1 Overview

The programs made available on this web page are sample programs for the computation of estimators introduced in Kelejian and Prucha (1999). In particular we provide sample programs for the estimation of the following regression model, where the disturbances are generated from a (Cliff-Ord type) first order spatial autoregressive model:

$$
\begin{aligned}
y_n &= X_n\beta + u_n, && (1)\\
u_n &= \rho W_n u_n + \varepsilon_n, \qquad |\rho| < 1 && (2)
\end{aligned}
$$

where y_n is the $n \times 1$ vector of observations on the dependent variable, X_n is the $n \times k$ matrix of observation on k exogenous variables, W_n is an $n \times n$ spatial weighting matrix of known constants, β is the $k \times 1$ vector of regression parameters, ρ is the spatial autoregressive parameter, u_n is the $n \times 1$ vector of regression disturbances, ε_n is an $n \times 1$ vector of innovations with mean zero and variance σ^2_ε. The variables $W_n u_n$ is typically referred to as the spatial lag of u_n.

A spatial Cochrane-Orcutt type transformation of the model yields

$$y_{n*}(\rho) = X_{n*}(\rho)\beta + \varepsilon_n, \tag{3}$$

with $y_{n*}(\rho) = (I_n - \rho W_n)y_n$ and $X_{n*}(\rho) = (I_n - \rho W_n)X_n$.

Note: We provide two sets of sample programs, one for TSP and one for Stata. In the following we only describe the use of the TSP programs. The use of the Stata programs is analogous.

Note: The consistency result for the generalized moments (GM) estimator for ρ given in the paper only requires that the disturbances can be estimated $n^{1/2}$-consistently, and thus the result also applies to situations where y_n is generated by a more general model than (1). The expression for the asymptotic variance covariance matrix of the GM estimator introduced below is specific to the above model. It is essentially obtained as a special case of a more general result for a wider class of models given in Kelejian and Prucha (2004).

2 Data Files

The sample program involves two exogenous variables and an idealized spatial weighting matrix. This matrix corresponds to the case where each unit has "one neighbor ahead and one neighbor behind" in a wrap around world, and the row sums of the weighting matrix are normalized to one; for a more detailed description of this idealized matrix see the Monte Carlo section of the paper. The sample size is taken to be 100. The actual estimation programs assume that the data for the exogenous variables and spatial weighting matrix are stored in files named VAR1.DAT and MMAT.DAT, respectively.

3 Estimation Programs

The main estimation program is contained in the file PROGRAM1.TSP. This program calls three "subroutines" contained in the files GMPROC1.TSP, GLSPROC1.TSP and VGMPROC1.TSP. Those subroutines compute the GM estimator for ρ, the feasible GLS estimator for β and its asymptotic variance covariance matrix, and the asymptotic variance of the GM estimator for ρ, respectively.

The program PROGRAM1.TSP first reads in the data for the exogenous variables and spatial weighting matrix from the files VAR1.DAT and MMAT.DAT. The actual estimation of the parameters of the model (1)-(2) is then performed in four steps.

Step 1: In the first step we estimate the regression model in (1) using ordinary least squares (OLS) to obtain

$$\begin{aligned}\widehat{\beta}_{OLS} &= (X_n'X_n)^{-1}X_n'y_n, \\ \widehat{u}_n &= y_n - X_n\widehat{\beta}_{OLS}.\end{aligned}$$

Step 2: In the second step the spatial autoregressive parameter ρ and σ_ε^2 are estimated via the generalized moments estimator introduced in Kelejian and Prucha (1999) utilizing the residuals obtained via the first step. More specifically, the estimators of ρ and σ_ε^2, $\widetilde{\rho}_n$ and $\widetilde{\sigma}_{\varepsilon,n}^2$, are defined as the nonlinear least squares estimators corresponding to the regression

$$g_n = G_n \begin{bmatrix} \rho \\ \rho^2 \\ \sigma_\varepsilon^2 \end{bmatrix} + \Delta_n, \tag{4}$$

where Δ_n can be viewed as a vector of regression residuals,

$$G_n = \frac{1}{n}\begin{bmatrix} 2\widehat{u}_n'\widehat{\overline{u}}_n & -\widehat{\overline{u}}_n'\widehat{\overline{u}}_n & n \\ 2\widehat{\overline{u}}_n'\widehat{\overline{\overline{u}}}_n & -\widehat{\overline{\overline{u}}}_n'\widehat{\overline{\overline{u}}}_n & Tr(W_n'W_n) \\ \widehat{u}_n'\widehat{\overline{\overline{u}}}_n + \widehat{\overline{u}}_n'\widehat{\overline{u}}_n & -\widehat{\overline{u}}_n'\widehat{\overline{\overline{u}}}_n & 0 \end{bmatrix}, \quad g_n = \frac{1}{n}\begin{bmatrix} \widehat{u}_n'\widehat{u}_n \\ \widehat{\overline{u}}_n'\widehat{\overline{u}}_n \\ \widehat{u}_n'\widehat{\overline{u}}_n \end{bmatrix},$$

and $\widehat{\overline{u}}_n = W_n\widehat{u}_n$, and $\widehat{\overline{\overline{u}}}_n = W_n^2\widehat{u}_n$. The code for computing the GM estimators is contained in the file GMPROC1.TSP.

Step 3: In the third step we first apply a spatial Cochrane-Orcutt type transformation to the original regression model (1) based on $\widetilde{\rho}_n$. The transformed model is as given in (3), but with ρ replaced by $\widetilde{\rho}_n$. We then reestimate β from the transformed model using OLS. This yields the following feasible GLS estimator

$$\widehat{\beta}_{FGLS} = (X_{n*}'(\widetilde{\rho}_n)X_{n*}(\widetilde{\rho}_n))^{-1}X_{n*}'(\widetilde{\rho}_n)y_{n*}(\widetilde{\rho}_n), \tag{5}$$

where $X_{n*}(\widetilde{\rho}_n) = (I_n - \widetilde{\rho}_nW_n)X_n$, and $y_{n*}(\widetilde{\rho}_n) = (I_n - \widetilde{\rho}_nW_n)y_n$. The variance covariance matrix of $\widehat{\beta}_{FGLS}$ is estimated as

$$\check{\sigma}_\varepsilon^2(X_{n*}'(\widetilde{\rho}_n)X_{n*}(\widetilde{\rho}_n))^{-1}$$

with

$$\begin{aligned}\check{\sigma}_\varepsilon^2 &= n^{-1}\check{\varepsilon}_n'\check{\varepsilon}_n, \\ \check{\varepsilon}_n &= y_{n*}(\widetilde{\rho}_n) - X_{n*}(\widetilde{\rho}_n)\widehat{\beta}_{FGLS} = (I_n - \widetilde{\rho}_nW_n)\check{u}_n, \\ \check{u}_n &= y_n - X_n\widehat{\beta}_{FGLS}.\end{aligned}$$

This estimator $\widehat{\beta}_{FGLS}$ and the estimator for it's variance covariance matrix is computed in GLSPROC1.TSP.

Step 4: In the fourth step we compute an estimate of the variance of $\widetilde{\rho}_n$. As discussed in more detail in the appendix, based on results in Kelejian and Prucha (2004), it is seen that

$$\widetilde{\rho}_n \dot{\sim} N(0, \widetilde{\Omega}_{\widetilde{\rho}_n}/n)$$

with

$$\widetilde{\Omega}_{\widetilde{\rho}_n} = (\widetilde{J}_n'\widetilde{J}_n)^{-1}\widetilde{J}_n'\widetilde{\Psi}_n\widetilde{J}_n(\widetilde{J}_n'\widetilde{J}_n)^{-1} = \widetilde{J}_n'\widetilde{\Psi}_n\widetilde{J}_n/(\widetilde{J}_n'\widetilde{J}_n)^2$$

and where

$$
\begin{aligned}
\underset{2\times 1}{\widetilde{J}_n} &= \frac{1}{n}\left[\begin{array}{cc} 2c_n\left[\widehat{\overline{\overline{u}}}'_n\widehat{\overline{u}}_n - a_n\widehat{\overline{u}}'_n\widehat{u}_n\right] & -c_n\left[\widehat{\overline{\overline{u}}}'\widehat{\overline{\overline{u}}} - a_n\widehat{\overline{u}}'\widehat{\overline{u}}\right] \\ \widehat{\overline{\overline{u}}}'_n\widehat{u}_n + \widehat{\overline{u}}'_n\widehat{\overline{u}}_n) & -\widehat{\overline{\overline{u}}}'_n\widehat{\overline{u}}_n \end{array}\right]\left[\begin{array}{c} 1 \\ 2\widetilde{\rho}_n \end{array}\right], \\
c_n &= \left[\frac{1}{1+a_n^2}\right]^{1/2}, \\
a_n &= n^{-1}Tr\left\{W_n'W_n\right\}
\end{aligned}
$$

and

$$
\underset{2\times 2}{\widetilde{\Psi}_n} = \left[\widetilde{\psi}_{rs,n}\right]_{r,s=1,2}
$$

with

$$
\begin{aligned}
\widetilde{\psi}_{rs,n} &= \check{\sigma}_\varepsilon^4(2n)^{-1}tr\left[\left(A_{r,n}+A'_{r,n}\right)\left(A_{s,n}+A'_{s,n}\right)\right], \\
A_{1,n} &= c_n\left[W_n'W_n - a_nI_n\right], \\
A_{2,n} &= W_n.
\end{aligned}
$$

We note that $\check{\sigma}_\varepsilon^2$ can be replaced by any other consistent estimator such as $\widetilde{\sigma}_\varepsilon^2$. Also observe that in computing $\widetilde{\psi}_{rs,n}$ we can utilize that

$$
\begin{aligned}
tr\left[\left(A_{1,n}+A'_{1,n}\right)\left(A_{1,n}+A'_{1,n}\right)\right] &= 4c_n^2[tr(W_n'W_nW_n'W_n) - na_n^2], \\
tr\left[\left(A_{1,n}+A'_{1,n}\right)\left(A_{2,n}+A'_{2,n}\right)\right] &= tr\left[\left(A_{2,n}+A'_{2,n}\right)\left(A_{1,n}+A'_{1,n}\right)\right] \\
&= 4c_ntr(W_n'W_nW_n), \\
tr\left[\left(A_{2,n}+A'_{2,n}\right)\left(A_{2,n}+A'_{2,n}\right)\right] &= 2[tr(W_nW_n) + na_n].
\end{aligned}
$$

The code for the computation of the asymptotic variance $\widetilde{\mathbf{\Omega}}_{\widetilde{\rho}_n}/n$ is contained in the file VGMPROC1.TSP.

A Appendix: Asymptotic Distribution of $\widetilde{\rho}_n$

In establishing the asymptotic distribution of the GM estimator $\widetilde{\rho}_n$ we first observe in the following that this estimator can be defined equivalently as the nonlinear least squares estimator of a regression corresponding to two moment conditions. As such, the estimator can then be seen to be a special case of a more general class of estimators considered in Kelejian and Prucha (2004). Given this observation, the asymptotic distribution of $\widetilde{\rho}_n$ can now be easily obtained as a special case of asymptotic normality results developed in this later paper.

A.1 Original Form of Moment Conditions

The GM estimators for ρ and σ_ε^2 in Kelejian and Prucha (1999) are based on the three moment conditions

$$\begin{aligned} E[\frac{1}{n}\varepsilon_n'\varepsilon_n] &= \sigma_\varepsilon^2, \\ E[\frac{1}{n}\overline{\varepsilon}_n'\overline{\varepsilon}_n] &= \sigma_\varepsilon^2 Tr\left\{W_n'W_n\right\}, \\ E[\frac{1}{n}\overline{\varepsilon}_n'\varepsilon_n] &= 0, \end{aligned} \tag{A.1}$$

where $\overline{\varepsilon}_n = W_n\varepsilon_n$. Let $\overline{u}_n = W_n u_n$ and $\overline{\overline{u}}_n = W_n^2 u_n$, then upon substitution of $\varepsilon_n = u_n - \rho\overline{u}_n$ and $\overline{\varepsilon}_n = \overline{u}_n - \rho\overline{\overline{u}}_n$ into (A.1) we obtain

$$\gamma_n = \Gamma_n \begin{bmatrix} \rho \\ \rho^2 \\ \sigma_\varepsilon^2 \end{bmatrix} \tag{A.2}$$

where

$$\begin{aligned} \Gamma_n &= \frac{1}{n}\begin{bmatrix} 2E(u_n'\overline{u}_n) & -E(\overline{u}_n'\overline{u}_n) & n \\ 2E(\overline{\overline{u}}_n'\overline{u}_n) & -E(\overline{\overline{u}}_n'\overline{\overline{u}}_n) & Tr(W_n'W_n) \\ E(u_n'\overline{\overline{u}}_n + \overline{u}_n'\overline{u}_n) & -E(\overline{u}_n'\overline{\overline{u}}_n) & 0 \end{bmatrix}, \\ \gamma_n &= \frac{1}{n}\begin{bmatrix} E(u_n'u_n) \\ E(\overline{u}_n'\overline{u}_n) \\ E(u_n'\overline{u}_n) \end{bmatrix}. \end{aligned}$$

The GM estimators $\widetilde{\rho}_n$ and $\widetilde{\sigma}_{\varepsilon,n}^2$ are defined as the nonlinear least squares estimators corresponding to the regression (4).. We note that (4) represents the sample analogue of (A.2), and this ample analogue was obtained by suppressing in (A.2) the expectations operator, and by replacing u_n, $\overline{u}_n$, $\overline{\overline{u}}_n$ by $\widehat{u}_n$, $\widehat{\overline{u}}_n$, $\widehat{\overline{\overline{u}}}_n$, respectively.

A.2 Alternative Form 1 of Moment Conditions

Let

$$\begin{aligned} c_n &= \left[\frac{1}{1+a_n^2}\right]^{1/2}, \\ a_n &= n^{-1}Tr\left\{W_n'W_n\right\}, \end{aligned}$$

then substitution of the first moment condition in (A.1) into the second, and multiplication of the second moment conditions by c_n yields the following two moment conditions:

$$\begin{aligned} c_n E[\frac{1}{n}\overline{\varepsilon}_n'\overline{\varepsilon}_n] &= c_n a_n E[\frac{1}{n}\varepsilon_n'\varepsilon_n], \\ E[\frac{1}{n}\overline{\varepsilon}_n'\varepsilon_n] &= 0. \end{aligned} \tag{A.3}$$

Upon substitution of $\varepsilon_n = u_n - \rho\overline{u}_n$ and $\overline{\varepsilon}_n = \overline{u}_n - \rho\overline{\overline{u}}_n$ into (A.3) we obtain

$$\gamma_{*n} = \Gamma_{*n}\begin{bmatrix} \rho \\ \rho^2 \end{bmatrix} \tag{A.4}$$

where

$$\begin{aligned} \Gamma_{*n} &= \frac{1}{n}\begin{bmatrix} 2c_n\left[E(\overline{\overline{u}}_n'\overline{u}_n) - a_n E(\overline{u}_n' u_n)\right] & -c_n\left[E(\overline{\overline{u}}_n'\overline{\overline{u}}_n) - a_n E(\overline{u}_n'\overline{u}_n)\right] \\ E(\overline{\overline{u}}_n' u_n + \overline{u}_n'\overline{u}_n) & -E(\overline{\overline{u}}_n'\overline{u}_n) \end{bmatrix}, \\ \gamma_{*n} &= \frac{1}{n}\begin{bmatrix} c_n\left[E(\overline{u}_n'\overline{u}_n) - a_n E(u_n' u_n)\right] \\ E(\overline{u}_n' u_n) \end{bmatrix}. \end{aligned}$$

Now consider the sample analogue of (A.4):

$$g_{*n} = G_{*n}\begin{bmatrix} \rho \\ \rho^2 \end{bmatrix} + \Delta_{*n} \tag{A.5}$$

where

$$\begin{aligned} G_{*n} &= \frac{1}{n}\begin{bmatrix} 2c_n\left[\widehat{\overline{\overline{u}}}_n'\widehat{\overline{u}}_n - a_n\widehat{\overline{u}}_n'\widehat{u}_n\right] & -c_n\left[\widehat{\overline{\overline{u}}}'\widehat{\overline{\overline{u}}} - a_n\widehat{\overline{u}}'\widehat{\overline{u}}\right] \\ \widehat{\overline{\overline{u}}}_n'\widehat{u}_n + \widehat{\overline{u}}_n'\widehat{\overline{u}}_n) & -\widehat{\overline{\overline{u}}}_n'\widehat{\overline{u}}_n \end{bmatrix}, \\ g_{*n} &= \frac{1}{n}\begin{bmatrix} c_n\left[\widehat{\overline{u}}_n'\widehat{\overline{u}}_n - a_n\widehat{u}_n'\widehat{u}_n\right] \\ \widehat{\overline{u}}_n'\widehat{u}_n \end{bmatrix}. \end{aligned}$$

and where the 2×1 vector Δ_{*n} can again be viewed as a vector of regression residuals. It is not difficult, although a bit tedious, to check that the nonlinear least squares estimator corresponding to the regression (A.5) is identical to the GM estimator $\widetilde{\rho}_n$.

A.3 Alternative Form 2 of Moment Conditions

To connect the GM estimator $\widetilde{\rho}_n$ to the class of estimators considered in Kelejian and Prucha (2004) observe that we can rewrite the moment conditions (A.3) as

$$n^{-1}E\begin{bmatrix} \varepsilon_n' A_{1,n}\varepsilon_n \\ \varepsilon_n' A_{2,n}\varepsilon_n \end{bmatrix} = 0.$$

with

$$\begin{aligned} A_{1,n} &= c_n\left[W_n'W_n - a_n I_n\right], \\ A_{2,n} &= W_n. \end{aligned}$$

Correspondingly we can also rewrite the matrix Γ_{*n} and vector γ_{*n} in (A.4) as

$$\begin{aligned}
\Gamma_{*n} &= \frac{1}{n}\left[\begin{array}{cc} 2Eu_n'W_n'A_{1,n}u_n & -Eu_n'W_n'A_{1,n}W_nu_n \\ Eu_n'W_n'(A_{2,n}+A_{2,n}')u_n & -Eu_n'W_n'A_{2,n}W_nu_n \end{array}\right], \\
\gamma_{*n} &= \frac{1}{n}\left[\begin{array}{c} Eu_n'A_{1,n}u_n \\ Eu_n'A_{2,n}u_n \end{array}\right],
\end{aligned}$$

and G_{*n} and vector g_{*n} in (A.5) as

$$\begin{aligned}
G_{*n} &= \frac{1}{n}\left[\begin{array}{cc} 2\widehat{u}_n'W_n'A_{1,n}\widehat{u}_n & -\widehat{u}_n'W_n'A_{1,n}W_n\widehat{u}_n \\ \widehat{u}_n'W_n'(A_{2,n}+A_{2,n}')\widehat{u}_n & -\widehat{u}_n'W_n'A_{2,n}W_n\widehat{u}_n \end{array}\right], \\
g_{*n} &= \frac{1}{n}\left[\begin{array}{c} \widehat{u}_n'A_{1,n}\widehat{u}_n \\ \widehat{u}_n'A_{2,n}\widehat{u}_n \end{array}\right].
\end{aligned}$$

A.4 Asymtotic Normality of $\widetilde{\rho}_n$

We have shown above that $\widetilde{\rho}_n$ can also be viewed as the nonlinear least squares estimator corresponding to (A.5). Using furthermore the alternative form of the moment conditions and expressions for Γ_{*n}, γ_{*n}, and G_{*n}, g_{*n} given in the previous subsection we can now use Theorem 2 in Kelejian and Prucha (2004) to establish the asymptotic distribution of $\widetilde{\rho}_n$. In particular it follows from that Theorem 2 that under the regularity conditions maintained by that theorem we have:

$$n^{1/2}(\widetilde{\rho}_n - \rho) = (J_n'J_n)^{-1}J_n'\Psi_n^{1/2}\xi_n + o_p(1)$$

where

$$J_n = \Gamma_{*n}\left[\begin{array}{c} 1 \\ 2\rho \end{array}\right]$$

and $\xi_n \xrightarrow{d} N(0, I_2)$. Furthermore $n^{1/2}(\widetilde{\rho}_n - \rho) = O_p(1)$ and

$$\Omega_{\widetilde{\rho}_n} = (J_n'J_n)^{-1}J_n'\Psi_nJ_n(J_n'J_n)^{-1} \geq const > 0.$$

The above result implies that the difference between the cumulative distribution functions of $n^{1/2}(\widetilde{\rho}_n - \rho)$ and $N\left[0, \Omega_{\widetilde{\rho}_n}\right]$ converges pointwise to zero, which justifies the use of the latter distribution as an approximation of the former.[1]

It follows furthermore from Theorem 3 in Kelejian and Prucha (2004) that $\widetilde{\Omega}_{\widetilde{\rho}_n} - \Omega_{\widetilde{\rho}_n} = o_p(1)$, and thus $\widetilde{\rho}_n \overset{\cdot}{\sim} N\left[0, \widetilde{\Omega}_{\widetilde{\rho}_n}/n\right]$. We can now test the hypothesis of zero spatial correlation, i.e., $H_0 : \rho = 0$, by comparing

$$\frac{\widetilde{\rho}_n}{[\widetilde{\Omega}_{\widetilde{\rho}_n}/n]^{1/2}}$$

with the fractiles of the standardized normal distribution.

[1] This follows from Corollary F4 in Pötscher and Prucha (1997). Compare also the discussion on pp. 86-87 in that reference.

References

[1] Kelejian, H. H. and I. R. Prucha, "A Generalized Moments of Estimator for the Autoregressive Parameter in a Spatial Model," *International Economic Review,* 40, 1999, 509-533.

[2] Kelejian, H. H. and I. R. Prucha, "Specification and Estimation of Spatial Autoregressive Models with Autoregressive and Heteroskedastic Disturbances", Department of Economics, University of Maryland, Revised October 2004, mimeo.

[3] Pötscher, B.M. and Prucha, I.R., *Dynamic Nonlinear Econometric Models, Asymptotic Theory.* New York: Springer Verlag, 1997.