

ECON 602 - Macroeconomic Analysis II

Comprehensive Exam

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Note that the timing convention of these questions are different from the one adopted in Spring 2008 and later.

a. (10 points)

Version 1:

$$m_t = \tilde{m}_{t-1} + p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} h_{t-1} - p_{t-1} c_{1t-1} \quad (1)$$

$$\tilde{m}_t = m_t - p_{t-1} c_{2t-1} - b_{t+1} + R_{t-1} b_t \quad (2)$$

Version 2:

$$m_t = \tilde{m}_{t-1} \quad (3)$$

$$\tilde{m}_t = m_t + p_t (1 - \tau_t^h) w_t h_t - p_t c_{1t} - p_t c_{2t} - b_{t+1} + R_{t-1} b_t \quad (4)$$

b. (20 points)

Version 1:

$$\max_{\{c_{1t}, c_{2t}, h_t, \tilde{m}_t, b_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, 1 - h_t) \quad (5)$$

$$\tilde{m}_t - p_t c_{1t} \geq 0 \quad (\text{with multiplier } \beta^t \mu_t) \quad (6)$$

$$R_{t-1} b_t + \tilde{m}_{t-1} + p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} h_{t-1} - b_{t+1} - \tilde{m}_t - p_{t-1} c_{1t-1} - p_{t-1} c_{2t-1} \geq 0 \quad (\text{with multiplier } \beta^t \lambda_t) \quad (7)$$

Note that the budget constraint follows from part (a).

FOC:

$$c_{1t} : u_1(t) - \mu_t p_t - \beta p_t E_t(\lambda_{t+1}) = 0 \quad (8)$$

$$c_{2t} : u_2(t) - \beta p_t E_t(\lambda_{t+1}) = 0 \quad (9)$$

$$h_t : u_3(t) - \beta p_t (1 - \tau_t^h) w_t E_t(\lambda_{t+1}) = 0 \quad (10)$$

$$\tilde{m}_t : \mu_t - \lambda_t + \beta E_t(\lambda_{t+1}) = 0 \quad (11)$$

$$b_{t+1} : -\lambda_t + \beta R_t E_t(\lambda_{t+1}) = 0 \quad (12)$$

(9) imply

$$u_2(t) = \beta p_t E_t(\lambda_{t+1}) \quad (13)$$

and using this we can simplify(8) and (10) to

$$u_1(t) - u_2(t) = \mu_t p_t \quad (14)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (15)$$

Next, multiply (11) by p_t and combine with (8) to get

$$\frac{u_1(t)}{p_t} = \lambda_t \quad (16)$$

which simplifies (12) to

$$\frac{u_1(t)}{p_t} = \beta R_t E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] \quad (17)$$

To sum up, we have

$$\frac{u_2(t)}{p_t} = \beta E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] \quad (18)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (19)$$

$$u_1(t) = R_t u_2(t) \quad (20)$$

These three equations, the budget constraint and the binding CIA constraint along with initial condition b_0 and \tilde{m}_0 and the TVC conditions for b and \tilde{m} completely characterize the solution to this problem for $\{c_{1t}, c_{2t}, h_t, \tilde{m}_t, b_t\}$.

Version 2:

$$\max_{\{c_{1t}, c_{2t}, h_t, m_t, b_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, 1 - h_t) \quad (21)$$

$$m_t - p_t c_{1t} \geq 0 \quad (\text{with multiplier } \beta^t \mu_t) \quad (22)$$

$$R_{t-1} b_t + m_t + p_t (1 - \tau_t^h) w_t h_t - p_t c_{1t} - p_t c_{2t} - b_{t+1} - m_{t+1} \geq 0 \quad (\text{with multiplier } \beta^t \lambda_t) \quad (23)$$

Again, the budget constraint follows from part (a).

FOC:

$$c_{1t} : u_1(t) - \mu_t p_t - \lambda_t p_t = 0 \quad (24)$$

$$c_{2t} : u_2(t) - \lambda_t p_t = 0 \quad (25)$$

$$h_t : u_3(t) - p_t (1 - \tau_t^h) w_t \lambda_t = 0 \quad (26)$$

$$m_{t+1} : \beta E_t (\mu_{t+1}) - \lambda_t + \beta E_t (\lambda_{t+1}) = 0 \quad (27)$$

$$b_{t+1} : -\lambda_t + \beta R_t E_t (\lambda_{t+1}) = 0 \quad (28)$$

(25) imply

$$u_2(t) = p_t \lambda_t \quad (29)$$

and using this on (24), (26) and (28) we get

$$u_1(t) - u_2(t) = \mu_t p_t \quad (30)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (31)$$

$$\frac{u_2(t)}{p_t} = \beta R_t E_t \left[\frac{u_2(t+1)}{p_{t+1}} \right] \quad (32)$$

Next, using the $t+1$ versions of (24) and (25), we get

$$\mu_{t+1} p_{t+1} + \lambda_{t+1} p_{t+1} = u_1(t+1) + u_2(t+1) - \lambda_{t+1} p_{t+1} \quad (33)$$

$$\mu_{t+1} p_{t+1} + \lambda_{t+1} p_{t+1} = u_1(t+1) \quad (34)$$

where the second equality follow from (29). Using this result on (27) we get

$$\beta E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] = \lambda_t \quad (35)$$

To sum up we have

$$\frac{u_2(t)}{p_t} = \beta E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] \quad (36)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (37)$$

$$E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] = R_t E_t \left[\frac{u_2(t+1)}{p_{t+1}} \right] \quad (38)$$

These three equations, the budget constraint and the binding CIA constraint along with initial condition b_0 and m_0 and the TVC conditions for b and \tilde{m} completely characterize the solution to this problem for $\{c_{1t}, c_{2t}, h_t, m_t, b_t\}$.

c. (5 points)

$$\max_{H_t} A_t H_t - w_t H_t \quad (39)$$

which yields $w_t = A_t$.

d. (15 points)

Version 1:

Equilibrium is a list of sequences $\{C_{1t}, C_{2t}, H_t, \tilde{M}_t, B_t, P_t, w_t, A_t\}$ such that

$$\frac{u_2(t)}{p_t} = \beta E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] \quad (40)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (41)$$

$$u_1(t) = R_t u_2(t) \quad (42)$$

$$\tilde{M}_t = p_t C_{1t} \quad (43)$$

$$w_t = A_t \quad (44)$$

$$R_{t-1} B_t + \tilde{M}_{t-1} + p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} H_{t-1} - B_{t+1} - \tilde{M}_t - p_{t-1} C_{1t-1} - p_{t-1} C_{2t-1} = 0 \quad (45)$$

$$A_t H_t = C_{1t} + C_{2t} + G_t \quad (46)$$

$$A_{t+1} = A(A_t) \quad (47)$$

along with the TVC and initial conditions. Note that combining (40) and (42) we get

$$\frac{u_1(t)}{p_t} = \beta R_t E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] \quad (48)$$

Note that we can also use the budget constraint of the government in this definition but one of the the following three would be redundant: household's budget constraint, government's budget constraint, resource constraint.

Version 2:

Equilibrium is a list of sequences $\{C_{1t}, C_{2t}, H_t, M_t, B_t, P_t, w_t, A_t\}$ such that

$$\frac{u_2(t)}{p_t} = \beta E_t \left[\frac{u_2(t+1)}{p_{t+1}} \right] \quad (49)$$

$$u_3(t) = u_2(t) (1 - \tau_t^h) w_t \quad (50)$$

$$E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] = R_t E_t \left[\frac{u_2(t+1)}{p_{t+1}} \right] \quad (51)$$

$$M_t = p_t C_{1t} \quad (52)$$

$$w_t = A_t \quad (53)$$

$$R_{t-1} B_t + M_t + p_t (1 - \tau_t^h) w_t H_t - p_t C_{1t} - p_t C_{2t} - B_{t+1} - M_{t+1} = 0 \quad (54)$$

$$A_t H_t = C_{1t} + C_{2t} + G_t \quad (55)$$

$$A_{t+1} = A(A_t) \quad (56)$$

along with the TVC and initial conditions.)

e. (10 points)

Version 1:

Remember that we have $\lambda_t = u_1(t)/p_t$ as one of the equilibrium conditions. We'll ignore expectations operators in this derivation. Also, realizing $c_{1,t-1} p_{t-1} = \tilde{M}_{t-1}$, by the (binding) CIA constraint, we simplify the budget constraint. Take (45) and multiply by $\sum_{t=0}^{\infty} \beta^t \lambda_t$ or equivalently $\sum_{t=0}^{\infty} \beta^t u_1(t)/p_t$ to get

$$\sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} R_{t-1} B_t + \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} H_{t-1} \quad (57)$$

$$- \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} B_{t+1} - \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} \tilde{M}_t - \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} C_{2,t-1} = 0 \quad (58)$$

Using (41) on the second term, (48) on the third term and (43) on the fourth term, we get

$$\sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} R_{t-1} B_t + \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} \frac{u_3(t-1)}{u_2(t-1)} H_{t-1} \quad (59)$$

$$- \sum_{t=0}^{\infty} \beta^{t+1} \frac{u_1(t+1)}{p_{t+1}} R_t B_{t+1} - \sum_{t=0}^{\infty} \beta^t u_1(t) C_{1,t} - \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} C_{2,t-1} = 0 \quad (60)$$

Using (40) the second term becomes

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} \frac{u_3(t-1)}{u_2(t-1)} H_{t-1} \\ &= \sum_{t=0}^{\infty} \beta^{t-1} u_3(t-1) H_{t-1} \end{aligned}$$

Using (40) the fifth term becomes

$$\begin{aligned} & - \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} p_{t-1} C_{2,t-1} \\ &= - \sum_{t=0}^{\infty} \beta^{t-1} u_2(t-1) C_{2,t-1} \end{aligned}$$

The first and the third term cancel out as follows

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} R_{t-1} B_t - \sum_{t=0}^{\infty} \beta^{t+1} \frac{u_1(t+1)}{p_{t+1}} R_t B_{t+1} \\ &= \sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} R_{t-1} B_t - \left[\sum_{t=0}^{\infty} \beta^t \frac{u_1(t)}{p_t} R_{t-1} B_t - \frac{u_1(0)}{p_0} R_{-1} B_0 \right] \\ &= \frac{u_1(0)}{p_0} R_{-1} B_0 \end{aligned}$$

We can expand the fourth term as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u_1(t) C_{1,t} &= \sum_{t=0}^{\infty} \beta^{t-1} u_1(t-1) C_{1,t-1} + \text{constant} \\ & \sum_{t=0}^{\infty} \beta^{t-1} u_1(t-1) C_{1,t-1} + \text{constant} \end{aligned}$$

where the last term comes from using the CIA constraint. Combining we have

$$- \sum_{t=0}^{\infty} \beta^{t-1} u_3(t-1) H_{t-1} + \sum_{t=0}^{\infty} \beta^{t-1} u_1(t-1) C_{1,t-1} + \sum_{t=0}^{\infty} \beta^{t-1} u_2(t-1) C_{2,t-1} = \text{constant} \quad (61)$$

or

$$\sum_{t=0}^{\infty} \beta^{t-1} [u_1(t-1) C_{1,t-1} + u_2(t-1) C_{2,t-1} - u_3(t-1) H_{t-1}] = \text{constant}$$

which can be also written as

$$\sum_{t=0}^{\infty} \beta^t [u_1(t) C_{1,t} + u_2(t) C_{2,t} - u_3(t) H_t] = \text{constant}$$

where we will collectively call the right hand side Ξ , which is a constant (given) from the viewpoint of the Ramsey planner. So we have

$$\sum_{t=0}^{\infty} \beta^t [u_1(t) C_{1,t} + u_2(t) C_{2,t} - u_3(t) H_t] = \Xi$$

Version 2:

Following the same steps we get

$$\sum_{t=0}^{\infty} \beta^t [\beta u_1(t+1) C_{1,t+1} + u_2(t) C_{2,t} - u_3(t) H_t] = \Psi \quad (62)$$

where Ψ is potentially a different constant.

f. (10 points)

Version 1:

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, 1 - h_t)$$

$$E_0 \sum_{t=0}^{\infty} \beta^t [u_1(t) C_{1,t} + u_2(t) C_{2,t} - u_3(t) H_t] = \Psi \text{ (with multiplier } \lambda)$$

$$A_t h_t - c_{1t} - c_{2t} - G \geq 0 \text{ (with multiplier } \beta^t \mu_t)$$

or

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_{1t}, c_{2t}, 1 - h_t) + \lambda [u_1(t) C_{1,t} + u_2(t) C_{2,t} - u_3(t) H_t - \Psi] + \mu_t [A_t h_t - c_{1t} - c_{2t} - G]\} \quad (63)$$

Version 2:

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}, 1 - h_t)$$

$$E_0 \sum_{t=0}^{\infty} \beta^t [\beta u_1(t+1) C_{1,t+1} + u_2(t) C_{2,t} - u_3(t) H_t] = \Psi \text{ (with multiplier } \lambda)$$

$$A_t h_t - c_{1t} - c_{2t} - G \geq 0 \text{ (with multiplier } \beta^t \mu_t)$$

or

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_{1t}, c_{2t}, 1 - h_t) + \lambda [\beta c_{1,t+1} u_1(t+1) + c_{2,t-1} u_2(t-1) - h_{t-1} u_3(t-1) - \Psi] + \mu_t [A_t h_t]\} \quad (64)$$

Note that in both versions, we also need to have a constraint (in terms of allocations) that makes sure $R_t \geq 1$ always holds. We drop this here for simplicity (since it will bind as we will find) but you should remember to put it at this stage, or at least talk about it.

g. (15 points)

Note that this is not a very nice utility function. $\log(c_1c_2)$ would be better.

Version 1:

The problem again

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{c_{1t}c_{2t} - Bh_t + \lambda [c_{1t}c_{2t} + c_{1t}c_{2t} - Bh_t - \Psi] + \mu_t [A_t h_t - c_{1t} - c_{2t} - G]\} \quad (65)$$

and FOC

$$c_{1t} : c_{2t} + 2\lambda c_{2t} = \mu_t \quad (66)$$

$$c_{2t} : c_{1t} + 2\lambda c_{1t} = \mu_t \quad (67)$$

$$h_t : -B - \lambda B + A_t \mu_t = 0 \quad (68)$$

or

$$(1 + 2\lambda) c_{2t} = \mu_t \quad (69)$$

$$(1 + 2\lambda) c_{1t} = \mu_t \quad (70)$$

$$A_t \mu_t = (1 + \lambda) B \quad (71)$$

and

$$c_{1t} = c_{2t} \quad (72)$$

Now remember the EE from Version 1

$$\frac{u_1(t)}{u_2(t)} = R_t \quad (73)$$

or with these functional forms

$$\frac{c_{2t}}{c_{1t}} = R_t \quad (74)$$

which imply the optimal policy is $R_t = 1$.

Version 2:

The problem again

$$\max_{\{c_{1t}, c_{2t}, h_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{c_{1t}c_{2t} - Bh_t + \lambda [\beta c_{1t+1}c_{2t+1} + c_{1t}c_{2t} - Bh_t - \Psi] + \mu_t [A_t h_t - c_{1t} - c_{2t} - G]\} \quad (75)$$

and FOC

$$c_{1t} : c_{2t} + \frac{\lambda}{\beta} \beta c_{2t} + \lambda c_{2t} = \mu_t \quad (76)$$

$$c_{2t} : c_{1t} + \frac{\lambda}{\beta} \beta c_{1t} + \lambda c_{1t} = \mu_t \quad (77)$$

$$h_t : -B - \lambda B + A_t \mu_t = 0 \quad (78)$$

or

$$(1 + 2\lambda) c_{2t} = \mu_t \quad (79)$$

$$(1 + 2\lambda) c_{1t} = \mu_t \quad (80)$$

$$A_t \mu_t = (1 + \lambda) B \quad (81)$$

From the Ramsey problem, we still get

$$c_{1t} = c_{2t} \quad (82)$$

and using the EE

$$E_t \left[\frac{u_1(t+1)}{p_{t+1}} \right] = R_t E_t \left[\frac{u_2(t+1)}{p_{t+1}} \right] \quad (83)$$

with these functional forms

$$E_t \left[\frac{c_{2t+1} - R_t c_{1t+1}}{p_{t+1}} \right] = 0 \quad (84)$$

and using the Ramsey solution

$$(1 - R_t) E_t \left[\frac{c_{1t+1}}{p_{t+1}} \right] = 0 \quad (85)$$

which imply $R_t = 1$ since that expectation can never be zero.

Note that we get $R_t = 1$ independent of the realization of A_t which means $R(A_t) = 1$.

h. (15 points)

The Ramsey problem gives us

$$A_t \mu_t = (1 + \lambda) B \quad (86)$$

and from the equilibrium conditions for both models we have

$$\frac{u_3}{u_2} = (1 - \tau_t^h) A_t \quad (87)$$

or

$$\frac{B}{c_{1t}} = (1 - \tau_t^h) A_t \quad (88)$$

Another Ramsey condition is

$$(1 + 2\lambda) c_{1t} = \mu_t \quad (89)$$

Combining this and the first one above, we get

$$c_{1t} = \frac{\mu_t}{1 + 2\lambda} \quad (90)$$

$$= \frac{(1 + \lambda) B}{A_t (1 + 2\lambda)} \quad (91)$$

and using this on the equilibrium condition

$$(1 - \tau_t^h) = \frac{B}{A_t c_{1t}} \quad (92)$$

$$= \frac{1 + 2\lambda}{1 + \lambda} \quad (93)$$

which shows that the labor income tax rate is constant.