ECON 602 - Macroeconomic Analysis II

Solutions to Comprehensive Exam

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Note that the timing convention of these questions are different from the one adopted in Spring 2008 and later.

1. (a) (15 points) **(Households)** *Consider the following problem of a household who takes as given prices* $\{p_t, w_t, R_t\}_{t=0}^{\infty}$ *and taxes* $\{\tau_t^h\}_{t=0}^{\infty}$

$$\max_{\{c_t,h_t,m_t,b_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t, z_t) \tag{1}$$

$$z_t \equiv \frac{m_{t+1}}{p_t} \tag{2}$$

$$p_t c_t + b_{t+1} + m_{t+1} \leq \left(1-\tau_t^h\right) p_t w_t h_t + R_{t-1} b_t + m_t \tag{3}$$

$$\lim_{z \to \bar{z}} u_3(c, 1-h, z) = 0 \text{ for } \bar{z} < \infty \tag{4}$$

$$u_1(c, 1-h, z) > 0 \text{ for all } z \tag{5}$$

where b_t is the stock of a one-period nominal bond that issued by the government and held by the households and R_t is its gross nominal interest which is paid in $t+1$. Use subscripts 1,2 and 3 to refer to marginal utilities with respect to consumption, leisure and real money balances.

i. (10 points) *Write down the conditions that characterize the solution to the household's problem.*

Following the same arguments as we did in class for the model with capital, we get the following four equations that characterize the solution to the household's problem.

$$u_2(t) = u_1(t)\left(1-\tau_t^h\right) w_t \tag{6}$$

$$\frac{u_1(t)}{p_t} = \beta u_1(t+1) \frac{R_t}{p_{t+1}} \tag{7}$$

$$\frac{u_1(t)}{p_t} = \frac{u_3(t)}{p_t} + \frac{\beta u_1(t+1)}{p_{t+1}} \tag{8}$$

$$p_t c_t + b_{t+1} + m_{t+1} = \left(1-\tau_t^h\right) p_t w_t h_t + R_{t-1} b_t + m_t \tag{9}$$

along with the transversality conditions

$$\lim_{t\to\infty} \beta^t u_1(t)\, k_{t+1} = 0 \tag{10}$$

$$\lim_{t\to\infty} \beta^t u_1(t)\, m_{t+1} = 0 \tag{11}$$

Rearranging (8), we get

$$1 = \frac{u_3(t)}{u_1(t)} + \frac{\beta p_t u_1(t+1)}{p_{t+1} u_1(t)} \tag{12}$$

$$1 - \frac{1}{R_t} = \frac{u_3(t)}{u_1(t)} \tag{13}$$

$$\frac{u_1(t)}{u_3(t)} = \frac{R_t}{R_t - 1} \tag{14}$$

which we will use below.

ii. (5 points) *What is the implication of (4) for the solution to the household's problem and therefore the equilibrium?*

This condition says it is not optimal for the household to choose a z that is greater than $\bar{z}$. As such,

$$\frac{m_{t+1}}{p_t} \leq \bar{z} \tag{15}$$

must be a constraint on the equilibrium.

(b) (5 points) **(Firms)** *A Neoclassical firm has access to production function*

$$Y_t = H_t \tag{16}$$

Write down the problem of the firm and solve it.

The problem of the firm is

$$\max_{H_t^D} p_t Y_t - p_t w_t H_t^D \tag{17}$$

$$= \max_{H_t^D} p_t H_t^D - p_t w_t H_t^D \tag{18}$$

whose first order condition yields

$$w_t = 1 \tag{19}$$

c (20 points) **(Equilibrium)** *There is also a government in this economy with a budget constraint*

$$p_t G_t + R_{t-1} B_t + M_t = \tau_t^h p_t w_t H_t + B_{t+1} + M_{t+1} \tag{20}$$

where $\{G_t\}_{t=0}^{\infty}$ *is a deterministic and known sequence of government expenditures that needs to be financed. Using all the information so far, carefully define the equilibrium. Your definition must be self-contained. At the expense of being redundant, write down all the equations that need to be in this definition. (No multipliers should appear in these definitions.)*

Definition 1 *A* ***competitive equilibrium*** *is a list of sequences* $\{c_t, h_t\}_{t=0}^{\infty}$ *[allocations],* $\{m_t, b_t\}_{t=0}^{\infty}$ *[portfolio choices],* $\{p_t, w_t\}_{t=0}^{\infty}$ *[prices],* $\{B_t\}_{t=0}^{\infty}$ *taking as given* $\left\{\tau_t^h, R_t\right\}_{t=0}^{\infty}$ *[policy variables] ,*$\{G_t\}_{t=0}^{\infty}$*,* m_0*,* B_0 *and* b_0 *such that*

1. *Households optimize : Given prices and policy variables, allocations and portfolio choices solve the household's problem by satisfying (6)-(9), the transversality conditions (10) and (11) and the constraint (15).*
2. *Firms optimize :* $w_t = 1$*.*
3. *Government's budget constraint (20) holds.*
4. *Consistency / Market clearing :* $b_t = B_t$*.*

In order to write down the equilibrium conditions, we should impose the conditions above to the optimality conditions of the household. So the equilibrium is a list of sequences $\{c_t, h_t, m_t, b_t, p_t\}_{t=0}^{\infty}$ that satisfy the following equations (plus the TVC) given $\left\{G_t, R_t, \tau_t^h\right\}_{t=0}^{\infty}$, m_0, b_0

$$u_2\left(c_t, 1-h_t, \frac{m_{t+1}}{p_t}\right) = u_1\left(c_t, 1-h_t, \frac{m_{t+1}}{p_t}\right)\left(1-\tau_t^h\right) \tag{21}$$

$$\frac{u_1\left(c_t, 1-h_t, \frac{m_{t+1}}{p_t}\right)}{u_3\left(c_t, 1-h_t, \frac{m_{t+1}}{p_t}\right)} = \frac{R_t}{R_t - 1} \tag{22}$$

$$\frac{u_1\left(c_t, 1-h_t, \frac{m_{t+1}}{p_t}\right)}{\beta u_1\left(c_{t+1}, 1-h_{t+1}, \frac{m_{t+2}}{p_{t+1}}\right)} = \frac{R_t}{\pi_{t+1}} \tag{23}$$

$$p_t c_t + b_{t+1} + m_{t+1} = \left(1-\tau_t^h\right) p_t h_t + R_{t-1} b_t + m_t \tag{24}$$

$$G_t + R_{t-1} b_t + m_t = \tau_t^h H_t + b_{t+1} + m_{t+1} \tag{25}$$

$$\frac{m_{t+1}}{p_t} \leq \bar{z} \tag{26}$$

where $\pi_{t+1} \equiv p_{t+1}/p_t$.

1. d (15 points) **(Ramsey Planner's Problem)** *Write down the Ramsey planner's problem with the objective function, things that he chooses, constraints and multipliers carefully specified. (It is very important that you remember your answer in question 1b above.) Do not proceed further, just write down the problem. [You do not need to derive the implementability constraint, unless you do not remember how it should look like.]*

Ramsey planner's problem is

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-h_t, z_t\right) \tag{27}$$

$$h_t - G_t - c_t \geq 0 \ \left[\beta^t \mu_t\right] \tag{28}$$

$$\sum_{t=0}^{\infty} \beta^t \left[u_1\left(t\right) c_t - h_t u_2\left(t\right) + z_t u_3\left(t\right)\right] - A_0 \geq 0 \ [\lambda] \tag{29}$$

$$\bar{z} - z_t \geq 0 \ \left[\beta^t \xi_t\right] \tag{30}$$

where he chooses $\{c_t, h_t, z_t\}$, by maximizing the lifetime utility of the households, respecting feasibility (with multiplier μ_t), the implementability constraint (with multiplier λ) and the constraint that real money balances cannot exceed $\bar{z}$ (with multiplier ξ_t) or in Lagrangian form

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t \left\{u\left(c_t, 1-h_t, z_t\right) + \lambda\left[u_1\left(t\right) c_t - h_t u_2\left(t\right) + z_t u_3\left(t\right) - A_0\right] + \mu_t\left[h_t - G_t - c_t\right] + \xi\left(\bar{z} - z_t\right)\right\} \tag{31}$$

e (45 points) **(Optimal Policy)** *Assume the utility function takes the form*

$$u\left(c, 1-h, z\right) = cz + \left(1-h\right) \tag{32}$$

Note that this is not a very well-behaved utility function. It would be better to consider $\log\left(cz\right)$ instead of simply $\left(cz\right)$.

1. (a) i. (15 points) *Solve the Ramsey problem in question 4. You need to write down all the equations and inequalities that characterize the solution. Do not proceed any further in this part.*

With this utility function, the Ramsey problem in Lagrangian form is

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t \left\{c_t z_t + \left(1-h_t\right) + \lambda\left[z_t c_t - h_t + z_t c_t - A_0\right] + \mu_t\left[h_t - G_t - c_t\right] + \xi_t\left(\bar{z} - z_t\right)\right\} \tag{33}$$

with first order conditions

$$c_t : z_t + \lambda 2 z_t - \mu_t = 0 \tag{34}$$

$$h_t : -1 - \lambda + \mu_t = 0 \tag{35}$$

$$z_t : c_t + \lambda 2 c_t - \xi_t = 0 \tag{36}$$

and the complementary slackness conditions

$$\lambda\left[z_t c_t - h_t + z_t c_t - A_0\right] = 0 \tag{37}$$

$$\mu_t\left[h_t - G_t - c_t\right] = 0 \tag{38}$$

$$\xi_t\left(\bar{z} - z_t\right) = 0 \tag{39}$$

and the multipliers must be nonnegative

$$\lambda \geq 0, \ \mu_t \geq 0 \text{ and } \xi_t \geq 0 \tag{40}$$

ii (10 points) *What is the optimal interest rate that comes out of the Ramsey problem?*

First, it's easy to show that the constraint on z_t should bind. If we assume $\xi_t = 0$ then from (36), we have $1 + 2\lambda > 0$, which means z_t should be as high as possible, i.e. $z_t = \bar{z}$. On the other hand, if $\xi_t > 0$, we already have $z_t = \bar{z}$ from the complementary slackness condition. So, regardless of the value of ξ_t, $z_t = \bar{z}$ in the Ramsey equilibrium.

Remembering the equilibrium condition (22)

$$\frac{u_1(t)}{u_3(t)} = \frac{R_t}{R_t - 1} \tag{41}$$

since $z_t = \bar{z}$ implies $u_3(t) = 0$, this implies $R_t = 1$. (The LHS goes to ∞ and the only way the RHS would also go to ∞ is when $R_t = 1$.)

iii (10 points) *What is the optimal labor income tax that comes out of the Ramsey problem?*

Remember the equilibrium condition

$$u_2(c_t, 1 - h_t, z_t) = u_1(c_t, 1 - h_t, z_t)\left(1 - \tau_t^h\right) \tag{42}$$

$$1 = z_t\left(1 - \tau_t^h\right) \tag{43}$$

which is what we will use to back out the tax rate. Since $z_t = \bar{z}$, we have

$$\left(1 - \tau_t^h\right) = \frac{1}{\bar{z}} \tag{44}$$

$$\tau_t^h = 1 - \frac{1}{\bar{z}} \tag{45}$$

$$\tau_t^h = \frac{\bar{z} - 1}{\bar{z}} \tag{46}$$

iv (10 points) *State the "uniform taxation" result as formally and generally as you can and relate it to your findings above.*

The uniform taxation results says, that if we have n consumption goods denoted by $c_1, ..., c_n$ and labor denoted by h, and the utility function takes the form

$$U(c, \ell) = W\left[G(c), \ell\right] \tag{47}$$

where $c = (c_1, ..., c_n)$ and $G(.)$ is homothetic (and labor enters separably from consumption goods) then consumption taxes under Ramsey policy satisfy $\tau_i = \tau_j$ for all i, j.

This result is relevant for the optimal monetary policy we found above. The utility function $cz + (1 - h)$ satisfies the conditions of the uniform taxation result. So, since c and z enter homothetically, the taxes on them has to be equal. In our model, consumption is not taxed and therefore optimal policy would dictate real money balances not to be taxed either.