1649	.1706	.0283
-.2	.5	.8139	.8264	.9507	.9542	.9253	.9303	.1007
-.2	.9	1	1	1	1	1	1	.3454
0	-.9	.9975	.9975	1	1	.9999	.9999	.1614
0	-.5	.7412	.7349	.9551	.9537	.8856	.8832	.0869
0	-.2	.1507	.1473	.3246	.3193	.1956	.1937	.0344
0	0	.0431	.0478	.0695	.073	.0467	.0495	0
0	.2	.3318	.3515	.4654	.4814	.3613	.3747	.0407
0	.5	.9737	.9756	.9949	.9956	.9893	.9899	.1433
0	.9	1	1	1	1	1	1	.6315
.2	-.9	.9868	.9859	1	1	.9998	.9998	.1563
.2	-.5	.482	.4765	.8725	.868	.7744	.771	.0815
.2	-.2	.0518	.0519	.1516	.1493	.1119	.1125	.0282
.2	0	.1738	.1883	.2036	.2179	.1481	.1598	.0284
.2	.2	.7101	.7268	.7928	.8044	.6983	.7127	.0806
.2	.5	.9986	.9988	.9999	.9999	.9995	.9995	.2336
.2	.9	1	1	1	1	1	1	.995
.5	-.9	.8953	.8928	.9993	.9992	.9981	.9981	.1482
.5	-.5	.1438	.1411	.6192	.6126	.591	.5888	.0723
.5	-.2	.1569	.1707	.1823	.1953	.2043	.2175	.0319
.5	0	.6959	.7116	.7215	.7358	.6324	.6493	.0857
.5	.2	.9822	.9834	.9885	.9895	.9767	.9787	.1816
.5	.5	1	1	1	1	1	1	.9969
.5	.9	1	1	1	1	1	1	.9849
.9	-.9	.4436	.4364	.9844	.9837	.986	.9857	.1358
.9	-.5	.1349	.143	.2919	.2977	.5339	.5446	.0592
.9	-.2	.8545	.863	.8578	.8662	.8354	.8454	.134
.9	0	.9974	.9977	.9979	.9982	.996	.9965	.452
.9	.2	1	1	1	1	1	1	.5403
.9	.5	1	1	1	1	1	1	.9988
.9	.9	1	1	1	1	1	1	.9117

Table 3: Power and Size of Tests: Location Pattern 3, $n = 97$, $W_1 = W_I$, $W_2 = W_C$, %NE= 26

ρ_1	ρ_2	Rejection Rate of Tests						Corr.
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	Measure
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	.7872	.7788	.9675	.9665	.9049	.9033	.2889
-.9	-.5	.4914	.4797	.746	.7416	.5366	.5347	.2511
-.9	-.2	.2871	.2766	.4394	.4316	.2521	.248	.2226
-.9	0	.1867	.1801	.262	.255	.1617	.1584	.2023
-.9	.2	.1154	.112	.1401	.1367	.1285	.1285	.1806
-.9	.5	.0595	.0569	.0887	.0895	.1831	.1856	.158
-.9	.9	.0907	.0926	.3084	.3233	.417	.43	.1819
-.5	-.9	.5672	.5549	.9183	.9163	.8099	.8088	.2456
-.5	-.5	.2563	.2491	.5697	.5645	.3653	.3635	.1838
-.5	-.2	.1182	.1135	.2474	.2425	.1305	.1296	.1473
-.5	0	.064	.0607	.1155	.1123	.0736	.0734	.1199
-.5	.2	.0386	.0373	.0606	.0596	.0715	.0729	.0895
-.5	.5	.0673	.0699	.1408	.1481	.1562	.1639	.0864
-.5	.9	.3518	.3581	.6046	.6195	.5714	.5871	.225
-.2	-.9	.3916	.3791	.8565	.8538	.7263	.7256	.2295
-.2	-.5	.1335	.1269	.4176	.4128	.2638	.2624	.1302
-.2	-.2	.046	.0436	.138	.1355	.0752	.0749	.0859
-.2	0	.0318	.0318	.0659	.0659	.0419	.0425	.0513
-.2	.2	.0514	.0526	.0791	.0833	.068	.0712	.0325
-.2	.5	.2222	.226	.3318	.343	.2733	.2829	.1212
-.2	.9	.7312	.7361	.8637	.8711	.8202	.8299	.481
0	-.9	.2847	.2748	.8049	.8024	.6675	.6668	.2188
0	-.5	.0774	.0736	.331	.3272	.2162	.2171	.1156
0	-.2	.029	.0282	.0962	.0946	.0625	.0634	.0454
0	0	.0455	.0463	.0745	.076	.053	.0543	0
0	.2	.1301	.1337	.1709	.1776	.1267	.1315	.0568
0	.5	.4465	.4526	.5476	.5582	.4761	.4866	.2268
0	.9	.9283	.9301	.9685	.9703	.9544	.9568	.891
.2	-.9	.1895	.1802	.7303	.7273	.6037	.6038	.2077
.2	-.5	.0405	.0385	.2472	.2439	.1714	.1721	.1041
.2	-.2	.0427	.0433	.0858	.0868	.0662	.0675	.0323
.2	0	.1171	.1208	.1422	.1473	.1107	.1132	.0638
.2	.2	.2848	.2885	.3339	.3402	.2658	.2728	.1579
.2	.5	.6819	.687	.7558	.7639	.6911	.6993	.4786
.2	.9	.9577	.959	.9825	.9833	.9737	.9751	.8903
.5	-.9	.0834	.0792	.6024	.5996	.5083	.5087	.1894
.5	-.5	.0382	.0392	.1664	.1652	.1455	.1473	.0842
.5	-.2	.1685	.1718	.1921	.1959	.169	.1719	.1072
.5	0	.3812	.3862	.4085	.4147	.3495	.3547	.2308
.5	.2	.6643	.6676	.697	.7018	.6398	.6455	.4749
.5	.5	.9969	.9969	.9978	.9979	.9968	.997	.9998
.5	.9	.956	.9576	.9859	.9867	.9745	.9764	.6875
.9	-.9	.0379	.0372	.4172	.4131	.4188	.4215	.1616
.9	-.5	.2263	.2307	.2735	.2773	.289	.2912	.1561
.9	-.2	.6176	.6221	.629	.6344	.5924	.5964	.465
.9	0	.8933	.895	.9008	.9032	.8791	.8811	.8869
.9	.2	.9363	.9374	.9466	.9479	.9271	.9289	.8934
.9	.5	.9185	.9211	.9494	.9519	.9197	.9228	.6292
.9	.9	.9985	.9985	.9995	.9996	.9991	.9993	.9794

Table 4: Power and Size of Tests: Location Pattern 4, $n = 485$, $W_1 = W_I$, $W_2 = W_C$, %NE= 25

ρ_1	ρ_2	Rejection Rate of Tests						Corr. Measure
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	1	1	1	1	1	1	.2165
-.9	-.5	.9882	.9879	.9984	.9983	.9895	.9892	.1811
-.9	-.2	.8368	.8356	.9101	.9084	.7641	.7638	.1521
-.9	0	.5597	.5581	.6216	.6185	.4208	.4227	.1309
-.9	.2	.2632	.2623	.2776	.2766	.2681	.2704	.1076
-.9	.5	.0479	.0499	.2055	.2153	.455	.4573	.0751
-.9	.9	.3665	.3885	.9448	.9477	.9615	.9623	.1415
-.5	-.9	.9992	.9992	1	1	1	1	.186
-.5	-.5	.9047	.9047	.9837	.9831	.9475	.9468	.1363
-.5	-.2	.4841	.4832	.6638	.6613	.4504	.4503	.1022
-.5	0	.1953	.1949	.2596	.2581	.1458	.1464	.0765
-.5	.2	.0494	.0498	.0779	.0797	.1114	.1136	.048
-.5	.5	.1628	.1781	.5159	.5299	.5295	.5369	.0726
-.5	.9	.8862	.8944	.9967	.9973	.9948	.9954	.2
-.2	-.9	.9926	.9926	1	1	.9994	.9994	.1763
-.2	-.5	.7101	.709	.9371	.9353	.8519	.8503	.1008
-.2	-.2	.2134	.2126	.4098	.4062	.2402	.2402	.062
-.2	0	.0505	.0515	.0902	.0897	.0543	.055	.0321
-.2	.2	.072	.0802	.1474	.1561	.1242	.1285	.0281
-.2	.5	.5567	.5778	.8105	.8204	.7469	.7564	.1004
-.2	.9	.9947	.9952	1	1	.9999	.9999	.354
0	-.9	.9727	.9722	1	1	.9992	.9991	.1702
0	-.5	.5059	.5046	.8742	.8722	.7526	.7516	.0914
0	-.2	.0973	.0973	.2528	.2503	.1485	.1499	.0361
0	0	.0446	.0497	.0741	.0795	.0496	.0535	0
0	.2	.2169	.2353	.3255	.3417	.2381	.2503	.0431
0	.5	.8182	.8309	.9378	.9419	.9013	.9061	.156
0	.9	.9997	.9997	1	1	1	1	.7369
.2	-.9	.9182	.9175	.9996	.9995	.9978	.9978	.1637
.2	-.5	.3175	.3164	.7645	.7619	.6507	.6494	.0845
.2	-.2	.0473	.0497	.1556	.156	.1117	.114	.0281
.2	0	.1361	.147	.1668	.1776	.1186	.1284	.0371
.2	.2	.5068	.5265	.6054	.6196	.4893	.5062	.097
.2	.5	.965	.969	.9913	.9924	.9816	.983	.29
.2	.9	1	1	1	1	1	1	.8694
.5	-.9	.7477	.7471	.9953	.995	.989	.9889	.1531
.5	-.5	.1002	.1005	.5412	.5374	.4949	.4968	.0722
.5	-.2	.1511	.1623	.1856	.1961	.1863	.1978	.0516
.5	0	.528	.5462	.56	.5769	.4721	.4928	.1203
.5	.2	.8971	.9049	.9272	.9321	.8811	.8882	.2575
.5	.5	1	1	1	1	1	1	.9984
.5	.9	1	1	1	1	1	1	.9248
.9	-.9	.346	.3448	.9658	.9647	.9653	.9654	.1359
.9	-.5	.1549	.1632	.333	.3385	.4835	.4944	.0628
.9	-.2	.7341	.747	.7411	.7536	.714	.7291	.2259
.9	0	.9778	.9805	.981	.9832	.9687	.9712	.6718
.9	.2	.9981	.9984	.9992	.9994	.9977	.9981	.7048
.9	.5	1	1	1	1	1	1	.9613
.9	.9	1	1	1	1	1	1	.9268

Table 5: Power and Size of Tests: Location Pattern 5, $n = 100$, $W_1 = W_B$, $W_2 = W_R$, %NE= 0

ρ_1	ρ_2	Rejection Rate of Tests						Corr. Measure
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	1	1	1	1	1	1	.779
-.9	-.5	1	1	1	1	1	1	.8167
-.9	-.2	1	1	1	1	1	1	.8283
-.9	0	1	1	1	1	1	1	.8304
-.9	.2	1	1	1	1	1	1	.8283
-.9	.5	1	1	1	1	1	1	.8167
-.9	.9	1	1	1	1	1	1	.779
-.5	-.9	.3788	.382	.9995	.9994	.9992	.9993	.3767
-.5	-.5	.8382	.8397	.9564	.9562	.9495	.951	.2636
-.5	-.2	.9172	.9184	.9244	.9251	.8801	.8815	.3004
-.5	0	.9367	.9371	.9375	.938	.8837	.8867	.3069
-.5	.2	.924	.9256	.9309	.9338	.8905	.8952	.3004
-.5	.5	.8425	.844	.9571	.9637	.9535	.9586	.2636
-.5	.9	.3979	.4006	.9997	.9998	.9998	.9998	.3767
-.2	-.9	.6679	.6792	.9997	.9997	.9996	.9996	.4715
-.2	-.5	.1107	.1136	.8494	.8486	.7609	.763	.2341
-.2	-.2	.2325	.2355	.3582	.359	.2391	.2435	.0995
-.2	0	.2757	.2793	.2907	.2956	.1732	.1779	.1106
-.2	.2	.2392	.2408	.353	.3697	.2415	.2565	.0995
-.2	.5	.1145	.117	.8491	.865	.7662	.7867	.2341
-.2	.9	.6508	.6634	1	1	1	1	.4715
0	-.9	.9955	.9958	1	1	1	1	.7743
0	-.5	.3208	.3344	.9459	.945	.8935	.8933	.2914
0	-.2	.0614	.0666	.3108	.3094	.214	.2151	.106
0	0	.0481	.051	.0906	.096	.044	.0482	0
0	.2	.0611	.0663	.2913	.3149	.1992	.218	.106
0	.5	.3047	.3162	.935	.9427	.8781	.8911	.2914
0	.9	.9936	.9938	1	1	1	1	.7743
.2	-.9	1	1	1	1	1	1	.9794
.2	-.5	.8744	.8814	.9941	.9944	.99	.9898	.4397
.2	-.2	.3915	.4055	.6106	.6157	.5532	.5568	.1515
.2	0	.272	.2876	.3251	.3411	.2463	.2605	.1106
.2	.2	.3692	.3853	.5797	.6027	.5292	.5529	.1515
.2	.5	.861	.8673	.9907	.9925	.9865	.9883	.4397
.2	.9	1	1	1	1	1	1	.9794
.5	-.9	1	1	1	1	1	1	.9999
.5	-.5	1	1	1	1	1	1	1
.5	-.2	.9711	.9732	.9811	.9824	.9742	.9747	.3867
.5	0	.9335	.9401	.9392	.9457	.9017	.9085	.3069
.5	.2	.9688	.9723	.9792	.982	.9707	.9754	.3867
.5	.5	1	1	1	1	1	1	1
.5	.9	1	1	1	1	1	1	.9999
.9	-.9	1	1	1	1	1	1	.9595
.9	-.5	1	1	1	1	1	1	.9541
.9	-.2	1	1	1	1	1	1	.975
.9	0	1	1	1	1	1	1	.8304
.9	.2	1	1	1	1	1	1	.975
.9	.5	1	1	1	1	1	1	.9541
.9	.9	1	1	1	1	1	1	.9595

Table 6: Power and Size of Tests: Location Pattern 6, $n = 484$, $W_1 = W_B$, $W_2 = W_R$, %NE= 0

ρ_1	ρ_2	Rejection Rate of Tests						Corr. Measure
		I Test	I_S Test	Uniform Rej I	Uniform Rej I_S	$I^2(2)$ Test	$I_S^2(2)$ Test	
		Tests based on W_1 only		Tests based on W_1 and W_2	Tests based on W_2 and W_1	Tests based W_1 and W_2		
-.9	-.9	1	1	1	1	1	1	.679
-.9	-.5	1	1	1	1	1	1	.7193
-.9	-.2	1	1	1	1	1	1	.7317
-.9	0	1	1	1	1	1	1	.734
-.9	.2	1	1	1	1	1	1	.7317
-.9	.5	1	1	1	1	1	1	.7193
-.9	.9	1	1	1	1	1	1	.679
-.5	-.9	.8414	.8407	1	1	1	1	.3688
-.5	-.5	1	1	1	1	1	1	.2405
-.5	-.2	1	1	1	1	1	1	.2769
-.5	0	1	1	1	1	1	1	.2833
-.5	.2	1	1	1	1	1	1	.2769
-.5	.5	.9999	.9999	1	1	1	1	.2405
-.5	.9	.8458	.8443	1	1	1	1	.3688
-.2	-.9	.9966	.9968	1	1	1	1	.4564
-.2	-.5	.188	.1863	1	1	1	1	.2281
-.2	-.2	.7668	.7646	.9545	.9538	.9488	.9469	.0929
-.2	0	.8567	.855	.8607	.859	.7653	.7636	.1035
-.2	.2	.773	.771	.9506	.9525	.9416	.9425	.0929
-.2	.5	.201	.1992	1	1	1	1	.2281
-.2	.9	.9958	.9958	1	1	1	1	.4564
0	-.9	1	1	1	1	1	1	.7167
0	-.5	.8364	.8403	1	1	1	1	.2801
0	-.2	.0864	.0885	.8766	.8747	.7921	.7887	.1025
0	0	.0504	.0514	.0939	.0952	.0474	.0476	0
0	.2	.0806	.0837	.863	.8693	.7763	.7853	.1025
0	.5	.8232	.8266	1	1	1	1	.2801
0	.9	1	1	1	1	1	1	.7167
.2	-.9	1	1	1	1	1	1	.9933
.2	-.5	1	1	1	1	1	1	.4088
.2	-.2	.9508	.9531	.9943	.9944	.9931	.9931	.143
.2	0	.8574	.8608	.8692	.8722	.7939	.7998	.1035
.2	.2	.9439	.9457	.9934	.9936	.9915	.9921	.143
.2	.5	1	1	1	1	1	1	.4088
.2	.9	1	1	1	1	1	1	.9933
.5	-.9	1	1	1	1	1	1	.9948
.5	-.5	1	1	1	1	1	1	1
.5	-.2	1	1	1	1	1	1	.3483
.5	0	1	1	1	1	1	1	.2833
.5	.2	1	1	1	1	1	1	.3483
.5	.5	1	1	1	1	1	1	1
.5	.9	1	1	1	1	1	1	.9948
.9	-.9	1	1	1	1	1	1	.9741
.9	-.5	1	1	1	1	1	1	.973
.9	-.2	1	1	1	1	1	1	1
.9	0	1	1	1	1	1	1	.734
.9	.2	1	1	1	1	1	1	1
.9	.5	1	1	1	1	1	1	.973
.9	.9	1	1	1	1	1	1	.9741